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Distortion analyticity for two-body Schrödinger operators

by

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ABSTRACT. — A complex distortion which is local in momentum is studied, and the analyticity with respect to this distortion is proved for a large class of potentials including analytic potentials and exponentially decaying potentials.

RÉSUMÉ. — Nous étudions une distorsion complexe qui est locale dans les moments. Nous prouvons l'analyticité par rapport à cette distorsion pour une grande classe de potentiels qui contient les potentiels analytiques et les potentiels qui décroissent exponentiellement.

1. INTRODUCTION

Since the concept of complex dilation was introduced by Aguilar, Combes and Balslev ([1], [4]), the method has been applied to various problems of Schrödinger operators. In particular, they proved that the resolvent can

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be continued to the second sheet of the complex plane as a form on a space of analytic vectors, and resonances were defined as the poles of the continued form. A dilation analytic potential is, roughly speaking, a function which is analytic in the radial variable $|x|$. On the other hand, it had been known that such a meromorphic continuation is possible for resolvents of Schrödinger operators with potentials decaying exponentially at infinity, without analyticity conditions. Cycon [7] and Sigal [14] proposed a complex distortion that is local in the momentum variables, to synthesize these two approaches, and they proved the distortion analyticity for potentials radially symmetric and dilation analytic, or sufficiently smooth and exponentially decaying at infinity.

The purpose of this paper is to extend their results to analytic but not necessarily radially symmetric potentials, and exponentially decaying potentials with singularities, using pseudodifferential operator techniques. Moreover, the method applies also to the semiclassical analysis, and we state our result in an h-dependent form where h is the Planck constant. In particular, the existence of shape resonances in the semiclassical limit ([6], [9], [17]) will be studied in another paper [12] based on our complex distortion method. The method enables us to consider semiclassical resonances for Schrödinger operators with exponentially decaying but not analytic potentials, to which usual complex scaling in configuration space is not applicable.

We study Schrödinger operator: $H=-h^2\Delta+V(x)$ on $L^2(\mathbf{R}^n)$. We employ a distortion of the following form (*cf.* [7], [14]):

$$T_\theta:\quad \xi\to T_\theta\xi=\xi-\theta v(\xi) \tag{1.1}$$

where ξ is a momentum, v is a vector field and θ is a distortion parameter. We suppose that v satisfies the following condition:

CONDITION (A). – $v\in C^\infty(\mathbf{R}^n,\mathbf{R}^n)$ such that

$$|v(\xi)|\leqq C,\qquad \left|\left(\frac{\partial}{\partial\xi}\right)^\alpha v(\xi)\right|\leqq C_\alpha\langle\xi\rangle^{1-|\alpha|} \tag{1.2}$$

for any α, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $|\alpha|=\sum_i\alpha_i$. Moreover v satisfies the following out-going property:

$$v(\xi).\xi\geqq 0,\qquad \xi\in\mathbf{R}^n. \tag{1.3}$$

The following is a typical example of $v(\xi)$ and this choice is usually sufficient for applications.

Example. – $v(\xi)=f(|\xi|).\xi$ where $f\in C^\infty([0,\infty))$ such that $f(t)\geqq 0$, $f(t)=1$ if $t<R$, and $f(t)=\text{Const.}/t$ for $t\gg R$ with some $R>0$.

If $|\theta|$ is sufficiently small, T_θ is invertible. We set $\theta_0>0$ so that if $\theta\in\mathbf{C}$ and $|\theta|\leqq\theta_0$ then $|J_\theta(\xi)|\geqq\varepsilon>0$ for $\xi\in\mathbf{R}^n$, where $J_\theta(\xi)$ is the Jacobian of

T_θ at ξ. If $\theta \in (-\theta_0, \theta_0)$, T_θ defines a diffeomorphism on $\mathbf{R}^n$. T_θ naturally induces the following unitary operator:

$$U_\theta \varphi = \mathscr{F}^{-1} \{ J_\theta(\xi)^{1/2} (\mathscr{F}\varphi)(T_\theta \xi) \} \tag{1.4}$$

where $\mathscr{F}$ is the Fourier transform:

$$(\mathscr{F}\varphi)(\xi) = (2\pi h)^{-n/2} \int e^{-ix\xi/h} \varphi(x)\, dx. \tag{1.5}$$

