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A natural graded Lie algebra sheaf over Riemann surfaces

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ABSTRACT. — I show that simultaneous uniformization of a family of Riemann surfaces corresponds to a sheaf of graded Lie algebras defined on the family. This sheaf is equivalent to the family of super Riemann surfaces defined in [4]. Since the Teichmüller deformations of these two constructions are the same, the present approach to super Riemann surfaces seems to be economical enough to obtain an explicit modular equivariant construction, and hence the global space of moduli of super Riemann surfaces.

RÉSUMÉ. — Nous montrons que à l'uniformisation simultanée d'une famille de surfaces de Riemann correspond un faisceau d'algèbres de Lie gradué défini sur la famille. Ce faisceau est équivalent à la famille de super surfaces de Riemann définie dans [4]. Puisque les déformations de Teichmüller de ces deux constructions sont identiques, cette approche des super surfaces de Riemann semble suffisamment efficace pour obtenir une construction modulaire équivariante, et donc l'espace global des modules des super surfaces de Riemann.

Classification A.M.S. : 30 F 10, 32 G 15, 17 B 70, 58 A 50.

1. INTRODUCTION

A fundamental construction in Riemann surface theory is the uniformization of a Riemann surface M, that is the possibility to represent M as the quotient of a domain in $\mathbb{C}P^1$, factored by the group of covering transformations.

Namely, let (U_α, z_α) be an atlas of M belonging to its complex structure, and $\{\varphi_\alpha\}$ an element of the space of projective connections of M, an affine space over the space of holomorphic quadratic differentials of M [1]. Then the local solutions f_α of the Schwarz problem

$$\Sigma(f_\alpha, z_\alpha) = \left(\frac{f''_\alpha}{f'_\alpha}\right)' - \frac{1}{2}\left(\frac{f''_\alpha}{f'_\alpha}\right)^2 = \varphi_\alpha \tag{1.1}$$

patch together on a universal covering space $\tilde{M}$ of M, defining a global map (the monodromy map)

$$f: \quad \tilde{M} \to \mathbb{C}P^1 \tag{1.2}$$

which is a local homeomorphism.

The monodromy group Γ of the map (1.2) is a subgroup of PL(2, $\mathbb{C}$) and M can be identified with the quotient $f(\tilde{M})/\Gamma$ (if $f(\tilde{M})$ is simply connected and different from $\mathbb{C}P^1$ [1]).

In particular, by an old result due to Klein and Poincaré, it is possible to find a projective connection φ_0 such that $M \cong U/\Gamma$ where U is the unit disc in $\mathbb{C}P^1$ and Γ a subgroup of PL(2, $\mathbb{R}$). If we allow the complex structure of M to vary, we can regard M as being the central fiber of a family of Riemann surfaces parametrized by points s in $\mathbb{C}^{3g-3}$ [2].

Then it is reasonable to state the problem of how to uniformize simultaneously the Riemann surfaces of the family.

Of course the problem can be solved by uniformizing each Riemann surface, fiber by fiber, setting $M = U/\Gamma$, as before.

But this uniformization process will not be holomorphic with respect to the parameter s.

Neverthless, a simultaneous uniformization holomorphic in s can be accomplished [3], a famous result in Riemann surface theory. In this paper I show that the uniformization process of either a single or a family of Riemann surfaces can be realized by a sheaf of $\mathbb{Z}_2$ graded Lie algebras defined over a single and a family of Riemann surfaces respectively.

Moreover, by such sheaves, I can define in a natural way a structure equivalent to that of the so-called super Riemann Surfaces ([4]-[7]). These objects have been recently introduced as the geometrical framework for the study of superstring theory. One advantage of the approach proposed here is its deep connection with the classical theory.

This should make it possible to solve certain open problems on super Riemann surfaces, such as the characterization of their moduli space.

This is an important matter for physical applications, since this moduli space is the domain of definition for the superstring partition function.

2. SHEAVES OF GRADED LIE ALGEBRAS OVER A SINGLE RIEMANN SURFACE

Projective uniformization of a Riemann surface M of genus $g \geqq 2$ is a process by which M is mapped onto the sphere $\mathbb{C}P^1$ in a locally homeomorphic way.

