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Quantum logics, vector-valued measures and representations

by

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ABSTRACT. — Relations between vector-valued measures and Hilbert space representations of quantum logics are studied. It is shown that a sum logic admits a faithful Hilbert space representation if and only if Segal product defined on bounded observables of the logic is distributive.

RÉSUMÉ. — Les relations entre les mesures aux valeurs dans un espace d'Hilbert et les représentations des logiques quantiques dans un espace d'Hilbert sont étudié. Il est montré, qu'une logique où pour tous les deux observables bornées leur somme existe, prend une représentation fidèle dans l'espace d'Hilbert si et seulement si le produit de Segal sur les observables bornées de la logique est distributif.

INTRODUCTION

Vector-valued measures on quantum logics have been studied by several authors, e. g. [7], [14], [16], [19], [12], [13]. In [16], there has been proved

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that if there exists a vector-valued measure ξ on a quantum logic L with values in a Hilbert space H, then there is a logic morphism Φ from L into the logic L (H) of all orthogonal projections on H and a vector $v_\xi \in H$ such that $\xi(a) = \Phi(a) v_\xi$ for all $a \in L$. In [7], there has been proved that a sum logic with distributive Segal product admits a rich family of vector-valued measures. These two facts are used in the present paper to prove that a sum logic admits a faithful lattice σ-morphism into a Hilbert space logic L (H) if and only if Segal product defined on bounded observables of the logic is distributive. In analogy with representations of C*-algebras we call a morphisms from a quantum logic L into a Hilbert space logic L (H) a representation of L in H. We show that any representation of a sum logic can be extended to a representation of observables by self-adjoint operators which preserves sums and Segal products of bounded observables. We also show that the existence of joint distribution of type 1 for a finite set of bounded observables on a sum logic with distributive Segal product implies the existence of joint distribution of type 2 for these observables in a given state m on L, and the latter joint distribution is identical with the joint distribution of type 1.

1. BASIC FACTS ABOUT LOGICS

A (quantum) logic L is a partially ordered set with 0 and 1 and with orthocomplementation $' : L \to L$ such that

(i) $(a')' = a$,

(ii) $a \leqq b \Rightarrow b' \leqq a'$,

(iii) $a \vee a' = 1$, $a \wedge a' = 0$,

(iv) $a \leqq b' \Rightarrow a \vee b$ exists in L,

(v) $a \leqq b \Rightarrow b = a \vee (a' \wedge b)$ (orthomodularity).

Elements a, $b \in L$ are orthogonal (written $a \perp b$) if $a \leqq b'$. A logic L is a σ-logic if $\bigvee_{i \in N} a_i$ exists in L for every sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of L.

A measure on L is a map $m : L \to \langle 0, \infty)$ such that $m(0) = 0$ and $m(a \vee b) = m(a) + m(b)$ for any a, $b \in L$, $a \perp b$. A measure m on L is σ-additive (or a σ-measure) if $m(\bigvee_{i \in N} a_i) = \sum_{i \in N} m(a_i)$ for any sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of L such that $\bigvee_{i \in N} a_i$ exists in L.

A measure m on L is faithful if $m(a) = 0$ implies $a = 0$, and a measure m is a state if $m(1) = 1$.

Let L_1, L_2 be logics. A map $\Phi : L_1 \to L_2$ is a morphism if

(i) $\Phi(1) = 1$,

(ii) $a, b \in L_1$, $a \perp b \Rightarrow \Phi(a) \perp \Phi(b)$ and $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$. A morphism $\Phi: L_1 \to L_2$ is a σ-morphism if $\Phi(\bigvee_{i \in N} a_i) = \bigvee_{i \in N} \Phi(a_i)$ for any sequence $(a_i)_{i \in N}$ of mutually orthogonal elements of L_1 such that $\bigvee_{i \in N} a_i$ exists. A morphism $\Phi: L_1 \to L_2$ is a lattice morphism if $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$ $(\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b))$ for every $a, b \in L_1$ such that $a \vee b$ $(a \wedge b)$ exists in L_1.

A subset A of a logic L is a compatible subset if there is a Boolean subalgebra B of L such that $A \subset B$. A two-element set $\{a, b\}$ is compatible (or a and b are compatible, written $a \leftrightarrow b$) iff there are $a_1, b_1, c \in L$ pairwise orthogonal and such that $a = a_1 \vee c$, $b = b_1 \vee c$. (We note that $a_1 = a \wedge b'$, $b_1 = a' \wedge b$, $c = a \wedge b$). If L is a lattice then a subset A of L is compatible iff $a \leftrightarrow b$ for every $a, b \in A$.

