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Post-Newtonian generation of gravitational waves. II. The spin moments

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ABSTRACT. – This paper extends the multipolar gravitational wave generation formalism of Blanchet and Damour [6] by deriving post-Newtonian-accurate expressions for the spin multipole moments. Both the algorithmic and the radiative spin multipole moments are expressed as well-defined compact-support integrals involving only the components of the stress-energy tensor of the material source. This result is obtained by combining three tools: (i) a multipolar post-Minkowskian algorithm for the external field, (ii) the results of a direct multipole analysis of linearized gravitational fields by means of irreducible cartesian tensors, and (iii) a study of various kernels appropriate for solving the quadratic nonlinearities of Einstein's field equations.

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RÉSUMÉ. — Cet article étend le formalisme multipolaire pour la génération d'ondes gravitationnelles de Blanchet et Damour [6] en dérivant des expressions pour les moments multipolaires de spin à une précision post-Newtonienne. Les moments de spin algorithmiques, ainsi que radiatifs, sont exprimés comme des intégrales à support compact ne contenant que les composantes du tenseur d'énergie-impulsion de la matière. Ce résultat est obtenu en combinant trois outils : (i) un algorithme multipolaire post-Minkowskien pour le champ externe, (ii) les résultats d'une analyse multipolaire directe des champs gravitationnels linéarisés en termes de tenseurs cartésiens irréductibles, et (iii) une étude de divers noyaux utiles pour résoudre les nonlinéarités quadratiques des équations d'Einstein.

I. INTRODUCTION AND OUTLINE OF THE METHOD

This paper deals with the problem of the generation of gravitational waves, *i.e.* the problem of relating the outgoing gravitational wave field to the structure and motion of the material source. The present investigation will be concerned with the generation of gravitational radiation by *semi-relativistic* sources. By this we mean isolated material systems, S, such that the dimensionless parameter

$$\varepsilon := \sup\left[\left(\frac{Gm}{c^2 r_0}\right)^{1/2}, \left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \frac{r_0}{\bar{\lambda}}\right], \tag{1.1}$$

is smaller, but possibly not much smaller, than one (say $\varepsilon \leqq 0.2$). In eq. (1.1) m denotes a characteristic mass and r_0 a characteristic size of the system S (we shall assume that r_0 is strictly greater than the radius of a sphere in which S can be completely enclosed), $T^{\mu\nu}$ denote the components of the stress-energy tensor in some coordinate system $x^\mu = (ct, x^i)$, regularly covering S (with the vertical bars denoting a suitable norm for tensors, and $\bar{\lambda} = \lambda/2\pi$ denotes a characteristic reduced wavelength of the gravitational radiation emitted by the system. If we choose physical units of mass, length and time adapted to the internal dynamics of the material system S, the parameter ε becomes proportional to the value, in these units, of the inverse of the velocity of light, c^{-1}. Hence, we can take c^{-1} as ordering parameter of the expansion in powers of ε.

When the parameter ε is much smaller than one ("non relativistic" sources), *i.e.* when considering material systems which are weakly self-gravitating, slowly moving, weakly stressed and located well within their gravitational near-zone, one can estimate the gravitational radiation emitted by the system S by means of the standard Einstein-Landau-Lifshitz "far-field quadrupole equation" ([1], [2]), which expresses the asymptotic gravitational radiation amplitude in terms of the second time derivative of the usual "Newtonian" trace-free quadrupole moment of the mass-energy distribution of the source. The original derivation of Einstein [1] was, in fact, restricted to the case of *negligibly* self-gravitating sources, *i. e.* systems whose motion is governed entirely by non-gravitational forces, and it was only the later work of Landau and Lifshitz [2] which showed that the "quadrupole equation" gave also the dominant gravitational wave emission from (*weakly*) self-gravitating sources. A few years later, Fock [3] introduced a different approach to the problem of the generation of gravitational waves, and obtained, with its help, a new derivation of the quadrupole equation.

In recent years, the *gravitational wave generation problem* has become of increasing interest, especially in view of the development of a world-wide network of gravitational wave detectors. This motivated several attempts at generalizing the quadrupole equation, valid only for non-relativistic sources, to the more general case of semi-relativistic sources. The first attempt at going beyond the lowest-order "Newtonian" quadrupole formalism was made by Epstein and Wagoner [4] ("post-Newtonian" generation formalism). Then, Thorne [5] clarified and extended this approach by introducing a systematic multipole decomposition of the gravitational wave amplitude, and by (formally) including higher-order post-Newtonian contributions. However, the Epstein-Wagoner-Thorne formalism is unsatisfactory because it involves, both through the intermediate steps, and in the final results, many undefined (divergent) integrals. The origin of these ill-defined integrals can be traced back to the fact that their formalism is a direct generalization of the Landau-Lifshitz approach in making an essential use of an effective stress-energy for the gravitational field which is not localized on the compact support of the material source, but extends, with a rather slow fall-off, up to infinity.

Recently, Blanchet and Damour [6] (hereafter referred to as paper I) have introduced a new post-Newtonian gravitational wave generation formalism which, instead of being a generalization of the Landau-Lifshitz derivation of the ("Newtonian") quadrupole formalism, implements ideas which can be traced back to the Fock derivation [3]. The main idea is to separate the problem into two sub-problems (one dealing with the gravitational field in the near zone of the source, and the other one dealing with the wave-zone field), and then to transfer information between these two sub-problems by "matching" the near-zone and the wave-zone fields. More precisely, the method of Ref. [6] (paper I), that we shall follow and extend here, can be decomposed into four steps:

Step 1. – Let $D_e = \{ (\mathbf{x}, t) \mid r > r_0 \}$ (*see* note [7] for our notation) denote the domain *external* to the material source S. In D_e we decompose the external gravitational field (taken in the densitized contravariant metric form) in a *multipolar post-Minkowskian* (MPM) expansion. This consists of combining a post-Minkowskian (or non-linearity) expansion

$$\mathcal{G}^{\alpha\beta}_{\text{ext}} := \sqrt{g_{\text{ext}}}\, g^{\alpha\beta}_{\text{ext}}$$
$$= f^{\alpha\beta} + G\, h_1^{\alpha\beta} + G^2 h_2^{\alpha\beta} + \ldots + G^n h_n^{\alpha\beta} + \ldots, \qquad (1.2\,a)$$

where $f^{\alpha\beta}$ denotes the flat (Minkowski) metric [7], with multipole expansions, associated with the SO(3) group of rotations of the spatial coordinates (which leaves invariant $r = |\mathbf{x}| = [\delta_{ij} x^i x^j]^{1/2}$), for $h_n^{\alpha\beta}(\mathbf{x}, t)$:

$$h_n^{\alpha\beta}(\mathbf{x}, t) = \sum_{l \geq 0} \hat{n}^{L}(\theta, \varphi)\, h_{nL}^{\alpha\beta}(r, t), \qquad (1.2\,b)$$

where $L := i_1 i_2 \ldots i_l$ denotes a (spatial) multi-index of order l, and $\hat{n}^L$ the symmetric trace-free (STF) part of $n^L := n^{i_1} n^{i_2} \ldots n^{i_l}$, with $n^i(\theta, \varphi) = x^i/r$ being the unit coordinate direction vector from the origin (located within the source) towards the external field point $\mathbf{x}$. The multipolar post-Minkowskian expansion (1.2) was recently put on a clear algorithmic basis in a series of publications by Blanchet and Damour ([8]-[12]) (thereby elucidating and perfecting earlier work by Bonnor and coworkers, and Thorne and coworkers). The essential outcome of this approach is to show that the most general (past-stationary and past-asymptotically-flat) MPM-expandable solution of the vacuum Einstein equations can be algorithmically constructed in terms of two sets of *algorithmic multipole moments*,

$$\mathcal{G}^{\alpha\beta}_{\text{ext}} = \mathcal{G}^{\alpha\beta}_{\text{can}}[\mathcal{M}], \qquad (1.3\,a)$$
$$\mathcal{M} = \{ M_L(t), S_L(t) \}, \qquad (1.3\,b)$$

where "can" refers to a convenient "canonical" coordinate system, and where $\{ M_L \} = \{ M, M_i, M_{i_1 i_2}, \ldots \}, \{ S_L \} = \{ S_i, S_{i_1 i_2}, \ldots \}$ denote, respectively, the "mass", and the "spin" algorithmic multipole moments. These moments are all symmetric and trace-free cartesian tensors which depend on one (time) variable (except M, M_i and S_i which are time-independent). It is important to keep in mind that the algorithmic moments have no (and need not have any) direct physical meaning [apart from M, which is the Arnowitt-Deser-Misner (ADM) rest-mass of the system]. They play the role of arbitrary functional parameters in the construction of the external metric, and will serve as go-betweens transferring information from the source to the radiation zone.

Step 2. – The general external MPM metric was shown [11] (under the assumption of past-stationarity) to admit a regular structure (in Penrose's conformal sense) in the asymptotic wave zone, which means in particular that there exist some algorithmically defined "radiative" coordinates, $X^\mu=(cT, X^i)$, with respect to which the metric coefficients, say $G^{ext}_{\alpha\beta}(X^\gamma)$, admit an asymptotic expansion in powers of R^{-1}, when $R=|\mathbf{X}|\to\infty$ with $T-R/c$ and $\mathbf{N}=\mathbf{X}/R$ being fixed ("future null infinity"). Following Thorne [5] one can then decompose the leading 1/R term in the radiative metric in a multipolar series, with expansion coefficients being written as derivatives of some *radiative multipole moments*, $\mathcal{M}^{rad}=\{M^{rad}_L, S^{rad}_L;\ l\geqq 2\}$:

$$(G^{ext}_{hk}-\delta_{hk})^{TT}=+\frac{4G}{c^2R}P_{hkij}(\mathbf{N})\sum_{l\geqq 2}\frac{1}{l!\,c^l}\left\{N_{L-2}\overset{(l)}{M}{}^{rad}_{ijL-2}(U)\right.$$
$$\left.-\frac{2l}{(l+1)c}N_{aL-2}\,\varepsilon_{ab(i}\overset{(l)}{S}{}^{rad}_{j)bL-2}(U)\right\}+O\left(\frac{1}{R^2}\right),\quad (1.4)$$

where

$$U:=T-R/c,\ \overset{(l)}{F}(U):=d^l F(U)/dU^l,\ T_{(ij)}:=(T_{ij}+T_{ji})/2,$$

and where

$$P_{hkij}(\mathbf{N})=(\delta_{hi}-N_hN_i)(\delta_{kj}-N_kN_j)-\frac{1}{2}(\delta_{hk}-N_hN_k)(\delta_{ij}-N_iN_j)$$

denotes the transverse-traceless (TT) projection operator onto the plane orthogonal to **N**.

Unlike the algorithmic moments, $\mathcal{M}=\{M_L, S_L\}$, the radiative moments, $\mathcal{M}^{rad}=\{M^{rad}_L, S^{rad}_L;\ l\geqq 2\}$ (or rather the l-th order derivatives of M^{rad}_L and S^{rad}_L) have a direct physical meaning in terms of quantities measurable, in principle, by an array of gravitational wave detectors around the source. However, the algorithmic construction of the external metric (in its original coordinates) and of the transformation to the asymptotically regular radiative coordinates yields, in principle, an algorithmic expression of $\mathcal{M}^{rad}$ in terms of $\mathcal{M}$:

$$\mathcal{M}^{rad}=\mathcal{M}^{rad}[\mathcal{M}].\quad (1.5)$$

The first two steps of the method give, in principle, a fairly complete picture of the nonlinear structure of the external gravitational field everywhere outside the source. However, this knowledge is not yet related to the actual source and must be complemented by a different, source-rooted, approach. The last two steps play precisely this role.

Step 3. – Let $D_i=\{(\mathbf{x}, t)\,|\,r<r_1\}$, where $r_0<r_1\ll\bar{\lambda}$, denote an *inner* domain which encloses the source S and defines, for our purpose, the near zone of S. In D_i we decompose the inner gravitational field in a *post-Newtonian* (PN) expansion, *i.e.* in a combined weak-field-slow-motion expansion in powers of $\varepsilon\sim c^{-1}$ (where "slow-motion" refers both to the smallness of the velocities in the source, $|T^{0i}|\leqq\varepsilon T^{00}$, and to a near-zone

expansion, $r_0 \leqq \varepsilon \bar{\lambda}$, or $\partial_0 g \sim c^{-1} \partial_i g$):

$$\mathscr{G}_{\text{in}}^{\alpha\beta} := \sqrt{g_{\text{in}}}\, g_{\text{in}}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{c} h_{(1)}^{\alpha\beta} + \frac{1}{c^2} h_{(2)}^{\alpha\beta} + \ldots + \frac{1}{c^n} h_{(n)}^{\alpha\beta} + \ldots \quad (1.6)$$

The post-Newtonian expansion scheme has been investigated by many authors, notably Fock, Chandrasekhar, Anderson and coworkers, Ehlers and coworkers (*see* references 29-40 in Ref. [12]), and the implementation of the first steps of the method leads to an explicit expression for the inner gravitational field in terms of the source variables, so that we can write for some p,

$$\mathscr{G}_{\text{in}}^{\alpha\beta} = \mathscr{F}^{\alpha\beta}[\text{source}] + O(c^{-p}). \quad (1.7)$$

Step 4. – Finally, we need to transfer information between the external (MPM) expansion scheme, eqs (1.2)-(1.3), and the inner (PN) one (1.6)-(1.7). This can be done by a variant of the method of matched asymptotic expansions [13], as first advocated in the gravitational radiation context by Burke [14] (*see* e. g. Ref. [15] for references to other works having made use of this technique in general relativity). The variant we shall use is the one put forward in Ref. [12] and consists of requiring the existence of a (post-Newtonian expanded) coordinate transformation such that the coordinate transform of the post-Newtonian expansion (1.6)-(1.7) coincides, in the external near-zone $D_i \cap D_e$, with the post-Newtonian re-expansion of the multipolar-post-Minkowskian expansion (1.2)-(1.3) (*see* Section VI of Ref. [12] for a more detailed definition of this matching requirement). The outcome of such a matching of the external and inner expansions is to provide the explicit expression of the algorithmic moments in terms of the source variables, symbolically

$$\mathscr{M} = \mathscr{M}[\text{source}] + O(c^{-q}), \quad (1.8)$$

where the order q depends on the order p within which the inner field is known in terms of the source, *see* eq. (1.7). Remembering now the outcome of *Step* 2, eq. (1.5), we see that we can finally eliminate the algorithmic moments and obtain the (physical) radiative moments in terms of the source variables.

We should warn the reader that we have been somewhat remiss in our presentation of *Step* 3. First, the expansion (1.6) contains not only powers of c^{-1}, but also $(lnc)^p c^{-q}$ terms. However, at the precision at which we shall work in the present paper these mixed power-logarithm terms will always stay buried in the error terms, like $O(c^{-p})$ in eq. (1.7) (*see* Ref. [12] for a careful treatment of these terms). And second, the procedure of expressing the post-Newtonian-expanded inner metric as a functional of the source variables necessitates, in principle, some transfer of information from the external metric. However, the combined work of Refs. [14], [16],

[9], [12] shows that the standard post-Newtonian schemes are justified at the precision at which we shall work.

Having outlined the four-step method (from paper I) that we shall use in the present paper, let us now indicate which steps have already been solved with sufficient precision, and which ones we shall improve upon. The two basic results of paper I were to obtain eq. (1.5) of *Step* 2 up to $O(c^{-3})$ relative error terms, and half of eq. (1.8) of *step* 4 up to $O(c^{-4})$. Namely, equations (3.32) of paper I show that one has simply,

$$M_L^{rad}(U) = M_L(U) + O(c^{-3}), \tag{1.9 a}$$

$$S_L^{rad}(U) = S_L(U) + O(c^{-3}), \tag{1.9 b}$$

while equation (3.25 *a*) of paper I shows that

$$M_L(U) = I_L[\text{source}] + O(c^{-4}), \tag{1.10 a}$$

where I_L[source] is a well-defined (compact-support) integral expression involving only the source variables [*see* eq. (2.27) of paper I, or eq. (5.12) below]. As for the other half of eq. (1.8), *i.e.* the expression of the algorithmic *spin* moments in terms of the source it was obtained in paper I (after Ref. [5]) only up to $O(c^{-2})$ error terms ("Newtonian accuracy"). The aim of the present paper is to solve the second half of eq. (1.8) with the same ("post-Newtonian") accuracy which was achieved in paper I for the first half, *i.e.* to find a well-defined (compact-support) integral expression [17] involving only the source variables, say J_L[source], such that the algorithmic spin moments can be simply written as

$$S_L(U) = J_L[\text{source}] + O(c^{-4}). \tag{1.10 b}$$

We shall leave an improvement of eq. (1.5), *i.e.* eqs (1.9 *a-b*), to future work.

Beyond the tools developed in Refs [8]-[12], the new tools that will allow us to obtain eq. (1.10 *b*) are: (i) the expressions for the linearized-gravity multipole moments recently derived by Damour and Iyer [18] (as emphasized below, the expressions derived by Thorne [5] are not adequate to our purpose), and (ii) the study, performed below, of various kernels appropriate for solving the quadratic nonlinearities of Einstein's field equations. The strategy for obtaining eq. (1.10 *b*), and thereby the plan of this paper, is the following: In Section II we shall solve *Step* 1 (*i.e.* determine $\mathscr{G}_{ext}[\mathscr{M}]$) with an improved accuracy with respect to paper I. In Section III we shall discuss the various kernels that will allow us to relate the exterior and the inner metrics. In section IV we shall solve *Step* 3 (inner metric) and *Step* 4 (matching) with an accuracy sufficient to get eq. (1.10 *b*). The end result of Section IV leads to a well-defined prescription for constructing the post-Newtonian source-spin-moments J_L. In Section V we make use of the recent reexamination of the radiative multipole moments in linearized gravity [18] to transform the prescription of Section IV into various explicit expressions for J_L. Section VI contains a brief summary and our concluding remarks. Finally, three appendices complete our paper: Appendix A discusses the quasi-conservation law of a compact-support distribution that plays the role of an effective source in our approach; Appendix B deals with the transformation laws, under a shift of the spatial origin, of our multipole moments; and Appendix C gives the (formal) point-particle limit of our results.

II. THE EXTERNAL GRAVITATIONAL FIELD

Using the results and notation of Refs [6], [9]–[12] the external "gothic" metric, $\mathcal{G}^{\alpha\beta}_{\text{ext}} := \sqrt{g_{\text{ext}}}\, g^{\alpha\beta}_{\text{ext}}$, as a functional of the algorithmic multipole moments $\mathcal{M} = \{ M_L, S_L \}$, reads

$$\mathcal{G}^{\alpha\beta}_{\text{ext}}[\mathcal{M}] = f^{\alpha\beta} + G\, h^{\alpha\beta}_1 + G^2\, h^{\alpha\beta}_2 + \ldots + G^n\, h^{\alpha\beta}_n + \ldots, \tag{2.1a}$$

with

$$h^{\alpha\beta}_n[\mathcal{M}] = p^{\alpha\beta}_n + q^{\alpha\beta}_n; \qquad n \geqq 2, \tag{2.1b}$$

where

$$p^{\alpha\beta}_n = \underset{B=0}{\text{Finite Part}} \{ \Box_R^{-1} [(r/\bar{\lambda})^B N^{\alpha\beta}_n (h_1, \ldots, h_{n-1})] \}, \tag{2.2}$$

and where $q^{\alpha\beta}_n$ is a particular solution of the homogeneous wave equation which is algorithmically defined from the multipole decomposition of

$$r^\alpha_n = \partial_\beta p^{\beta\alpha}_n = \underset{B=0}{\text{Residue}} \{ \Box_R^{-1} [(r/\bar{\lambda})^B r^{-1} n^i N^{\alpha i}_n] \}. \tag{2.3a}$$

Namely, defining the multipole moments, A_L, B_L, C_L and D_L (label n suppressed for simplicity) of r^α_n by

$$r^0_n = \sum_{l \geqq 0} \partial_L \left(\frac{A_L(u)}{r} \right), \tag{2.3b}$$

$$r^i_n = \sum_{l \geqq 0} \left\{ \partial_{iL} \left(\frac{B_L(u)}{r} \right) + \partial_L \left(\frac{C_{iL}(u)}{r} \right) + \varepsilon_{iab}\, \partial_{aL} \left(\frac{D_{bL}(u)}{r} \right) \right\}, \tag{2.3c}$$

one defines

$$q_n^{00} = -c \frac{\overset{(-1)}{\mathrm{A}}(u)}{r} - c\,\partial_a\left(\frac{\overset{(-1)}{\mathrm{A}}_a(u)}{r}\right) + c^2\,\partial_a\left(\frac{\overset{(-2)}{\mathrm{C}}_a(u)}{r}\right), \qquad (2.4a)$$

$$q_n^{0i} = -c \frac{\overset{(-1)}{\mathrm{C}}_i(u)}{r} - c\,\varepsilon_{iab}\,\partial_a\left(\frac{\overset{(-1)}{\mathrm{D}}_b(u)}{r}\right) - \sum_{l \geqq 1} \partial_{\mathrm{L}}\left(\frac{\mathrm{A}_{i\mathrm{L}}(u)}{r}\right), \qquad (2.4b)$$

$$\begin{aligned} q_n^{ij} = -\delta_{ij}\left[\frac{\mathrm{B}(u)}{r} + \partial_a\left(\frac{\mathrm{B}_a(u)}{r}\right)\right] \\ + \sum_{l \geqq 0}\Bigg\{\partial_{\mathrm{L}}\left[\frac{1}{r}\left(\frac{1}{c}\overset{(1)}{\mathrm{A}}_{ij\mathrm{L}}(u) + \frac{3}{c^2}\overset{(2)}{\mathrm{B}}_{ij\mathrm{L}}(u) - \mathrm{C}_{ij\mathrm{L}}(u)\right)\right] \\ + 2\,\delta_{ij}\,\partial_{\mathrm{L}+2}\left(\frac{\mathrm{B}_{\mathrm{L}+2}(u)}{r}\right) - 6\,\partial_{\mathrm{L}+1\,(i}\left[\frac{\mathrm{B}_{j)\,\mathrm{L}+1}(u)}{r}\right] \\ - 2\,\partial_{a\mathrm{L}}\left[\frac{\varepsilon_{ab\,(i}\mathrm{D}_{j)\,b\mathrm{L}}(\mathrm{u})}{\mathrm{r}}\right]\Bigg\}. \qquad (2.4c) \end{aligned}$$

See Ref. [10] or Ref. [12] for the definition of the various symbols appearing in eqs (2.2)-(2.4), and eqs (2.9)-(2.10) below for the expression of the "seed" term for the algorithm: $h_1^{\alpha\beta}[\mathscr{M}]$.

