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Consequences of the validity of Huygens' principle for the conformally invariant scalar wave equation, Weyl's neutrino equation and Maxwell's equations on Petrov type II space-times

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ABSTRACT. — We show that a necessary condition for the validity of Huygens' principle for the conformally invariant scalar wave equation, Weyl's neutrino equation or Maxwell's equations on a Petrov type II space-time is that the repeated principal null congruence of the Weyl tensor be geodesic, shear free and hypersurface orthogonal.

RÉSUMÉ. — On démontre que la validité du principe de Huygens pour l'équation invariante conforme des ondes scalaires, l'équation de neutrino de Weyl ou les équations de Maxwell sur un espace-temps de type II de Petrov implique que la congruence isostrope caractéristique double du tenseur de Weyl est géodésique, sans distortion et intégrable.

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1. INTRODUCTION

This paper is the fifth in a series devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation, Weyl's neutrino equation and Maxwell's equations. These equations may be written respectively as

$$\square u + \frac{1}{6} R u = 0, \tag{1.1}$$

$$\nabla^{A\dot{A}} \varphi_A = 0, \tag{1.2}$$

$$\nabla^{A\dot{A}} \varphi_{AB} = 0. \tag{1.3}$$

The conventions and formalisms in this paper are those of Carminati and McLenaghan [2]. All considerations in this paper are entirely local.

According to Hadamard [9], Huygens' principle (in the strict sense) is valid for equation (1.1) if and only if for every Cauchy initial value problem and every $x_0 \in V_4$, the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$ where S denotes the initial surface and $C^-(x_0)$ the past null conoid from x_0. Analogous definitions of the validity of the principle for Weyl's equation (1.2) and Maxwell's equations (1.3) have been given by Wünsch [18] and Günther [7] respectively in terms of appropriate formulations of the initial value problems for these equations. Hadamard's problem for the equations (1.1), (1.2) or (1.3), originally posed only for scalar equations, is that of determining *all* space-times for which Huygens' principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens' principle, the determination may only be effected up to an arbitrary conformal transformation of the metric on V_4

$$\tilde{g}_{ab} = e^{2\varphi} g_{ab}, \tag{1.4}$$

where φ is an arbitrary real function.

Huygens' principle is valid for (1.1), (1.2) and (1.3) on any conformally flat space-time and also on any space-time conformally related to the exact plane wave space-time ([6], [10], [19]), the metric of which has the form

$$ds^2 = 2\,dv\,[du + [D(v)\,z^2 + \bar{D}(v)\,\bar{z}^2 + e(v)\,z\bar{z}]\,dv] - 2\,dz\,d\bar{z}, \tag{1.5}$$

in a special co-ordinate system, where D and e are arbitrary functions. These are the only *known* space-times on which Huygens' principle is valid for these equations. Furthermore, it has been shown ([8], [11], [19]) that these are the only conformally empty space-times on which Huygens' principle is valid. In the non-conformally empty case several results are known. In particular for Petrov type N space-times Carminati and McLenaghan ([1], [2]) have proved the following result: *Every Petrov type* N *space-time on which the conformally invariant scalar wave equation* (1.1) *satisfies Huygens' principle is conformally related to an exact plane wave space-time* (1.5). This result together with Günther's [6] solves Hadamard's problem for the conformally invariant scalar wave equation on type N space-times. An analogous result has been proved for the non-self adjoint scalar wave equation on type N space-times by McLenaghan and Walton [13]. For the case of Petrov type D space-times Carminati and McLenaghan [3], Wünsch [20] and McLenaghan and Williams [14] established the following: *There exist no Petrov type* D *space-times on which the conformally invariant scalar wave equation, Weyl's neutrino equation or Maxwell's equations* (1.1) *satisfy Huygens' principle.* A similar result also holds for the conformally invariant scalar wave equation on space-times of Petrov type III under a certain mild assumption [4]. These results lend weight to the conjecture that *every space-time on which the conformally invariant scalar wave equation satisfies Huygens' principle is conformally related to the plane wave space-time* (1.5) *or is conformally flat* ([1], [2]).