This process is conveniently described by a choice of spin structure L (that is one of the 2^{2g} consistent ways to choose a square root of the canonical bundle K) and a projective connection over M [1]. Projective connections over M can be defined as holomorphic connections of a rank two vector bundle E which is the jet bundle $E = J^1(L^{-1})$, that is, E is an extension of L by its inverse [1]

$$0 \to L \to E \to L^{-1} \to 0$$

The holonomy group of any such holomorphic connection on E gives rise to a representation of the homotopy group of the surface $\pi_1(M)$ onto $SL(2, \mathbb{C})$, that is we have a map:

$$p: \begin{matrix} \text{equivalence classes of} \\ \text{projective connections} \\ \text{over M} \end{matrix} \to \mathrm{Hom}(\pi_1(M), SL(2, \mathbb{C}))_{/SL(2, \mathbb{C})}$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad ||$$
$$\left\{ \begin{matrix} \text{holomorphic connections} \\ \text{over M} \end{matrix} \right\}_{/\mathrm{Aut}\, E} \qquad H^1(M, SL(2, \mathbb{C})) \tag{2.1}$$

The image of the map (2.1) is a cocycle

$$p(\varphi) = T_{\alpha\beta} \tag{2.2}$$

The cocycle (2.2) represents the monodromy for the solutions of the Hill's equation on M [8]

$$D_\varphi^2 u_\alpha = u_\alpha'' + \frac{1}{2}\varphi_\alpha u_\alpha = 0. \tag{2.3}$$

In fact, if $(G_{1/2}^\alpha, G_{-1/2}^\alpha), (G_{1/2}^\beta, G_{-1/2}^\beta)$ are pairs of linearly independent solutions of the equation (2.3) in $z_\alpha(U_\alpha)$, $z_\beta(U_\beta)$ respectively, we have:

$$\begin{pmatrix} G_{1/2}^\alpha \\ G_{-1/2}^\alpha \end{pmatrix} = T_{\alpha\beta}\, \xi_{\alpha\beta} \begin{pmatrix} G_{1/2}^\beta \\ G_{-1/2}^\beta \end{pmatrix} \tag{2.4}$$

An interesting result in [8] is that $\xi_{\alpha\beta}$ are the bundle transition functions of L^{-1}, which means that $(G_{1/2}, G_{-1/2})$ are germs of holomorphic spinors.

In φ-projective coordinates on M, $w_\alpha = f_\alpha \circ z_\alpha$ with f_α solution of (1.1), we have that (2.3) becomes $u''_\alpha = 0$.

Therefore the space of solutions of (2.3) is the subsheaf $\mathscr{A}_1(\varphi)$ of $\mathcal{O}(L^{-1})$, whose elements are polynomial of degree 1 in φ-coordinates. In other words $\mathscr{A}_1(\varphi)$ is defined via the exact sequence of sheaves over M:

$$0 \to \mathscr{A}_1(\varphi) \overset{i}{\hookrightarrow} \mathcal{O}(L^{-1}) \overset{D^2_\varphi}{\to} \mathcal{O}(K^{3/2}) \to 0 \tag{2.5}$$

A basis on each stalk of $\mathscr{A}_1(\varphi)$ is given by

$$(G_{1/2}, G_{-1/2}) = (z\, dz^{-1/2}, dz^{-1/2}) \tag{2.6}$$

Linearly independent solutions u_1^α, u_2^α of the Hill's equation, projected onto $\mathbb{C}P^1$ by $(u_1^\alpha, u_2^\alpha) \mapsto f_\alpha = u_1^\alpha / u_2^\alpha$, generate solutions of the Schwarz equation (1.1).