As the *Step* 2 of our method (*see* Introduction above), which involves controlling the transition between the near zone and the wave zone, has already been solved [by eqs (1.9)], it will be sufficient for our present purpose to control the near-zone expansion of the external metric. For any external field quantity Q, let us denote by $\bar{\mathrm{Q}}$ its near-zone (or post-Newtonian) expansion, *i.e.* its asymptotic expansion along the gauge functions $(lnc)^n c^{-p}$ when c tends to infinity keeping fixed the time, $t = x^0/c$, and space, x^i, external coordinates. For the sake of simplicity, let us denote for the scalar, vector and tensorial quantities respectively

$$\bar{\mathrm{Q}} = O(c^{-p}) \;\Leftrightarrow\; \bar{\mathrm{Q}} = O(p), \qquad (2.5a)$$

$$\bar{\mathrm{Q}}^0 = O(c^{-p}), \bar{\mathrm{Q}}^i = O(c^{-q}) \;\Leftrightarrow\; \bar{\mathrm{Q}}^\alpha = O(p, q), \qquad (2.5b)$$

$$\begin{aligned} \bar{\mathrm{Q}}^{00} = O(c^{-p}), \qquad \bar{\mathrm{Q}}^{0i} = O(c^{-q}), \\ \bar{\mathrm{Q}}^{ij} = O(c^{-r}) \;\Leftrightarrow\; \bar{\mathrm{Q}}^{\alpha\beta} = O(p, q, r). \end{aligned} \qquad (2.5c)$$

We have seen in paper I that, in order to express the algorithmic mass moments in terms of the source modulo $O(4)$, we needed to know the near-zone (or PN) expansion of the external covariant metric, $g_{\alpha\beta}^{\mathrm{ext}}$, modulo $\delta\bar{g}_{\alpha\beta}^{\mathrm{ext}} = O(6, 5, 4)$ (see eqs (3.22) of paper I). It can be seen in advance that, in order to express also the algorithmic spin moments modulo $O(4)$, we need to know $\bar{g}_{\alpha\beta}^{\mathrm{ext}}$ modulo $O(6, 7, 6)$, or equivalently the PN-expanded gothic metric $\bar{\mathscr{G}}_{\mathrm{ext}}^{\alpha\beta}$ modulo

$$\delta\bar{\mathscr{G}}_{\mathrm{ext}}^{\alpha\beta} = O(6, 7, 6) \qquad (2.6)$$

[*see* the paragraph before eqs (5.10)].

It has been remarked in paper I that a consequence of the eqs I (3.10) and I (3.15) [19] was that the PN expansion of the n^{th} PM approximation of the gothic metric was of order

$$\bar{h}_n^{\alpha\beta} = O(2n, 2n+1, 2n). \tag{2.7}$$

The comparison with eq. (2.6) shows that it is sufficient to control the linear and the quadratic approximations (second-post-Minkowskian, or 2 PM, level)

$$\bar{\mathscr{G}}_{\text{ext}}^{\alpha\beta}[\mathscr{M}] = f^{\alpha\beta} + \mathrm{G}\,\bar{h}_1^{\alpha\beta}[\mathscr{M}] + \mathrm{G}^2\,\bar{h}_2^{\alpha\beta}[\mathscr{M}] + O(6, 7, 6). \tag{2.8}$$

The linearized approximation to the external metric, $\mathrm{G}\,h_1^{\alpha\beta}[\mathscr{M}]$, reads [eqs I (3.2)]

$$\mathrm{G}\,h_1^{00}[\mathscr{M}] = -\frac{4}{c^2}\mathrm{V}^{\text{ext}}, \tag{2.9 a}$$

$$\mathrm{G}\,h_1^{0i}[\mathscr{M}] = -\frac{4}{c^3}\mathrm{V}_i^{\text{ext}}, \tag{2.9 b}$$

$$\mathrm{G}\,h_1^{ij}[\mathscr{M}] = -\frac{4}{c^4}\mathrm{V}_{ij}^{\text{ext}}, \tag{2.9 c}$$

where the "scalar", "vector" and "tensor" external potentials, V^{ext}, $\mathrm{V}_i^{\text{ext}}$, $\mathrm{V}_{ij}^{\text{ext}}$, are given in terms of the algorithmic mass and spin moments by

$$\mathrm{V}^{\text{ext}} = \mathrm{G}\sum_{l\geqq 0}\frac{(-)^l}{l!}\partial_{\mathrm{L}}\left(r^{-1}\,\mathrm{M}_{\mathrm{L}}\left(t-\frac{r}{c}\right)\right), \tag{2.10 a}$$

$$\mathrm{V}_i^{\text{ext}} = -\mathrm{G}\sum_{l\geqq 1}\frac{(-)^l}{l!}\left\{\partial_{\mathrm{L}-1}\left(r^{-1}\,\overset{(1)}{\mathrm{M}}_{i\mathrm{L}-1}\left(t-\frac{r}{c}\right)\right) + \frac{l}{l+1}\varepsilon_{iab}\,\partial_{a\mathrm{L}-1}\left(r^{-1}\,\mathrm{S}_{b\mathrm{L}-1}\left(t-\frac{r}{c}\right)\right)\right\}, \tag{2.10 b}$$

$$\mathrm{V}_{ij}^{\text{ext}} = \mathrm{G}\sum_{l\geqq 2}\frac{(-)^l}{l!}\left\{\partial_{\mathrm{L}-2}\left(r^{-1}\,\overset{(2)}{\mathrm{M}}_{ij\mathrm{L}-2}\left(t-\frac{r}{c}\right)\right) + \frac{2l}{l+1}\partial_{a\mathrm{L}-2}\left(r^{-1}\,\varepsilon_{ab(i}\,\overset{(1)}{\mathrm{S}}_{j)b\mathrm{L}-2}\left(t-\frac{r}{c}\right)\right)\right\}. \tag{2.10 c}$$

The PN expansion, $\mathrm{G}\,\bar{h}_1$, of the linearized approximation is obtained from eqs (2.9)-(2.10) (making use of the useful identity (A33), or (A36), of Ref. [10]) and has the following structure

$$\mathrm{G}\,\bar{h}_1^{00} = -\frac{4}{c^2}\mathrm{U}^{\text{ext}} + O(4), \tag{2.11 a}$$

$$\mathrm{G}\,\bar{h}_1^{0i} = -\frac{4}{c^3}\mathrm{U}_i^{\text{ext}} + O(5), \tag{2.11 b}$$

$$\mathrm{G}\,\overline{h}_1^{ij}=O(4), \tag{2.11c}$$

where we have introduced

$$\mathrm{U}^{\mathrm{ext}}[\mathscr{M}]:=\mathrm{G}\sum_{l\geq 0}\frac{(-)^l}{l!}\mathrm{M}_{\mathrm{L}}(t)\,\partial_{\mathrm{L}}\left(\frac{1}{r}\right), \tag{2.12a}$$

$$\mathrm{U}_i^{\mathrm{ext}}[\mathscr{M}]:=-\mathrm{G}\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\overset{(1)}{\mathrm{M}}_{i\mathrm{L}-1}(t)\,\partial_{\mathrm{L}-1}\left(\frac{1}{r}\right)+\frac{l}{l+1}\varepsilon_{iab}\,\mathrm{S}_{b\mathrm{L}-1}(t)\,\partial_{a\mathrm{L}-1}\left(\frac{1}{r}\right)\right\}. \tag{2.12b}$$

Let us now consider the second approximation, $h_2^{\alpha\beta}=p_2^{\alpha\beta}+q_2^{\alpha\beta}$, which is generated by the quadratic nonlinearities $\mathrm{N}_2^{\alpha\beta}(h_1)$, through eqs (2.2)-(2.4). The expression for $\mathrm{N}_2^{\alpha\beta}(h)$ in *harmonic coordinates* can be found, e. g., in the Appendix A of Ref. [20], and reads

$$\begin{aligned}\mathrm{N}_2^{\alpha\beta}(h)=&-h^{\mu\nu}\,\partial_{\mu\nu}h^{\alpha\beta}+\frac{1}{8}f^{\alpha\beta}(\partial_\mu h)(\partial^\mu h)\\&-\frac{1}{4}(\partial^\alpha h)(\partial^\beta h)-\frac{1}{4}f^{\alpha\beta}(\partial_\mu h_{\nu\rho})(\partial^\mu h^{\nu\rho})\\&+\frac{1}{2}f^{\alpha\beta}(\partial_\mu h_{\nu\rho})(\partial^\nu h^{\mu\rho})+\frac{1}{2}(\partial^\alpha h^{\mu\nu})(\partial^\beta h_{\mu\nu})\\&-(\partial^\alpha h^{\mu\nu})(\partial_\mu h_\nu^\beta)-(\partial^\beta h^{\mu\nu})(\partial_\mu h_\nu^\alpha)\\&+(\partial_\nu h^{\alpha\mu})(\partial^\nu h_\mu^\beta)+(\partial_\nu h^{\alpha\mu})(\partial_\mu h^{\beta\nu})\end{aligned} \tag{2.13}$$

(*see* e. g. eq. (3.5) of Ref. [12] for the expression of $\mathrm{N}_2(h)$ in arbitrary coordinates). Actually, we need to control only the PN expansion of $h_2^{\alpha\beta}$, *i. e.* $\overline{h}_2^{\alpha\beta}=\overline{p}_2^{\alpha\beta}+\overline{q}_2^{\alpha\beta}$. From eqs I (3.10) and I (3.15) taken for $n=2$, we know that the PN expansion of $\overline{p}_2^{\alpha\beta}$ is essentially generated by the PN expansion of $\mathrm{N}_2^{\alpha\beta}(h_1)$, *i. e.* by $\overline{\mathrm{N}_2(h_1)}=\overline{\mathrm{N}}_2(\overline{h}_1)$:

$$\overline{p}_2^{\alpha\beta}=\mathrm{FP}_{\mathrm{B}=0}\sum_{k\geq 0}\frac{1}{c^{2k}}\frac{\partial^{2k}}{\partial t^{2k}}\Delta^{-k-1}[(r/\lambda)^{\mathrm{B}}\,\overline{\mathrm{N}}_2^{\alpha\beta}(\overline{h}_1)]+O(10, 9, 8). \tag{2.14}$$

Inserting eqs (2.11) into eq. (2.13), and taking advantage of the fact that in near-zone expansions $\partial_0=c^{-1}\partial_t=O(1)$, we find the following structure for $\overline{\mathrm{N}}_2(\mathrm{G}\,\overline{h}_1)$

$$\overline{\mathrm{N}}_2^{00}(\mathrm{G}\,\overline{h}_1)=-\frac{14}{c^4}\partial_s\mathrm{U}^{\mathrm{ext}}\,\partial_s\mathrm{U}^{\mathrm{ext}}+O(6), \tag{2.15a}$$

$$\overline{\mathrm{N}}_2^{0i}(G\,\overline{h}_1)=\frac{1}{c^5}[16\,\partial_s\mathrm{U}^{\mathrm{ext}}(\partial_i\mathrm{U}_s^{\mathrm{ext}}-\partial_s\mathrm{U}_i^{\mathrm{ext}})-12\,\partial_i\mathrm{U}^{\mathrm{ext}}\,\partial_s\mathrm{U}_s^{\mathrm{ext}}]+O(7), \tag{2.15b}$$

$$\overline{\mathrm{N}}_2^{ij}(\mathrm{G}\,\overline{h}_1)=\frac{1}{c^4}[4\,\partial_i\mathrm{U}^{\mathrm{ext}}\,\partial_j\mathrm{U}^{\mathrm{ext}}-2\,\delta_{ij}\,\partial_s\mathrm{U}^{\mathrm{ext}}\,\partial_s\mathrm{U}^{\mathrm{ext}}]+O(6). \tag{2.15c}$$

Inserting now eqs (2.15) into eq. (2.14) we see that, in order to reach the accuracy (2.6), it is sufficient to retain the first term ($k=0$) in the right-hand side of eq. (2.14).

Let us now consider the complementary contribution $q_2^{\alpha\beta}$ to $h_2^{\alpha\beta}$. Its definition, eqs (2.4), shows that $q_2^{\alpha\beta}$ can be decomposed in two parts: (i) a "(semi-) hereditary" part (in the terminology of [12]), *i.e.* a part which contains anti-derivatives of the multipole moments, $A_L(u)$, $B_L(u)$, ..., of the vector r_2^α, eqs (2.3), namely

$$q_{\text{hered}}^{00}=-c\frac{\overset{(-1)}{A}(u)}{r}-c\,\partial_a\left(\frac{\overset{(-1)}{A}_a(u)}{r}\right)+c^2\,\partial_a\left(\frac{\overset{(-2)}{C}_a(u)}{r}\right), \qquad (2.16a)$$

$$q_{\text{hered}}^{0i}=-c\frac{\overset{(-1)}{C}_i(u)}{r}-c\,\varepsilon_{iab}\,\partial_a\left(\frac{\overset{(-1)}{D}_b(u)}{r}\right), \qquad (2.16b)$$

$$q_{\text{hered}}^{ij}=0, \qquad (2.16c)$$

and, (ii) an "instantaneous" (in the retarded sense) part which is algebraic in $A_L(u)$, $B_L(u)$, etc. and their time derivatives. As seen in eqs (2.16) the hereditary part is more delicate in that it can *decrease* the post-Newtonian order of the multipoles of r_2^α. However, for this part we can use the same argument that was used in paper I to bound $q_2^{00}+q_2^{ss}$.

Indeed the "factorization" result

$$h_n^{\alpha\beta}(\mathbf{x},t,c)=\sum_{E_n}c^{-(3n+\Sigma \underline{l}_i)}\hat{h}_n^{\alpha\beta}(\mathbf{x}/c,t) \qquad (2.17)$$

(eq. (5.3) of [10], the notation being the one of Ref. [12] and paper I), together with the fact that the most negative power of r in eqs (2.16) is r^{-2}, implies that

$$q_{n\,\text{hered}}^{\alpha\beta}=O\left(3n-2+\sum_{i=1}^{n}\underline{l}_i\right), \qquad (2.18)$$

uniformly for all the components $\alpha\beta$.

As in paper I the $\underline{l}_i$ values required to make the vectors C_a or D_a are constrained by $\underline{l}_1\geqq 2$ (because $q_n^{\alpha\beta}$ is identically zero for stationary moments) and $\underline{l}_2=\underline{l}_1\pm 1$ (to make a vector). Hence $\Sigma\underline{l}_i=\underline{l}_1+\underline{l}_2\geqq 3$ and eq. (2.18) with $n=2$ gives

$$q_{2\,\text{hered}}^{\alpha\beta}=O(7,7,7), \qquad (2.19)$$

which can be ignored because of eq. (2.6). We need now to control the "instantaneous" part of $\bar{q}_2^{\alpha\beta}$. We have seen above that

$$\bar{p}_2^{\alpha\beta}=\mathrm{FP}_{B=0}\{\Delta^{-1}[(r/\lambda)^B\,\bar{N}_2^{\alpha\beta}(\bar{h}_1)]\}+O(6,7,6), \qquad (2.20)$$

from which one deduces (using $\partial_\beta N_2^{\alpha\beta}=0$)

$$\bar{r}_2^\alpha=\partial_\beta\bar{p}_2^{\alpha\beta}=\mathrm{Res}_{B=0}\{\Delta^{-1}[(r/\lambda)^B r^{-1}n^i\,\bar{N}_2^{\alpha i}]\}+O(7,6). \qquad (2.21)$$

Inserting eqs (2.15) into eq. (2.21) will yield a $\bar{r}_2^\alpha$ equal to a $O(5,4)$ contribution coming from the explicit terms in the righthand sides of eqs (2.15*b*) and (2.15*c*), plus a $O(7,6)$ error term. If we prove that the explicit $O(5,4)$ contribution vanishes, this will imply that $\bar{r}_2^\alpha = O(7,6)$ and therefore [from eqs (2.3)] that $A_L = O(7)$, while B_L, C_L, D_L will be $O(6)$. From eqs (2.4) it will then follow that the "instantaneous" part of $q_2^{\alpha\beta}$ is

$$q^{00}_{2\,\text{inst}} \equiv 0, \qquad q^{0i}_{2\,\text{inst}} = O(7), \qquad q^{ij}_{2\,\text{inst}} = O(6). \tag{2.22}$$

To prove the vanishing of the explicit contribution to $\bar{r}_2^\alpha$ coming from the terms appearing in the right-hand side of eqs (2.15) let us first remark that, from eqs (2.12) and (2.15), $\bar{N}_2^{\text{explicit}}$ can be decomposed in a series of products of two multi-spatial gradients of $1/r$ [the explicit simple gradients appearing in eqs (2.15) combining themselves with the multi-gradients in eqs (2.12)]. Therefore the dependence on spatial coordinates of $\bar{N}_2^{\text{explicit}}$ has the form

$$\bar{N}_2^{\text{explicit}} \sim \sum_{l_1, l_2} \partial_{L_1}\left(\frac{1}{r}\right) \partial_{L_2}\left(\frac{1}{r}\right) \sim \sum_{l_1, l_2} \frac{\hat{n}^{L_1}}{r^{l_1+1}} \cdot \frac{\hat{n}^{L_2}}{r^{l_2+1}}. \tag{2.23}$$

Each product $\hat{n}^{L_1}\hat{n}^{L_2}$ can be written as a sum of terms of the form $\hat{n}^{L_1+L_2}$, $\hat{n}^{L_1+L_2-2}$, $\hat{n}^{L_1+L_2-4}$, ... down to $\hat{n}^{L_1-L_2}$ (when $l_1 \geqq l_2$), as seen e.g., from eq. (A.22*b*) of Ref. [10]. Therefore the spatial dependence of $\bar{N}_2$ can be written as

$$\bar{N}_2^{\text{expl.}}(\mathbf{x}) \sim \sum_{l, k \geqq 0} \frac{\hat{n}^L}{r^{l+2k+2}}, \tag{2.24}$$

which implies (by adding an extra n^i together with an extra r^{-1})

$$\frac{n^i}{r} (\bar{N}_2^{i\alpha})^{\text{expl.}} \sim \sum_{l', k'} \frac{\hat{n}^{L'}}{r^{l'+2k'+2}}, \tag{2.25}$$

with $l' = l \pm 1$.