We now extend our analysis to Petrov type II space-times continuing the program outlined in [1] and [2] for the solution of Hadamard's problem based on the conformally invariant Petrov classification. We recall that a Petrov type II space-time is characterized by the existence of a null vector field l^a satisfying the following conditions

$$C_{abc[d}\, l_{e]}\, l^b\, l^c = 0, \tag{1.6}$$

$$C_{abc[d}\, l_{e]}\, l^c \neq 0. \tag{1.7}$$

THEOREM. — *The validity of Huygens' principle for the conformally invariant scalar wave equation* (1.1), *Weyl's equation* (1.2), *or Maxwell's equations* (1.3) *on a Petrov type II space-time implies that the repeated principal null congruence (defined by the null vector field l^a) of the Weyl tensor is geodesic, shear free and hypersurface orthogonal, that is*

$$l^b\, \nabla_b\, l_a = f l_a, \tag{1.8}$$

$$[\nabla^{(b}\, l^{a)}]\, \nabla_b\, l_a = \frac{1}{2}[\nabla^a\, l_a]^2, \tag{1.9}$$

$$l_{[a}\, \nabla_b\, l_{c]} = 0. \tag{1.10}$$

2. PROOF OF THEOREM

The proof of the theorem will follow the same procedure as that used to obtain the analogous result for the case of type D space-times ([3], [14]). The procedure in both the type D case and the type II case relies on the existence of repeated principal null congruences. In the case of type D space-times the argument is applied to both of the two repeated principal null congruences enabling a complete solution of the problem. In the case of type II space-times however only the one repeated principal null congruence is available.

The spinor form of the third necessary condition [5] (III's) that we shall use is the stronger form of it obtained by McLenaghan and Williams [14].

$$\nabla^{K}{}_{\dot{A}}\nabla^{L}{}_{\dot{B}}\Psi_{ABKL}+\Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL}=0. \tag{2.1}$$

The spinor form of the fifth necessary condition ([12], [16], [17], [18]) (V's) is

$$\begin{aligned}
&k_1[\nabla^{K\dot{K}}\Psi_{ABCD}][\nabla_{K\dot{K}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}]\\
&\quad+k_2[\nabla_{(D(\dot{D}}\Psi^{K}{}_{ABC)}][\nabla_{K}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}]+k_2[\nabla_{(D(\dot{D}}\bar{\Psi}^{\dot{K}}{}_{\dot{A}\dot{B}\dot{C})}][\nabla^{L}{}_{\dot{K}}\Psi_{ABC)L}]\\
&\quad-2(8k_1-k_2)[\nabla^{K}{}_{(\dot{D}}\Psi_{K(ABC}][\nabla_{D)}{}^{\dot{K}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{K}}]\\
&\quad-k_2\Psi^{K}{}_{(ABC}\nabla_{D)(\dot{D}}\nabla_{K}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}-k_2\bar{\Psi}^{\dot{K}}_{(\dot{A}\dot{B}\dot{C}}\nabla_{(D\dot{D})}\nabla^{L}{}_{\dot{K}}\Psi_{ABC)L}\\
&\quad+4k_1\Psi^{K}{}_{(ABC}\nabla_{|K|(\dot{D}}\nabla_{D}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}+4k_1\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\nabla_{(D|\dot{K}|}\nabla^{L}{}_{\dot{D})}\Psi_{ABC)L}\\
&\quad+2(k_2-4k_1)\Psi^{K}{}_{(ABC}\Phi_{D)K\dot{K}(\dot{D}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}}{}^{\dot{K}}\\
&\quad-2(4k_1+k_2)\Lambda\Psi_{ABCD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}=0,
\end{aligned} \tag{2.2}$$

where k_1 takes the values 3, 8, 5 and k_2 the values 4, 13, 16 depending on whether the equation under consideration is the conformally invariant scalar wave equation, Weyl's equation or Maxwell's equations respectively.

In order to refer to the individual components of conditions III's and V's we shall subscript the roman numerals III and V in a manner analogous to that used to refer to the components of the Ricci spinor.

A Petrov type II space-time is characterised by the Weyl spinor having one two-fold principal spinor, that is, there exists spinors o_A, α_A, β_A none of which are multiples of any of the others, such that

$$\Psi_{ABCD}=o_{(A}\,o_B\,\alpha_C\,\beta_{D)}. \tag{2.3}$$

If the repeated principal spinor o_A is chosen as the first of the basis spinors of the spin frame $\{o_A, \iota_A\}$, then the Weyl spinor can be written in the form

$$\Psi_{ABCD}=o_{(A}\,o_B\,(o_C+\eta_0\,\iota_C)(\eta_1\,o_{D)}+\eta_2\,\iota_{D)}), \tag{2.4}$$

where the η_i are three unspecified functions. Consequently we have

$$\Psi_0=\Psi_1=0. \tag{2.5}$$

If the space-time is to be a proper (*i.e.* non-degenerate) type II space-time then we must also have

