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JOÃO-PAULO DIAS
MÁRIO FIGUEIRA

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The simplified Wheeler-DeWitt equation: The Cauchy problem and some spectral properties

by

João-Paulo DIAS and Mário FIGUEIRA

C.M.A.F., Av. Prof. Gama Pinto, 2
1699 Lisboa Codex, Portugal

ABSTRACT. – In this paper we consider the simplified Wheeler-DeWitt equation which describes the minisuperspace model for the wave function ψ of a closed universe (*cf.* [4], [1]). We study this equation as an evolution equation in the scalar field $y \in \mathbf{R}$ with a scale factor $x \in]0, R[$. We solve the Cauchy problem for the initial data $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$ and we study the spectrum of a differential operator related to the equation (*cf.* [4]).

RÉSUMÉ. – Dans ce papier nous considerons l'équation de Wheeler-DeWitt simplifiée qui décrit le modèle de mini-superespace pour la fonction d'onde ψ d'un univers fermé (*cf.* [4], [1]). Nous étudions cette équation comme une équation d'évolution dans le champ scalaire $y \in \mathbf{R}$ avec le facteur d'échelle $x \in]0, R[$. Nous résolvons le problème de Cauchy pour des données initiales $\psi(x, 0)$ et $\frac{\partial \psi}{\partial y}(x, 0)$ et nous étudions le spectre d'un opérateur différentiel associé à l'équation (*cf.* [4]).

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1. INTRODUCTION

The simplified Wheeler-DeWitt equation can be written as follows (*cf.* [4], [1]):

$$\frac{1}{x^2}\frac{\partial^2 \psi}{\partial y^2}-\frac{\partial^2 \psi}{\partial x^2}-\frac{p}{x}\frac{\partial \psi}{\partial x}+x^2\psi-k^2x^4\psi=0 \tag{1.1}$$

where $y \in \mathbf{R}$ is the scalar field, $x \in]0, R[$ $(R>0)$ is a scale factor, $p \in \mathbf{R}$ and $k^2>0$ are given constants (p reflects the factor-ordering ambiguity and k^2 is a mass factor) and $\psi :]0, R[\times \mathbf{R} \rightarrow \mathbf{C}$ is the wave function of the universe for the minisuperspace model. The equation (1.1) is equivalent, in the sense of distributions, to the following equation:

$$\frac{\partial^2 \psi}{\partial y^2}-x^2\frac{\partial^2 \psi}{\partial x^2}-px\frac{\partial \psi}{\partial x}+x^4\psi-k^2x^6\psi=0 \tag{1.2}$$

which can be considered as an evolution equation in $y \in \mathbf{R}$.

Given $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$, we will solve the corresponding Cauchy problem, assuming that the initial data belong to some suitable weighted Sobolev spaces.

In second part of the paper we study the following eigenvalue problem (*cf.* [4], V):

$$-\frac{\partial^2 \psi}{\partial x^2}-\frac{p}{x}\frac{\partial \psi}{\partial x}+x^2\psi-k^2x^4\psi=\lambda\psi \tag{1.3}$$

for ψ in a suitable domain.

In order to solve these problems we must introduce some weighted Sobolev spaces related to those introduced by P. Grisvard (*cf.* [2]) in the case where $]0, R[$ is replaced by $\mathbf{R}_+$.

2. FUNCTION SPACES

Let $\varphi \in \mathscr{D}(]0, R[)$ (extended by zero to $\mathbf{R}_+$) and put

$$F(x)=\int_0^x |\varphi'(t)|\,dt, \qquad G(x)=\int_x^{+\infty} |\varphi'(t)|\,dt, \qquad \varphi'=\frac{d\varphi}{dt}.$$

We have, by theorem 330 in [3],

$$\int_0^{+\infty} x^{-r}F^2(x)\,dx \leqq \frac{4}{|r-1|^2}\int_0^{+\infty} x^{-r}(x|\varphi'(x)|)^2\,dx, \qquad \text{if} \quad r>1,$$

$$\int_0^{+\infty} x^{-r}G^2(x)\,dx \leqq \frac{4}{|r-1|^2}\int_0^{+\infty} x^{-r}(x|\varphi'(x)|)^2\,dx, \qquad \text{if} \quad r<1.$$