From eq. (2.21) we see that $(\bar{r}_2^\alpha)^{\text{expl.}}$ will be nonzero only when $\Delta^{-1}[r^B r^{-1} n^i \bar{N}_2^{i\alpha}] \sim \Sigma \Delta^{-1}[r^B \hat{n}^{L'} r^{-p'}]$, with $p' = l' + 2k' + 2$, has a pole. However, by definition (*see* eq. (3.9) of Ref. [10])

$$\Delta^{-1}\left(r^B \frac{\hat{n}^{L'}}{r^{p'}}\right) := \frac{r^B}{(B+2-l'-p')(B+3+l'-p')} \frac{\hat{n}^{L'}}{r^{p'-2}}, \tag{2.26}$$

which means that a pole can appear only when $p' = l' + 3$ or $p' = 2 - l'$. The first alternative is excluded by the result $p' = l' + 2k' + 2$, and the second by the easily checked fact that $p' \geqq 5$ in $r^{-1} n^i (\bar{N}_2^{i\alpha})^{\text{expl.}}$. We have therefore demonstrated that [21]

$$(\bar{r}_2^\alpha)^{\text{explicit}} = 0, \tag{2.27}$$

which proves, as said above, that $\bar{r}_2^\alpha = O(7,6)$, and thereby that eqs (2.22) hold. Combining eqs (2.19) and (2.22) we can therefore conclude that

$$q_2^{\alpha\beta} = O(7,7,6). \tag{2.28}$$

To sum up, we have proved that the post-Newtonian (or "near-zone") expansion of the external gravitational field (considered as a functional of the algorithmic moments $\mathscr{M} = \{ M_L, S_L \}$) has the form

$$\mathscr{G}^{\alpha\beta}_{\rm ext}[\mathscr{M}] = f^{\alpha\beta} + G\,\bar{h}_1^{\alpha\beta}[\mathscr{M}] + {\rm FP}\,\Delta^{-1}\,N^{\alpha\beta}(U^{\rm ext}) + O(6,7,6), \tag{2.29}$$

where $G\,\bar{h}_1^{\alpha\beta}[\mathscr{M}]$ is obtained by expanding in powers of c^{-1} eqs (2.9)-(2.10) [beyond the lowest-order terms shown in eqs (2.11)], where (simplifying the notation) $N^{\alpha\beta}(U^{\rm ext})$ denotes the explicit quadratic expressions in the gradients of $U^{\rm ext}$ and $U_i^{\rm ext}$ appearing in the right-hand side of eqs (2.15) [remembering the definitions (2.12)], and where the operator ${\rm FP}\,\Delta^{-1}$ is defined as being the finite part at $B=0$ of the meromorphic function of the complex number B defined by its action (2.26) on each term of the (orbital angular momentum) multipole expansion of $N(U^{\rm ext})$. In fact, as a by-product of the proof above we see from eq. (2.24) (and the fact that the minimum power of r^{-1} in $N(U^{\rm ext})$ is four) that there are no poles at $B=0$ and therefore that the definition of ${\rm FP}\,\Delta^{-1}$ yields simply, when acting on the orbital multipole expansion obtainable from inserting eqs (2.12) into (2.15) (l and k denoting natural integers)

$$N^{\alpha\beta}(U^{\rm ext}) = \sum_{l+2k\geq 2} F^{[k]}_{\alpha\beta L}\,\frac{\hat{n}^L}{r^{l+2k+2}}, \tag{2.30 a}$$

$${\rm FP}\,\Delta^{-1}\,N^{\alpha\beta}(U^{\rm ext}) = \sum_{l+2k\geq 2} F^{[k]}_{\alpha\beta L}\,\frac{\hat{n}^L}{(2k+2l)(2k-1)\,r^{l+2k}}. \tag{2.30 b}$$

III. KERNELS FOR SOLVING THE QUADRATIC NONLINEARITIES

In the previous section we have exhibited the explicit results for the near-zone expanded external metric, $\mathscr{G}^{\alpha\beta}_{\rm ext}[\mathscr{M}]$, that followed from the application of the MPM algorithm of Refs [8]-[12]. The main difficulty came from the need to solve the quadratic nonlinearities. This finally gave rise to a special inverse Laplacian of the *external* quadratic effective source $N^{\alpha\beta}(U^{\rm ext})$. Here, after simplification of the notation, we mean by $N^{\alpha\beta}(U)$ the following quadratic forms in the gradients of both a scalar potential, U, and a vector potential, U_i:

$$N^{00}(U) := -\frac{14}{c^4}\,\partial_s U\,\partial_s U, \tag{3.1 a}$$

$$N^{0i}(U):=\frac{1}{c^5}[16\,\partial_s U(\partial_i U_s-\partial_s U_i)-12\,\partial_i U\,\partial_s U_s], \qquad (3.1\,b)$$

$$N^{ij}(U):=\frac{1}{c^4}[4\,\partial_i U\,\partial_j U-2\,\delta_{ij}\,\partial_s U\,\partial_s U]. \qquad (3.1\,c)$$

We shall see in the next section that, when solving for the (post-Newtonian-expanded) inner metric, $\mathcal{G}^{\alpha\beta}_{in}$, the same quadratic effective source arises, but now computed in terms of some *inner* potentials (U^{in}, U^{in}_i), instead of the *external* potentials (U^{ext}, U^{ext}_i) appearing in the previous section. The central difficulty of the present investigation will be to relate the usual Poisson integral of $N(U^{in})$,

$$\{\Delta^{-1} N(U^{in})\}(\mathbf{x}):=-\frac{1}{4\pi}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}N(U^{in}(\mathbf{x}')) \qquad (3.2\,a)$$

[which is well-defined because $N(U^{in})$ falls off like $|\mathbf{x}'|^{-4}$ at spatial infinity] to the special inverse-Laplacian introduced by the algorithm which applied to the *external* nonlinearities:

$$\mathrm{FP}\,\Delta^{-1}N(U^{ext}):=\underset{B=0}{\text{Finite Part}}\,\Delta^{-1}\times\left\{\left(\frac{r}{b}\right)^B \text{Multipole Expansion }[N(U^{ext})]\right\}, \qquad (3.2\,b)$$

where b is some length scale (we had taken $b=\bar{\lambda}$ above) and where Multipole Expansion refers to the expansion in orbital multipoles $\hat{n}^L$ (*i.e.* in eigenfunctions of the angular Laplacian).

In the inner problem, the inner potentials, U^{in}, will be the Newtonian potentials generated by some compact support scalar or vectorial densities, say

$$U^{in}(\mathbf{x}, t)=G\int d^3y\,\frac{\sigma(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}, \qquad (3.3\,a)$$

$$U^{in}_i(\mathbf{x}, t)=G\int d^3y\,\frac{\sigma^i(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}. \qquad (3.3\,b)$$

Inserting eqs (3.3) into eqs (3.1) shows that $N(U^{in})$ can be decomposed as a sum of terms of the form

$$N^{\alpha\beta}(U^{in})=\sum C^{\alpha\beta}_{ab\,(c)}\times\iint d^3y_1\,d^3y_2\,\sigma_1(\mathbf{y}_1)\,\sigma^{(c)}_2(\mathbf{y}_2)\frac{\partial}{\partial y^a_1}\frac{\partial}{\partial y^b_2}\left(\frac{1}{r_1 r_2}\right), \qquad (3.4)$$

where the C's are some constant coefficients, where σ_1, $\sigma^{(c)}_2$ denote $\sigma(\mathbf{y}_1)$ and $\sigma(\mathbf{y}_2)$ when $\alpha\beta=00$ or ij, or $\sigma(\mathbf{y}_1)$ and $\sigma^c(\mathbf{y}_2)$ when $\alpha\beta=0i$, where (anticipating on future needs)

$$r_1:=|\mathbf{x}-\mathbf{y}_1|, \qquad r_2:=|\mathbf{x}-\mathbf{y}_2|, \qquad r_{12}:=|\mathbf{y}_1-\mathbf{y}_2|, \qquad (3.5)$$

and where we have replaced the gradients with respect to $\mathbf{x}$ in eqs (3.1) by gradients with respect to $\mathbf{y}_1$ or $\mathbf{y}_2$ (shifts of source points instead of shifts of field points, using $\partial_{\mathbf{x}} r_1^{-1} \equiv -\partial_{\mathbf{y}_1} r_1^{-1}$). For notational simplicity the whole complicated operation of summation, source-points double integration, and source-points double differentiation appearing in eq. (3.4) will be abbreviated as

$$N^{in}(\mathbf{x}) = \sum\nolimits_{12}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_1||\mathbf{x}-\mathbf{y}_2|}\right), \tag{3.6}$$

where it is essential to keep in mind that Σ_{12} is a complicated but linear operator acting only on the pair of source variables $(\mathbf{y}_1, \mathbf{y}_2)$ and totally independent of the field point $\mathbf{x}$.

Let us now define [using the notation (3.5)] the following *kernels*, considered as functions of $\mathbf{x}$ on the one hand, and the pair $(\mathbf{y}_1, \mathbf{y}_2)$ on the other hand

$$g(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) := \ln(r_1 + r_2 + r_{12}), \tag{3.7a}$$

$$k(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) := \frac{1}{2}\ln[(r_1 + r_2)^2 - r_{12}^2], \tag{3.7b}$$

$$h(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) := \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right), \tag{3.7c}$$

as well as the following distribution with respect to the field point $\mathbf{x}$ (for fixed source points)

$$\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) := r_{12}\int_0^1 d\alpha\, \delta(\mathbf{x} - \mathbf{y}_\alpha), \tag{3.8a}$$

$$\mathbf{y}_\alpha := \alpha\,\mathbf{y}_1 + (1-\alpha)\,\mathbf{y}_2. \tag{3.8b}$$

In eq. (3.8*a*) $\delta(\mathbf{x} - \mathbf{y}_\alpha)$ denotes the three-dimensional Dirac distribution with respect to $\mathbf{x}$, so that $\delta_{12}(\mathbf{x})$ is a distribution with respect to $\mathbf{x}$ which represents just a homogeneous linear density (with density $= 1$ per unit length) distributed along the segment joining $\mathbf{y}_1$ and $\mathbf{y}_2$.

The above introduced quantities satisfy the following relations

$$g \equiv k + \frac{1}{2}h, \tag{3.9a}$$

$$\Delta_x g = \frac{1}{r_1 r_2}, \tag{3.9b}$$

$$\Delta_x k = \frac{1}{r_1 r_2} + 2\pi\delta_{12}, \tag{3.9c}$$

$$\Delta_x h = -4\pi\delta_{12}, \tag{3.9d}$$

which are satisfied in the sense of distributions with respect to $\mathbf{x}$ (*i.e.* for all values of $\mathbf{x}$, including possibly singular ones). The Laplace operators appearing in eqs (3.9) are all taken with respect to **x** (in the sense of distributions).

The relations (3.9) can be conveniently checked by introducing, instead of **x**, elliptic coordinates (ξ, η, φ), linked to the two focal points $\mathbf{y}_1$, $\mathbf{y}_2$ (and still working consistently in the framework of distributions),

$$\xi=\frac{r_2+r_1}{r_{12}}, \qquad \eta=\frac{r_2-r_1}{r_{12}}, \tag{3.10 a}$$
$$\varphi = \text{longitude around the segment } \mathbf{y}_1\,\mathbf{y}_2,$$

such that $\xi \geqq 1$, $-1 \leqq \eta \leqq 1$, $0 \leqq \varphi \leqq 2\pi$ and

$$d\mathbf{x}^2=\left(\frac{r_{12}}{2}\right)^2\left[(\xi^2-\eta^2)\left(\frac{d\xi^2}{\xi^2-1}+\frac{d\eta^2}{1-\eta^2}\right)+(\xi^2-1)(1-\eta^2)\,d\varphi^2\right], \tag{3.10 b}$$

$$r_1 r_2 \Delta_x=\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} +\left(\frac{1}{\xi^2-1}+\frac{1}{1-\eta^2}\right)\frac{\partial^2}{\partial\varphi^2}. \tag{3.10 c}$$

The fact that the kernel g satisfies eq. (3.9 *b*) everywhere (including at the points $\mathbf{y}_1$, $\mathbf{y}_2$ and along the segment $\mathbf{y}_1\,\mathbf{y}_2$) implies that

$$\Delta_x[\textstyle\sum_{12}(g(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2))]=\sum_{12}(\Delta_x g)=\sum_{12}\left(\frac{1}{r_1 r_2}\right)=\mathrm{N}^{\mathrm{in}}(\mathbf{x}). \tag{3.11}$$

As it is easy to check that $\Sigma_{12}(g)$ falls off as $1/|\mathbf{x}|$ when $|\mathbf{x}|\to\infty$, we conclude that $\Sigma_{12}(g)$ is the *unique* everywhere regular, tending to zero at infinity, solution of the Poisson equation with source $\mathrm{N}^{\mathrm{in}}(\mathbf{x})$. In other words

$$\textstyle\sum_{12}(g)=\Delta^{-1}\,\mathrm{N}(\mathrm{U}^{\mathrm{in}}), \tag{3.12}$$

in the sense of eq. (3.2 *a*).

Having exhibited the relation of the kernel g to the usual Poisson operator, let us now show how the kernel k is linked to the algorithmic inverse Laplacian $\mathrm{FP}\,\Delta^{-1}$ of eq. (3.2 *b*).

Let us from now on consider the situation where the field point, **x**, is in the external domain D_e, *i. e.* outside a sphere which encloses the support of the source densities σ_1 and σ_2 which appear in the operator Σ_{12} of eqs (3.4) or (3.6), and let us study the quantity [indices $\alpha\beta\,(c)$ suppressed]

$$\textstyle\sum_{12}(k)(\mathbf{x})=\sum \mathrm{C}_{ab} \times \displaystyle\iint d^3y_1\,d^3y_2\,\sigma_1(y_1)\,\sigma_2(y_2)\frac{\partial}{\partial y_1^a}\frac{\partial}{\partial y_2^b}k(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2). \tag{3.13}$$

It is easy to see that, for **x** fixed in D_e, $k(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ is jointly analytic in y_1^i and y_2^j (this property distinguishes k from the other kernels which are not regular when $|\mathbf{y}_1-\mathbf{y}_2|\to 0$). Expanding k into a double Maclaurin series in $(\mathbf{y}_1, \mathbf{y}_2)$ and organizing the coefficients along the orbital harmonics $\hat{n}^{\mathrm{L}}$ leads to a (convergent) series having the following structure (coefficients being suppressed)

$$\mathrm{M}_{12}(k(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2))=\ln(2r)+\sum_{l+2k=l_1+l_2\geq 1} y_1^{\mathrm{L}_1}y_2^{\mathrm{L}_2}\frac{\hat{n}^{\mathrm{L}}}{r^{l+2k}}, \qquad (3.14)$$

where M_{12} stands for "Maclaurin expansion with respect to $(\mathbf{y}_1, \mathbf{y}_2)$" where $r:=|\mathbf{x}|$, $n^i:=x^i/r$, where the condition $l+2k=l_1+l_2$ follows immediately from dimensional considerations, and where the (very useful) information that the power of r^{-1} differs from the order of multipolarity by an even natural integer, $2k$, is proven by inspection of the general structure of the expansion. If we similarly expand $(r_1 r_2)^{-1}$ in a double Maclaurin series in $(\mathbf{y}_1, \mathbf{y}_2)$, we find a structure of the type (coefficients suppressed)

$$\mathrm{M}_{12}\left(\frac{1}{r_1 r_2}\right)=\frac{1}{r^2}+\sum_{l+2k=l_1+l_2\geq 1} y_1^{\mathrm{L}_1}y_2^{\mathrm{L}_2}\frac{\hat{n}_{\mathrm{L}}}{r^{l+2k+2}}. \qquad (3.15)$$

If we now apply the operator $\mathrm{FP}\,\Delta^{-1}$ defined by eq. (3.2 b) to the right-hand side of eq. (3.15), we can prove that

$$\mathrm{M}_{12}(k(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2))=\mathrm{FP}_{\mathrm{B}=0}\,\Delta^{-1}\times\left\{\left(\frac{r}{b}\right)^{\mathrm{B}}\mathrm{M}_{12}\left(\frac{1}{r_1 r_2}\right)\right\}+\ln(2b)+1. \qquad (3.16)$$

The proof that eq. (3.16) holds does not necessitate a precise computation of the unwritten coefficients in eqs (3.14) and (3.15). Indeed, it suffices to remark on the one hand, that eq. (3.9 c) guarantees that the (usual) Laplacian of both sides of eq. (3.16) coincide in D_e, and on the other hand that none of the $\hat{n}^{\mathrm{L}}r^{-(l+2k)}$ terms appearing on both sides of eq. (3.16) vanish under the action of the Laplacian (except the constant term which is easily treated separately and which yields the strange, but unimportant, $\ln(2b)+1$ additive constant).

Let us now apply the Σ_{12} operator on both sides of eq. (3.16). Because of the explicit terms $\sim\hat{n}^{\mathrm{L}}r^{-p}$ in the right-hand side of eq. (3.14), $\Sigma_{12}(\mathrm{M}_{12}(k))$ will yield simply the usual (orbital) *Multipole Expansion* (ME) of $\Sigma_{12}(k)(\mathbf{x})$. In the right-hand side of eq. (3.16) we can notice that $M_{12}((r_1 . r_2)^{-1}) = M_1(r_1^{-1}) M_2(r_2^{-1})$ (where M_1 is a Maclaurin expansion with respect to $\mathbf{y}_1$, etc.), and that

$$\begin{aligned}\int d^3 y_1\, \sigma_1(y_1) \frac{\partial}{\partial y_1^a} M_1\left(\frac{1}{r_1}\right) &= -\frac{\partial}{\partial x^a} \int d^3 y_1\, \sigma_1(y_1) M_1\left(\frac{1}{r_1}\right) \\ &= -\frac{\partial}{\partial x^a} \int d^3 y_1\, \sigma_1(y_1) \sum_{l \geqq 0} \frac{(-1)^l y_1^L}{l!} \partial_L \frac{1}{r} \\ &= -\frac{\partial}{\partial x^a} \sum_{l \geqq 0} \frac{(-)^l}{l!} \hat{M}_L(\sigma_1) \partial_L \frac{1}{r} \\ &= -\frac{\partial}{\partial x^a} \mathrm{ME}\left(\int d^3 y_1 \frac{\sigma_1(\mathbf{y}_1)}{|\mathbf{x}-\mathbf{y}_1|} \right),\end{aligned} \qquad (3.17)$$

where we recognize (in its various forms, including fully explicitly with $\hat{M}_L(\sigma_1) := \int d^3 y_1\, \sigma_1(\mathbf{y}_1) y_1^{\langle L \rangle}$) the (orbital) multipole expansion of the Newtonian potential of σ_1. Finally this leads us to

$$\mathrm{ME}\{\textstyle\sum_{12}(k)\} = \mathrm{FP}\,\Delta^{-1}\{\mathrm{N}[\mathrm{ME}(\mathrm{U}^{\mathrm{in}})]\}, \qquad (3.18)$$

where ME, on both sides, means (orbital) "Multipole Expansion of", where N is as defined in eqs (3.1), and where we recall that the $(\mathbf{y}_1, \mathbf{y}_2)$-integro-differential operator Σ_{12} was defined by eqs (3.4), (3.6). Let us note also that the length scale, b, introduced in the definition of the "external" inverse Laplacian [*see* eq. (3.2*b*)] drops out from the final result (3.18).

In summary, the two main results of this section are eq. (3.12) and eq. (3.18) which we shall put to use in the next section to, respectively, solve the quadratic nonlinearities of the inner metric, and connect this inner solution to our previous result (2.29) for the corresponding external solution.

IV. THE INNER GRAVITATIONAL FIELD AND ITS MATCHING TO THE EXTERNAL ONE

As said above our aim is to find the (PN expanded) gothic metric, $\mathscr{G}_{\mathrm{in}}^{\alpha\beta}$, in the inner domain D_i, as a functional of the source variables up to post-Newtonian error terms

$$\delta \mathscr{G}_{\mathrm{in}}^{\alpha\beta} = O(6, 7, 6). \qquad (4.1)$$

This level of precision is readily achieved by starting from the knowledge of the following PN-truncated linearized solution (in harmonic coordinates) of Einstein's inhomogeneous equations,

$$\mathcal{G}_{\text{in}}^{00} = -1 - \frac{4}{c^2}\text{U}^{\text{in}} + O(4), \tag{4.2a}$$

$$\mathcal{G}_{\text{in}}^{0i} = -\frac{4}{c^3}\text{U}_i^{\text{in}} + O(5), \tag{4.2b}$$

$$\mathcal{G}_{\text{in}}^{ij} = \delta^{ij} + O(4), \tag{4.2c}$$

with

$$\text{U}^{\text{in}}(\mathbf{x}, t) := \text{G}\int d^3y \frac{\sigma(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}, \tag{4.3a}$$

$$\text{U}_i^{\text{in}}(\mathbf{x}, t) := \text{G}\int d^3y \frac{\sigma^i(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}, \tag{4.3b}$$

where we have defined, following paper I, an "active gravitational mass density", σ, and an "active mass current density", σ^i by

$$\sigma := c^{-2}(\text{T}^{00} + \text{T}^{ss}), \tag{4.4a}$$

$$\sigma^i := c^{-1}\text{T}^{0i}. \tag{4.4b}$$

When comparing eqs (4.2) with the corresponding (more accurate) results I (2.8)-(2.10) of paper I one must remember that the near-zone expansion of eqs I (2.8), I (2.9) introduces terms apparently of order $O(3)$ in $\mathcal{G}_{\text{in}}^{00}$ and $O(4)$ in $\mathcal{G}_{\text{in}}^{0i}$, but that these terms are down by a factor c^{-2} because of the conservation laws for mass and linear momentum.