Note that $\{ U_\theta \}$ does *not* form a one parameter group, but it works as remarked by Cycon [7]. Then the distortion analyticity with respect to this distortion is defined in a standard way:

DEFINITION. – Let A be an operator on $L^2(\mathbf{R}^n)$. A is called *distortion analytic* (with respect to $v(\xi)$) if $A_\theta = U_\theta A U_\theta^{-1}$ $(\theta \in I = (-\theta_0, \theta_0))$ can be extended to an analytic family of operators on a neighborhood of 0 in $\mathbf{C}$.

Of course, $H_0 = -h^2 \Delta$ is distortion analytic. In fact, it is easy to see that

$$H_{0,\theta} = \mathscr{F}^{-1} (\xi - \theta v(\xi))^2 \mathscr{F} \tag{1.6}$$

and $\{ H_{0,\theta} \}$ forms an analytic family of type A in any neighborhood of I.

CONDITION (B). – $V(x)$ is a sum of two potentials $V(x) = V_1(x) + V_2(x)$ such that:

(i) There exists a holomorphic function $\tilde{V}_1(z)$ on

$$D(a,b) = \{ z \in \mathbf{C}^n \,\|\, \operatorname{Im} z | < a | \operatorname{Re} z | \text{ or } | \operatorname{Im} z | < b \}$$

for some a, $b > 0$ such that for any α,

$$\left| \left(\frac{\partial}{\partial z} \right)^\alpha \tilde{V}_1(z) \right| \leqq C_\alpha \langle z \rangle^{-|\alpha|}, \qquad z \in D(a,b) \tag{1.7}$$

and $\tilde{V}_1(x) = V_1(x)$ for $x \in \mathbf{R}^n$.

(ii) There exists $\delta > 0$ such that $\exp(\delta \langle x \rangle) . V_2(x) \in H^s(\mathbf{R}^n)$ for some $s \geqq 0$, $s > n/2 - 2$.

We write $V_1(z)$ instead of $\tilde{V}_1(z)$ for simplicity. To obtain the H_0-compactness of $V(x)$, we need an additional assumption.

CONDITION (C). – In Condition (B)-(i), $|V_1(x)| \to 0$ as $|x| \to \infty$.

Then our main result is the following:

THEOREM 1.1. – *Suppose* (A) *and* (B). *Then as a* $B(H^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$-*valued analytic function,* $V_\theta = U_\theta V U_\theta^{-1}$ $(\theta \in I)$ *is extended to* $\theta \in D_d^h = \{ z \in \mathbf{C} \,\|\, z | < dh \}$ *for some* $d > 0$ *and any* $h \in (0, 1]$, *and* V_θ *is infinitesimally* H_0-*bounded. Furthermore, if* (C) *holds,* V_θ *is* H_0-*compact.*

Remark 1.2. – Condition (B)-(ii) allows V_2 to have the optimum local singularity and we need no short range condition on V_1 (*cf.* [3], [5]). But the above potential class does *not* include the class of dilation analytic potentials ([1], [4]).

The next corollary follows immediately from Theorem 1.1 by a standard perturbation argument.

COROLLARY 1.3. — *Suppose* (A) *and* (B). *Then* H *is distortion analytic. More precisely,* $\{H_\theta\}$ *is a self-adjoint family of type* A *on* D_d^h.

The above result can be applied to obtain the analytic continuation of the resolvent on a class of analytic vectors defined by

$$\mathscr{A} = \{ \varphi \in L^2(\mathbf{R}^n) \mid \forall c, s > 0, \exp(c\langle x \rangle)\varphi(x) \in H^s(\mathbf{R}^n) \}. \qquad (1.8)$$

PROPOSITION 1.4. — *For any* $\varphi \in \mathscr{A}$, $\theta \to U_\theta \varphi$ *can be extended to an* $L^2(\mathbf{R}^n)$*-valued analytic function on* $\{\theta \in \mathbf{C} \mid |\theta| < \theta_0\}$, *and the range* $U_\theta \mathscr{A}$ *is dense in* $L^2(\mathbf{R}^n)$ *for each* θ.