Hence, we obtain

$$\int_0^R x^{p-2}\,|\varphi(x)|^2\,dx \leqq \frac{4}{|p-1|^2}\int_0^R x^p\,|\varphi'(x)|^2\,dx, \qquad \text{if} \quad p\neq 1 \tag{2.1}$$

In the special case $p=1$, applying (2.1) to $p=1+\varepsilon$, $\varepsilon>0$, we get

$$\int_0^R x^{-1+\varepsilon}\,|\varphi(x)|^2\,dx \leqq \frac{4}{\varepsilon^2}\int_0^R x^{1+\varepsilon}\,|\varphi'(x)|^2\,dx \leqq \frac{4\,R^\varepsilon}{\varepsilon^2}\int_0^R x\,|\varphi'(x)|^2\,dx \tag{2.2}$$

We define, for $p\in\mathbf{R}$,

$$H_p^1 = H_p^1(]0,\,R[) = \left\{ u\in L_p^2 \;\middle|\; \frac{du}{dx}\in L_p^2 \right\}$$

with its natural norm, where $L_p^2(]0,\,R[)$ is the L^2 space for the measure $d\mu = x^p\,dx$. We denote by $H_{p,0}^1$ the closure of $\mathscr{D}(]0,\,R[)$ in H_p^1. By (2.1), (2.2), we get that $\left|\dfrac{du}{dx}\right|_{L_p^2}$ is, in $H_{p,0}^1$, an equivalent norm to $|u|_{H_p^1}$ denoted by $|u|_{H_{p,0}^1}$.

By (2.1) we can also conclude that, if $p\neq 1$, we have $H_{p,0}^1 \hookrightarrow L_{p-2}^2$, where $\hookrightarrow$ means continuous injection.

We can finally remark that, following the ideas of [2], the functions belonging to $H_{p,0}^1$, for $p<1$, are the functions of H_p^1 that are equal to zero at the boundary of]0, R[.

We need the following lemmas:

LEMMA 2.1. – *We have* $\mathscr{D}(]0,\,R[)$ *dense in* $H_{1,0}^1\cap L_{-1}^2$ *for its natural norm.*

Proof. – Let $u\in H_{1,0}^1\cap L_{-1}^2$, $\varphi_n\in\mathscr{D}(]0,\,R[)$ such that $\varphi_n\to u$ in $H_{1,0}^1$. We can choose φ_n such that

$$|\varphi_n - u|_{H_{1,0}^1} \leqq \frac{1}{n^5}, \qquad n=1,\,2,\,\ldots \tag{2.3}$$

Let $\psi_n = x^{1/n}\varphi_n$, $n=1,\,2,\,\ldots$ We have $\psi_n\in\mathscr{D}(]0,\,R[)$.

With $v_n=\varphi_n-u$, we get, by (2.1) and (2.3),

$$\begin{aligned}
\int_0^R x^{-1}\,|x^{1/n}v_n|^2\,dx &\leqq R^{1/n^2}\int_0^R x^{-1-(1/n^2)}\,|x^{1/n}v_n|^2\,dx \\
&\leqq R^{1/n^2}\,4n^4\left[\int_0^R x^{1-(1/n^2)}\,|x^{1/n}v_n'|^2\,dx + \frac{1}{n^2}\int_0^R x^{1-(1/n^2)}\,|x^{(1/n)-1}v_n|^2\,dx\right] \\
&\leqq R^{1/n^2}\,4n^4\left[\int_0^R x\,|v_n'|^2\,dx + \frac{1}{n^2}\int_0^R x^{-1+(2/n)-(1/n^2)}\,|v_n|^2\,dx\right] \\
&\leqq R^{1/n^2}\,4n^4\left(\frac{1}{n^5}+\frac{c}{n^2}\int_0^R x\,|v_n'|^2\,dx\right)\to 0.
\end{aligned}$$