The infinitesimal version of (1.1) is

$$D^3_\varphi f^*_\alpha = (f^*_\alpha)''' + 2\varphi_\alpha f^{*\prime}_\alpha + f^*_\alpha \varphi'_\alpha = 0 \tag{2.7}$$

The sheaf $\mathscr{A}_0(\varphi)$ of solutions of this equation can be defined by the exact sequence [1]

$$0 \to \mathscr{A}_0(\varphi) \overset{i}{\hookrightarrow} \mathcal{O}(k^{-1}) \overset{D^3_\varphi}{\to} \mathcal{O}(k^2) \to 0 \tag{2.8}$$

We can define on each stalk of the sheaf $\mathscr{A}_0(\varphi)$ a basis

$$(L_{-1}, L_0, L_1) = \left(\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right). \tag{2.9}$$

Now it is easy to prove the following

PROPOSITION 1.1. – *The sheaf* $\mathscr{A}(\varphi) = \mathscr{A}_1(\varphi) \oplus \mathscr{A}_0(\varphi)$ *is a graded Lie algebra sheaf, all the fibers being isomorphic to the graded Lie algebra* osp (2.1).

Proof. – We define the $\mathbb{Z}_2$-graded algebra structure by the following products:

$[.,.]: \mathscr{A}_0 \times \mathscr{A}_0 \to \mathscr{A}_0$ is the commutator,

$\{.,.\}: \mathscr{A}_1 \times \mathscr{A}_1 \to \mathscr{A}_0$ is twice the tensor product,

$[.,.]: \mathscr{A}_0 \times \mathscr{A}_1 \to \mathscr{A}_1$ is the Lie derivative of vectors on fields,

that is the last product is defined by

$$\left[f\frac{d}{dz}, \xi\, dz^{-1/2}\right] = \left(f\frac{d\xi}{dz} - \frac{1}{2}\frac{df}{dz}\xi\right) dz^{-1/2}.$$

The above is a graded Lie algebra, that is the graded Jacobi identity are satisfied, because it verifies the following condition which characterizes graded Lie algebra [9]:

Let A be a $\mathbb{Z}_2$-graded algebra $A=A_0+A_1$, with A_0 Lie algebra. If $[A_1, \{A_1, A_1\}]=0$, then A is a graded Lie algebra.

Namely $\mathscr{A}_0(\varphi)$ has stalks isomorphic to the Lie algebra $sl(2, \mathbb{C})$, and $\mathscr{A}(\varphi)$ is a sheaf of graded Lie algebra isomorphic to $osp(2, 1)$.

I refer to $\mathscr{A}(\varphi)$ as the graded Lie algebra sheaf over M related to the projective uniformization φ of M.

Two remarks are in order:

(i) $\mathscr{A}(\varphi)$ is fully generated by $\mathscr{A}_1(\varphi)$; this reflects the classical fact that the space of solutions of (2.3) generates all the space of solutions of (2.7) by tensor products.

(ii) if φ_0 is the projective connection which identifies the universal covering of M with the unit disc in $\mathbb{C}P^1$, (2.2) gives $p(\varphi_0)\in H^1(M, SL(2, \mathbb{R}))$.

In this case $\mathscr{A}(\varphi_0)$ is a sheaf of real graded Lie algebras.

3. THE GRADED LIE ALGEBRA SHEAF OVER A FAMILY

Let us now consider deformations the complex structure of M. A Riemann surface M can be regarded as union of open sets of the complex plane $V_\alpha=z_\alpha(U_\alpha)$, modulo the identifications in V_α, V_β: $z_\alpha=f_{\alpha\beta}(z_\beta)$, where $f_{\alpha\beta}=z_\alpha\circ z_\beta^{-1}$.

Then deformation of the complex structure of M will mean a variation of the identifications $f_{\alpha\beta}$.

Let us parametrize different $f_{\alpha\beta}$'s by points s in a parameter space S, so that, for any $s\in S$, $f_{\alpha\beta}(z_\beta, s)$ are holomorphic in the first argument and $f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, s), s)=f_{\alpha\gamma}(z_\gamma, s)$.

Moreover, let $f_{\alpha\beta}(z_\beta, 0)$ be $f_{\alpha\beta}$, the transition function of M. Since for each s, $f_{\alpha\beta}(z_\beta, s)$ define a Riemann surface M_s, a family of Riemann surfaces is defined, having M as central fiber: namely the family is the fiber space over S: $V=\bigcup_{s\in S} M_s\to S$ and π is the obvious projection.