Let L be a σ-logic. An observable on L is a σ-morphism $x: B(R) \to L$, where B (R) is the σ-algebra of Borel subsets of the real line R. If x is an observable and $f: R \to R$ is a Borel function, then $x.f^{-1}$ is also an observable. If x is an observable and m is a σ-additive state on L, then $m_x: B(R) \to \langle 0, 1 \rangle$, $m_x(E) = m(x(E))$ is a probability measure on B (R), which is called the probability distribution of the observable x in the state m. The expectation of x in m is then given by

$$m(x) = \int_R t m_x(dt),$$

if the integral exists. An observable x is bounded if there is a compact subset $E \subset R$ such that $x(E) = 1$. If x is bounded then $m(x)$ is finite for every state m on L.

A set $\{x_i \mid i \in I\}$ of observables on L is compatible if $\bigcup_{i \in I} x_i(B(R))$ is a compatible set. We note that the range $x(B(R))$ of an observable x is a Boolean sub-σ-algebra of L. If $x_1, x_2, \ldots, x_n$ are compatible observables on L, then there is an observable u and Borel functions $f_1, \ldots, f_n$ such that $x_i = u.f_i^{-1}$, $i \leq n$.

Let L be a σ-logic and let S be a set of σ-additive states on L. We shall say that S is strongly ordering if for any $a, b \in L$, $a \nleq b$ there is $m \in S$ such that $m(a) = 1$, $m(b) \neq 1$.

Let L be a σ-logic and let the set $\mathscr{S}(L)$ of all σ-additive states on L be strongly ordering. We shall say that L is a sum logic if for any two bounded observables x, y on L there is a unique bounded observable z such that $m(x) + m(y) = m(z)$ for every $m \in \mathscr{S}(L)$. The observable z is called the sum of x and y and we write $z = x + y$. If x and y are bounded observables on a sum logic, then

$$x \circ y = \frac{1}{4}[(x+y)^2 - (x-y)^2]$$

defines Segal product of x and y. Segal product is distributive if for any bounded observables x, y, z we have $(x+y)\circ z = x\circ y + x\circ z$.

We note that a sum logic is always a lattice and $(q_a+q_b)\{2\}=a\wedge b$ for any a, $b\in L$, where q_d denotes the (unique) observable such that $q_d\{1\}=d$, $q_d\{0\}=d'$.

For more details about quantum logics *see* ([2], [11], [21]). Sum logics have been introduced and studied in [11], where they are called logics with Uniqueness and Existence properties. Due to Christensen-Yeadon-Paszkiewicz-Matveichuk theorem ([5], [22], [20], [18]), there is a rich class of W*-algebras whose projection logics are sum logics with distributive Segal product.

2. VECTOR-VALUED MEASURES ON QUANTUM LOGICS

Let H be a Hilbert space (real or complex). An H-valued measure on a logic L is a map $\xi: L\to H$ such that $a, b\in L$, $a\perp b \Rightarrow (\xi(a), \xi(b))=0$ and $\xi(a\vee b)=\xi(a)+\xi(b)$. An H-valued measure ξ on L is σ-additive if for any sequence $(a_i)_{i\in N}$ of pairwise orthogonal elements of L such that $\bigvee_{i\in N} a_i$ exists in L we have $\xi(\bigvee_{i\in N} a_i)=\sum_{i\in N}\xi(a_i)$, where the series on the right converges in norm in H. If $\xi: L\to H$ is an H-valued measure, then the map $a\to\|\xi(a)\|^2$ is a measure on L which is σ-additive iff ξ is σ-additive. We shall say that ξ is an H-valued state if $a\to\|\xi(a)\|^2$ is a state. The problem of existence of H-valued (σ-) states on a logic is not trivial in general, since the existence of such state entails the existence of a state on L. It is well-known that there are logics with no states, and hence no H-valued states [10]. The quotient algebra B (R)/I of the Borel algebra B (R) with respect to the σ-ideal I of all subsets of the first category is an example of a logic with no σ-states [3], hence no H-valued σ-states. But there exist finitely additive states on B (R)/I, and to every finitely additive state m, the function $K_m(a, b)=m(a\wedge b)$, a, $b\in B(R)/I$ is positive definite, which implies the existence of an H-valued state ξ such that $m(a)=\|\xi(a)\|^2$, $a\in B(R)/I$ (*see* [17], [7]).