If we introduce an (exact) "gothic deviation" for the inner metric,

$$h_{\text{in}}^{\alpha\beta} := \mathcal{G}_{\text{in}}^{\alpha\beta} - f^{\alpha\beta} \equiv \sqrt{g_{\text{in}}}\, g_{\text{in}}^{\alpha\beta} - f^{\alpha\beta}, \tag{4.5}$$

then in harmonic coordinates,

$$\partial_\beta h_{\text{in}}^{\alpha\beta} = 0, \tag{4.6}$$

and the Einstein equations can be written as

$$\Box\, h_{\text{in}}^{\alpha\beta} = \frac{16\pi\text{G}}{c^4} g_{\text{in}}\,\text{T}^{\alpha\beta} + \Lambda^{\alpha\beta}(h_{\text{in}}), \tag{4.7}$$

where $\Box = f^{\mu\nu}\partial_{\mu\nu}$, $g = -\det(g_{\alpha\beta})$, and where $\Lambda^{\alpha\beta}(h)$ is the total "effective nonlinear source" for the gravitational field, whose lowest-order (quadratic) piece is given by eq. (2.13) above [note in eq. (4.7) h_{in} is not expanded in a nonlinearity series, contrarily to the external expansion, eq. (2.1)].

If we insert the lowest-order approximation (4.2) in the right-hand side of eq. (4.7), this allows us to determine the (near-zone) value of the right-hand side, as a functional of the source, modulo the error (4.1). Explicitly we find

$$\Box h_{\rm in}^{\alpha\beta}=\frac{16\pi G}{c^4}\left(1+\frac{4}{c^2}U^{\rm in}\right)T^{\alpha\beta}+N^{\alpha\beta}(U^{\rm in})+O(6,7,6), \qquad (4.8)$$

where $N^{\alpha\beta}(U^{\rm in})$ denotes, as above, the value taken by the quadratic expressions (3.1) upon inserting $(U, U_i)=(U^{\rm in}, U_i^{\rm in})$ as defined by eqs (4.3). Note that we followed paper I in using the near-zone assumption $\partial_0 h^{\mu\nu}/\partial_i h^{\mu\nu}=O(c^{-1})$ only in the already algebraically small nonlinear terms of the field equations. In the left-hand side of eq. (4.8) we keep, for simplifying the writing, the wave operator, as well as $h_{\rm in}^{\alpha\beta}$ itself, unexpanded, although we shall remember when needed that we are considering only the near-zone expansion of the metric.

A solution of eq. (4.8), with the required accuracy, is

$$h_{\rm in}^{\alpha\beta}=\frac{16\pi G}{c^4}\bar{\Box}_R^{-1}\left[\left(1+\frac{4}{c^2}U^{\rm in}\right)T^{\alpha\beta}\right]+\Delta^{-1}N^{\alpha\beta}(U^{\rm in})$$
$$+\frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3x\, N^{\alpha\beta}(U^{\rm in})+O(6,7,6), \qquad (4.9)$$

where the bar over the retarded potential operator $\Box_R^{-1}$ means (as in Section II) that we are working, in discussing the inner metric, with near-zone expanded (in powers of c^{-1}) quantities. As discussed in the previous section the second term in the right-hand side of eq. (4.9) denotes the usual Poisson integral of $N^{\alpha\beta}(U^{\rm in})$ (both this Poisson integral and the next spatial integral converge because of the r^{-4} fall off of $N(U^{\rm in})$). Using the notation [notably the abbreviation Σ_{12} of eq. (3.6)] and the result (3.12) of Section III, we can write the solution (4.9) as the following explicit functional of the source:

$$h_{\rm in}^{\alpha\beta}[\text{source}]=\frac{16\pi G}{c^4}\bar{\Box}_R^{-1}\left[\left(1+\frac{4}{c^2}U^{\rm in}\right)T^{\alpha\beta}\right]+\sum\nolimits_{12}^{\alpha\beta}(g)$$
$$+\frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3x \sum\nolimits_{12}^{\alpha\beta}\left(\frac{1}{r_1 r_2}\right)+O(6,7,6). \qquad (4.10)$$

Let us remark that the last term that we added to eq. (4.9) and (4.10) is a function only of the time coordinate, and that we could, *a priori*, have added other such solutions, regular in D_i, of the Laplace equation. However, we shall prove directly below that the solution (4.10) matches completely the general radiative external metric discussed in Section II, which means that if we were to add a homogeneous solution to eq. (4.10) it would have to be "pure gauge" modulo $O(6, 7, 6)$. In fact, the last term in eq. (4.9) and eq. (4.10) is already pure gauge modulo a $\delta g_{00}=O(c^{-7})$, as are, in fact, some of the first explicit "radiative" terms in the canonical MPM external metric which are not lost in the $O(6,7,6)$

error term: $\delta^{\mathrm{rad}} h^{0i}_{\mathrm{ext}} = O(6)$ and $\delta^{\mathrm{rad}} h^{ij}_{\mathrm{ext}} = O(5)$, see Ref. [9] (case $l=2$). The retention of this last term in eq. (4.9), although not strictly necessary at the precision at which we are working (because it could be absorbed in the coordinate transformation connecting the inner to the external metrics) will clarify the way in which the inner and the external metrics match.

Eq. (4.10) solves the *Step* 3 of our strategy [*see* eq. (1.7) above]. Let us now turn to the (last) *Step* 4, namely the matching between the external and the inner gravitational fields. Our procedure for doing this matching is presented in detail in Section VI of Ref. [12], it can be summarized by the equation

$$\mathrm{PN}\{\mathscr{G}^{\mathrm{ext}}[\mathscr{M}[\mathrm{source}]]\} \equiv \mathrm{PN}\{\mathrm{T} * \mathscr{G}^{\mathrm{in}}[\mathrm{source}]\}, \tag{4.11}$$

where PN means "Post Newtonian Expansion of", where $\mathscr{M}$ [source] are the sought for expressions of the algorithmic multipole moments as functionals of the source variables, and where T denotes some (PN-expanded) coordinate transformation connecting the inner and the external metrics. Note that in the left-hand side of eq. (4.11) the PN expansion applies both to $\mathscr{G}^{\mathrm{ext}}[\mathscr{M}]$ (for fixed $\mathscr{M}$), which is what we denoted by $\bar{\mathscr{G}}^{\mathrm{ext}}[\mathscr{M}]$ above and to the functional $\mathscr{M}$[source] which contains power (and logarithms) of c^{-1} (*see* Ref. [12] for more details).

Let us start by asserting the effect of a PN-expanded coordinate transformation on the metric. From the work of Refs [9], [12] and of paper I we know beforehand that the PN-expanded transformation, T, connecting the inner and the external coordinates will have the form

$$\mathrm{T}\left\{\begin{aligned} x'^0 &= x^0 + \varphi^0 = x^0 + \frac{1}{c^3}\varphi^0_{(3)} + \frac{1}{c^5}\varphi^0_{(5)} + \frac{1}{c^6}\varphi^0_{(6)} + O(7) \\ x'^i &= x^i + \varphi^i = x^i + \frac{1}{c^4}\varphi^i_{(4)} + \frac{1}{c^5}\varphi^i_{(5)} + O(6) \end{aligned}\right. \tag{4.12}$$

where all the terms written out are, in principle, explicitly needed for proving the matching between the two metrics. In view of the high accuracy of the expansions (4.12), one expects that it will no longer be possible to treat the effect of T by the standard linearized expressions ($\delta g_{\mu\nu} = \partial_\mu \varphi_\nu + \partial_\nu \varphi_\mu$; as was sufficient in paper I). However, it turns out (somewhat surprisingly) that this *is* possible, if one uses the gothic metric components as basic variables. Indeed, the exact formula

$$\mathscr{G}'^{\mu\nu}(x'^\lambda) = \left[\det\left(\frac{\partial x'^\lambda}{\partial x^\gamma}\right)\right]^{-1} \frac{\partial x'^\mu}{\partial x^\alpha}\,\frac{\partial x'^\nu}{\partial x^\beta}\,\mathscr{G}^{\alpha\beta}(x^\gamma) \tag{4.13}$$

for the transformation $x'^\mu = x^\mu + \varphi^\mu(x^\alpha)$, with $\varphi^0 = O(3)$ and $\varphi^i = O(4)$, and using the known PN structure, eq. (4.2), of the metrics of interest (as well as the PN assumptions for φ^μ) leads, after a straightforward calculation, to

$$\mathscr{G}'^{00}(x) = \mathscr{G}^{00}(x) + \partial_i \varphi^i - \partial_0 \varphi^0 + O(6), \tag{4.14a}$$

$$\mathscr{G}'^{0i}(x)=\mathscr{G}^{0i}(x)-\partial_0\varphi^i+\partial_i\varphi^0+O(7), \qquad (4.14\,b)$$

$$\mathscr{G}'^{ij}(x)=\mathscr{G}^{ij}(x)+\partial_i\varphi^j+\partial_j\varphi^i-\delta_{ij}(\partial_k\varphi^k+\partial_0\varphi^0)+O(8), \quad (4.14\,c)$$

where the error terms are even better than what we need [$O(6,7,8)$ instead of $O(6,7,6)$].

One should note that with $\varphi^0=O(3)$ and $\varphi^i=O(4)$, the sole effect of a coordinate transformation (4.12) on the lowest-order "linearized" metric (4.2) is to change the "vector potential" U_i by a gradient $\sim\partial_i(c^3\varphi^0)$ [leaving U fixed modulo $O(2)$]. Moreover, if the coordinate transformation connects two harmonic charts, the Laplacian of $c^3\varphi^0$ will be of post-Newtonian order, $O(c^{-2})$. Therefore, although U_i changes, both its curl, $\partial_i U_j-\partial_j U_i$, and it divergence, $\partial_s U_s$, are invariant [modulo $O(2)$] under a harmonic coordinate transformation (4.12). As the explicit quadratic nonlinearities $N^{\alpha\beta}(U, U_i)$, eq. (3.1), depend only on U, and the curl and the divergence of U_i, we conclude that $N^{\alpha\beta}(U, U_i)$ is invariant, modulo $O(2)$ relative errors, under the transformation connecting the inner and the external metrics:

$$N^{\alpha\beta}(U^{\text{ext}})=\mathcal{N}^{\alpha\beta}(U^{\text{in}})+O(6, 7, 6). \qquad (4.15)$$

Let us take the (orbital) multipole expansion of the right-hand side of eq. (4.15) and apply the algorithmic inverse Laplacian operator, $\text{FP}\,\Delta^{-1}$. Thanks to our previous result (3.18), we obtain,

$$\text{FP}\,\Delta^{-1}\,N(U^{\text{ext}})=\text{ME}\left\{\sum\nolimits_{12}(k)\right\}. \qquad (4.16)$$

The left-hand side of eq. (4.16) is precisely the (quadratically) nonlinear piece of $\mathscr{G}_{\text{ext}}$[*see* eq. (2.29)], while its right-hand side differs from the nonlinear piece of $\mathscr{G}_{\text{in}}$ [in the sense of the one generated by $\Lambda^{\alpha\beta}$ in eq. (4.7)] by the sole but crucial fact that the kernel g appearing in $\mathscr{G}_{\text{in}}$ [*see* eq. (4.10)] gets replaced by the kernel k.

We can now perform the matching between the inner and the external metrics. Inserting in our basic matching equation (4.11) the information contained in eqs (2.29), (3.12), (4.14) and (4.16) we get

$$\begin{aligned} f^{\alpha\beta}+G\,\bar{h}_1^{\alpha\beta}[\mathscr{M}]+\text{ME}\left\{\sum\nolimits_{12}(k)\right\} &= f^{\alpha\beta}+\frac{16\pi G}{c^4}\,\Box_R^{-1}\left[\left(1+\frac{4}{c^2}U^{\text{in}}\right)T^{\alpha\beta}\right] \\ &\quad+\sum\nolimits_{12}^{\alpha\beta}(g)+\frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3x\sum\nolimits_{12}^{\alpha\beta}\left(\frac{1}{r_1 r_2}\right) \\ &\quad+f^{\alpha\mu}\partial_\mu\varphi^\beta+f^{\beta\mu}\partial_\mu\varphi^\alpha-f^{\alpha\beta}\partial_\mu\varphi^\mu+O(6,7,6), \end{aligned} \qquad (4.17)$$

which should hold in $D_i \cap D_e$, *i.e.* in the external near zone (where field quantities can be, simultaneously, multipole expanded and near-zone expanded).

Inserting $g \equiv k + h/2$ in the right-hand side, and simplifying the $\Sigma(k)$ terms we are left with an extra-term which, using eq. (3.9 *d*), can be written as

$$\sum_{12}^{\alpha\beta}\left(\frac{1}{2}h\right) = \Delta_x^{-1}\left\{\sum_{12}^{\alpha\beta}(-2\pi\delta_{12})\right\}, \tag{4.18}$$

where we recall that Σ_{12} acts only on the $(\mathbf{y}_1, \mathbf{y}_2)$ source points (as an integro-differential operator) and that the dependence on the field point $\mathbf{x}$ is contained entirely in $h(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$ and $\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$. The term (4.18) is followed by $(4\pi c)^{-1}\partial/\partial t$ of the following integral

$$\int d^3x \sum_{12}^{\alpha\beta}\left(\frac{1}{r_1 r_2}\right) = \sum_{12}^{\alpha\beta}(-2\pi r_{12}) = \int d^3x \sum_{12}^{\alpha\beta}(-2\pi\delta_{12}). \tag{4.19}$$

Note that all the integrals appearing in eq. (4.19) are convergent. The verification of (4.19) rests on the validity of the well-defined integral relation

$$\int d^3x\, \partial_{y_1^a}\left(\frac{1}{r_1}\right)\partial_{y_2^b}\left(\frac{1}{r_2}\right) = -2\pi\,\partial_{y_1^a}\partial_{y_2^b}(r_{12}), \tag{4.20}$$

which can be checked in various ways (a straightforward one being to use the integral formula (2) in section 12 of the first Ref. [22]).

On combining eq. (4.18) + $(4\pi c)^{-1}\partial/\partial t$ eq. (4.19) we recognize the first two terms in the following near-zone expansion

$$\Box_R^{-1}\left\{\sum_{12}^{\alpha\beta}(-2\pi\delta_{12})\right\} = \Delta^{-1}\left\{\sum_{12}^{\alpha\beta}(-2\pi\delta_{12})\right\} + \frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3x \sum_{12}^{\alpha\beta}(-2\pi\delta_{12}) + O\left(\frac{1}{c^2}\sum_{12}^{\alpha\beta}\right). \tag{4.21}$$

The term $\sum_{12}^{\alpha\beta}(-2\pi\delta_{12})$ is a distribution in $\mathbf{x}$ which is of *compact* support (because of the definition of δ_{12}) and which appears now in the right-hand side of eq. (4.17) preceded by the same operator as the material source term, $16\pi G c^{-4}(1 + 4U^{\text{in}} c^{-2}) T^{\alpha\beta}$. We are thereby naturally led to introducing the following quantity

$$\tau_c^{\alpha\beta}(\mathbf{x}, t) := \left(1 + \frac{4}{c^2}U^{\text{in}}\right)T^{\alpha\beta} - \frac{c^4}{8G}\sum_{12}^{\alpha\beta}(\delta_{12}). \tag{4.22}$$

This quantity is a distribution in $\mathbf{x}$ with a *spatially compact* support, and plays, in the external near-zone $D_i \cap D_e$, the role of an effective stress-energy tensor in the precise sense that eq. (4.17) is written now as

$$G \bar{h}_1^{\alpha\beta}[\mathcal{M}] = \frac{16\pi G}{c^4} \bar{\Box}_R^{-1}\{\tau_c^{\alpha\beta}\} + f^{\alpha\mu}\partial_\mu \varphi^\beta + f^{\beta\mu}\partial_\mu \varphi^\alpha - f^{\alpha\beta}\partial_\mu \varphi^\mu + O(6,7,6). \quad (4.23)$$

The result (4.23) is remarkably simple. In words, it says that, in the external near zone $D_e \cap D_i$, the *linearized* algorithmic gothic metric is equal, modulo $O(6,7,6)$, to the *linearized* coordinate transform of the *linearized* harmonic gothic metric generated (via the retarded Green's function) by the spatially compact effective stress-energy tensor $\tau_c^{\alpha\beta}$. Moreover as, by construction, both the inner and the external coordinates are harmonic (in the full curved spacetime sense $0 = \Box_g x^\alpha = \Box_g x'^\alpha$), we know that the four functions $\varphi^\alpha(x)$ are also harmonic in the curved sense, which implies

$$0 = \sqrt{g}\,\Box_g \varphi^\alpha = \partial_\mu(\mathcal{G}^{\mu\nu}\partial_\nu \varphi^\alpha) = \mathcal{G}^{\mu\nu}\partial_{\mu\nu}\varphi^\alpha. \quad (4.24a)$$

This, on using the fact that $\mathcal{G}^{\mu\nu} = f^{\mu\nu} + O(2,3,4)$, gives in the near zone [where $\partial_0 = O(1)$]

$$f^{\mu\nu}\partial_{\mu\nu}\varphi^\alpha = O(c^{-4}\varphi^\alpha) = O(7,8). \quad (4.24b)$$

Let us now define, in D_e, the following quantities

$$h_c^{\alpha\beta}(\mathbf{x}, t) := \frac{16\pi G}{c^4}\Box_R^{-1}\{\tau_c^{\alpha\beta}\}. \quad (4.25)$$

We can conclude from eqs (4.23), (4.24b) and the fact that, by construction, $\partial_\beta h_1^{\alpha\beta} = 0$ in D_e (independently of the higher-order corrections $h_n^{\alpha\beta}$) that, in D_e,

$$\partial_\beta h_c^{\alpha\beta} = -f^{\mu\nu}\partial_{\mu\nu}\varphi^\alpha + O(7,6) = O(7,6). \quad (4.26)$$

With this notation, eq. (4.23) transcribes simply to

$$G \bar{h}_1^{\alpha\beta}[\mathcal{M}] = \overline{\mathrm{ME}}\{h_c^{\alpha\beta}[\text{source}] + f^{\alpha\mu}\partial_\mu \varphi^\beta + f^{\beta\mu}\partial_\mu \varphi^\alpha - f^{\alpha\beta}\partial_\mu \varphi^\mu\} + O(6,7,6), \quad (4.27)$$

with the additional information that both "gothic deviations" are harmonic solutions of the linearized Einstein equations (*i.e.* that they satisfy both $\Box_f h^{\alpha\beta} = 0$ and $\partial_\beta h^{\alpha\beta} = 0$ modulo the indicated error terms). One should beware of the fact that, by its derivation, the equality asserted by eq. (4.27) holds only after having performed the (orbital) multipole expansion of the right-hand side (as indicated by the ME), as well as an implicit or explicit near-zone expansion of both sides (as symbolized by the overbar). We shall take up in the next section the task of deducing from eq. (4.27) explicit expressions for the algorithmic multipole moments in terms of the source. As a final comment let us emphasize that we have been able to reduce the problem to what is essentially a linearized gravity one only because we had already gone through a detailed analysis of nonlinearities, both for the external field (Section II) and for the inner one (present section). Even after having derived our final linear-looking result we do not see how it could have been obtained (or even heuristically guessed) without the foregoing careful analysis of the nonlinearities of Einstein's field equations.