Hence, since $u \in L^2_{-1}$, we obtain

$$|\psi_n - u|^2_{L^2_{-1}} \leqq 2[|x^{1/n}(\varphi_n - u)|^2_{L^2_{-1}} + |(x^{1/n}-1)u|^2_{L^2_{-1}}] \to 0.$$

Furthermore, we have

$$|\psi_n - u|^2_{H^1_{1,0}} \leqq 2\left[\int_0^R x\,|u' - x^{1/n}\varphi_n'|^2\,dx + \int_0^R x\left|\frac{1}{n}x^{(1/n)-1}\varphi_n\right|^2 dx\right]$$

$$\leqq 4\left[\int_0^R x(1-x^{1/n})^2\,|u'|^2\,dx + \int_0^R x\,|x^{1/n}(u'-\varphi_n')|^2\,dx + \frac{1}{n^2}\int_0^R x^{-1}\,|x^{1/n}\varphi_n|^2\,dx\right].$$

But

$$\int_0^R x\,|x^{1/n}(u'-\varphi_n')|^2\,dx \leqq R^{1/n^2}\,|u-\varphi_n|^2_{H^1_{1,0}} \to 0$$

and

$$\int_0^R x^{-1}\,|x^{1/n}\varphi_n|^2\,dx = |\psi_n|^2_{L^2_{-1}} \quad \text{is bounded.}$$

Hence, $|\psi_n - u|^2_{H^1_{1,0}} \to 0$ and we conclude that

$$\psi_n \to u \quad \text{in } H^1_{1,0} \cap L^2_{-1} \quad \square$$

LEMMA 2.2. — *We have* $\{u \mid x^{p/2} u \in H^1_0\} \hookrightarrow H^1_{p,0} \cap L^2_{p-2}$, *where* $H^1_0 = H^1_{0,0}$.

Proof. — Let u be such that $x^{p/2} u \in H^1_0$ and take $\varphi_n \in \mathscr{D}(]0, R[)$, $\psi_n \to x^{p/2} u$ in H^1_0.

Hence, $x^{-1}\varphi_n \to x^{(p/2)-1} u$ in L^2 and, if $u_n = x^{-p/2}\varphi_n$, we have

$$u_n \in \mathscr{D}(]0, R[), \qquad u_n \to u \quad \text{in } L^2_p$$

$$x^{-p/2}\varphi_n' \to \frac{p}{2}x^{-1}u + u' \quad \text{in } L^2_p$$

$$x^{-1}u_n = x^{-1-(p/2)}\varphi_n \to x^{-1}u \quad \text{in } L^2_p.$$

Hence,

$$u_n' = -\frac{p}{2}x^{-(p/2)-1}\varphi_n + x^{-(p/2)}\varphi_n' \to -\frac{p}{2}x^{-1}u + \frac{p}{2}x^{-1}u + u' = u' \quad \text{in } L^2_p.$$

We get $u \in H^1_{p,0}$ and $|u|_{H^1_{p,0}} \leqq c\,|x^{p/2} u|_{H^1_0}$.

By (2.1) (with $p=0$) it follows that $\{u \mid x^{1/2} u \in H^1_0\} \hookrightarrow L^2_{-1}$. $\square$

LEMMA 2.3. — *We have*

$$H^1_{p,0} \cap L^2_{p-2} \hookrightarrow \{u \mid x^{p/2} u \in H^1_0\}$$

and hence by lemma 2.2

$$H^1_{p,0} \cap L^2_{p-2} = \{u \mid x^{p/2} u \in H^1_0\}.$$

For $p=1$, *we have* $H^1_{1,0} \subsetneq \{u \mid x^{q/2} u \in H^1_0\}$, $\forall q>1$.