Another example of a logic possessing no σ-additive H-valued state, is the logic E (V) of all splitting subspaces of an incomplete inner-product space V of $\aleph_0$-orthogonal dimension (we recall that a subspace $M\subset V$ is splitting if $M+M'=V$), since V is complete iff E (V) possesses at least one σ-state (*see* [9]). On the other hand, there exist many finitely additive H-valued states. Indeed, let $H=\bar{V}$, where $\bar{V}$ is the completion of V and for any $x\in V$, $\|x\|=1$, define an H-valued mapping $\xi_x: E(V)\to H$ via $\xi_x(M)=x_M$, $M\in E(V)$, where $x=x_M+x_{M^\perp}$, $x_M\in M$, $x_{M^\perp}\in M^\perp$. Then ξ_x is an H-valued state on L.

In [13], an example of a finite logic is constructed, which possesses ordinary states, but does not have any H-valued state in any Hilbert space H.

The following principal criterion has been proved in [7]. Here we present its more compact form.

THEOREM 2.1. — *Let* L *be a* (σ-) *logic and* m *be a* (σ-) *measure on* L. *Then there is a Hilbert space* H *and an* H-*valued* (σ-) *measure* ξ *on* L *such that* $\|\xi(a)\|^2 = m(a)$, $a \in \mathrm{L}$, *if and only if there is a map* $\mathrm{K}_m : \mathrm{L} \times \mathrm{L} \to \mathrm{C}$ (*or* R) *such that*

(i) $\mathrm{K}_m(a, b) = m(a \wedge b)$ *if* $a \leftrightarrow b$,

(ii) $\sum_{i,\, j \leqq n} \alpha_i \bar{\alpha}_j \mathrm{K}_m(a_i, a_j) \geqq 0$ for all $\alpha_i \in \mathrm{C}$ (or $\alpha_i \in \mathrm{R}$), $a_i \in \mathrm{L}$, $i \leqq n$,

$\mathrm{K}_m(a, b) = \mathrm{K}_m(b, a)$ in the real case.

Proof. — If ξ exists, we put $\mathrm{K}_m(a, b) = (\xi(a), \xi(b))$. If K with properties (i) and (ii) is given, the proof follows by a well-known theorem (*see* e.g. [17], p. 489) using the same ideas as in [7].

We note that if an H-valued state on a logic L exists, then there exists an H-valued state in an infinite dimensional, real Hilbert space. Indeed, due to (i), (ii) in the above theorem, there is a probability space $(\Omega, \mathscr{S}, \mathrm{P})$ and a Gaussian process $\{\xi(a) \mid a \in \mathrm{L}\} \subset \mathscr{L}_2(\Omega, \mathscr{S}, \mathrm{P})$ such that $\mathrm{K}(a, b) = (\xi(a), \xi(b))$ is the covariance function, and $\mathrm{H} = \mathscr{L}_2(\Omega, \mathscr{S}, \mathrm{P})$ is that. Moreover, if K is a covariance function, then the real part of K is also a covariance function (*see* [17]), hence we may choose a real H.

Two H-valued measures ξ, η on L are said to be biorthogonal if for every a, $b \in \mathrm{L}$, $a \perp b$ we have $(\xi(a), \eta(b)) = 0$. Following statement is straightforward.

LEMMA 2.2. — *Let* ξ, η *be* H-*valued measures on* L. *Following conditions are equivalent:*

(i) ξ, η *are biorthogonal,*

(ii) *for every* α, $\beta \in \mathrm{C}$ (*or* α, $\beta \in \mathrm{R}$ *if* H *is real*) *the* map $a \to \alpha\xi(a) + \beta\eta(a)$ *from* L *into* H *is an* H-*valued measure.*

A family $\mathscr{N}$ of H-valued measures on L is said to be biorthogonal if every two measures ξ, $\eta \in \mathscr{N}$ are biorthogonal. A biorthogonal family $\mathscr{N}$ is a maximal biorthogonal family if every H-valued measure on L, which is biorthogonal to every member of $\mathscr{N}$, necessarily belongs to $\mathscr{N}$. By Lemma 2.4, every maximal biorthogonal family is a linear space over C (or over R). Clearly, every biorthogonal family is contained in a maximal one.

Following theorem shows that the family of all H-valued measures on L (and also every maximal biorthogonal family) is sequentially closed.