It is easy to see the former assertion because $\mathscr{F}\varphi$ is an entire function with rapid decreasing. The latter assertion is essentially due to Cycon [7] and Hunziker [10], and we omit the proof.

Now suppose (A), (B) and (C), and let $\theta \in D_d^h$ and $\operatorname{Im}\theta > 0$. By Theorem 1.1 and Weyl's theorem on the essential spectrum, we have

$$\sigma_{\text{ess}}(H_\theta) = \sigma(H_{0,\theta}) = \{ (\xi - \theta v(\xi))^2 \mid \xi \in \mathbf{R}^n \}. \qquad (1.9)$$

If $|\theta|$ is sufficiently small, the out-going property (1.3) implies

$$\sigma(H_{0,\theta}) \subset \mathbf{C}^{\pm} = \{ z \in \mathbf{C} \mid \operatorname{Re} z \geqq 0, \operatorname{Im} z < 0 \}. \qquad (1.10)$$

Thus Theorem 1 of Cycon [7] can be applied to H_θ and we obtain the following corollary [note that $\sigma_{\text{ess}}(H_{0,\theta})$ is not necessarily one-dimensional manifold in our situation, but it makes no difference]:

COROLLARY 1.5. — *Suppose* (A), (B) *and* (C), *and let* $\theta \in D_d^h$ *with* $\operatorname{Im}\theta > 0$. *Then for any* $\varphi, \psi \in \mathscr{A}$, $f_{\varphi\psi}(z) = (\varphi, (H - z)^{-1}\psi)$ *has a meromorphic continuation from* $\mathbf{C}^{++} = \{ z \in \mathbf{C} \mid \operatorname{Re} z \geqq 0,\ \operatorname{Im} z > 0 \}$ *to* S_θ: *the union of connected components of* $\mathbf{C}^{+-} \setminus \sigma_{\text{ess}}(H_\theta)$ *having an open intersection with* $\mathbf{R}^+$. *Moreover*

$$\bigcup_{\varphi, \psi \in \mathscr{A}} \{ \textit{poles of } f_{\varphi\psi}(z) \textit{ in } S_\theta \} = \sigma_{\text{disc}}(H_\theta) \cap S_\theta. \qquad (1.11)$$

The latter assertion is a consequence of the density of $U_\theta \mathscr{A}$ (*cf.* Remark 2 to Theorem 1 of [7] and Theorem 3 of [10]). The nonreal eigenvalues of H_θ are called resonances, and this definition depends only on $\mathscr{A}$ by virtue of (1.11).

The method of complex scaling or distortion was introduced by Aguilar-Combes [1] and Balslev-Combes [4]. Since then, many papers have appeared on generalizations of their method. Our approach is due to Cycon [7] and Sigal [14], but also similar to Hunziker [10] and Babbit-Balslev

[2]. In particular, Hunziker studied a method of distortion local in configuration space, and it can be considered as a generalization of the exterior scaling (Simon [15], Combes-Duclos-Klein-Seiler [6]). For a survey of these methods including applications to Stark effect, we refer a monograph by Cycon-Froese-Kirsch-Simon [8]. On the other hand, Balslev and Skibsted proved that the resolvent can be meromorphically continued (*cf.* Corollary 1.5) if the potential is a sum of a short range dilation analytic potential and an exponentially decaying one. Their proof is based on the stationary theory of scattering and the perturbation of the scattering matrix ([3], [5]).

In Section 2 and 3, we study the distortion analyticity of $V_1(x)$ and $V_2(x)$ respectively. Theorem 1.1 follows immediately from the results of these sections.

2. ANALYTIC POTENTIALS

THEOREM 2.1. – *Suppose* V *satisfies* (B)-(i) *and* v *satisfies*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} v(\xi)\right| \leqq C_{\alpha}\langle\xi\rangle^{1-|\alpha|} \tag{2.1}$$

for any α. *Then* $V_\theta = U_\theta V U_\theta^{-1}$ *is extended to* $D_d^1 = \{\theta \in \mathbf{C} \,\|\, z\,| < d\}$ *as a* $B(L^2(\mathbf{R}^n))$*-valued analytic function for some* $d > 0$. *Moreover if* $|V(x)| \to 0$ *as* $|x| \to \infty$, *then* V_θ *is* H_0*-compact.*

Remark 2.2. – As is seen from the proof, V_θ is a pseudodifferential operator. Furthermore, V_θ is in a nice class of h-pseudodifferential operators (*cf.* [12]).