There is an important characterization of the parameter space S, due to Bers [3], which makes the family holomorphic (that is V, S are complex manifolds and π an holomorphic map) and universal (that is any other family $X\to S'$ can be realized as pull-back of the given one via a unique map $\psi: S'\to S$). The Bers construction is here briefly recalled, referring to [10] for all the details.

Define the Teichmüller space as

$$T_g=\{f_\mu:\mathbb{C}P^1\to\mathbb{C}P^1 \text{ which are } \mu \text{ quasi conformal}\}/\sim \qquad (3.1)$$

where:

1. the μ's are measurable functions on the upper half plane $\mathbb{H}$ with L^∞ norm less than 1 and verifying

$$\mu(\gamma z)=\mu(z)\frac{\overline{\gamma'(z)}}{\gamma(z)} \qquad \text{for} \quad \gamma \text{ in } \Gamma.$$

Moreover, such μ's are extended to $\mathbb{C}P^1$ by setting $\mu=0$ on the lower half plane $\mathbb{L}$.

2. for any μ, f_μ is the unique solution of the Beltrami equation $\partial f_\mu=\mu\,\bar\partial f_\mu$ which fixes the points $0, 1, \infty$.

3. the equivalence relation in definition (3.1) is:

$$f_\mu \sim f_\nu \qquad \text{iff} \quad f_\mu=f_\nu \quad \text{on } \mathbb{L}.$$

There exists an injective map from the Teichmüller space to the complex normed space of quadratic differentials on M:

$$B: \quad T_g \to H^0(M, K^2) \quad \text{(Bers embedding)}$$

B is defined by the Schwarzian derivative:

$$B([f_\mu])=\sum(f_\mu, z), \qquad \text{with} \quad z\in\mathbb{L}.$$

One can take $S=B(T_g)$.

To construct the family V_g, note that the domain $D_\tau=f_\mu(\mathbb{H})$ depends only on the equivalence class $\tau=[f_\mu]$, and its boundary $\mathcal{G}(\tau, x)=f_\mu(x)$, $x\in\mathbb{R}$, is a holomorphic function of $\tau\in T_g$.

Moreover, the uniformization group of $M=U/\Gamma$ is transformed into the "quasi-fuchsian group" $\Gamma_\tau=f_\mu \Gamma f_\mu^{-1}$.

Now, $F_g=\{(z, \tau)\in D_\tau\times T_g\}$ is a fiber space over T_g, and the Γ action on $F_g: \Gamma\times F_g \to F_g$

$$(\gamma, z, \tau)\mapsto(\gamma_\tau z, \tau)$$

(where $\gamma_\tau=f_\mu \gamma f_\mu^{-1}$) produces a complex fiber space $V_g \to T_g$ whose holomorphically varying fibers are the Riemann surfaces D_τ/Γ_τ. Stressing the dependence on $s=(s_1, \ldots, s_{3g-3})$, we have that the family of Riemann surfaces $V_g=\bigcup_{s\in S} D(s)/\Gamma(s)\to S$ is holomorphic and universal [10].

Note that, for each s, $D(s)$ is a universal covering space of M_s and we have the covering map $\pi_s: D(s)\to M_s$.

The inverse map is a linearly polymorphic function satisfying the Schwarz equation on $D(s): \sum(\pi_s^{-1}, z)=\varphi_s$ for some quadratic differential of M_s, and these vary holomorphically with s. In other words the Bers construction defines a universal family of Riemann surfaces together with a universal uniformizing connection.

To define the graded Lie algebras sheaf associated to such simultaneous uniformization, a square root of the dual of the canonical bundle over the universal Teichmüller family $V_g\to S$ has to be defined. Recall the

DEFINITION 3.1. – The canonical bundle of the universal Teichmüller curve is a holomorphic family of line bundles $K(V_g) \to V_g \to S$. The fiber over $s \in S$ is the canonical bundle of M_S [11].