THEOREM 2.3 (Nikodym theorem). — *Let* L *be a* σ*-logic and let* ξ_n, $n \in \mathrm{N}$ *be* H*-valued* σ*-measures on* L. *If for any* $a \in \mathrm{L}$ *there is* $\xi(a) = \lim \xi_n(a)$ (i. e. $\|\xi_n(a) - \xi(a)\| \to 0$), *then* ξ *is an* H*-valued measure on* L.

Proof. — Let $\xi(a) = \lim \xi_n(a)$. If $a \perp b$, then

$$(\xi(a), \xi(b)) = \lim (\xi_n(a), \xi_n(b)) = 0$$

and

$$\xi(a \vee b) = \lim \xi_n(a \vee b) = \lim \xi_n(a) + \lim \xi_n(b) = \xi(a) + \xi(b).$$

We claim to show $\xi(a) = \sum_{i \in \mathrm{N}} \xi(a_i)$ if $a_i \perp a_j$, $i \neq j$, and $a = \vee_{i \in \mathrm{N}} a_i$. The functions $m(b) = \|\xi(b)\|^2$, $m_n(b) = \|\xi_n(b)\|^2$, $b \in \mathrm{L}$, are additive and σ-additive measures on L. Moreover, for any $b \in \mathrm{L}$,

$$|m(b) - m_n(b)| = |\|\xi_n(b)\|^2 - \|\xi(b)\|^2| = |\|\xi_n(b)\| - \|\xi(b)\||.$$
$$|\|\xi_n(b)\| + \|\xi(b)\|| \leqq \|\xi_n(b) - \xi(b)\| . \mathrm{K} \to 0,$$

where

$$\mathrm{K} = \sup \{ m_n(1), m(1) \,|\, n \in \mathrm{N} \} < \infty.$$

Hence, by [6], $m(a) = \sum_{i \in \mathrm{N}} m(a_i)$. Therefore,

$$\|\xi(a) - \sum_{i \leqq n} \xi(a_i)\|^2 = \|\xi(a \wedge (\vee_{i \leqq n} a_i)')\|^2 = m(a \wedge (\vee_{i \leqq n} a_i)')$$
$$= m(a) - \sum_{i \leqq n} m(a_i) \to 0,$$

hence $\xi(a) = \sum_{i \in \mathrm{N}} \xi(a_i)$.

We note that if ξ_n, $n \in \mathrm{N}$ belong to a maximal orthogonal family $\mathcal{N}$, then ξ also belongs to $\mathcal{N}$. Indeed, if $a \perp b$, then for any $\eta \in \mathcal{N}$, $(\xi(a), \eta(b)) = \lim (\xi_n(a), \eta(b)) = 0$.

Following theorem has been proved in [16] for lattice logics and complex Hilbert spaces, but the method of the proof can be applied to logics which are not necessarily lattices and real Hilbert spaces as well.

THEOREM 2.4. — *Let* L *be a logic and let* H *be a Hilbert space (real or complex). Let* $\mathcal{N}$ *be a maximal biorthogonal family of* H*-valed measures on* L. *For every* $a \in \mathrm{L}$ *put* $\mathcal{N}(a) = \{ \xi(a) \,|\, \xi \in \mathcal{N} \}$. *Then following statements hold.*

(i) *For every* $a \in \mathrm{L}$, $\mathcal{N}(a)$ *is a closed linear subspace of* H.

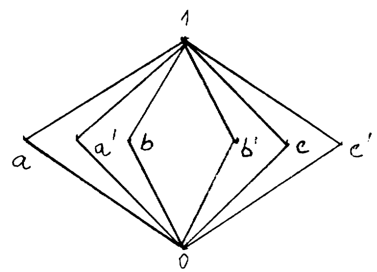
(ii) *For every* $a, b \in \mathrm{L}$, $a \perp b$, *we have* $\mathcal{N}(a) \perp \mathcal{N}(b)$ *and* $\mathcal{N}(a \vee b) = \mathcal{N}(a) \vee \mathcal{N}(b)$, i. e. $\Phi(a \vee b) = \Phi(a) + \Phi(b)$, *where* $\Phi(a)$ *denotes the projection on* $\mathcal{N}(a)$. *If, in addition, all the measures in* $\mathcal{N}$ *are* σ*-additive, then for every sequence* $(a_i)_{i \in \mathrm{N}}$ *of mutually orthogonal elements of* L *such that* $\vee_{i \in \mathrm{N}} a_i$ *exists in* L *we have* $\Phi(\vee_{i \in \mathrm{N}} a_i) = \sum_{i \in \mathrm{N}} \Phi(a_i)$, *where the sum converges in the strong operator topology on* H.