V. POST-NEWTONIAN (MASS AND) SPIN MULTIPOLE MOMENTS

To sum up the results of the previous sections, we have shown that the linearized algorithmic external gothic metric $h_1^{\alpha\beta}[\mathcal{M}]$ was equal, in the external near zone $D_e \cap D_i$, modulo a linearized coordinate transformation and $O(6, 7, 6)$ near-zone error terms, to the multipole expansion of the linearized harmonic gothic metric $h_c^{\alpha\beta}$ generated via retarded potentials by an effective stress-energy tensor $\tau_c^{\alpha\beta}$. The latter effective source is a distribution in $\mathbf{x}$ having compact support and defined in the following way. Let us start from the usual pseudo stress-energy tensor for the matter and gravitational field system taken at the approximation we need:

$$\tau^{\alpha\beta} := \left(1 + \frac{4}{c^2} U^{in}\right) T^{\alpha\beta} + \frac{c^4}{16\pi G} N^{\alpha\beta}(U^{in}). \tag{5.1}$$

The quantity $\tau^{\alpha\beta}$ is not of (spatially) compact support because of the quadratically nonlinear contribution $N^{\alpha\beta}(U^{in})$. On inserting the explicit Newtonian-potential expressions (3.3) for U^{in} and U_i^{in} in the definition (3.1) of $N^{\alpha\beta}(U)$ one sees that $N^{\alpha\beta}(U^{in})$ can be written as a sum of double (in fact, sextuple) integrals (over two source points $\mathbf{y}_1$, $\mathbf{y}_2$) of derivatives (with respect to $\mathbf{y}_1$ and $\mathbf{y}_2$) of the product of the kernels of the two Newtonian potentials : $|\mathbf{x}-\mathbf{y}_1|^{-1} . |\mathbf{x}-\mathbf{y}_2|^{-1}$. Let us, as above, denote symbolically this structure as

$$N^{\alpha\beta}(U^{in}) = \sum\nolimits_{12}^{\alpha\beta}\left(\frac{1}{|\mathbf{x}-\mathbf{y}_1||\mathbf{x}-\mathbf{y}_2|}\right) \tag{5.2}$$

[*see* eqs (3.4) and (3.6)]. It is clear from the structure (5.2) (where Σ_{12} contains two spatial derivatives), that $N(U^{in})$ falls off only as $|\mathbf{x}|^{-4}$ at spatial infinity.

Let us now replace the product of two kernels, $r_1^{-1} r_2^{-1}$, in eq. (5.2) by a compact-support distribution according to the following prescription

$$|\mathbf{x}-\mathbf{y}_1|^{-1} |\mathbf{x}-\mathbf{y}_2|^{-1} \to -2\pi\,\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2), \tag{5.3}$$

where $\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$ is the distribution describing a uniform linear density along the segment joining $\mathbf{y}_1$ and $\mathbf{y}_2$ (with density = 1 per unit length). In explicit mathematical terms one has

$$\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) = r_{12} \int_0^1 d\alpha\, \delta(\mathbf{x} - \mathbf{y}_\alpha), \tag{5.4}$$

where

$$r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|, \qquad \mathbf{y}_\alpha = \alpha \mathbf{y}_1 + (1-\alpha)\mathbf{y}_2, \tag{5.5}$$

and where $\delta(\mathbf{x}-\mathbf{y})$ denotes the three-dimensional Dirac distribution in $\mathbf{x}$ (*i.e.* a unit mass located at $\mathbf{x}=\mathbf{y}$). The application of the prescription (5.3) in eq. (5.2) defines a compact-support distribution [23], say

$$\mathrm{N}_c^{\alpha\beta}(\mathbf{x}) = \textstyle\sum_{12}^{\alpha\beta}(-2\pi\,\delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)), \tag{5.6}$$

and the work of the previous sections shows that the compact-support effective stress-energy tensor $\tau_c^{\alpha\beta}$ which generates $h_1^{\alpha\beta}[\mathcal{M}]$ modulo a linearized coordinate transformation and $O(6, 7, 6)$ error terms is [*see* eq. (4.22)]

$$\tau_c^{\alpha\beta} := \left(1 + \frac{4}{c^2}\mathrm{U}^{\mathrm{in}}\right)\mathrm{T}^{\alpha\beta} + \frac{c^4}{16\pi \mathrm{G}}\mathrm{N}_c^{\alpha\beta}. \tag{5.7}$$

Explicitly, we find from eqs (3.1)

$$c^{-2}\tau_c^{00} = \left(1 + \frac{4}{c^2}\mathrm{U}^{\mathrm{in}}\right)\frac{\mathrm{T}^{00}}{c^2} + \frac{7\mathrm{G}}{4c^2}\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2 \frac{\partial^2}{\partial y_1^s\, \partial y_2^s}(\delta_{12}), \tag{5.8a}$$

$$\begin{aligned} c^{-1}\tau_c^{0i} = {} & \left(1 + \frac{4}{c^2}\mathrm{U}^{\mathrm{in}}\right)\sigma^i \\ & - \frac{\mathrm{G}}{2c^2}\Bigg[4\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2^s \frac{\partial^2}{\partial y_1^s\, \partial y_2^i}(\delta_{12}) \\ & - 4\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2^i \frac{\partial^2}{\partial y_1^s\, \partial y_2^s}(\delta_{12}) \\ & - 3\int d^3y_1\, d^3y_2\, \sigma_1^s \sigma_2 \frac{\partial^2}{\partial y_1^s\, \partial y_2^i}(\delta_{12})\Bigg], \end{aligned} \tag{5.8b}$$

$$\begin{aligned} \tau_c^{ij} = \mathrm{T}^{ij} - \frac{\mathrm{G}}{4}\Bigg[& 2\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2 \frac{\partial^2}{\partial y_1^i\, \partial y_2^j}(\delta_{12}) \\ & - \delta_{ij}\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2 \frac{\partial^2}{\partial y_1^s\, \partial y_2^s}(\delta_{12})\Bigg], \end{aligned} \tag{5.8c}$$

where

$$\sigma := c^{-2}(\mathrm{T}^{00} + \mathrm{T}^{ss}), \tag{5.9a}$$

$$\sigma^i := c^{-1} \mathrm{T}^{0i}, \tag{5.9b}$$

and where σ_1 (resp. σ_2) is a short-hand notation for $\sigma(\mathbf{y}_1)$ [resp. $\sigma(\mathbf{y}_2)$].

Let us now consider the retarded potential generated by $\tau_c^{\alpha\beta}$, say $h_c^{\alpha\beta}$ given by eq. (4.25), and study its multipole expansion outside the (compact) support of $\tau_c^{\alpha\beta}$. This question has been considered by us recently [18]. Let us first emphasize the important subtle differences between the problem treated in Ref. [18] and the present problem. Indeed, in Ref. [18] we were working within the framework of linearized gravity, *i. e.* we were assuming that the compact-support source (denoted $\mathrm{T}^{\alpha\beta}$) of the gothic deviation (denoted $-\bar{h}^{\alpha\beta}$) satisfied the local conservation law $\partial_\beta \mathrm{T}^{\alpha\beta} = 0$. By contrast, in the present problem the effective compact-support source $\tau_c^{\alpha\beta}$ is not locally conserved, but satisfies only an (approximate) "quasi-conservation" law which is discussed in Appendix A. This quasi-conservation law is weaker than the full conservation law and therefore, among all the integral identities used for $\mathrm{T}^{\alpha\beta}$ in Ref. [18], only a sub-class of them are still satisfied [modulo some additional error terms compatible with the error terms in eqs (4.26) and (4.27) above] in our present problem where $\mathrm{T}^{\alpha\beta}$ is replaced by $\tau_c^{\alpha\beta}$. This sub-class of "allowed" integral identities coincides (modulo the just alluded addition of error terms) with the class of identities among the STF tensors appearing in the original (unreduced) multipole expansion of the gothic deviation which can be derived from the harmonicity condition $\partial_\beta h^{\alpha\beta} = 0$. This class was shown in Refs [5] and [10] to consist of the relations (5.27)-(5.28) of Ref. [18]. Now, it is easy to check that the exact results (5.33) and (5.35) in Ref. [18] for the mass and spin multipole moments have been derived by using only the allowed identities (5.27)-(5.28) (with a special use of eq. (5.17), with eq. (5.27*b*), to simplify the final expression of the mass moments). The main conclusion is therefore that we can apply the "linearized" results (5.31)-(5.35) of Ref. [18] to our "quasi-linearized" problem, with the simple replacement $\mathrm{T}^{\alpha\beta} \to \tau_c^{\alpha\beta}$. As we shall emphasize below, this conclusion does *not* apply to other forms of the multipole moments of linearized gravity, in particular to the expressions derived by Thorne [5]. This again shows that $\tau_c^{\alpha\beta}$ is only an effective source for describing the external gravitational field after a multipole expansion in $\mathrm{D}_e \cap \mathrm{D}_i$ [remember the symbol ME in the right-hand side of eq. (4.27)], and that non-linear effects are still playing an important role in our final linear-looking results.

The final result of Ref. [18] is that the multipole-expanded external linearized gravitational field can be written, modulo a coordinate transformation w^α [source], as the linearized algorithmic gothic deviation [*see* eqs (2.9), (2.10) above] computed for some "source multipole moments" given as some explicit integrals of the right-hand side of the linearized field equations. In keeping with the notation of the Introduction (and of paper I) we shall denote these explicit integral expressions of source variables as I_L (for the mass moments) and J_L (for the spin moments). Beware of the change of notation from Ref. [18] where GI_L and GJ_L were denoted respectively as M_L and S_L (which we keep for *algorithmic* moments in this paper), and where the linearized gothic deviation $h^{\alpha\beta}$ (of the present paper) was denoted $-\bar{h}^{\alpha\beta}$.

In Ref. [18] we have obtained both the exact closed-form expressions of I_L [source] and J_L [source] and the first two terms of their near-zone expansion. Let us consider now the effect of the $O(6,7,6)$ error terms contained in eq. (4.23). It is easy to see (for instance by following paper I in eliminating the coordinate-transform terms φ^α and w^α [source] by computing the linearized curvature of both sides) that eq. (4.23) allows one to conclude that the *algorithmic* multipole moments, M_L, and S_L, of the left-hand side must coincide with the *source* multipole moments, I_L and J_L, of the right-hand side modulo fractional error terms $O(c^{-4})$ for both types of moments:

$$M_L = I_L[\text{source}] + O(4); \; l \geqq 0, \tag{5.10a}$$

$$S_L = J_L[\text{source}] + O(4); \; l \geqq 1. \tag{5.10b}$$

It is therefore sufficient to insert into the near-zone expanded forms derived in Ref. [18] [eqs (5.38) and (5.40) there] $\tau_c^{\alpha\beta}$ instead of $T^{\alpha\beta}$ to define the source multipole moments

$$(l \geqq 0);$$

$$I_L[\text{source}] := \int d^3x \left[\hat{x}^L \frac{\tau_c^{00} + \tau_c^{ss}}{c^2} + \frac{1}{2(2l+3)c^2} \mathbf{x}^2 \hat{x}^L \frac{\partial^2}{\partial t^2}\left(\frac{\tau_c^{00}}{c^2}\right) - \frac{4(2l+1)}{(l+1)(2l+3)c^2} \hat{x}^{La} \frac{\partial}{\partial t}\left(\frac{\tau_c^{0a}}{c}\right) \right] + O(4), \tag{5.11a}$$

$$(l \geqq 1);$$

$$J_L[\text{source}] := \underset{L}{\text{STF}}\, \varepsilon_{i_l ab} \times \int d^3x \left[\hat{x}^{aL-1} \left(\frac{\tau_c^{0b}}{c} + \frac{1}{2(2l+3)} \frac{\mathbf{x}^2}{c^2} \frac{\partial^2}{\partial t^2}\left(\frac{\tau_c^{0b}}{c}\right)\right) - \frac{(2l+1)}{(l+2)(2l+3)c^2} \hat{x}^{asL-1} \frac{\partial}{\partial t}(\tau_c^{bs}) \right] + O(4). \tag{5.11b}$$

Both sides of equations (5.11) are evaluated at the same coordinate time $t = x^0/c$. Let us recall that, for the multi-index $L = i_1 i_2 \ldots i_l$, $\underset{L}{\text{STF}}$ denotes the projection operation onto the symmetric and trace-free (STF) part of some cartesian tensor of rank l, and that $\hat{x}^L$ denotes the STF part of $x^L \equiv x^{i_1} x^{i_2} \ldots x^{i_l}$. The subscript "$c$" (for "compact") on $\tau_c^{\alpha\beta}$ should not be confused with the velocity of light which appears explicitly in eqs (5.11).

Finally, let us emphasize that one would obtain incorrect results if one would perform the replacement $T^{\alpha\beta} \to \tau_c^{\alpha\beta}$ in the "Thorne form" of the linearized multipole moments (eqs (5.39) and (5.41) of Ref. [18]), or, in other words, the replacement $\tau^{\alpha\beta} \rightarrow \tau_c^{\alpha\beta}$ in the formal expressions (5.32 *a*), (5.32 *b*) of Ref. [5] (*see* Appendix A).

V. A. The post-Newtonian mass moments

The main aim of the present paper was to obtain the post-Newtonian spin source moments, $J_L[\text{source}] + O(4)$, because the post-Newtonian mass source moments, $I_L[\text{source}] + O(4)$, have already been derived in paper I. However, as the method we used here is different we should first check that our new form (5.11 *a*) is equivalent to the one derived in paper I which was simply [eq. I (2.27)]

$$(l \geqq 0);\quad I_L[\text{source}] = \int d^3x \left[\hat{x}^L \sigma + \frac{1}{2(2l+3)c^2} \mathbf{x}^2 \hat{x}^L \frac{\partial^2 \sigma}{\partial t^2} - \frac{4(2l+1)}{(l+1)(2l+3)c^2} \hat{x}^{La} \frac{\partial \sigma^a}{\partial t} \right] + O(4), \quad (5.12)$$

where σ and σ^i are defined as in eqs (5.9) above. Remembering that the (nonlinear) δ_{12}-terms in $\tau_c^{\alpha\beta}$, eqs (5.8), bring additional terms of relative order $O(c^{-2})$ in τ_c^{00} and τ_c^{0i}, but of order $O(c^0)$ in τ_c^{ij}, we see that eq. (5.11 *a*) will be consistent with eq. (5.12) if the additional δ_{12}-terms present in $\int d^3x\, \hat{x}^L (\tau_c^{00} + \tau_c^{ss})/c^2$ cancel with the $4\,U^{\text{in}}\sigma/c^2$ present in the same integral. Adding the trace of eq. (5.8 *c*), we see that it reduces to checking the validity of

$$\int d^3x\, \hat{x}^L \left[4\,U^{\text{in}}(\mathbf{x})\, \sigma(\mathbf{x}) + 2G \int\int d^3y_1\, d^3y_2\, \sigma(\mathbf{y}_1)\, \sigma(\mathbf{y}_2) \frac{\partial^2}{\partial y_1^s\, \partial y_2^s} \delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) \right] = 0. \quad (5.13)$$

In the second term we can interchange the d^3x with the $d^3y_1\, d^3y_2$ integration, letting the distribution $\delta_{12}(\mathbf{x})$ act on the "test" function $\hat{x}^L$ to produce from eqs (5.4), (5.5)

$$\int d^3x\, \hat{x}^L \delta_{12}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) = r_{12} \int_0^1 d\alpha\, \hat{y}_\alpha^L,$$

where $\hat{y}_\alpha^L = \text{STF}(y_\alpha^{i_1} \ldots y_\alpha^{i_l})$, with each vector y_α^i being given by eq. (5.5) (with a common value of α). It is then easy to see that eq. (5.13), being a quadratic functional of $\sigma(\mathbf{y})$, is equivalent to

$$\frac{\partial^2}{\partial y_1^s\, \partial y_2^s} \left[r_{12} \int_0^1 d\alpha\, \hat{y}_\alpha^L \right] = -\frac{1}{r_{12}} [\hat{y}_1^L + \hat{y}_2^L]. \quad (5.14\,a)$$

To see that eq. (5.14 *a*) holds it is convenient to work with the variables

$$\mathbf{y} := \frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2), \quad (5.15a)$$

$$\mathbf{z} := \frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_2), \quad (5.15b)$$

$$\gamma := 2\alpha - 1, \quad (5.15c)$$

such that

$$\mathbf{y}_\alpha := \mathbf{y} + \gamma\,\mathbf{z}. \quad (5.15d)$$

With these variables eq. (5.14*a*) is equivalent (taking into account $\Delta_y[(y+\gamma z)^{\langle L \rangle}] = 0$) to

$$\Delta_z \left\{ \frac{|\mathbf{z}|}{2} \int_{-1}^{+1} d\gamma\, (y+\gamma z)^{\langle L \rangle} \right\} = \frac{1}{|\mathbf{z}|} \{ (y+z)^{\langle L \rangle} + (y-z)^{\langle L \rangle} \}, \quad (5.14b)$$

which is easily verified to hold. For completeness, let us note that the integral identity (5.13) can also be proven by verifying the following differential identity among distributions:

$$\frac{\partial^2}{\partial y_1^s\, \partial y_2^s} \delta_{12}(\mathbf{x};\, \mathbf{y}_1,\, \mathbf{y}_2) = -\frac{1}{r_{12}} [\delta(\mathbf{x}-\mathbf{y}_1) + \delta(\mathbf{x}-\mathbf{y}_2)] + \Delta_x \left[r_{12} \int_0^1 d\alpha\, \alpha (1-\alpha)\, \delta(\mathbf{x}-\mathbf{y}_\alpha) \right]. \quad (5.16)$$

Having checked that the post-Newtonian mass moments agree with the results of paper I, let us next turn our attention to the spin multipole moments.

V. B. The post-Newtonian spin moments

The calculations done for the mass moments indicate the route to obtaining an explicit expression for the spin source moments. First one notices that one needs to keep the δ_{12}-additional terms only in the first and the third terms in the right-hand side of eq. (5.11 *b*), but that the replacement $\ddot{\tau}_c^{0b} \to \ddot{\sigma}^b$ is accurate enough in the second term. Let us also introduce a special notation for the integrals that appear when one performs the d^3x integration first:

$$\mathrm{Y}^L(\mathbf{y}_1,\, \mathbf{y}_2) := r_{12} \int_0^1 d\alpha\, y_\alpha^{\langle L \rangle} \quad (5.17a)$$

(where $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, $y_\alpha^i = \alpha y_1^i + (1-\alpha) y_2^i$), and for their derivatives

$$\mathrm{Y}^L_{,ab} := \frac{\partial^2}{\partial y_1^a\, \partial y_2^b} \mathrm{Y}^L. \quad (5.17b)$$

Note that Y^L is a STF tensor that depends on two source points (and on the choice of origin for the spatial coordinates x^i), and that $Y^L(\mathbf{y}_1, \mathbf{y}_2) = Y^L(\mathbf{y}_2, \mathbf{y}_1)$ while the order of indices after the comma in $Y^L_{,ab}$ is important : $Y^L_{,ab}(\mathbf{y}_1, \mathbf{y}_2) = Y^L_{,ba}(\mathbf{y}_2, \mathbf{y}_1)$. With the notation (5.17), we can now write down from eqs (5.11 *b*) and (5.8) the main new result of this paper, *i. e.* the explicit expression of the post-Newtonian spin source moments, J_L [source] + $O(4)$:

$$(l \geqq 1);$$

$$J_L\,[\text{source}] = \underset{L}{\text{STF}} \left\langle \varepsilon_{i_l ab} \left\{ \int d^3 x\, \hat{x}^{aL-1} \left(1 + \frac{4\,U^{\text{in}}}{c^2}\right) \sigma^b(x) \right.\right.$$
$$- \frac{G}{2c^2} \int d^3 y_1\, d^3 y_2 \,[\sigma_1^s\, \sigma_2\,(4\,Y^{aL-1}_{,bs} - 3\,Y^{aL-1}_{,sb}) - 4\,\sigma_1^b\,\sigma_2\,Y^{aL-1}_{,ss}]$$
$$+ \frac{1}{2(2l+3)c^2} \frac{d}{dt} \left[\int d^3 x\, \hat{x}^{aL-1}\, \mathbf{x}^2 \frac{\partial \sigma^b}{\partial t} - \frac{2(2l+1)}{(l+2)} \int d^3 x\, \hat{x}^{asL-1}\, T^{bs} \right.$$
$$\left.\left.\left. + \frac{G(2l+1)}{(l+2)} \int d^3 y_1\, d^3 y_2\, \sigma_1\, \sigma_2\, Y^{asL-1}_{,bs} \right] \right\} \right\rangle + O(4), \quad (5.18)$$

where all quantities on both sides are evaluated at the same coordinate time, t, and where $\sigma_i = \sigma(\mathbf{y}_i, t)$; $i = 1,2$ [we recall also the definitions (4.3 *a*) and (4.4)].