Proof. – Assume $p \neq 1$ and let $u \in H^1_{p,0} \cap L^2_{p-2} = H^1_{p,0}$, $\varphi_n \in \mathscr{D}(]0, R[)$, $\varphi_n \to u$ in $H^1_{p,0}$. By (2.1) we get $\varphi_n \to u$ in L^2_{p-2} and so $x^{(p/2)-1}\varphi_n \to x^{(p/2)-1} u$ in L^2.

Since

$$x^{p/2}\psi'_n \to x^{p/2} u' \quad \text{in } L^2$$

we get

$$(x^{p/2}\varphi_n)' = \frac{p}{2} x^{(p/2)-1}\varphi_n + x^{p/2}\varphi'_n \to \frac{p}{2} x^{(p/2)-1} u + x^{p/2} u' = (x^{p/2} u)' \quad \text{in } L^2.$$

Hence, $x^{p/2} u \in H^1_0$ and $|x^{p/2} u|_{H^1_0} \leq c_1 |u|_{H^1_{p,0}}$.

For $p=1$, by lemma 2.1, we can choose $\varphi_n \in \mathscr{D}(]0, R[)$ such that $\varphi_n \to u$ in $H^1_{1,0} \cap L^2_{-1}$ and the proof is the same. Furthermore,

$$H^1_{1,0} \subsetneq H^1_{q,0} \subsetneq \{u \mid x^{q/2} u \in H^1_0\}, \qquad \forall q>1. \quad \square$$

We can now prove the following:

PROPOSITION 2.1. – *The injection* $H^1_{p,0} \subsetneq L^2_p$ *is compact, for every* $p \in \mathbf{R}$.

Proof. – Assume first $p \neq 1$. Since the injection $H^1_0 \subsetneq L^2$ is compact, the result is a consequence of lemmas 2.2 and 2.3 and of the mappings

$$\begin{gathered} u \to x^{p/2} u \to x^{p/2} u \to u \\ (H^1_{p,0} \to H^1_0 \to L^2 \to L^2_p). \end{gathered}$$

Suppose now $p=1$ and let $u \in H^1_{1,0}$: we have

$$\int_0^R x|u|^2 dx = \int_0^R x^{-1/2} x^{3/2} |u|^2 dx \leq \left(\int_0^R x^{-3/4} dx\right)^{2/3} \left(\int_0^R x^{9/2} |u|^6 dx\right)^{1/3}$$

Hence $|u|_{L^2_1} \leq c\,|x^{3/4} u|_{L^6}$. Furthermore, by applying lemma 2.3 with $q=\frac{3}{2}$, we get

$$|x^{3/4} u|_{H^1_0} \leq c_1 |u|_{H^1_{1,0}}.$$

The result is now a consequence of the compact injection $H^1_0 \subsetneq L^6$ and of the mappings

$$\begin{gathered} u \to x^{3/4} u \to x^{3/4} u \to u \\ (H^1_{1,0} \to H^1_0 \to L^6 \to L^2_1). \quad \square \end{gathered}$$

3. THE CAUCHY PROBLEM

In order to study the Cauchy problem for the equation (1.2) let us put $\psi_1 = \frac{\partial \psi}{\partial y}$,

$$\mathrm{B}\,\psi = x^2 \frac{\partial^2 \psi}{\partial x^2} + p x \frac{\partial \psi}{\partial x} \tag{3.1}$$

$$\mathrm{V}\,\psi = -x^4 \psi + k^2 x^6 \psi \tag{3.2}$$

The equation (1.2) can be written as follows:

$$\frac{\partial}{\partial y}\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathrm{B} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathrm{V} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \tag{3.3}$$

We consider the Hilbert space $\mathrm{H} = \mathrm{H}^1_{p,0} \times \mathrm{L}^2_{p-2}$.

If $p \neq 1$ we have $\mathrm{H}^1_{p,0} \subsetneq \mathrm{L}^2_{p-2}$.