$$\Psi_2\neq 0. \tag{2.6}$$

For the purposes of the proof it is not necessary to impose any other constraints on the spin frame. We could however at this point completely specify the spin frame by imposing the requirements, for example

$$\Psi_3=0 \qquad \text{and} \qquad \Psi_4=\Psi_2. \tag{2.7}$$

Alternatively we could require

$$\Psi_3=0 \qquad \text{qnd} \qquad \Psi_4\bar{\Psi}_4=1, \tag{2.8}$$

a choice which leaves one real degree of freedom remaining in the choice of the spin frame. Both these choices of spin frame appear attractive for investigating the conjecture that there exists no type II space-times on which any of the three wave equations satisfy Huygens' principle. The requirements of (2.7) are also invariant under a conformal transformation while those of (2.8) are easily modified to retain the unimodular behaviour of Ψ_4 after a conformal transformation.

The conformal invariance of Huygens' principle gives us some freedom to choose the form of the Weyl spinor. In particular the conformal transformation (2.4) in the appropriate tetrad form induces the transformation

$$\tilde{\Psi}_i=e^{-2\varphi}\Psi_i. \tag{2.9}$$

Since Ψ_2 is non-zero we are able to apply a conformal transformation to set

$$\Psi_2\bar{\Psi}_2=1. \tag{2.10}$$

We then write

$$\Psi_2=e^{\mathrm{H}}, \qquad \bar{\Psi}_2=e^{-\mathrm{H}}. \tag{2.11}$$

Note that H is imaginary. Examining condition V_{00} we find that

$$k_1\,\kappa\bar{\kappa}=0, \tag{2.12}$$

and so

$$\kappa=0, \tag{2.13}$$

since $k_1\neq 0$. It follows that the repeated principal null congruence of the space-time is *geodesic*.

With $\kappa=0$ condition V_{02} now reads

$$(8\,k_1-k_2)\,\sigma\,\mathrm{DH}-\frac{1}{2}(4\,k_1+k_2)(\mathrm{D}+3\,\rho-\bar{\rho}-3\,\varepsilon+\bar{\varepsilon})\,\sigma+24\,k_1\,\bar{\rho}\sigma=0. \tag{2.14}$$

At this point we will assume that $\sigma\neq 0$. From the Newman-Penrose (NP) equations we have

$$(\mathrm{D}-\rho-\bar{\rho}-3\,\varepsilon+\bar{\varepsilon})\,\sigma=0, \tag{2.15}$$

whence equation (2.14) becomes, on dividing by σ

$$(8\,k_1-k_2)\,\mathrm{DH}-2\,(4\,k_1+k_2)\,\rho+24\,k_1\,\bar{\rho}=0. \tag{2.16}$$

The real part of this gives

$$(8\,k_1-k_2)\,(\rho+\bar{\rho})=0. \tag{2.17}$$

and so ρ is imaginary since $8k_1 \neq k_2$ for all three of the wave equations under consideration. Solving for DH in (2.16) then gives

$$\mathrm{DH} = \frac{2(16k_1 + k_2)}{(8k_1 - k_2)}\rho. \qquad (2.18)$$

From the NP equation

$$(\mathrm{D} - \rho - \varepsilon - \bar{\varepsilon})\rho = \sigma\bar{\sigma} + \Phi_{00}, \qquad (2.19)$$

and its complex conjugate we obtain

$$(\mathrm{D} - \varepsilon - \bar{\varepsilon})\rho = 0 \qquad \text{and} \qquad \Phi_{00} = -\rho^2 - \sigma\bar{\sigma}. \qquad (2.20)$$

We now examine condition III_{00}, with $\kappa = 0$ and $\rho + \bar{\rho} = 0$. It reads

$$\mathrm{D}^2\mathrm{H} + (\mathrm{DH})^2 - (6\rho + \varepsilon + \bar{\varepsilon})\mathrm{DH} - 3(\mathrm{D} - \varepsilon - \bar{\varepsilon})\rho + \Phi_{00} + 9\rho^2 - 3\sigma\bar{\sigma} = 0. \qquad (2.21)$$

Applying the equations (2.20) we obtain

$$\mathrm{D}^2\mathrm{H} + (\mathrm{DH})^2 - (6\rho + \varepsilon + \bar{\varepsilon})\mathrm{DH} + 4(2\rho^2 - \sigma\bar{\sigma}) = 0. \qquad (2.22)$$