Our central result (5.18) can be cast in another equivalent form (for the isolated systems that we are considering) by using, in the 1 PN correction terms, the Newtonian momentum balance equation:

$$\frac{\partial \sigma^b}{\partial t} + \frac{\partial T^{bj}}{\partial x^j} = \sigma \frac{\partial U^{\text{in}}}{\partial x^b} + O(2). \quad (5.19)$$

Replacing $\partial\sigma^b/\partial t$ in eq. (5.18) using eq. (5.19), performing an integration by parts in the term containing $\partial T^{bj}/\partial x^j$, and employing (*see* Ref. [18])

$$\hat{x}_{ac\,L-1} = \underset{L-1}{\text{STF}} \left[x_a\, x_c\, \hat{x}_{L-1} - \frac{l-1}{2l+1} r^2 (\hat{x}_{a\,L-2}\, \delta_{ci_{l-1}} + \hat{x}_{c\,L-2}\, \delta_{ai_{l-1}}) \right.$$
$$\left. - \frac{\delta_{ac}\, \hat{x}_{L-1}\, r^2}{2l+1} - \frac{(l-1)(l-2)}{(2l-1)(2l-3)} r^4\, \hat{x}_{L-3}\, \delta_{ai_{l-2}}\, \delta_{ci_{l-1}} \right] \quad (5.20)$$

(where r^2 denotes $\mathbf{x}^2$) it follows after some computation that the $\partial\sigma^b/\partial t$ and T^{bs} terms reduce to

$$\frac{1}{2(2l+3)c^2} \underset{L}{\text{STF}}\, \varepsilon_{i_l ab} \frac{d}{dt} \left\{ \int d^3 x\, x^{a\,L-1}\, r^2\, \sigma\, U_{,b} \right.$$
$$\left. + \frac{l-1}{l+2} \int d^3 x\, [(l+4)\, x^{a\,L-2}\, r^2\, T^{bi_{l-1}} - 2\, x^{as\,L-1}\, T^{bs}] \right\}.$$

Thus J_L can equivalently be rewritten as

$$(l \geqq 1);\quad J_L(t) = \underset{L}{\mathrm{STF}} \Big\langle \varepsilon_{i_l ab} \Big[\int d^3 x\, x^{aL-1} \left(1 + \frac{4U^{\mathrm{in}}}{c^2}\right) \sigma^b$$
$$- \frac{G}{2c^2} \int d^3 y_1\, d^3 y_2 \{ \sigma_1^s \sigma_2 (4 Y^{aL-1}_{,bs} - 3 Y^{aL-1}_{,sb}) - 4 \sigma_1^b \sigma_2 Y^{aL-1}_{,ss} \}$$
$$+ \frac{1}{2(2l+3)c^2} \frac{d}{dt} \Big\{ \int d^3 x\, x^{assL-1} \sigma \frac{\partial U^{\mathrm{in}}}{\partial x^b}$$
$$+ \frac{G(2l+1)}{(l+2)} \int d^3 y_1\, d^3 y_2\, \sigma_1 \sigma_2 Y^{asL-1}_{,bs}$$
$$+ \frac{(l-1)}{(l+2)} \int d^3 x [(l+4) x^{assL-2} T^{bi_{l-1}} - 2 x^{asL-1} T^{bs}] \Big\} \Big] \Big\rangle. \tag{5.21}$$

In the form above the x^L's are not STF projected, and the explicit contribution of the stress term disappears for $l=1$ (but not for $l \geqq 2$). It may be noted however that, both in eq. (5.18) and in eq. (5.21) only the STF part of the stress tensor, $\hat{T}^{ab} \equiv T^{ab} - \delta^{ab} T^{ss}/3$, contributes.

The integral forms (5.18) or (5.21) constitute our complete and general answers to the problem of finding explicit expressions for the post-Newtonian spin algorithmic moments in terms of source variables. However, we shall next indicate how the forms (5.18) or (5.21), which contain explicit double integrals $\left(\iint d^3 y_1\, d^3 y_2\right)$, can be transformed into some *"potential"* forms whose explicit appearance will contain only *simple integrals* $\left(\int d^3 y\right)$, the second integral being absorbed into the definition of some intermediate potentials $\left(\text{e.g. } U_{(\alpha)}(\mathbf{y}) = \int d^3 y'\, \sigma(\mathbf{y}') \,|\mathbf{y} - \mathbf{y}'|^{\alpha}\right)$. The algorithm for going to the potential form is most conveniently implemented by expressing all quantities in terms of the pair of variables $(\mathbf{y}_1, \mathbf{y}_{12})$, with

$$y_{12}^i := y_1^i - y_2^i, \tag{5.22}$$

instead of the original pair of source points $(\mathbf{y}_1, \mathbf{y}_2)$.

Since one needs to evaluate expressions of the form

$$Y^L_{,ab} := \frac{\partial^2}{\partial y_1^a\, \partial y_2^b} \left(r_{12} \int_0^1 d\alpha\, y_\alpha^{\langle L \rangle} \right)$$

one begins by expanding $\int_0^1 d\alpha\, y_\alpha^{\langle L \rangle}$ in terms of the new variables. This is easily done by starting from

$$(y+z)^{L} = \underset{L}{\mathrm{S}} \left\{ \sum_{p=0}^{l} \binom{l}{p} y^{L-P} z^{P} \right\}, \tag{5.23}$$

where S denotes the symmetrization operator over the l indices present on $z\,(P = i_1 \ldots i_p)$ or $y\,(L-P := i_{p+1} \ldots i_l)$. One finds

$$\int_0^1 d\alpha\, y_\alpha^{\langle L \rangle} = \sum_{p=0}^{l} \binom{l}{p} \frac{(-)^p}{(p+1)} y_1^{\langle L-P} y_{12}^{P \rangle}. \tag{5.24}$$

One thus ends up with having to evaluate expressions of the form $\frac{\partial}{\partial y_1^b\, \partial y_2^c}(r_{12} f(\mathbf{y}_1, \mathbf{y}_{12}))$ which yields

$$-\frac{\partial^2}{\partial y_1^b\, \partial y_2^c}(r_{12} f(\mathbf{y}_1, \mathbf{y}_{12})) = r_{12,\,cb} f + r_{12,\,c} \frac{\partial f}{\partial y_1^b} + r_{12,\,c} \frac{\partial f}{\partial y_{12}^b} + r_{12,\,b} \frac{\partial f}{\partial y_{12}^c} + r_{12} \frac{\partial^2 f}{\partial y_1^b\, \partial y_{12}^c} + r_{12} \frac{\partial^2 f}{\partial y_{12}^b\, \partial y_{12}^c}$$

(where $r_{12,\,b}$ denotes $\partial r_{12}/\partial y_{12}^b$, and where the partial derivatives in the left-hand side refer to the pair $(\mathbf{y}_1, \mathbf{y}_2)$, while the ones in the right-hand side refer to $(\mathbf{y}_1, \mathbf{y}_{12})$).

Consequently at the end of the computations for $Y^{L}_{,ab}$ one is left with a sum of terms whose general structure is

$$\mathrm{G} \int d^3 y_1\, \sigma_1^k\, y_1^{M} \int d^3 y_2\, \sigma_2\, r_{12}^{\alpha}\, n_{12}^{N}.$$

The integral over $\mathbf{y}_2$ introduces some generalized tensorial "potential' which we designate as $P_{(\alpha)}^{N}(\mathbf{y}_1)$:

$$P_{(\alpha)}^{i_1 \ldots i_n}(\mathbf{x}) \equiv P_{(\alpha)}^{N}(\mathbf{x}) := \mathrm{G} \int d^3 y\, \sigma(\mathbf{y})\, r_{xy}^{\alpha}\, n_{xy}^{i_1} \ldots n_{xy}^{i_n}, \tag{5.23}$$

where $r_{xy} := |\mathbf{x} - \mathbf{y}|$ and $n_{xy}^i := (x^i - y^i)/r_{xy}$ (the unit vector being oriented from the "source point" $\mathbf{y}$ to the "field point" $\mathbf{x}$).

Note that the scalar potential $P_{(-1)}$ is just the usual Newtonian potential of σ, eq. (4.3 *a*) (we shall henceforth drop the label "in" on U)). Note also that all our potentials are generated by the scalar source σ, we shall not introduce any potential generated by the vector source σ^i.

In terms of the P's the spin moments are thus expressible as a sum of terms of the form

$$\int d^3 x\, \sigma^k(\mathbf{x})\, x^M\, P^N_{(\alpha)}(\mathbf{x}).$$

We illustrate the above procedure for $l=1$ and discuss it in detail. The result for $l=2$ is also explicitly given below. From our discussion and the examples it is clear that the procedure is generic and the general structure of the terms constituting the spin moments will be written down. The relevant numerical coefficiens may be computed if needed and the only complication is a proliferation of terms.

For $l=1$ eq. (5.21) becomes

$$J_i = J_i^{LL}[\tau_c] + \frac{1}{10\,c^2}\frac{d}{dt}Q_i,$$

where

$$\begin{aligned} J_i^{LL}[\tau_c] &= \varepsilon_{iab}\int d^3 x\, x^a \frac{\tau_c^{0\,b}}{c} \\ &= \varepsilon_{iab}\left[\int d^3 x\, x^a\left(1+\frac{4}{c^2}U\right)\sigma^b \right.\\ &\quad \left. - \frac{G}{2\,c^2}\int d^3 y_1\, d^3 y_2\,\{\sigma_1^s\,\sigma_2\,(4\,Y^a_{,bs} - 3\,Y^a_{,sb}) - 4\,\sigma_1^b\,\sigma_2\,Y^a_{,ss}\}\right], \end{aligned} \quad (5.26)$$

denotes the familiar looking "Landau-Lifshitz" form of the total spin, with the replacement $\tau^{\alpha\beta}\to\tau_c^{\alpha\beta}$, and where

$$Q_i = \varepsilon_{iab}\left[\int d^3 x\,(x^a \mathbf{x}^2 \sigma\,\partial_b U) + G\int d^3 y_1\, d^3 y_2\,\sigma_1\,\sigma_2\,Y^{as}_{,bs}\right]. \quad (5.27)$$

A straightforward calculation gives

$$Y^a_{,bs} = -\left[r_{12,\,bs}\left(y_1^a - \frac{1}{2}y_{12}^a\right) + \frac{1}{2}r_{12,\,s}\,\delta_{ab} - \frac{1}{2}r_{12,\,b}\,\delta_{as}\right],$$

$$\varepsilon_{iab}\,Y^{as}_{,bs} = \varepsilon_{iab}\,n_{12}^b\left(\frac{n_{12}^s\,y_1^{\langle as\rangle}}{r_{12}} + \frac{1}{3}y_1^a\right),$$

and eqs (5.26) and (5.27) become respectively

$$J_i^{LL}[\tau_c] = \varepsilon_{iab}\int d^3 x\left[x^a\sigma^b + \frac{1}{2\,c^2}\left(x^a\sigma^b P_{(-1)} - x^a\sigma^s P^{sb}_{(-1)} + 7\,\sigma^b P^a_{(0)}\right)\right],$$

$$Q_i = \varepsilon_{iab}\int d^3 x\left(x^{ass}\,\partial_b P_{(-1)} + x^{as} P^{bs}_{(-1)} + \frac{1}{3}x^a\,\partial_b P_{(1)}\right)\sigma.$$

The above forms look more involved than necessary. This however is easy to remedy if one realizes that the potentials $\partial_b \mathrm{P}_{(-1)}$, $\mathrm{P}^{sb}_{(-1)}$ and $\partial_b \mathrm{P}_{(1)}$ are not all independent but linked by algebraic relations. Using,

$$\mathrm{P}^{bc}_{(-1)} = \delta^{bc}\,\mathrm{P}_{(-1)} - \partial_{bc}\,\mathrm{P}_{(1)}, \qquad (5.28\,a)$$

$$\Delta \mathrm{P}_{(1)} = 2\,\mathrm{P}_{(-1)}, \qquad (5.28\,b)$$

the expression of Q_i becomes

$$\mathrm{Q}_i = \varepsilon_{iab} \int d^3 x \left\{ \frac{1}{2}\, x^{ass}\, \Delta \partial_b \mathrm{P}_{(1)} - x^{ac}\, \partial_{bc} \mathrm{P}_{(1)} + \frac{1}{3}\, x^a\, \partial_b \mathrm{P}_{(1)} \right\} \sigma. \quad (5.29)$$

An explicit calculation then shows that Q_i identically vanishes as a consequence of the following identity

$$\varepsilon_{iab}\,[3\,(y_1^a\,|\mathbf{y}_1|^2 - y_2^a\,|\mathbf{y}_2|^2)\,\partial_{bcc} - 6\,(y_1^a\,y_1^c + y_2^a\,y_2^c)\,\partial_{bc} + 2\,y_{12}^a\,\partial_b]\,r_{12} = 0, \qquad (5.30)$$

where the differentiations above are with respect to $\mathbf{y}_1$. Thus for an isolated system we have proven that

$$\mathrm{J}_i = \mathrm{J}_i^{\mathrm{LL}}[\tau_c]$$
$$= \varepsilon_{iab} \int d^3 x \left[x^a\,\sigma^b + \frac{1}{2\,c^2}\,(x^a\,\sigma^b\,\mathrm{P}_{(-1)} - x^a\,\sigma^s\,\mathrm{P}^{sb}_{(-1)} + 7\,\sigma^b\,\mathrm{P}^a_{(0)}) \right]. \quad (5.31)$$

It may be worth pointing out that the above relation may be rewritten entirely in terms of the potential $\mathrm{P}_{(1)}$ as

$$\mathrm{J}_i = \varepsilon_{iab} \int d^3 x \left[x^a\,\sigma^b + \frac{1}{2\,c^2}\,(x^a\,\sigma^s\,\partial_{sb}\,\mathrm{P}_{(1)} + 7\,\sigma^b\,\partial_a\,\mathrm{P}_{(1)}) \right]. \quad (5.32)$$

It may be then transformed to the following "Fock" form of the conserved total spin of an isolated system at the post-Newtonian approximation [3]

$$\mathrm{J}_i = \varepsilon_{iab} \int d^3 x\, x^a \left\{ \sigma^b \left(1 + \frac{4\,\mathrm{U}}{c^2} \right) - \frac{\sigma}{c^2} \left(4\,\mathrm{U}^b - \frac{1}{2}\,\mathrm{W}^b \right) \right\}, \quad (5.33)$$

where U^b is the vector potential (3.3 b) and

$$\mathrm{W}^b(\mathbf{x}) := \mathrm{G} \int d^3 y\, \sigma^a(\mathbf{y})\, \frac{(\delta^{ab} - n^a_{xy}\, n^b_{xy})}{r_{xy}}.$$

This identity between our algorithm-derived $l=1$ spin moment and the usual conserved 1 PN spin is a necessary check on our analysis because we knew in advance that the algorithmic spin S_i is exactly conserved, and, we had shown that $\mathrm{J}_i = \mathrm{S}_i + O(4)$ [eq. (5.10 b)]. We can note in passing that, contrarily to the Fock formula (5.33), or our "compactified Landau-Lifshitz" formula (5.26), the original Landau-Lifshitz one,

$$J_i^{LL}[\tau]=\varepsilon_{iab}\int d^3x\, x^a\,\frac{\tau^{0b}}{c},$$

with τ given by eq. (5.1), is mathematically ambiguous as the integral on the right-hand side is not absolutely convergent. This convergence problem evidently worsens for higher values of l, and shows again the necessity of a proper treatment of nonlinear gravitational effects.

For completeness let us note that if one starts from eq. (5.18) and does not use the momentum balance equation (5.19), but makes use of the Newtonian mass conservation law,

$$\frac{\partial\sigma}{\partial t}+\partial_i\sigma^i=O(c^{-2}),$$

to transform the last (1 PN) term in eq. (5.18), one derives yet another "potential" form of the total spin:

$$\begin{aligned}J_i=\varepsilon_{iab}\Big\langle\int d^3x\Big[x^a\sigma^b+\frac{1}{5c^2}\{18\,\sigma^b P^a_{(0)}+2x^a\sigma^b P_{(-1)}\\ -x^a\sigma^k P^{kb}_{(-1)}-x^c\sigma^b P^{ca}_{(-1)}-3x^{ac}\sigma^k P^{kcb}_{(-2)}\\ +x^{ac}\sigma^b P^c_{(-2)}+x^{ak}\sigma^k P^b_{(-2)}\}\Big]\\ +\frac{1}{10c^2}\frac{d^2}{dt^2}\int d^3x\,x^a\mathbf{x}^2\sigma-\frac{1}{5c^2}\frac{d}{dt}\int d^3x\,x^{ac}T^{bc}\Big\rangle.\end{aligned}\quad(5.34)$$

Let us now discuss the "potential" form of the spin quadrupole ($l=2$). For $l=2$ eq. (5.21) becomes

$$\begin{aligned}J_{ij}=\underset{ij}{\mathrm{STF}}\Big\langle\varepsilon_{jab}\Big[\Big[\int d^3x\,x^{ai}\Big(1+\frac{4}{c^2}U\Big)\sigma^b\\ -\frac{G}{2c^2}\int d^3y_1\,d^3y_2\{\sigma_1^s\sigma_2(4Y^{ai}_{,bs}-3Y^{ai}_{,sb})-4\sigma_1^b\sigma_2 Y^{ai}_{,ss}\}\\ +\frac{1}{14c^2}\frac{d}{dt}\Big\{\int d^3x\,x^{assi}\sigma\partial_b U+\frac{5G}{4}\int d^3y_1\,d^3y_2\,\sigma_1\sigma_2 Y^{asi}_{,bs}\\ +\frac{1}{4}\int d^3x[6x^{ass}T^{bi}-2x^{asi}T^{bs}]\Big\}\Big]\Big\rangle.\end{aligned}\quad(5.35)$$

Implementing the procedure outlined above for introducing potentials is long but straightforward and yields

$$\begin{aligned}J_{ij}=\underset{ij}{\mathrm{STF}}\Big\langle\varepsilon_{jab}\int d^3x\{\sigma^b x^{ai}\\ +\frac{1}{c^2}\Big[\frac{1}{2}\sigma^b x^{ai}P_{(-1)}-\frac{1}{2}\sigma^s x^{ai}P^{bs}_{(-1)}+\frac{7}{4}\sigma^b x^a P^i_{(0)}\end{aligned}$$

$$+\frac{1}{4}\,\sigma^s x^a \mathrm{P}^{bsi}_{(0)}+\frac{7}{2}\,\sigma^b x^i \mathrm{P}^a_{(0)}+\frac{11}{4}\,\sigma^a \mathrm{P}^{bi}_{(1)}-\frac{7}{4}\,\sigma^i x^a \mathrm{P}^b_{(0)}\Bigg]$$

$$+\frac{1}{c^2}\frac{d}{dt}\Bigg[\frac{5}{56}\,\sigma\, x^{ais}\mathrm{P}^{sb}_{(-1)}-\frac{1}{56}\,\sigma\, x^{ass}\mathrm{P}^{ib}_{(-1)}-\frac{3}{112}\,\sigma\, x^{as}\mathrm{P}^{sbi}_{(0)}$$

$$+\frac{9}{112}\,\sigma\, x^{ai}\mathrm{P}^{b}_{(0)}-\frac{9}{112}\,\sigma\, x^{a}\mathrm{P}^{bi}_{(1)}-\frac{1}{14}\,\sigma\, x^{aiss}\mathrm{P}^{b}_{(-2)}$$

$$+\frac{3}{28}\,x^{ass}\mathrm{T}^{bi}-\frac{1}{28}\,x^{asi}\mathrm{T}^{bs}\Bigg]\Bigg\}\Bigg\rangle. \quad (5.36)$$

It may be worth pointing out that, contrarily to the $l=1$ case, the stress terms have not been eliminated in the expression for J_{ij} by our use of the momentum balance equation (5.19). In fact, if we had not used eq. (5.19) the expression for J_{ij} would have been slightly simpler, the last three terms in the right-hand side of eq. (5.36) getting replaced by

$$\frac{1}{14\,c^2}\,\underset{ij}{\mathrm{STF}}\,\varepsilon_{jab}\,\frac{d}{dt}\left[\int d^3x\left(\hat{x}^{ai}\,\mathbf{x}^2\,\frac{\partial\sigma^b}{\partial t}-\frac{5}{2}\,\hat{x}^{asi}\,\mathrm{T}^{bs}\right)\right]. \quad (5.37)$$

As a check on the algebra which led to eq. (5.36) we have verified that the right-hand side of eq. (5.36) transforms under a shift of the origin of the spatial coordinate system by the formulas discussed in Appendix B below.

Let us end by outlining the structure of the potential form of the general spin multipole moment.