We put

$$\mathrm{D}(\mathrm{A}) = \mathrm{D}(\mathrm{B}) \times (\mathrm{H}^1_{p,0} \cap \mathrm{L}^2_{p-2}),$$

where

$$\mathrm{D}(\mathrm{B}) = \{ u \in \mathrm{H}^1_{p,0} \mid \mathrm{B}\,u \in \mathrm{L}^2_{p-2} \}.$$

THEOREM 3.1. — *The operator* $\mathrm{A} : \mathrm{D}(\mathrm{A}) \to \mathrm{H}$ *defined by*

$$\mathrm{A}\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathrm{B} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \mathrm{B}\,\psi \end{pmatrix}$$

is skew-self-adjoint in H.

Proof. — Let $(w, h) \in \mathrm{D}(\mathrm{A}^*)$: there exists $(q, l) \in \mathrm{H}$ such that, for every $(u, v) \in \mathrm{D}(\mathrm{A})$, we have

$$(\mathrm{A}(u, v), (w, h)) = ((u, v), (q, l))$$

that is

$$(v, w)_{\mathrm{H}^1_{p,0}} + (\mathrm{B}\,u, h)_{\mathrm{L}^2_{p-2}} = (u, q)_{\mathrm{H}^1_{p,0}} + (v, l)_{\mathrm{L}^2_{p-2}} \tag{3.4}$$

In particular, with $u = 0$, we get

$$\int_0^{\mathrm{R}} x^p v' \bar{w}' \, dx = \int_0^{\mathrm{R}} x^{p-2} v \bar{l} \, dx, \qquad \forall v \in \mathscr{D}(]0, \mathrm{R}[)$$

which implies

$$-\mathrm{B}\,w = l \quad \text{in } \mathscr{D}'(]0, \mathrm{R}[)$$

Since $w \in \mathrm{H}^1_{p,0}$, we get $w \in \mathrm{D}(\mathrm{B})$.

Now, put $v = 0$ in (3.4). We obtain

$$(\mathrm{B}\,u, h)_{\mathrm{L}^2_{p-2}} = \int_0^{\mathrm{R}} x^p u' \bar{q}' \, dx, \qquad \forall u \in \mathrm{D}(\mathrm{B})$$

We have

$$\int_0^R x^p |f'|^2 dx = |f|^2_{H^1_{p,0}}, \qquad f \in H^1_{p,0}.$$

Hence, if $\varphi \in \mathscr{D}(]0, R[)$, there exists $u \in H^1_{p,0}$ such that

$$\int_0^R x^p u' \bar{f}' \, dx = \int_0^R x^{p-2} \varphi \bar{f} dx, \qquad \forall f \in H^1_{p,0},$$

that is $Bu = \varphi$ in $\mathscr{D}'(]0, R[)$. Then $u \in D(B)$ and

$$\langle \bar{h}, x^{p-2} \varphi \rangle_{\mathscr{D}' \times \mathscr{D}} = (\varphi, h)_{L^2_{p-2}}$$
$$= \int_0^R x^p u' \bar{q}' \, dx = \int_0^R x^{p-2} \varphi \bar{q} dx = \langle \bar{q}, x^{p-2} \varphi \rangle_{\mathscr{D}' \times \mathscr{D}}.$$

Since φ is arbitrary we conclude that $h = q \in H^1_{p,0}$. Hence, $(w, h) \in D(A)$ and so $D(A^*) \subset D(A)$. Furthermore, if $w \in D(B)$ and $v \in H^1_{p,0} \cap L^2_{p-2}$, it is easy to see that $(v, w)_{H^1_{p,0}} = -(v, Bw)_{L^2_{p-2}}$, by lemma 2.1.

This implies $D(A) \subset D(A^*)$ and that, in $D(A)$, $A^* = -A$. This achieves the proof of theorem 3.1. □

PROPOSITION 3.1. – *The operator* $D : H \to H$ *defined by*

$$D \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ V\psi \end{pmatrix}$$

is continuous.