(iv) *For every* $\xi \in \mathcal{N}$ *there is a vector* $v_\xi \in \mathrm{H}$ *such that* $\xi(a) = \Phi(a)\, v_\xi$, $a \in \mathrm{L}$.

COROLLARY 2.5. – *Let* ξ *be an* H*-valued measure on* L. *Then there is a closed subspace* H_0 *of* H, *a morphism* Φ *from* L *into* $\mathrm{L}(\mathrm{H}_0)$ *and a vector* $v_\xi \in \mathrm{H}_0$ *such that* $\xi(a) = \Phi(a)\, v_\xi$ *for every* $a \in \mathrm{L}$. *If, in addition,* L *is a* σ*-logic and* ξ *is* σ*-additive, then* Φ *is a* σ*-morphism.*

Proof. – The measure ξ is contained in at least one maximal biorthogonal family of H-valued measures on L, so that we can apply Theorem 2.4. We put $\mathrm{H}_0 = \mathcal{N}(1)$. Then H_0 is a closed subspace of H. From $\Phi(a)\,\mathcal{N}(1) = \Phi(a)\,(\mathcal{N}(a) \vee \mathcal{N}(a')) = \Phi(a)\,\mathcal{N}(a)$ we get $\mathcal{N}(a) \subset \mathcal{N}(1) = \mathrm{H}_0$ for every $a \in \mathrm{L}$, and $\Phi(a) + \Phi(a') = \Phi(1)$, $\Phi(a) \perp \Phi(a')$ imply that $\Phi(a') = \Phi(a)' \wedge \Phi(1)$. Hence Φ is a morphism from L into $\mathrm{L}(\mathrm{H}_0)$. For every $a \in \mathrm{L}$, $\Phi(a)\,\xi(1) = \Phi(a)\,(\xi(a) + \xi(a')) = \xi(a) = \Phi(a)\, v_\xi$, hence we may put $v_\xi = \xi(1) \in \mathrm{H}_0$.

Following example shows that the morphism Φ need not be a lattice morphism. Let $\mathrm{L} = \mathrm{MO}(3)$ be "Chinese lanterne" (*Fig.*). Let $\mathrm{H} = \mathrm{R}^3$ and



let $\{x, y, z\}$ be an orthogonal base in H. For every $t = (\alpha, \beta, \gamma) \in \mathrm{R}^3$ define $\xi_t(a) = \alpha x$, $\xi_t(b) = \beta y$, $\xi_t(c) = \gamma z$, $\xi_t(a') = \beta y + \gamma z$, $\xi_t(b') = \alpha x + \gamma z$, $\xi_t(c') = \alpha x + \beta y$, $\xi_t(0) = 0$, $\xi_t(1) = \alpha x + \beta y + \gamma z$. It is easy to check that $\mathcal{N} = \{\xi_t \mid t \in \mathrm{R}^3\}$ is a biothogonal family of H-valued measures on L. Now let ζ be an H-valued measure on L which is biorthogonal to all members of $\mathcal{N}$. Then $\zeta(a) \perp \xi_t(a')$ for all $t \in \mathrm{R}^3$ implies that $\zeta(a) = \alpha_1 x$ for some $\alpha_1 \in \mathrm{R}$. Similarly, $\zeta(b) = \beta_1 y$, $\zeta(c) = \gamma_1 z$ for some $\beta_1, \gamma_1 \in \mathrm{R}$. Further, $\zeta(a') \perp \xi_t(a)$ for every t implies that $\zeta(a') = \beta_2 y + \gamma_2 z$, and analogically $\zeta(b') = \alpha_2 x + \gamma_3 z$, $\zeta(c') = \alpha_3 x + \beta_3 y$ for some $\alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3 \in \mathrm{R}$. From

$$\zeta(1) = \zeta(a) + \zeta(a') = \zeta(b) + \zeta(b') = \zeta(c) + \zeta(c')$$

we obtain

$$\alpha_1 x + \beta_2 y + \gamma_2 z = \alpha_2 x + \beta_1 y + \gamma_3 z = \alpha_3 x + \beta_3 y + \gamma_1 z,$$

and independence of x, y, z entails $\alpha_1 = \alpha_2 = \alpha_3$, $\beta_1 = \beta_2 = \beta_3$, $\gamma_1 = \gamma_2 = \gamma_3$. Hence $\zeta \in \mathcal{N}$, *i.e.* $\mathcal{N}$ is a maximal biorthogonal family. Now $\mathcal{N}(a) = [x]$,

$\mathcal{N}(b)=[y]$, $\mathcal{N}(c)=[z]$, $\mathcal{N}(a')=[y, z]$, $\mathcal{N}(b')=[x, z]$, $\mathcal{N}(c')=[x, y]$, $\mathcal{N}(1)=\mathrm{H}$, where $[u, v, \ldots]$ denotes the linear subspace of H generated by the vectors $u, v, \ldots$ in H. But $\mathcal{N}(a \vee b)=\mathcal{N}(1) \neq \mathcal{N}(a) \vee \mathcal{N}(b)=[x, y]$.