From an examination of the form of $\mathrm{J_L}$, eq. (5.21), it is easy to demonstrate that in the $\mathrm{Y}^{a\mathrm{L}-1}_{,cd}$ 1 PN terms (with c, $d\neq a$) the implicit STF operation on $a\mathrm{L}-1$ [*see* the definition (5.17)] can be replaced by a mere symmetrization since the trace terms do not contribute. Further in the terms containing an inner contraction between up and down indices, $\mathrm{Y}^{as\mathrm{L}-1}_{,bs}$, only the terms involving a single trace over either as or si_k contribute beyond the untraced symmetrized ones. With these comments it is easy to show that for arbitrary l, $\mathrm{J_L}$ may be written in the following potential form

$$\mathrm{J_L}\sim\underset{\mathrm{L}}{\mathrm{STF}}\,\varepsilon_{i_l ab}\int d^3x\Bigg[x^{a\mathrm{L}-1}\sigma^b$$

$$+\frac{1}{c^2}\Bigg\{\sum_{l_1=0}^{l}x^{\mathrm{L}_1}\Bigg[\sigma^k\,\mathrm{P}^{\mathrm{L}-\mathrm{L}_1}_{(l-l_1-1)}+\sigma^s\,\mathrm{P}^{s\,\mathrm{L}-\mathrm{L}_1+1}_{(l-l_1-1)}$$

$$+\frac{d}{dt}\left(\sigma\,\mathrm{P}^{\mathrm{L}-\mathrm{L}_1+1}_{(l-l_1)}+x^s\,\sigma\,\mathrm{P}^{s\,\mathrm{L}-\mathrm{L}_1+1}_{(l-l_1-1)}+x^{ss}\,\sigma\,\mathrm{P}^{\mathrm{L}-\mathrm{L}_1+1}_{(l-l_1-2)}\right)\Bigg]$$

$$+\frac{l-1}{2(2l+3)(l+2)}\,\frac{d}{dt}\left((l+4)\,x^{ass\,\mathrm{L}-2}\,\mathrm{T}^{bi_{l-1}}-2\,x^{as\,\mathrm{L}-1}\,\mathrm{T}^{bs}\right)\Bigg\}\Bigg], \quad (5.38)$$

where $k=b, i_{l-1}$, and the symbol $\sim$ means that the numerical coefficients of (most of) the terms are omitted. Finally let us remark that, similarly to eq. (5.32), one could express J_L in terms of only scalar potentials $P_{(\alpha)}(x)=\int d^3 y\, \sigma(y)\,|\mathbf{x}-\mathbf{y}|^\alpha$, and their derivatives [using the generalizations of eq. (5.28 *a*) to arbitrary values of α and arbitrary number of derivatives].

VI. SUMMARY AND CONCLUSION

This paper is the second one in a series of papers that develop a new formalism for investigating the generation of gravitational waves by semi-relativistic sources. The main idea of this formalism dates back to Fock [3] and consists of splitting the problem into two sub-problems (near-zone problem and external problem) whose solutions are then "matched". The main tool used for implementing this formalism is a recently developed multipolar post-Minkowskian (MPM) algorithm ([8] − [12]) which is summarized in Section II above. This algorithm (which implemented ideas put forward by Bonnor and Thorne) makes an essential use of a skeleton of time-dependent "algorithmic multipole moments", M_L, S_L, which play the role of arbitrary functional parameters in the construction of the external gravitational field. The first paper in this series [6] had achieved two results:

(i) to show that the "radiative multipole moments" that parametrize the asymptotic outgoing gravitational wave amplitude [5], say $M_L^{\rm rad}$ and $S_L^{\rm rad}$ [*see* eq. (1.4) above], are related to the algorithmic multipole moments by

$$(l\geqq 2);\quad M_L^{\rm rad}=M_L(U)+O(c^{-3}), \tag{6.1 a}$$
$$(l\geqq 2);\quad S_L^{\rm rad}=S_L(U)+O(c^{-3}), \tag{6.1 b}$$

where $U=T-R/c$ is the retarded time in some "radiative" coordinate system; and

(ii) to show that the algorithmic mass multipole moment, M_L, is related to the distribution of energy, momentum and stress in the material source by a relation of the form

$$(l\geqq 0);\quad M_L=I_L[\text{source}]+O(c^{-4}), \tag{6.2}$$

where I_L [source] denotes a well-defined spatial integral over the compact support of the material source [*see* eq. (5.12) above].

In the present paper we have succecded in finding the analogue of eq. (6.2) for the spin multipole moments, namely

$$(l\geqq 1);\quad S_L=J_L[\text{source}]+O(c^{-4}), \tag{6.3}$$

where S_L is the algorithmic spin multipole moment and J_L[source] a well-defined compact-support integral involving only the source variables [given by eq. (5.18) or by eq. (5.21)]. The new tools that have allowed us to succeed in getting eq. (6.3) were: (i) the results of a recent reexamination [18] of the multipole analysis of the linearized gravitational field emitted by a compact source, and (ii) a study of the various kernels appropriate for solving the quadratic nonlinearities of Einstein equations (Section III above).

Our post-Newtonian spin-dipole source moment J_i[source] has been shown to coincide [modulo $O(c^{-4})$] with the usual post-Newtonian total spin vector for an isolated system [3], eq. (5.33), for which our method provides several new expressions, eqs (5.26), (5.31), (5.32) and (5.34).

We have shown explicitly how to express our post-Newtonian spin-quadrupole source moment, J_{ij}[source], originally given in the (compact-support) double-integral form (5.35), in an alternative (compact-support) simple-integral form (5.36) involving some auxiliary potentials, eq. (5.25). This transformation to a "potential" form works for all higher moments as outlined in eq. (5.38) for the lth-order post-Newtonian spin moment originally obtained in the double-integral forms (5.18) or (5.21).

Our new results are important in two separate respects. On the one hand, they bring (in conjunction with the other explicit results of Section II above) an explicit knowledge of the multipole expansion of the external near-zone field entirely in terms of the source at the level of accuracy $\delta\mathcal{G}^{00}=O(c^{-6})$, $\delta\mathcal{G}^{0i}=O(c^{-7})$, $\delta\mathcal{G}^{ij}=O(c^{-6})$ (which is well beyond the 1 PN level of Ref. [6], being nearly at the 2 PN level). On the other hand, they bring [in conjunction with eqs (6.1)] an explicit knowledge of the "magnetic part" of the gravitational wave amplitude in terms of the source with unprecedented accuracy. This knowledge can be useful for different purposes. It can allow one to refine the recent formalism of Ref. [24] in which a consistent post-Newtonian formulation of the dynamics and gravitational wave generation of a semi-relativistic source has been given in a form suitable for numerical calculations. Indeed, the contribution to the gravitational wave amplitude (1.4) of the post-Newtonian terms in J_L and thereby $S_L^{\rm rad}$ is of order $O(c^{-3})$ (because of the extra c^{-1} in front of spin moments) relatively to the dominant (Newtonian) quadrupole contribution and this is bigger than the fractional error $(O(c^{-4}))$ in the treatment of the source dynamics in Ref. [24]. Moreover, as emphasized in Ref. [25], an accurate knowledge of the radiative spin-quadrupole moment is important for improving the estimate of the physically important "rocket effect" induced in a black-hole binary system by the emission of gravitational radiation, and this is provided by the present work. For such applications, we give in Appendix C below the point-particle limit of our result for the post-Newtonian spin quadrupole.

Finally, let us note that the improvement brought about by the present paper in the relation between algorithmic moments and source moments [$O(c^{-4})$ error terms in both eqs (6.2) and (6.3)] obliges us now to refine the link between the radiative moments and the algorithmic ones [only $O(c^{-3})$ accuracy in eqs (6.1)]. This refinement is, for instance, necessary in eq. (6.1 *a*) if we want to control the full gravitational wave amplitude modulo errors $O(c^{-4})$ relative to the dominant quadrupole contribution. This problem will be the subject of a subsequent paper.

REFERENCES

[1] A. ENSTEIN, *Preuss. Akad. Wiss. Sitzber, Berlin*, 1918, p. 154.

[2] L. D. LANDAU and E. M. LIFSHITZ, *Teoriya Polya*, Nauka, Moscow, 1941.

[3] V. A. FOCK, *Teoriya prostranstva vremeni i tyagoteniya*, Fizmatgiz, Moscow, 1955.

[4] R. EPSTEIN and R. V. WAGONER, *Astrophys. J.*, Vol. **197**, 1975, p. 717; R. V. WAGONER, in *Gravitazione Sperimentale*, B. BERTOTTI Ed., Accademia Nazionale dei Lincei, Rome, 1977, p. 117.

[5] K. S. THORNE, *Rev. Mod. Phys.*, Vol. **52**, 1980, p. 299.

[6] L. BLANCHET and T. DAMOUR, *Ann. Inst. Henri Poincaré, Phys. théor.*, Vol. **50**, 1989, p. 377. Referred to as paper I.

[7] Our conventions and notation are the following: signature $-+++$; greek indices $=0$, 1, 2, 3; latin indices $=1$, 2, 3; $g=-\det(g_{\mu\nu})$; $\mathcal{G}^{\alpha\beta}=\sqrt{g}\,g^{\alpha\beta}$; $f_{\alpha\beta}=f^{\alpha\beta}=$ flat metric $=\mathrm{diag}(-1, +1, +1, +1)$; $h^{\alpha\beta}:=\mathcal{G}^{\alpha\beta}-f^{\alpha\beta}$; L denotes a multi-index with l spatial indices, $i_1 i_2 \ldots i_l$, $\mathrm{L}-1$ denoting $i_1 \ldots i_{l-1}$; $x^{\mathrm{L}}:=x^{i_1}x^{i_2}\ldots x^{i_l}$, $\partial_{\mathrm{L}}:=\partial_{i_1}\ldots\partial_{i_l}$; the symmetric and trace-free (STF) part of the tensor $\mathrm{T_L}$ is denoted equivalently by $\hat{\mathrm{T}}_{\mathrm{L}}\equiv\mathrm{T}_{\langle \mathrm{L}\rangle}\equiv\mathrm{T}_{\langle i_1\ldots i_l\rangle}\equiv\underset{\mathrm{L}}{\mathrm{STF}}\,\mathrm{T_L}$; spatial indices are freely raised and lowered by means of $f^{ij}=f_{ij}=\delta_{ij}$; ε_{ijk} (with $\varepsilon_{123}=+1$) denotes the totally antisymmetric Levi-Civita symbol.

[8] L. BLANCHET and T. DAMOUR, *C. R. Acad. Sci. Paris*, T. **298**, Série II, 1984, p. 431.

[9] L. BLANCHET and T. DAMOUR, *Phys. Lett.*, Vol. **104A**, 1984, p. 82.

[10] L. BLANCHET and T. DAMOUR, *Phil. Trans. R. Soc. Lond. A.*, Vol. **320**, 1986, p. 379.

[11] L. BLANCHET, *Proc. R. Soc. Lond. A.*, Vol. **409**, 1987, p. 383.

[12] L. BLANCHET and T. DAMOUR, *Phys. Rev. D*, Vol. **37**, 1988, p. 1410.

[13] P. A. LAGERSTRÖM, L. N. HOWARD and C. S. LIU, *Fluid Mechanics and Singular Perturbations; a Collection of Papers by Saul Kaplun*, Academic Press, New York, 1967; M. VAN DYKE, *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford, California, 1975 (annotated edition).

[14] W. L. BURKE, *J. Math. Phys.*, Vol. **12**, 1971, p. 401.

[15] T. DAMOUR, in *Gravitation in Astrophysics*, B. CARTER and J. B. HARTLE Eds., Plenum Press, New York, 1987, pp. 3-62.

[16] J. L. ANDERSON, R. E. KATES, L. S. KEGELES and R. G. MADONNA, *Phys. Rev. D.*, Vol. **25**, 1982, p. 2038; and references therein.

[17] The formal end results of Refs [4] and [5] do not provide satisfactory answers because they are given by divergent integrals.

[18] T. DAMOUR and B. R. IYER, *Multipole analysis for electromagnetism and linearized gravity with irreducible cartesian tensors, Phys. Rev. D*, Vol. **43**, n° 8, 1991, in press.

[19] For simplicity's sake we refer to the equation, say, (2.3) of paper I (Ref. [6]) as eq. I (2.3), etc.

[20] L. BEL, T. DAMOUR, N. DERUELLE, J. IBAÑEZ and J. MARTIN, *Gen. Rel. Grav.*, Vol. **13**, 1981, p. 963.
[21] For completeness, let us note that one can also prove Eq. (2.27) by the same method of proof used in the Appendix C of Ref. [10] for the stationary case. Indeed, the only new terms we need to consider are the ones in $\bar{N}_2^{i0}$ coming from the time-derivative terms, say dM_{L_2}/dt, in U_i^{ext}. However, it is impossible to construct a non-zero (STF) multipole contribution, A_L, to $\bar{r}_2^0$ from a tensorial product $M_{L_1} \otimes dM_{L_2}/dt$. Indeed, in units where $G=c=1$, the length-dimension of A_L must be l from eq. (2.3 *b*), while the length-dimension of the tensorial product is $(l_1+1)+(l_2+1-1)=l_1+l_2+1$ which is striclty greater than l because of the inequality $l \leq l_1+l_2$ satisfied by the addition of angular momenta.
[22] T. DAMOUR, in *Gravitational Radiation*, N. DERUELLE and T. PIRAN Eds, North-Holland, Amsterdam, 1983, *see* p. 125; *see* also Appendix A of T. DAMOUR and G. SCHÄFER, *Gen. Rel. Grav.*, Vol. **17**, 1985, p. 879.
[23] Because of the segment-support property of δ_{12}, the support of $N_c^{\alpha\beta}$, and $\tau_c^{\alpha\beta}$, extends (in general) outside the support of the material stress-energy to the convex hull of the body.
[24] L. BLANCHET, T. DAMOUR and G. SCHÄFER, *Mon. Not. R. Astron. Soc.*, Vol. **242**, 1990, p. 289.
[25] C. W. LINCOLN and C. M. WILL, *Phys. Rev. D.*, Vol. **42**, 1990, p. 1123.
[26] T. DAMOUR, M. SOFFEL and C. XU, *General Relativistic Celestial Mechanics I. Method and Definition of Reference Systems*, *Phyvs. Rev. D*, 1991, in press.
[27] L. BLANCHET and G. SCHÄFER, *Mon. Not. R. Astron. Soc.*, Vol. **239**, 1989, p. 845.

APPENDIX A
QUASI-CONSERVATION LAWS FOR QUASI-LINEARIZED METRICS

Let us call a tensorial field $h_q^{\alpha\beta}$ a "quasi-linearized" (gothic) metric if it is generated via retarded potentials by a spatially compact source $\tau_q^{\alpha\beta}$, and if it is divergence-free *outside* some spatially compact domain. The first condition reads

$$h_q^{\alpha\beta} = \Box_R^{-1} \left\{ \frac{16\pi G}{c^4} \tau_q^{\alpha\beta} \right\}. \tag{A.1}$$

If we then define, possibly in the sense of distributions, for all values of x^μ

$$\Sigma_q^\alpha (x^\mu) := \frac{c^4}{16\pi G} \partial_\beta h_q^{\alpha\beta} (x^\mu), \tag{A.2}$$

the second condition states that $\Sigma_q^\alpha (x^\mu)$ has, like $\tau_q^{\alpha\beta} (x^\mu)$, a spatially compact support. Combinings eqs. (A.1) and (A.2), it is easy to deduce that $\tau_q^{\alpha\beta}$ satisfies the following "quasi-conservation" law:

$$\partial_\beta \tau_q^{\alpha\beta} = \Box_x \Sigma_q^\alpha. \tag{A.3}$$

Reciprocally, if some spatially compact source $\tau_q^{\alpha\beta}$ satisfies (in a distributional sense) the quasi-conservation law (A.3) for some spatially compact Σ_q^α the retarded field (A.1) satisfies (A.2), and is therefore divergence-free outside some spatially compact domain. From the results of Refs [5] or [10], the various coefficients, A_L, B_L, ..., J_L of the exterior multipole decomposition, eqs (2.25) of Ref. [10], of $h_q^{\alpha\beta}$ satisfy the relations (2.27) and (2.28) of Ref. [10]. This proves that the quasi-conservation law (A.3) is sufficient to entail the latter relations (which are numbered (5.27) and (5.28) in Ref. [18] and which were explicitly verified there by assuming the full conservation law: $\partial_\beta T^{\alpha\beta}=0$). This proves therefore that the equations (5.31), (5.32) of Ref. [18] apply also to any quasi-linearized metric, with mass and spin multipole moments given by the remplacement $T^{\alpha\beta}\to\tau_q^{\alpha\beta}$ in eqs (5.33)-(5.35) there.

Let us apply these considerations to our effective, compact-support, source $\tau_c^{\alpha\beta}$. Starting from its definition (5.7), and using eqs. (3.9) and (5.1), (5.2), (5.6), we see that

$$\tau_c^{\alpha\beta}=\tau^{\alpha\beta}-\frac{c^4}{16\pi G}\Delta_x[\Sigma_{12}^{\alpha\beta}(k)], \tag{A.4}$$

where the linear operation Σ_{12} was defined in Section III. We know that $\tau^{\alpha\beta}$, defined by eq. (5.1), satisfies the following (approximate) strong conservation law

$$\partial_\beta\tau^{\alpha\beta}=0(3, 2), \tag{A.5}$$

but we need to compute (in a distributional sense) the divergence of $\Sigma_{12}^{\alpha\beta}(k)$. From eq. (3.13) it has the form

$$\partial_\beta\Sigma_{12}^{\alpha\beta}(k)=\sum C_{ab\,(c)}^{\alpha\beta}\int\int d^3y_1\,d^3y_2\frac{\partial}{\partial x^\beta}\left[\sigma_1\,\sigma_2^{(c)}\frac{\partial}{\partial y_1^a}\frac{\partial}{\partial y_2^b}k(\mathbf{x};\,\mathbf{y}_1,\,\mathbf{y}_2)\right]. \tag{A.6}$$

Using the translational invariance of the kernel k we can replace the spatial x-derivatives appearing in eq. (A.6) (which act only on k) by y_1- and y_2-derivatives according to

$$\frac{\partial}{\partial x^j}k=-\left(\frac{\partial}{\partial y_1^j}+\frac{\partial}{\partial y_2^j}\right)k. \tag{A.7}$$

As for the time derivative, $\partial/\partial x^0$, they are either negligible (when $\alpha=i$) or they act (when $\alpha=0$) only on $\sigma_1\sigma_2$ according to

$$\frac{\partial}{\partial t}(\sigma_1\,\sigma_2)=\frac{\partial\sigma_1}{\partial t}\sigma_2+\sigma_1\frac{\partial\sigma_2}{\partial t}=-\frac{\partial\sigma_1^c}{\partial y_1^c}\sigma_2-\sigma_1\frac{\partial\sigma_2^c}{\partial y_2^c}+O(2). \tag{A.8}$$

Using eqs (A.7) and (A.8) the calculation (A.6) is reduced to computing some combinations of third-order y_1- and y_2-derivatives of the kernel k.

After several simplifications one is left with terms that contain only the y_1- or y_2-Laplacians of k. The problem is then reduced to calculating, say the y_1-Laplacian of $k(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$, considered as a distribution in $\mathbf{x}$, depending on the parameters $\mathbf{y}_1$ and $\mathbf{y}_2$. This calculation is somewhat delicate. One way of doing it is to take the y_1-Laplacian of both sides of eq. (3.9 *c*). This leads to

$$\Delta_x(\Delta_{y_1} k) = 2\pi \left[\Delta_{y_1} \delta_{12} - \frac{2}{r_{12}} \delta(\mathbf{x} - \mathbf{y}_1) \right]. \qquad (A.9)$$

In the calculation of the right-hand side of eq. (A.9) from the definition (3.8) of δ_{12} there appears the following $\mathbf{x}$-distribution

$$\delta_{12}^{(2)}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) := r_{12} \int_0^1 d\alpha\, \alpha^2\, \delta(\mathbf{x} - \alpha \mathbf{y}_1 - (1-\alpha)\mathbf{y}_2). \qquad (A.10)$$

More precisely, thanks to some simplifications the right-hand side of eq. (A.9) is found to be simply $2\pi\, \Delta_x \delta_{12}^{(2)}$. This $\mathbf{x}$-distributional identity allows one to conclude that

$$\Delta_{y_1} [k(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)] = 2\pi\, \delta_{12}^{(2)}(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2), \qquad (A.11)$$

where it is important to note that $\delta_{12}^{(2)}$ is, like δ_{12}, a distribution which is entirely concentrated on the segment connecting $\mathbf{y}_1$ and $\mathbf{y}_2$ (but, contrarily to δ_{12}, the linear mass density of $\delta_{12}^{(2)}$ is not uniform).

Finally, these explicit calculations lead to the proof of the following (approximate) quasi-conservation law for $\tau_c^{\alpha\beta}$:

$$\partial_\beta \tau_c^{\alpha\beta} = \Delta_x [\Sigma^\alpha] + O(3, 2) = \Box_x [\Sigma^\alpha] + O(3, 2), \qquad (A.12)$$

with

$$\Sigma^0 = \frac{G}{2c} \int\!\!\int d^3 y_1\, d^3 y_2\, \sigma_1\, \sigma_2^i \left[4 \frac{\partial}{\partial y_1^i} \delta_{21}^{(2)} - 3 \frac{\partial}{\partial y_2^i} \delta_{12}^{(2)} \right], \qquad (A.13\,a)$$

$$\Sigma^i = \frac{G}{2} \int\!\!\int d^3 y_1\, d^3 y_2\, \sigma_1\, \sigma_2 \left[4 \frac{\partial}{\partial y_1^i} \delta_{21}^{(2)} \right], \qquad (A.13\,b)$$

where $\delta_{21}^{(2)}$ is defined by exchanging $\mathbf{y}_1$ and $\mathbf{y}_2$ in the definition (A.10) (it comes from $\Delta_{y_2} k = 2\pi\, \delta_{21}^{(2)}$).