Proof. – We have

$$|V\psi|_{L^2_{p-2}} \leqq |x^4 \psi|_{L^2_{p-2}} + k^2 |x^6 \psi|_{L^2_{p-2}} \leqq c |\psi|_{H^1_{p,0}}$$

by (2.1) and (2.2). □

By applying a well known result (*cf.* [5]) we get

COROLLARY 3.1. – *The operator* $T = A + D : D(A) \to H$ *is the infinitesimal generator of a strongly continuous group of operators in* H, *denoted by* e^{yT}, $y \in \mathbf{R}$.

Hence, for

$$\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in D(A) = D(B) \times (H^1_{p,0} \cap L^2_{p-2}),$$

there exists a unique

$$\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \in \mathscr{C}(\mathbf{R}; D(A)) \cap \mathscr{C}^1(\mathbf{R}; H),$$

such that $\begin{pmatrix} \psi(0) \\ \psi_1(0) \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix}$ and verifying (3.3) for $y \in \mathbf{R}$.

We have $\begin{pmatrix} \psi(y) \\ \psi_1(y) \end{pmatrix} = e^{y\mathrm{T}} \begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix}$, $y \in \mathbf{R}$.

Returning to the equation (1.2) we obtain

THEOREM 3.2. — *Assume* $\psi_0 \in \mathrm{D}(\mathrm{B})$ *and* $\psi_{1,0} \in \mathrm{H}^1_{p,0} \cap \mathrm{L}^2_{p-2}$. *Then, there exists a unique*

$$\psi \in \mathscr{C}(\mathbf{R};\, \mathrm{D}(\mathrm{B})) \qquad \text{with} \quad \frac{\partial \psi}{\partial y} \in \mathscr{C}(\mathbf{R};\, \mathrm{H}^1_{p,0} \cap \mathrm{L}^2_{p-2}) \cap \mathscr{C}^1(\mathbf{R};\, \mathrm{L}^2_{p-2})$$

such that

$$\psi(x, 0) = \psi_0(x), \qquad \frac{\partial \psi}{\partial y}(x, 0) = \psi_{1,0}(x), \quad x \in]0, \mathrm{R}[,$$

and verifying (1.2).

We have also the following

PROPOSITION 3.2. — *Assume* $p \neq 1$. *Then we have*

$$\mathrm{D}(\mathrm{B}) = \{ u \in \mathrm{H}^1_{p,0} \,|\, x^{(p/2)+1} u \in \mathrm{H}^2 \}.$$

Proof. — Suppose $u \in \mathrm{D}(\mathrm{B})$: we have $u \in \mathrm{H}^1_{p,0}$ (and hence $x^{p/2} u \in \mathrm{H}^1_0$ by lemma 2.3),

$$x^2 u'' + pxu' \in \mathrm{L}^2_{p-2}$$

Hence

$$\begin{aligned} \frac{d^2}{dx^2}(x^{(p/2)+1} u) &= x^{(p/2)+1} u'' + (p+2) x^{p/2} u' + \left(\frac{p}{2}+1\right)\frac{p}{2} x^{(p/2)-1} u \\ &= (x^{(p/2)+1} u'' + x^{p/2} p u') + 2 x^{p/2} u' + \left(\frac{p}{2}+1\right)\frac{p}{2} x^{(p/2)-1} u \in \mathrm{L}^2, \end{aligned}$$

and

$$x^{(p/2)+1} u \in \mathrm{H}^2.$$

Conversely, if $u \in \mathrm{H}^1_{p,0}$ and $x^{(p/2)+1} u \in \mathrm{H}^2$, we get

$$x^{(p/2)+1} u'' + p x^{p/2} u' = \frac{d^2}{dx^2}(x^{(p/2)+1} u) - \left(\frac{p}{2}+1\right)\frac{p}{2} x^{(p/2)-1} u - 2 x^{(p/2)} u' \in \mathrm{L}^2$$

Hence,

$$x^{(p-2)/2} (x^2 u'' + pxu') \in \mathrm{L}^2,$$

that is $x^2 u'' + pxu' \in \mathrm{L}^2_{p-2}$ and so $u \in \mathrm{D}(\mathrm{B})$. □

4. SPECTRAL PROPERTIES OF A RELATED OPERATOR

We want to find $\lambda \in \mathbf{C}$ such that there exists u verifying

$$u \in \mathrm{D}(\mathrm{B}), \qquad u \neq 0, \qquad -\mathrm{B} u - \mathrm{V} u = \lambda (x^2 u) \quad \text{in } \mathscr{D}'(]0, \mathrm{R}[) \tag{4.1}$$

where B and V are defined by (3.1), (3.2).