3. REPRESENTATIONS OF QUANTUM LOGICS

In analogy with representations of C*-algebras, we shall call every (σ-) morphism from a (σ-) logic into a Hilbert space logic L (H) a (σ-) representation of L in H. We shall say that a representation Φ of L in H is faithful if $\Phi(a)=0$ implies $a=0$. If L is a lattice logic and Φ is a faithful lattice representation, then Φ is one-to-one.

Let Φ_i be representations of L in H_i, $i \in \mathrm{I}$. Put $\mathrm{H}=\bigoplus_{i \in \mathrm{I}} \mathrm{H}_i$, $\Phi=\bigoplus_{i \in \mathrm{I}} \Phi_i$, $\Phi(a)=(\Phi_i(a))_{i \in \mathrm{I}}$. Then clearly, Φ is a representation of L in H, which is the direct sum of the representations Φ_i, $i \in \mathrm{I}$.

Let m be a measure on L. If there is a representation Φ of L in a Hilbert space H such that $m(a)=\|\Phi(a) v\|^2$, $a \in \mathrm{L}$, for a vector $v \in \mathrm{H}$, we shall call Φ the representation associated with the measure m. Clearly, a representation Φ associated with a measure m is a σ-representation iff m is σ-additive, and if m is faithful, then also Φ is faithful.

LEMMA 3.1. – *Let Φ be a representation of a logic* L *in a Hilbert space* H. *Then*

(i) $a, b \in \mathrm{L}$, $a \leftrightarrow b \Rightarrow \Phi(a \wedge b)=\Phi(a)\Phi(b)=\Phi(b)\Phi(a)=\Phi(a) \wedge \Phi(b)$. *In particular, if* B *is a Boolean subalgebra of* L, *then* $\Phi(\mathrm{B})$ *is a Boolean subalgebra of* L (H),

(ii) *for every state* m *on* L (H) *there is a state* m^{L} *on* L *such that* $m^{\mathrm{L}}(a)=m(\Phi(a))$, $a \in \mathrm{L}$.

If Φ *is a* σ*-representation of a* σ*-logic* L, *then*

(iii) *to every observable* x *on* L *there is a self-adjoint operator* $\Phi(x)$ *on* H *with the spectral measure* $\mathrm{E} \mapsto \Phi(x)(\mathrm{E})=\Phi(x(\mathrm{E}))$, $\mathrm{E} \in \mathrm{B}(\mathrm{R})$. *If* x *is an observable and* $f: \mathrm{R} \to \mathrm{R}$ *is a Borel function, then* $\Phi(x . f^{-1})=\Phi(x) . f^{-1}=f(\Phi(x))$. *In particular, if* x *and* y *are compatible observable on* L, *then* $\Phi(x)$ *and* $\Phi(y)$ *commute and* $\Phi(x+y)=\Phi(x)+\Phi(y)$, $\Phi(x . y)=\Phi(x) . \Phi(y)$, *where the operations* + *and* . *are defined by the functional calculus for compatible observables on* L, *resp. on* L (H).

Proof of the lemma is standard and we omit it.

THEOREM 3.2. – *Let* L *be a* σ*-logic,* $\mathcal{S}(\mathrm{L})$ *be the set of all* σ*-additive states on* L *and let* (L, $\mathcal{S}$(L)) *be a sum logic. Then there is a faithful lattice* σ*-representation of* L *in a* (*real*) *Hilbert space* H *iff Segal product on bounded observables on* L *is distributive.*

Proof. – Let (L, $\mathscr{S}$ (L)) be a sum logic with distributive Segal product. Let $m \in \mathscr{S}(\mathrm{L})$ and put $\mathrm{K}_m(a, b) = m(q_a \circ q_b)$, a, $b \in \mathrm{L}$. Due to distributivity of Segal product, $\mathrm{K}_m(a, b)$ satisfies the conditions of Theorem 2.1. Indeed, let $(\alpha_i)_{i \leqq n} \subset \mathrm{R}$. We have

$$\sum_{i, j \leqq n} \alpha_i \alpha_j \mathrm{K}_m(a_i, a_j) = \sum_{i, j \leqq n} \alpha_i \alpha_j m(q_{a_i} \circ q_{a_i}) = m((\sum_{i \leqq n} \alpha_i q_{a_i})^2) \geqq 0 \quad (\textit{see} [7]).$$