Reconizing that the transition functions of the complex manifold V are $\gamma(z, s=[f_\mu]) = (\gamma_s z = f_\mu \circ \gamma \circ f_\mu^{-1} z, s)$, the inverse K_g^{-1} of the canonical bundle is defined by the multiplier [15]

$$\frac{\partial \gamma_s}{\partial z}(z). \tag{3.1}$$

Denote by L_g^{-1} a square root of K_g^{-1}.

There are 2^{2g} choices of such square root: in the following T_g will denote the trivial 2^{2g}-fold cover of the Teichmüller space, each sheet corresponding to a choice of L_g^{-1} on the universal curve. Note that for a fixed s, bundle transition functions of L_g^{-1} give a square root of the tangent bundle of M_s, L_s^{-1}.

Two more bundles on S are now introduced:

(*a*) The bundle of projective connections, $\mathscr{P}_g \xrightarrow{\tilde{\pi}} V_g \xrightarrow{\pi} S$, having as fiber over s the space of projective connections on M_s. The Bers construction determines a holomorphic section of $\mathscr{P}_g$, $s \mapsto \varphi_s$.

(*b*) The bundle $p: \Psi_g \to V_g \to S$, having fiber over s the space of 3/2-differential forms over M.

Ψ_g is also defined as the 0-th direct image functor $R^0_*(K_g^{3/2})$, a sheaf over S induced by the presheaf $\mathscr{U} \mapsto H^0(\pi^{-1}(U), K_g^{3/2})$ [14], and $\mathscr{P}_g$ is an affine bundle over $R^0_*(K_g^2)$.

Then I define, as in section 2, the sheaves of graded Lie algebras

$$\mathscr{A}(s) = \mathscr{A}(\varphi_s, L_s^{-1}).$$

Since all is varying holomorphically with s, it is clear that $\mathscr{A} = \bigcup_{s \in S} \mathscr{A}(s)$ defines an analytic sheaf of graded Lie algebras over the universal Teichmüller curve $V_g \to S_g$. $\mathscr{A}$ is the graded Lie algebras sheaf related to simultaneous uniformization of Riemann surfaces.

I have shown that the well known uniformization theory of Riemann surfaces is in deep connection with graded Lie algebras. Then it is possible to realize the classical uniformization process for Riemann surfaces as encoded in a bigger process, which could contribute new problems and ideas on the classical theory; on the other hand new constructions like super Riemann surfaces ([4]-[7]) can then be understood in classical terms, hence they can enjoy the wealth of precise results available in the classical theory.

This may help, for instance, in understanding the global structure of the moduli space of such new objects.

Motivated by that, I shall consider a particular type of deformation of the sheaf $\mathscr{A}$ over V_g.

Let us recall a definition [12]:

DEFINITION 3.2. — Given a sheaf F over X, a local deformation of F is a sheaf G over $X \times T$ such that $G(t_0) \cong F$, as sheaves over X, where t_0 is some preferred point of the parameter space T.

Universality for the deformation of Definition 3.2 is ensured by the (Kodaira-Spencer) condition that the tangent space of T is isomorphic to the space of infinitesimal deformations of F [12]. Let us remark that there are two possible kinds of deformation for the sheaf $\mathscr{A}$ defined on the family $V_g \to S$: one is the deformation of $\mathscr{A}$ leaving the fibers of V_g fixed; this corresponds to deformation of $\mathscr{A}$ over V_g in the sense of Definition 3.2. The other one is the more general deformation of the sheaf $\mathscr{A}(s)$ over the fiber M_s, in which both the sheaf and its base surface are allowed to vary (joint deformation).

Let us consider now this latter case, since it includes all the possible infinitesimal deformations of the sheaf $\mathscr{A} \to V_g \to S$. The joint infinitesimal deformations of the sheaf $\mathscr{A}(s) \to M_s$ are given by the Eichler cohomology $H^1(M, \mathscr{A}_1(s)) + H^1(M, \mathscr{A}_0(s))$ [13] defined by applying the cohomology sequence to the exact sequences of sheaves (2.5), (2.8).

$$O \to H^0(M_s, K^{1+(j/2)}) \xrightarrow{\delta^*} H^1(M_s, \mathscr{A}_{j \bmod 2}(s)) \xrightarrow{i} H^1(M_s, K^{-j/2}) \to 0 \quad (3.2)$$