Since $u \in H^1_{p,0}$ implies Vu, $x^2 u \in L^2_{p-2}$, we can replace condition (4.1) by the following

$$u \in H^1_{p,0}, \qquad u \neq 0, \qquad -\frac{1}{x^2}(B+V)u = \lambda u \quad \text{in } \mathscr{D}'(]0, R[) \tag{4.2}$$

The equation in (4.2), which is the same in (1.3), has an important physical meaning, namely if we can find sufficient conditions such that $\lambda = 0$ is not a possible eigenvalue (*cf.* [4], V).

In order to solve this problem, we consider the following hermitian continuous sesquilinear form over $H^1_{p,0}$:

$$a_\gamma(u, v) = \int_0^R x^p u' \bar{v}' \, dx + \int_0^R x^{p+2} u \bar{v} \, dx - k^2 \int_0^R x^{p+4} u \bar{v} \, dx + \gamma \int_0^R x^p u \bar{v} \, dx, \qquad \text{where} \quad \gamma \geqq 0 \tag{4.3}$$

By (2.1), (2.2), we have

$$\beta = \sup_{v \in H^1_{p,0},\, v \neq 0} \frac{\int_0^R x^p |v|^2 \, dx}{\int_0^R x^p |v'|^2 \, dx} < +\infty \tag{4.4}$$

and $\beta \leqq \dfrac{4R^2}{|p-1|^2}$ for $p \neq 1$, $\beta \leqq R^2$ for $p = 1$.

It is easy to check that if

$$\theta = \gamma - k^2 R^4 > -\beta^{-1} \tag{4.5}$$

then, with $\alpha = \min[t, (1 + \beta\theta - t)\beta^{-1}]$, $0 < t < \min(1, 1 + \beta\theta)$, we have

$$a_\gamma(u, u) \geqq \alpha |u|^2_{H^1_{p,0}}, \qquad \forall u \in H^1_{p,0} \tag{4.6}$$

Furthermore, by proposition 2.1, the injection $H^1_{p,0} \hookrightarrow L^2_p$ is compact. Hence, by a well known result we obtain:

PROPOSITION 4.1. – *Let* $\gamma \geqq 0$ *be such that* (4.5) *is verified. Then there exists an increasing sequence* $\{\theta_n\}$, $\theta_n > 0$, $\theta_n \to +\infty$ *such that, for each* n, *the equation*

$$a_\gamma(u, v) = \theta_n \int_0^R x^p u \bar{v} \, dx, \qquad \forall v \in H^1_{p,0} \tag{4.7}$$

has a nontrivial solution $u \in H^1_{p,0}$.

It is easy to prove that (4.7) is equivalent to

$$u \in D(B), \qquad -Bu - Vu = (\theta_n - \gamma)(x^2 u).$$

Hence, we can find a nontrivial solution $u \in D(B)$ for (4.1) if

$$\lambda = \lambda_n = \theta_n - \gamma, \qquad n = 1, 2, \ldots$$

Then, since $\theta_n > 0$, we have $\lambda_n > -\gamma$, $n = 1, 2, \ldots$

If $k^2 R^4 < \beta^{-1}$ we can choose $\gamma = 0$ $\left(\text{a sufficient condition is } k^2 < \frac{|p-1|^2}{4R^6}, \text{ if } p \neq 1, \text{ or } k^2 < \frac{1}{R^6}, \text{ if } p = 1 \right)$.

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