Therefore, there is a (real) Hilbert space H_m and a σ-additive H_m-valued state $\xi_m : \mathrm{L} \to \mathrm{H}_m$ such that $\mathrm{K}_m(a, b) = (\xi_m(a), \xi_m(b))$ for any a, $b \in \mathrm{L}$. By Theorem 2.4, there is a σ-morphism $\Phi_m : \mathrm{L} \to \mathrm{L}(\mathrm{H}_m^0)$, $\mathrm{H}_m^0 \in \mathrm{L}(\mathrm{H}_m)$, and a vector $v_m \in \mathrm{H}_m^0$ such that $m(a) = \| \Phi(a) v_m \|^2$ for every $a \in \mathrm{L}$. Without any loss of generality we may assume that $\mathrm{H}_m = \mathrm{H}_m^0$. Now construct the direct sum $\Phi = \oplus \{ \Phi_m \mid m \in \mathscr{S}(\mathrm{L}) \}$. Since $\mathscr{S}(\mathrm{L})$ is strong, Φ is a faithful representation of L in $\mathrm{H} = \underset{m}{\oplus} \mathrm{H}_m$. By Lemma 3.1, to every observable x on L, there corresponds a s. a. operator $\Phi(x)$ on H. Let x, y be bounded observables on L. Then $\Phi(x)$, $\Phi(y)$ are bounded. Let $v \in \mathrm{H}$, $\| v \| = 1$, and let s_v be the corresponding state on L (H). From

$$s_v(\Phi(x+y)(\mathrm{E})) = s_v^{\mathrm{L}}((x+y)(\mathrm{E})), \qquad \mathrm{E} \in \mathrm{B}(\mathrm{R}),$$

we obtain

$$s_v(\Phi(x+y)) = s_v^{\mathrm{L}}(x+y) = s_v^{\mathrm{L}}(x) + s_v^{\mathrm{L}}(y) = s_v(\Phi(x)) + s_v(\Phi(y)),$$

as $s_v^{\mathrm{L}} \in \mathscr{S}(\mathrm{L})$. Therefore $(\Phi(x+y) v, v) = ((\Phi(x) + \Phi(y)) v, v)$ for every $v \in \mathrm{H}$, and hence $\Phi(x+y) = \Phi(x) + \Phi(y)$. As $\Phi(x . f^{-1}) = \Phi(x) . f^{-1}$, we have $\Phi(x^2) = \Phi(x)^2$. This entails that Φ preserves sums and Segal products of bounded observables. Let a, $b \in \mathrm{L}$, then $(q_a + q_b)\{2\} = a \wedge b$ implies $\Phi((q_a + q_b)\{2\}) = \Phi(a \wedge b)$. On the other hand, $q_a = q_a^2$ and $\Phi(q_a^2) = \Phi(q_a)^2$ imply that $\Phi(q_a)$ is a projection, and we have $\Phi(q_a)\{1\} = \Phi(q_a\{1\}) = \Phi(a)$. Therefore

$$\Phi((q_a + q_b)\{2\}) = \Phi(q_a + q_b)\{2\} = (\Phi(q_a) + \Phi(q_b))\{2\} = \Phi(a) \wedge \Phi(b).$$

Hence $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$, *i. e.* Φ is a lattice morphism.

Now suppose that (L, $\mathscr{S}$ (L)) is a sum logic which admits a faithful σ-representation in a Hilbert space H. Similarly as in the first part of this proof, we show that $\Phi(x+y) = \Phi(x) + \Phi(y)$ and $\Phi(x^2) = \Phi(x)^2$ for any bounded observables x, y on L, so that Φ preserves sums and Segal products, and Φ preserves lattice operations. Let x, y, z be bounded observables on L. We have

$$\Phi[(x+y) \circ z - (x \circ z + y \circ z)] = (\Phi(x) + \Phi(y)) \circ \Phi(z) - (\Phi(x) \circ \Phi(z) + \Phi(y) \circ \Phi(z)) = 0,$$

and as Φ is faithful, this entails that $(x+y) \circ z = x \circ z + y \circ z$.