$j=1, 2.$

Let us write a cocycle $\psi_{\alpha\beta}$ in $H^1(M_s, \mathscr{A}_{j \bmod 2}(s))$ as

$$\psi_{\alpha\beta} = \eta_\beta - \frac{1}{[f'_{\alpha\beta}(s)]^{j/2}} \eta_\alpha$$

then $\psi_{\alpha\beta}$ represents the following four possible infinitesimal deformations:

$j=2.$

(*a*) if η is holomorphic, then $\psi_{\alpha\beta}$ represents an infinitesimal displacement from φ_s along the fiber of the bundle $\mathscr{P}_g$ of projective connections defined on S.

In other words $\psi_{\alpha\beta}$ is an Eichler cocycle defined by a quadratic differential $\varphi = (\varphi_\alpha)$ via the equation $D^3_{\varphi_s} \eta_\alpha = \varphi_\alpha$ [*i.e.* $\psi_{\alpha\beta} = \delta^*(\varphi)$ in the sequence 3.2];

(*b*) if η is meromorphic, then $\psi_{\alpha\beta}$ represents an infinitesimal deformation of the complex structure of M_s: these are parametrized by elements of $H^1(M_s, K_s^{-1})$, and we can associate to $\psi_{\alpha\beta}$, the cocycle $i^*(\psi_{\alpha\beta}) \in H^1(M_s, K_s^{-1})$.

Deformations (*a*), (*b*), correspond to infinitesimal deformations of $\mathscr{A}_0$.

$j=1.$

(*c*) if η is holomorphic, then $\psi_{\alpha\beta}$ represents an infinitesimal displacement along the fiber $p^{-1}(s)$, of the bundle Ψ_g over S. Hence $\psi_{\alpha\beta}$ is defined by some 3/2-differential

$$\psi = (\psi_\alpha) \quad \text{via the equation } D^2_{\varphi_s} u_\alpha = \psi_\alpha$$

[*i. e.* $\delta^*(\psi) = \psi_{\alpha\beta}$ in (3.2)].

(*d*) if η is meromorphic, then $\psi_{\alpha\beta}$ represents a non trivial affine bundle over L^{-1}. Hence we can associate to it a generalized Beltrami differential of conformal weight $(-1/2, 1)$, which is $i^*(\psi_{\alpha\beta})$ in (3.2).

Deformations (*c*), (*d*) correspond to infinitesimal deformations of $\mathcal{A}_1$.

I shall consider deformations of the sheaf $\mathcal{A}$, leaving the complex structure of M and the bundle L^{-1} fixed (body fixed deformations in the terminology of [6]).

Therefore only type (*c*) deformations are allowed, representating the infinitesimal deformations of a certain structure.

The structure in question is what I call graded Riemann surface structure, for a reason which will become apparent in the following section.

Note that the joint deformations of the sheaf $\mathcal{A}$ are to some extent reseamblant of the effect of the action of the Lie algebra $\mathfrak{g} = \mathrm{Vect}\, S^1$ of vector fields on the circle, over its dual.

Briefly, from the adjoint action of $G = \mathrm{Diff}\, S^1$ on its Lie Algebra $\mathfrak{g} = \mathrm{Vect}\, S^1$, one derives the adjoint action of G on the dual of $\mathfrak{g}$, that is the space of quadratic differentials over S^1

$$\left.\begin{array}{c} G \times \mathfrak{g}^* \to \mathfrak{g}^* \\ (f, q) \mapsto q(f)(f')^2 + \sum(f, z),\ z \in S^1 \end{array}\right\} \qquad (3.3)$$

and the coadjoint action of $\mathfrak{g}$ over $\mathfrak{g}^*$ which is

$$\left.\begin{array}{c} \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \\ (f, q) \mapsto q + D^3_q f^* \end{array}\right\} \qquad (3.4)$$

The isotropy group of a point $q \in \mathfrak{g}^*$, $G_q = \{ g \in G \text{ s. t. } g^*(q) = q \}$ hat its Lie algebra which is equal to the stabilizer of q with respect to the coadjoint action.