LEMMA 2.3. – *For* $\theta \in I$, V_θ *is a pseudodifferential operator with the symbol* $W_\theta(\xi, x, \eta)$ *given by*

$$\begin{aligned} W_\theta(\xi, x, \eta) = {} & J_\theta(\xi)^{1/2} J_\theta(\eta)^{1/2} \\ & \times \det(1-\theta\, w(\xi,\eta))^{-1} V((1-\theta\, w(\xi,\eta))^{-1} x) \end{aligned} \tag{2.2}$$

where

$$w(\xi,\eta) = (w_{ij}(\xi,\eta)) = \left(\int_0^1 dt \left(\frac{\partial v_j}{\partial \xi_i}(t\xi + (1-t)\eta)\right)\right). \tag{2.3}$$

Moreover, for any α, β, γ,

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial\eta}\right)^{\gamma} W_\theta(\xi, x, \eta)\right| \leqq C_{\alpha\beta\gamma}. \tag{2.4}$$

Proof: For $\varphi \in \mathscr{S}$, we have

$$\mathscr{F} U_\theta V U_\theta^{-1} \mathscr{F}^{-1} \varphi(\xi)$$
$$= (2\pi h)^{-n} \int e^{-i(\xi - \theta v(\xi))x/h} J_\theta(\xi)^{1/2} V(x)$$
$$J_\theta(\eta)^{1/2} e^{i(\eta - \theta v(\eta))x/h} \varphi(\eta)\, d\eta\, dx$$
$$= (2\pi h)^{-n} \int \exp\{-i(\xi - \eta - \theta(v(\xi) - v(\eta)))x/h\}$$
$$J_\theta(\xi)^{1/2} J_\theta(\eta)^{1/2} V(x) \varphi(\eta)\, d\eta\, dx \quad (2.5)$$

where the integrals are taken to be oscillatory integrals. Noting that

$$(\xi - \eta - \theta(v(\xi) - v(\eta))).x$$
$$= \sum_{i,j} (\xi_i - \eta_i) \left\{ \delta_{ij} - \theta \int_0^1 dt \frac{\partial v_j}{\partial \xi_i}(t\xi + (1-t)\eta) \right\} x_j$$
$$= (\xi - \eta).(1 - \theta w(\xi, \eta)).x \quad (2.6)$$

and changing the integral variables: $y = (1 - \theta w(\xi, \eta))\, x$, we obtain

$$\mathscr{F} V_\theta \mathscr{F}^{-1} \varphi = (2\pi h)^{-n} \int e^{-i(\xi - \eta)y/h} V((1 - \theta w(\xi, \eta))^{-1} y)$$
$$J_\theta(\xi)^{1/2} J_\theta(\eta)^{1/2} \varphi(\eta) \left(\frac{\partial y}{\partial x}\right)^{-1} d\eta\, dy$$
$$= (2\pi h)^{-n} \int e^{-i(\xi - \eta)y/h} W_\theta(\xi, x, \eta) \varphi(\eta)\, dy\, d\eta. \quad (2.7)$$

By easy computations using (2.1) and Condition (B)-(i), we obtain (2.4). So the expression (2.7) is meaningful as a pseudodifferential operator with a double symbol W_θ, by virtue of the Calderon-Vaillancourt theorem [16]. ■

Proof of Theorem 2.1. — Under the assumption (B)-(i), $W_\theta(\xi, x, \eta)$ can be defined by (2.2) for complex θ if $|\theta| < \theta_0$ and

$$(1 - \theta w(\xi, \eta))^{-1} \mathbf{R}^n \subset D(a', b), \qquad a' < a \quad (2.8)$$

for all ξ, $\eta \in \mathbf{R}^n$. This condition is fulfilleld if $|\theta|$ is sufficiently small since $w(\xi, \eta)$ is uniformly bounded. We set $d > 0$ sufficiently small so that any $\theta \in D_d^1$ satisfies (2.8). By the exactly same computation as in Lemma 2.3, we obtain (2.4) for $\theta \in D_d^1$. Thus V_θ can be defined by (2.7) and it is bounded in $H^2(\mathbf{R}^n)$. Analyticity in θ can be proved by direct computations.