The explicit expressions (A.12), (A.13) allow us to investigate the effect of using various forms for the post-Newtonian-accurate multipole moments in linearized gravity as starting points for the replacement $T^{\alpha\beta} \to \tau_c^{\alpha\beta}$. We have seen in the text that this replacement gave the correct moments when starting from the BD (Ref. [6]) or DI (Ref. [18]) forms. Let us now consider the "T form", defined as the linearized-gravity limit of the results of Ref. [5] (which are given as formal integrals of the non-compact effective source $\tau^{\alpha\beta}$ in the post-Newtonian case). As exhibited in Ref. [18], one needs to use the full conservation law to go from the BD and DI forms to the T forms. However, the compact effective source $\tau_c^{\alpha\beta}$ satisfies only the quasi-conservation law (A.12). This means that extra-terms appear in going from the BDI to the T form. An explicit calculation shows that

$$I_L^{BD}[\tau_c] = I_L^T[\tau_c] + i_L, \tag{A.14a}$$

$$J_L^{DI}[\tau_c] = J_L^T[\tau_c] + j_L, \tag{A.14b}$$

with

$$i_L = \frac{1}{c^2} \underset{L}{\mathrm{STF}} \left[l \int d^3x\, x^{L-1} \Sigma^{i_l} \right] + O(4), \tag{A.15a}$$

$$j_L = \frac{1}{c^2} \frac{d}{dt} \underset{L}{\mathrm{STF}} \left[\varepsilon_{i_l ab} \int d^3x\, x^{aL-1} \Sigma^b \right] + O(4). \tag{A.15b}$$

Using the explicit expressions (A.13) one can show that the extra-terms i_L and j_L are zero when $l \leqq 1$. For J_i one thereby recovers the result (5.31) of the text, while for I_L one gets new expressions for the BD mass and dipole moments:

$$I^{BD} = \int d^3x \frac{\tau_c^{00}}{c^2}, \tag{A.16a}$$

$$I_i^{BD} = \int d^3x\, x^i \frac{\tau_c^{00}}{c^2}. \tag{A.16b}$$

On the other hand, it seems that i_L and j_L are not identically zero for $l \geqq 2$. Indeed, using eqs (A.13) one finds an extra-term in the mass quadrupole equal to

$$i_{ab} = -\frac{G}{3c^2} \int\int d^3y_1\, d^3y_2\, \sigma_1 \sigma_2 r_{12} n_{12}^{\langle a} n_{12}^{b\rangle} + O(4), \tag{A.17}$$

which is generically non zero.

Let us emphasize that the conclusion reached here of a "superiority" of the BD and DI forms over the T forms depends on our imposing as rule of the game a replacement $T^{\alpha\beta} \to \tau_c^{\alpha\beta}$ or $(\tau^{\alpha\beta} \to \tau_c^{\alpha\beta})$ in expressions derived in contexts where the use of a conservation law was allowed. When one does not change the rule of the game, and when all the various forms are mathematically well defined (which means when one restricts oneself to *linearized* gravity) they are all equivalent.

APPENDIX B
TRANSFORMATION OF MULTIPOLE MOMENTS UNDER SHIFTS OF THE SPATIAL ORIGIN

Let us put ourselves in the same general framework as Appendix A, *i. e.* let us consider a quasi-linearized metric $h_q^{\alpha\beta}$ generated by a spatially compact source $\tau_q^{\alpha\beta}$ satisfying the quasi-conservation law (A.3) for some spatially compact Σ_q^α. Note that all this setting is covariant under the Poincaré group. However, when we wish to analyze $h_q^{\alpha\beta}$ in multipoles we need to break the Poincaré invariance [down to an $O(3)$ invariance] by choosing a particular central worldline, say $\mathscr{L}$. Thereby, the values of the multipoles, in particular the "physical" (*i. e.* gauge invariant) mass and spin multipoles of $h_q^{\alpha\beta}$, say M_L and S_L, depend on the choice of $\mathscr{L}$. For simplicity's sake, we shall restrict ourselves to considering a three-parameter family of parallel central wordlines (*i. e.* to a free choice of an origin in space). Then, given an infinitesimal vector in Euclidean space, say $\boldsymbol{\varepsilon}$, we shall define

$$\delta_\varepsilon M_L := M_L[\mathscr{L}_\varepsilon] - M_L[\mathscr{L}], \tag{B.1 a}$$

$$\delta_\varepsilon S_L := S_L[\mathscr{L}_\varepsilon] - S_L[\mathscr{L}], \tag{B.1 b}$$

where $\mathscr{L}_\varepsilon$ is obtained by shifting $\mathscr{L}$ by $-\boldsymbol{\varepsilon}$ in space. The minus sign introduced in the shift is chosen to ensure that the spatial vector going from the central worldline to some given field point P, say $\mathbf{r}_{\mathscr{L}} := \mathscr{L}\mathbf{P}$ (projected onto the three-space orthogonal to the common time direction defined by the family of parallel central worldlines) changes by

$$\delta_\varepsilon \mathbf{r}_{\mathscr{L}} = \mathbf{r}_{\mathscr{L}_\varepsilon} - \mathbf{r}_{\mathscr{L}} = \mathscr{L}_\varepsilon \mathscr{L} = +\boldsymbol{\varepsilon}. \tag{B.2}$$

Let us recall from Refs [5] and [10] that the full multipole decomposition of $h_q^{\alpha\beta}(P)$ can be uniquely written in the form (indices omitted)

$$h_q(P) = \mathscr{F}_{\text{can}}[\mathscr{M}, \mathbf{r}_{\mathscr{L}}] + \mathscr{H}[\mathscr{W}, \mathbf{r}_{\mathscr{L}}], \tag{B.3}$$

where $\mathscr{M} = \{M_L, S_L\}$ denotes the set of "physical" mass and spin moments, $\mathscr{W} = \{W_L, X_L, Y_L, Z_L\}$ the set of "gauge" moments, $\mathscr{F}_{\text{can}}$ the "canonical" multipole series expressing a gothic metric in terms of physical moments [explicitly given by eqs (2.9), (2.10) above] and $\mathscr{H}^{\alpha\beta} = \partial^\alpha w^\beta + \partial^\beta w^\alpha - f^{\alpha\beta}\partial_\mu w^\mu$ a "gauge" part (*see* Refs [5], [10] or [18] for the explicit expression of $w^\alpha[\mathscr{W}]$). Let us note that we have taken advantage of our considering only parallel wordlines to forget the time dependence in the right-hand side of eq. (B.3).

Under a shift of the spatial origin, *i. e.* under a translation of the central worldline $\mathscr{L}$, keeping the geometrical field-point P fixed the left-hand side of eq. (B.3) must stay invariant, while in the right-hand side both the various multipole moments and the relative vector shift by:

$$\delta_\varepsilon \mathscr{M} = \{\delta_\varepsilon M_L, \delta_\varepsilon S_L\}, \qquad \delta_\varepsilon \mathscr{W} = \{\delta_\varepsilon W_L, \ldots\}, \qquad \delta_\varepsilon \mathbf{r}_{\mathscr{L}} = \boldsymbol{\varepsilon}. \tag{B.4}$$

Expressing that the shifts (B.4) must compensate themselves ($\delta_\varepsilon h_q(\mathrm{P})=0$) we find that $\delta_\varepsilon \mathscr{M}$ must be equal to

$$\delta_\varepsilon \mathscr{M} = \mathscr{M}\left\{-\boldsymbol{\varepsilon}\,.\,\frac{\partial}{\partial \mathbf{r}_{\mathscr{L}}}\mathscr{F}_{\mathrm{can}}[\mathscr{M}, \mathbf{r}_{\mathscr{L}}]\right\}, \tag{B.5}$$

where the letter $\mathscr{M}$ appearing in front of the right-hand side of eq. (B.5) denotes the operation of extracting the physical multipoles out of the (quasi-) linearized metric enclosed in curly brackets (it being clear that for any exterior solution $h^{\alpha\beta}$ of the linearized Einstein equations, $-\nabla_\varepsilon h^{\alpha\beta}$ defines another such solution which can also be analyzed in multipoles).

The explicit formulas for computing $\mathscr{M}\{k^{\alpha\beta}\}$ for any linearized exterior metric $k^{\alpha\beta}$ are given by eqs (2.26 *a*), (2.26 *b*) of Ref. [10]. By applying these formulas to $k^{\alpha\beta}=-\varepsilon^i\,\partial/\partial x^i\, h^{\alpha\beta}_{\mathrm{can}}$ one could directly derive the following "shift" formulas

$$\delta_\varepsilon \mathrm{M_L} = l\,\varepsilon_{\langle i_l}\mathrm{M}_{\mathrm{L}-1\rangle} - \frac{4l}{(l+1)^2 c^2}\varepsilon_a \overset{(1)}{\mathrm{S}}_{b\langle \mathrm{L}-1}\varepsilon_{i_l\rangle ab} + \frac{(l-1)(l+3)}{(l+1)^2(2l+3)c^2}\varepsilon_s \overset{(2)}{\mathrm{M}}_{s\mathrm{L}}, \tag{B.6 a}$$

$$\delta_\varepsilon \mathrm{S_L} = \frac{(l-1)(l+1)}{l}\varepsilon_{\langle i_l}\mathrm{S}_{\mathrm{L}-1\rangle} + \frac{1}{l}\varepsilon_a \overset{(1)}{\mathrm{M}}_{b\langle \mathrm{L}-1}\varepsilon_{i_l\rangle ab} + \frac{(l-1)(l+3)}{l(l+2)(2l+3)c^2}\varepsilon_s \overset{(2)}{\mathrm{S}}_{s\mathrm{L}}, \tag{B.6 b}$$

where one should not confuse the shift vector $\varepsilon_i \equiv \varepsilon^i$ with the Levi-Civita symbol ε_{ijk}.

In fact, we have used a more indirect, but simpler, method for proving eqs (B.6). Indeed, it is rather easy to prove that the algorithm $\mathscr{M}\{-\nabla_\varepsilon h_{\mathrm{can}}\}$ will, because of tensorial, dimensional, parity and time-reversal considerations, necessarily lead to shift formulas having the structure (B.6) with some *universal* l-dependent coefficients. It then suffices to determine these coefficients (*i.e.* l, $-4l/(l+1)^2$, etc.) in some particular cases. We have determined the coefficients of $\varepsilon_{\langle i_l}\mathrm{M}_{\mathrm{L}-1\rangle}$ and $\varepsilon_{\langle i_l}\mathrm{S}_{\mathrm{L}-1\rangle}$ by considering the particular case of the linearized gravitational field generated by a time-independent (fully conserved) source $\mathrm{T}^{\alpha\beta}_{\mathrm{stationary}}$, and by introducing a shift $\delta x^i=\varepsilon^i$ in the stationary-limit of the exact expressions (5.33) and (5.35) of Ref. [18]. Then we have determined the other coefficients by considering the wave-zone limit of the invariance requirements $\delta_\varepsilon h^{\alpha\beta}_q(\mathrm{P})=0$. In this limit, one is considering two different sections of Minkowskian future null infinity related by a shift of the retarded time variable equal to $\delta_\varepsilon u = -\boldsymbol{\varepsilon}.\mathbf{n}$ (where $\mathbf{n}=\mathbf{r}_{\mathscr{L}}/|\mathbf{r}_{\mathscr{L}}|$). Writing the geometrical invariance of the time-antiderivative of the news function, $-m^i m^j \lim_{r=\infty}(rh^{ij})$, with $\mathbf{m}=(\mathbf{e}_\theta - i\,\mathbf{e}_\varphi)/\sqrt{2}$, one finds, after some amount of algebra, the shift formulas for the l^{th}-time-derivatives of the (radiative) multipole moments, which determines the other coefficients in eqs (B.6). One can also conclude from this last argument that the l^{th}-time derivative of eqs (B.6) will give very generally the transformation formulas of the differentiated radiative moments, $\overset{(l)}{M}{}^{rad}_L$, $\overset{(l)}{S}{}^{rad}_L$, for any asymptotically simple curved Einstein metric, under the sub-group of the BMS group defined by $\delta_\varepsilon u = -\boldsymbol{\varepsilon}.\mathbf{n}(\theta, \varphi)$, where u, θ, φ are some Bondi coordinates. In particular, these transformation formulas will apply to the differentiated radiative moments of eq. (1.4) above, under a shift $\delta_\varepsilon U = -\boldsymbol{\varepsilon}.\mathbf{N}$ of the radiative coordinate $U = T - R/c$. However, in the present context we are not interested in applying the time-differentiated formulas (B.6) to the radiative moments of our physical metric $\mathcal{G}^{\alpha\beta}$ [as given by eqs (1.9) with a loss of information at the $O(c^{-3})$ level], but to our 1PN-accurate source multipole moments I_L and J_L [known up to $O(c^{-4})$].

We have set up the proof of the shift formulas (B.6) in a way which makes it clear that they apply to any quasi-linearized metric in the sense of Appendix A, *i. e.* a tensor which satisfies the linearized Einstein equations outside some world-tube, and which is generated by an effective source $\tau_q^{\alpha\beta}$ which might be only quasi-conserved in the sense of eq. (A.3). This situation applies to the metric $h_c^{\alpha\beta}$ of eq. (4.25) above. Therefore, we can conclude that our 1PN *source* multipole moments, I_L [source], J_L [source], defined by eqs (5.11) above transform according to eqs (B.6) [modulo additional $O(c^{-4})$ error terms in both equations] under the transformation $x'^0 = x^0$, $x'^i = x^i + \varepsilon^i$, of the harmonic coordinate system used in Section IV to describe the source and the curved metric in the inner region. We have explicitly verified, by a long but straightforward calculation, that the potential form, eq. (5.36), of our source spin quadrupole indeed transforms according to eq. (B.6 *b*) for $l=2$, *i. e.*

$$\delta_\varepsilon J_{ij} = \frac{3}{2}\varepsilon_{\langle i} J_{j\rangle} + \frac{1}{2}\varepsilon_a \overset{(1)}{I}_{b\langle i}\varepsilon_{j\rangle ab} + \frac{5}{56c^2}\varepsilon_s \overset{(2)}{J}_{sij} + O\left(\frac{1}{c^4}\right). \tag{B.7}$$

Note that in the right-hand side of eq. (B.7) both the 1PN spin vector J_j and the 1PN (BD) quadrupole moment I_{bi} appear, while it is enough to use the Newtonian value of the spin octupole. In evaluating the time derivative of the 1PN source quadrupole moment one needs to use a 1PN-accurate continuity equation. The relevant equation follows from $\nabla_\mu T^{0\mu} = 0$ and reads

$$\frac{\partial}{\partial t}\left(\frac{T^{00}}{c^2}\right) + \frac{\partial}{\partial x^i}\left(\frac{T^{0i}}{c}\right) = -\frac{1}{c^2}\sigma\frac{\partial U}{\partial t} + O\left(\frac{1}{c^4}\right), \tag{B.8 a}$$

or, in the notation (4.4),

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma^i}{\partial x^i} = \frac{1}{c^2} \frac{\partial T^{ss}}{\partial t} - \frac{1}{c^2} \sigma \frac{\partial U}{\partial t} + O\left(\frac{1}{c^4}\right). \tag{B.8 b}$$

APPENDIX C
THE POINT-PARTICLE LIMIT

The multipolar gravitational wave generation formalism introduced in paper I and extended here considers weakly self-gravitating sources. This still leaves the possibility of considering systems of well-separated extended bodies. A good theoretical description, at the 1PN approximation, of such N-body systems by means of a set of individual 1PN moments for each extended body has been achieved only recently [26]. As such an analysis is rather intricate, we shall content ourselves in this Appendix to use the familiar *formal* description [2] of N well-separated bodies, at the 1PN approximation, when neglecting all information about the inner structure of the bodies, that consists of taking a stress-energy tensor of the form,

$$T^{\mu\nu}(\mathbf{x}, t) = \sum_A m_A \frac{dx_A^\mu}{dt} \frac{dx_A^\nu}{dt} \frac{1}{\sqrt{g}} \frac{dt}{d\tau} \delta(\mathbf{x} - \mathbf{x}_A(t)), \tag{C.1}$$

and of neglecting all the ill-defined (formally infinite) terms. *See* also the appendix of Ref. [27] for a treatment of intermediate rigour (between (C.1) and Ref. [26]) of this point-particle limit for the 1PN quadrupole moment. Modulo $O(c^{-4})$ error terms, eq. (C.1) yields

$$\sigma = \sum_A m_A \left(1 - \frac{U_A}{c^2} + \frac{3}{2} \frac{\mathbf{v}_A^2}{c^2}\right) \delta(\mathbf{x} - \mathbf{x}_A), \tag{C.2 a}$$

$$\sigma^i = \sum_A m_A v_A^i \left(1 - \frac{U_A}{c^2} + \frac{1}{2} \frac{\mathbf{v}_A^2}{c^2}\right) \delta(\mathbf{x} - \mathbf{x}_A), \tag{C.2 b}$$

$$T^{ij} = \sum_A m_A v_A^i v_A^j \left(1 - \frac{U_A}{c^2} + \frac{1}{2} \frac{\mathbf{v}_A^2}{c^2}\right) \delta(\mathbf{x} - \mathbf{x}_A), \tag{C.2 c}$$

where $A = 1, \ldots, N$ labels the N separate bodies, $\mathbf{v}_A = dx_A(t)/dt$, and

$$U_A = G \sum_{B \neq A} \frac{m_B}{|\mathbf{x}_A - \mathbf{x}_B|}. \tag{C.3}$$

Ref. [27] has obtained the point-particle limit of the 1PN mass quadrupole. We shall here compute the (formal) point-particle limit of the 1PN spin quadrupole, starting from the potential form (5.36). It is then clear that the $O(c^{-2})$ contributions appearing in eqs (C.2) need to be kept only in σ^b and when evaluating the leading term in eq. (5.36). Noting that (formally)

$$P^{N}_{(\alpha)}(\mathbf{x}_A) = G \sum_{B \neq A} m_B \, |\mathbf{x}_A - \mathbf{x}_B|^{\alpha} n^{N}_{AB}, \tag{C.4}$$

where $n^i_{AB} = (x^i_A - x^i_B)/|\mathbf{x}_A - \mathbf{x}_B|$, we obtain our final point-particle limit for the spin quadrupole moment of a N-body system:

$$\begin{aligned}
J_{ij} = \underset{ij}{\mathrm{STF}}\, \varepsilon_{jab} \sum_{A} m_A \Bigg[& x^{ai}_A v^b_A \left(1 + \frac{1}{2}\frac{\mathbf{v}^2_A}{c^2}\right) \\
& + \frac{G}{c^2} \sum_{B \neq A} m_B \Bigg\{ \Bigg(-\frac{1}{2} \frac{v^b_A x^{ai}_A}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{1}{2} \frac{v^s_A x^{ai}_A n^{bs}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} \\
& + \frac{7}{4} v^b_A x^a_A n^i_{AB} + \frac{1}{4} v^s_A x^a_A n^{bsi}_{AB} + \frac{7}{2} v^b_A x^i_A n^a_{AB} \\
& + \frac{11}{4} v^a_A n^{bi}_{AB} |\mathbf{x}_A - \mathbf{x}_B| - \frac{7}{4} v^i_A x^a_A n^b_{AB} \Bigg) \\
& + \frac{d}{dt}\Bigg(\frac{5}{56} \frac{x^{ais}_A n^{sb}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{1}{56} \frac{x^{ass}_A n^{ib}_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|} - \frac{3}{112} x^{as}_A n^{sbi}_{AB} + \frac{9}{112} x^{ai}_A n^b_{AB} \\
& - \frac{9}{112} x^a_A n^{bi}_{AB} |\mathbf{x}_A - \mathbf{x}_B| - \frac{1}{14} \frac{x^{aiss}_A n^b_{AB}}{|\mathbf{x}_A - \mathbf{x}_B|^2} \Bigg) \Bigg\} \\
& + \frac{1}{c^2} \frac{d}{dt}\left(\frac{3}{28} x^{ass}_A v^{bi}_A - \frac{1}{28} x^{asi}_A v^{bs}_A \right) \Bigg].
\end{aligned} \tag{C.5}$$

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