The classification of the orbits of $\mathfrak{g}^*$ under the action of G: $X_q = \{ q' \in \mathfrak{g}^* \text{ s. t. } q' = g . q,\ g \in G \}$ and the Lie algebras $\mathfrak{g}^*_q$ provide informations about the representation of the group G (*see* e. g. [17]).

Formula (3.3) can be regarded as the action of local biholomorphisms on a fixed Riemann surface M, and (3.4) as the action over $\tilde{\pi}^{-1}(0)$ of the holomorphic sections of the tangent bundle of M, K^{-1}.

The stabilizer of $\varphi \in \tilde{\pi}^{-1}(0)$ with respect to the action (3.4) is just the sheaf $\mathcal{A}_0(\varphi)$.

More generally, let us define the graded Lie algebras sheaf $A = A_0 + A_1 = \mathcal{O}(K^{-1}) + \mathcal{O}(L^{-1})$, using Lie brackets, tensor products, and Lie derivatives as in Proposition 1.1

Consider the bundle $p: \Psi_g \to V_g \to T_g$ and its fiber

$$p^{-1}(0) = H^0(M, K^{3/2}).$$

Then A acts on $\tilde{\pi}^{-1}(0) + p^{-1}(0)$ as follows:

$$\begin{gathered} A \times [\tilde{\pi}^{-1}(0) + p^{-1}(0)] \to C^\infty [\tilde{\pi}^{-1}(0) + p^{-1}(0)] \\ ((f^*, u), (\varphi, \psi)) \mapsto (\varphi + D_\varphi^3 f^*, \psi + D_\varphi^2 u) \end{gathered} \tag{3.5}$$

The stabilizer of (φ, ψ) in A with respect to the action (3.5) is just the sheaf of osp(2,1) graded Lie algebras $\mathscr{A}(\varphi)$.

4. SUPER RIEMANN SURFACES AND THEIR MODULI

In this section I define graded Riemann surfaces structures using the classical constructions of the previous sections, then the present definition will be shown to be equivalent to another one given in the graded manifold context [4].

DEFINITION 4.1. — A graded Riemann surface structure is a holomorphic family of Riemann surfaces $X \to S$ together with an analytic sheaf $\mathscr{A}$ of osp(2,1) graded Lie algebras over it.

DEFINITION 4.2. — Two graded Riemann surface structures $\mathscr{A}$, $\mathscr{A}'$ are equivalent if the sheaves $\mathscr{A}(s)$, $\mathscr{A}'(s)$ are stabilizers of the same point with respect to the action (3.5).

Analyticity of the sheaves $\mathscr{A}$ over S implies that the even part of the graded Lie algebras sheaf $\mathscr{A}_0(s)$ has to be the stabilizer of the projective connection φ_s given by the Bers construction (or the pull-back of it if the family $X \to S$ is not the Teichmüller one).

It follows that $\mathscr{A}$ and $\mathscr{A}'$ are equivalent if and only if their odd sectors are stabilizers of the same point in $\Psi_g \ni p^{-1}(s)$. We have then:

PROPOSITION 4.1. — *Inequivalent graded Riemann surface structures are parametrized by* Ψ_g.

To make contact with the graded manifold definition of super Riemann surfaces of reference [4], note that each sheaf $\mathscr{A}(s)$ can be regarded as (sub) sheaf of derivations of a graded manifold. Roughly, a graded manifold is a manifold with a sheaf of $\mathbb{Z}_2$-graded algebras over it, which is locally the exterior algebra over some vector bundle (for an overview on graded manifold theory *see* [16]).

The definition of super Riemann surface of reference [4] follows:

DEFINITION 4.3. — A super Riemann surface over a space $\tilde{S}$, $\tilde{X} \to \tilde{S}$ is a family of graded manifolds of relative dimension (1, 1) (that is the fibers are (1, 1) graded manifolds), together with a subsheaf $\mathscr{D}$ of the sheaf of relative derivations of the family. $\mathscr{D}$ is a locally free subsheaf of rank (0/1) absolutely non integrable (that is, if $\mathscr{D}$ is generated by D, then D, and $\{D, D\} = 2\, D \otimes D = 2\, D^2$ is a local basis for the relative derivations).