Substituting for DH using equation (2.18) and applying (2.19) we find

$$\sigma\bar{\sigma} - \frac{6k_2(4k_1 + k_2)}{(8k_1 - k_2)^2}\rho^2 = 0. \qquad (2.23)$$

Both the terms appearing here are real and non-negative and so in particular we must have

$$\sigma = 0, \qquad (2.24)$$

contradicting our assumption that $\sigma \neq 0$. This implies that the repeated principal null congruence of the space-time is *shear free.*

With $\kappa = \sigma = 0$ condition III_{00}, after applying the NP equation (2.19), reads

$$\mathrm{D}^2\mathrm{H} + (\mathrm{DH})^2 - (6\rho + \varepsilon + \bar{\varepsilon})\mathrm{DH} + 2(3\rho^2 - \Phi_{00}) = 0. \qquad (2.25)$$

The real part of this equation yields

$$(\mathrm{DH})^2 - 3(\rho - \bar{\rho})\mathrm{DH} + 3\rho^2 + 3\bar{\rho}^2 - 2\Phi_{00} = 0. \qquad (2.26)$$

We now consider condition V_{11}, with $\kappa = \sigma = 0$:

$$\begin{aligned}(4k_1 - k_2)[2(\mathrm{DH})^2 + 2(\rho - \bar{\rho})\mathrm{DH} + 2\Phi_{00} \\ - 3(\mathrm{D} - \varepsilon - \bar{\varepsilon} + \rho + \bar{\rho})(\rho + \bar{\rho}) + 30\rho\bar{\rho}] \\ - 16k_1[(\mathrm{DH})^2 - 2(\rho - \bar{\rho})\mathrm{DH} - 4\rho\bar{\rho}] = 0. \qquad (2.27)\end{aligned}$$

Using equation (2.26) to remove the terms in $(\mathrm{DH})^2$ and applying (2.19) we obtain

$$(2k_1 - k_2)(\rho - \bar{\rho})DH + 20k_1\rho\bar{\rho} + \frac{3}{2}k_2(\rho - \bar{\rho})^2 - 4k_1\Phi_{00} = 0. \qquad (2.28)$$

Under the assumption that $\rho \neq \bar{\rho}$, we may solve for DH giving

$$\mathrm{DH} = \frac{(8 k_1 \Phi_{00} - 3 k_2 (\rho - \bar{\rho})^2 - 40 k_1 \rho \bar{\rho}) \mathrm{H}}{2 (2 k_1 - k_2) (\rho - \bar{\rho})}. \tag{2.29}$$

Returning to equation (2.26) we now substitute for DH from the above. Collecting the terms in Φ_{00} on one side of the equation and then completing the square we find

$$\begin{aligned}&[16 k_1^2 \Phi_{00} - (16 k_1^2 - 4 k_1 k_2 + k_2^2) (\rho + \bar{\rho})^2 - 4 (4 k_1^2 + 4 k_1 k_2 - k_2^2) \rho \bar{\rho}]^2 \\ &= (2 k_1 - k_2)^2 (\rho - \bar{\rho})^2 [(16 k_1^2 - 4 k_1 k_2 + k_2^2) (\rho + \bar{\rho})^2 + 4 k_2 (4 k_1 - k_2) \rho \bar{\rho}].\end{aligned} \tag{2.30}$$

We first note that if $\rho \neq \bar{\rho}$, then $\rho \neq 0$, and therefore the right hand side of the above equation is real and strictly negative provided $4 k_1 > k_2 > 0$ and $2 k_1 \neq k_2$, which is the case for all three of the wave equations under consideration. However, the left hand side of the above equation is clearly real and non-negative, we therefore must have

$$\rho = \bar{\rho}, \tag{2.31}$$

in contradiction to our assumption that $\rho \neq \bar{\rho}$. Placing $\rho = \bar{\rho}$ in equations (2.26) and (2.28) we immediately obtain

$$(\mathrm{DH})^2 + 6 \rho^2 = 2 \Phi_{00} \quad \text{and} \quad 5 \rho^2 = \Phi_{00}, \tag{2.32}$$

whence it follows that

$$(\mathrm{DH})^2 - 4 \rho^2 = 0. \tag{2.33}$$

Both the terms appearing here are real and non-positive and so we must have

$$\mathrm{DH} = \rho = 0, \tag{2.34}$$

which implies that the repeated principal null congruence of the space-time is *hypersurface orthogonal* and *non-expanding* in our particular choice of conformal gauge (2.10). Since the hypersurface orthogonal property is conformally invariant the theorem is proved.

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