The last assertion follows from Lemma 5 of [13] Section XIII-5, because $V(H_0 + 1)^{-1}$ is compact. ■

3. EXPONENTIALLY DECAYING POTENTIALS

THEOREM 3.1. – *Suppose* $V(x)$ *satisfies* (B)-(ii) *and* $v(\xi)$ *satisfies*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} v(\xi)\right| \leqq C_{\alpha} \tag{3.1}$$

for any α. *Then* $V_\theta = U_\theta V U_\theta^{-1}$ *is extended to* $D_d^h = \{\theta \in \mathbf{C} \,\|\, \theta| \leqq dh\}$ *as a* $B(H^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$*-valued analytic function for some* $d > 0$. *Moreover,* V_θ *is compact from* $H^2(\mathbf{R}^n)$ *to* $L^2(\mathbf{R}^n)$ *for each* $\theta \in D_d^h$.

We set

$$\hat{V}(\xi) = \int e^{-ix\xi} V(x)\,dx \tag{3.2}$$

and we show that $\mathscr{F} V_\theta \mathscr{F}^{-1}$ has the integral kernel:

$$G_\theta(\xi, \eta) = (2\pi h)^{-n} J_\theta(\xi)^{1/2} J_\theta(\eta)^{1/2} \hat{V}(\{\xi - \eta - \theta(v(\xi) - v(\eta))\}/h) \tag{3.3}$$

and that G_θ defines a compact operator from $H^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$. The next lemma follows immediatley from (B)-(ii):

LEMMA 3.2. – $\hat{V}$ *is analytic in* $\{\zeta \in \mathbf{C}^n \,\|\, \operatorname{Im}\zeta| < \delta\}$ *and bounded on* $\{\zeta \in \mathbf{C}^n \,\|\, \operatorname{Im}\zeta| \leqq \gamma\}$ *for any* $\gamma < \delta$.

Now, $G_\theta(\xi, \eta)$ is well-defined by (3.3) if $|\theta| \leqq \theta_0$ and

$$|\operatorname{Im}[\xi - \eta - \theta(v(\xi) - v(\eta))]| < \delta h.$$

The latter condition is satisfied if $|\operatorname{Im}\theta| \leqq dh$ with $\tilde{d} = (2 \sup |v(\xi)|)^{-1}\delta$. We can suppose $d < \tilde{d}$ changing d if necessary. Then $G_\theta(\xi, \eta)$ is well-defined for $\theta \in D_d^h$.

PROPOSITION 3.3. – $G_\theta(\xi, \eta)$ *defines a bounded operator from* $H^2(\mathbf{R}^n)$ *to* $L^2(\mathbf{R}^n)$ *for each* $\theta \in D_d^h$.

Proof. – We set $G_\theta^{(i)}(\xi, \eta)$, $i = 1, 2$, as

$$G_\theta^{(1)}(\xi, \eta) = \hat{V}(\{\xi - \eta - \theta(v(\xi) - v(\eta))\}/h) \times \langle \{\xi - \eta - \theta(v(\xi) - v(\eta))\}/h \rangle^{s}. \tag{3.4}$$

$$G_\theta^{(2)}(\xi, \eta) = \langle \{\xi - \eta - \theta(v(\xi) - v(\eta))\}/h \rangle^{-s} \tag{3.5}$$

where s is the constant in (B)-(ii). Since $J_\theta(\xi)^{1/2}$ is bounded and smooth, it is sufficient to show that $G_\theta^{(1)}(\xi, \eta) . G_\theta^{(2)}(\xi, \eta) . \langle \eta/h \rangle^{-2}$ defines a bounded operator on $L^2(\mathbf{R}^n)$. Let p be a constant such that $p = \infty$ if $s = 0$, and $n/s < p < 2n/(n-4)$ if $s > 0$. We claim the following estimates:

LEMMA 3.4. – *For each* h *and* θ,

$$\sup_{\eta \in \mathbf{R}^n} \| G_\theta^{(1)}(.\,, \eta) \|_{L^2(\mathbf{R}^n)} < \infty, \tag{3.6}$$

$$\sup_{\xi \in \mathbf{R}^n} \| G_\theta^{(2)}(\xi, .) \|_{L^p(\mathbf{R}^n)} < \infty. \tag{3.7}$$

Postponing the proof of Lemma 3.4, we proceed the proof of Proposition 3.3. By Theorem 11.7 of Jörgens [11], Lemma 3.4 implies that $G_\theta^{(1)} G_\theta^{(2)}$ defines a bounded operator from $L^q \mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ where q is the conjugate of p *i.e.* $1 = p^{-1} + q^{-1}$. On the other hand, since $q > 2n/(n+4)$, $\langle \eta/h \rangle^{-2}$ is a bounded operator from $L^2(\mathbf{R}^n)$ to $L^a(\mathbf{R}^n)$. Hence $G_\theta^{(1)}(\xi, \eta) G_\theta^{(2)}(\xi, \eta) \langle \eta/h \rangle^{-2}$ defines an L^2-bounded operator. ■

Proof of Lemma 3.4. – (3.7) is an easy consequence of $n < sp$ and

$$C^{-1} \langle (\xi - \eta)/h \rangle \leqq \langle \{ \xi - \eta - \theta (v(\xi) - v(\eta)) \}/h \rangle \leqq C \langle (\xi - \eta)/h \rangle \tag{3.8}$$

By (3.2) and (3.4), we have

$$\begin{aligned} G_\theta^{(1)}(\xi, \eta) &= \int e^{i(\xi - \eta - \theta(v(\xi) - v(\eta)))\, x/h} (\langle D \rangle^s V)(x)\, dx \\ &= \int e^{ix\,\xi/h} a(\xi, x) f_\eta(x)\, dx \end{aligned} \tag{3.9}$$

where $a(\xi, x)$ and $f_\eta(x)$ are defined by

$$a(\xi, x) = \exp\{ -i(\theta/h) v(\xi) . x - (\delta/2) \langle x \rangle \}, \tag{3.10}$$

$$f_\eta(x) = \exp\{ i(\theta/h) v(\eta) . x - (\delta/2) \langle x \rangle \} \times e^{\delta \langle x \rangle} e^{-ix\eta/h} (\langle D \rangle^s V)(x). \tag{3.11}$$

Since $\theta/h \leqq d < \tilde{d}$, the following estimates follows from (3.1) and (B)-(ii):

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\beta a(\xi, x) \right| \leqq C_{\alpha\beta N} \langle x \rangle^{-N}, \tag{3.12}$$

$$\| f_\eta(x) \|_{L^2(\mathbf{R}^n)} \leqq C. \tag{3.13}$$

Note that the estimate (3.13) is independent of η. By (3.12), the expression (3.9) can be considered as a pseudodifferential operator. Estimate (3.6) now follows from the L^2-boundedness theorem for pseudodifferential operators and (3.13). ■

Proof of Theorem 3.1. – Because $\mathscr{F} V \mathscr{F}^{-1}$ is a convolution operator: $(2\pi h)^{-n} \hat{V}(\xi/h)^*$, it is easy to see that $\mathscr{F} V_\theta \mathscr{F}^{-1}$ has the weak integral kernel given by (3.3) on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ if $\theta \in I$. Hence $(\varphi, (\mathscr{F} V_\theta \mathscr{F}^{-1}) \psi)$, $\varphi, \psi \in C_0^\infty$. has an analytic continuation to $\theta \in D_d^h$ with the integral kernel $G_\theta(\xi, \eta)$. Thus the former assertion of Theorem 3.1 follows from Proposition 3.3 and the density argument. Since $V = V_0$ is compact from $H^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$, the former assertion follows from Lemma 5 of [13] Section XIII-5, again. ■

Theorem 1.1 is a direct consequence of Theorems 2.1 and 3.1.

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