The space $\tilde{S}$ can be a graded manifold or an ordinary manifold, in this case the super Riemann surface is defined over a "reduced" space.

PROPOSITION 4.2. – Super Riemann surfaces over reduced spaces and graded Riemann surface structures correspond.

Proof. – Consider a graded Riemann surface structure. This is a sheaf of osp (2, 1) algebras with a basis (L_{-1}, L_0, L_1, $G_{1/2}$, $G_{-1/2}$) which varies holomorphically on z and s. For each s, this basis can be mapped into the sheaf of derivations of the graded manifold (M_s, $\wedge L_s$) as follow:

$$i(L_{-1}) = \frac{d}{dz}$$

$$i(L_0) = z\frac{d}{dz} + \frac{1}{2}\theta\frac{\partial}{\partial\theta}$$

$$i(L_1) = z^2\frac{d}{dz} + z\theta\frac{\partial}{\partial\theta}$$

$$i(G_{1/2}) = z\left(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}\right)$$

$$i(G_{-1/2}) = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}.$$

It is easy to verify that the above map is a graded Lie algebras homomorphism: $i\langle X, Y\rangle = \langle i(X), i(Y)\rangle$.

Namely, the image of i is given by those derivations of (M_s, $\wedge L_s$) which preserve the sheaf $\mathscr{D}$ generated by $D = \partial/\partial\theta + \theta\,\partial/\partial z$ (with (z, θ) local coordinates on $\wedge L_s$) and satisfies the equation (written in φ_s-coordinates) $D\,\partial^2 X = 0$ [18].

In particular, $i(G(s))$ is the generator over $\mathscr{O}_{M_s}$ of a sheaf $\mathscr{D}_s$ which determines a super Riemann surface over S.

Conversely, consider a super Riemann surface over S.

By definition, the structure $\mathscr{D}$ is holomorphic on moduli, so that the (local) generator D of $\mathscr{D}$ varies holomorphically with s. Then we can define the analytic sheaf of solutions of the equation $D_s\,\partial_s^2 X = 0$, which is a sheaf $\mathscr{A}(s)$ of osp (2, 1) algebras holomorphic on s.

Therefore a graded Riemann surface structure is determined.

In [5] the universal space of parameters of (local) deformations of super Riemann surfaces is found using graded manifold techniques.

The result is similar to the one presented here: it is possible to get the parameter space of super Riemann surfaces assocating to the parameter space of graded Riemann surfaces structures its related graded manifold, that is $(T_g, \wedge \Psi_g)$.

The only difference is in the existence, for any graded manifold, of a canonical automorphism which makes the Teichmüller moduli space of super Riemann surfaces not a graded manifold but a canonical super-orbifold modelled on $(T_g, \wedge \Psi_g)$ [5].

Finally, two comments are in order:

(i) to exploit the actual moduli space of super Riemann surfaces one should consider the action of the appropriate modular group in both the present and the graded manifold approach. This is not an easy matter, but it should be easier to work out a modular equivariant construction on the lines proposed here and eventually to translate it in terms of graded manifold, using the equivalence between the two approaches.

(ii) a contact with the supermanifold version of super Riemann surfaces ([6], [7]) is possible as well. In that context the central object is again the operator D, and a super Riemann surface is a (1/1) supermanifold with transition functions defined by the condition that the operator D transforms homogeneously. These "super-conformal transformations" are interesting. Namely, note that the joint infinitesimal deformations of the $sl(2, \mathbb{C})$-Lie algebra sheaf $\mathscr{A}_0(s) \to M_s$ are parametrized by the $6g$-6 dimensional cohomology group $H^1(M_s, \mathscr{A}_0(s))$ of deformations of all the Riemann surfaces together with all projective structures [1].

It is probable that the tangent space of superconformal deformations sits inside $H^1(M_s, \mathscr{A}_0(s))$ in a way dictated by the superconformal transformations of reference [6].

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