COROLLARY 3.3. – *Let* (L, $\mathscr{S}$ (L)) *be a sum logic. Then every σ-representation of* L *in a Hilbert space* H *is a lattice representation. In addition, the* *extension of the representation to bounded observables on* L *preserves sums and Segal products.*

Proof. – It follows immediately from the proof of Theorem 3.2.

THEOREM 3.4. – *Let* $(\mathrm{L}, \mathscr{S}(\mathrm{L}))$ *be a sum logic and let* H *be a Hilbert space. Let* $m \in \mathscr{S}(\mathrm{L})$ *and let there be an* H*-valued state* ξ *on* L *such that* $m(a) = \| \xi(a) \|^2$, $a \in \mathrm{L}$. *Then* $\operatorname{Re} \mathrm{K}_m(a, b) = m(q_a \circ q_b)$, $a, b \in \mathrm{L}$.

Proof. – By Corollary 2.5 and Corollary 3.3, to any bounded observables x, y on L there correspond bounded self-adjoint operators $\Phi(x)$, $\Phi(y)$ on H and $\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$, $\Phi(x+y) = \Phi(x) + \Phi(y)$. We have

$$\begin{aligned} \mathrm{K}_m(a, b) = (\xi(a), \xi(b)) = (\Phi(a) v_\xi, \Phi(b) v_\xi) = \overline{(\Phi(b) \Phi(a) v_\xi, v_\xi)} \\ = (\Phi(q_b) \Phi(q_a) v_\xi, v_\xi) = \overline{(\Phi(q_a) \Phi(q_b) v_\xi, v_\xi)} = \overline{\mathrm{K}_m(b, a)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} m(q_a \circ q_b) &= (\Phi(q_a) \circ \Phi(q_b) v_\xi, v_\xi) \\ &= \left(\frac{1}{2} (\Phi(q_a) \Phi(q_b) + \Phi(q_b) \Phi(q_a)) v_\xi, v_\xi \right) = \operatorname{Re} \mathrm{K}_m(a, b). \end{aligned}$$

We note that the result obtained by Hamhalter in [12] that every σ-additive state on a projection logic $\mathrm{L}(\mathfrak{U})$ of a W*-algebra $\mathfrak{U}$ without any type I_2 direct summand on a Hilbert space H with $\dim \mathrm{H} = \infty$ can be represented by an H-valued state (with values in the same Hilbert space H), follows directly from our criterion in Theorem 2.1 for $\mathrm{K}_m(\mathrm{P}, \mathrm{Q}) = m(\mathrm{PQ}) = \operatorname{Tr}(\mathrm{TPQ})$, $m \in \mathscr{S}(\mathrm{L})$, P, $\mathrm{Q} \in \mathrm{L}(\mathfrak{U})$, where $\mathrm{T} = \sum_i c_i(\,.\,, e_i)\, e_i$. It suffices to put $\xi(\mathrm{P}) = \bigoplus_i c_i^{1/2} \mathrm{P} e_i$. Nevertheless, the method of proof he used is very interesting.

THEOREM 3.5. – *Let* $(\mathrm{L}, \mathscr{S}(\mathrm{L}))$ *be a sum logic and let* $m \in \mathscr{S}(\mathrm{L})$.

(i) *There is an* H*-valued state* ξ *on* L *such that* $m(a) = \| \xi(a) \|^2$, $a \in \mathrm{L}$, *iff* $m([(x+y) \circ z - (x \circ z + y \circ z)]^2) = 0$ *for any three bounded observables* x, y, z *on* L.

(ii) *If* $s \in \mathscr{S}(\mathrm{L})$, $s \ll m$ [*in the sense that* $m(a) = 0 \Rightarrow s(a) = 0$, $a \in \mathrm{L}$] *and if there is an* H*-valued state* ξ *such that* $m(a) = \| \xi(a) \|^2$, $a \in \mathrm{L}$, *then there is an* H*-valued state* η *on* L *such that* $s(a) = \| \eta(a) \|^2$, $a \in \mathrm{L}$.

Proof. – (i) Let $m(a) = \| \xi(a) \|^2$, $a \in \mathrm{L}$, and let $\xi(a) = \Phi(a) v_\xi$. Then

$$\begin{aligned} m([(x+y) \circ z - (x \circ z + y \circ z)]^2) \\ = \| [(\Phi(x) + \Phi(y)) \circ \Phi(z) - (\Phi(x) \circ \Phi(z) + \Phi(y) \circ \Phi(z))] v_\xi \|^2 = 0. \end{aligned}$$

On the other hand, if the above condition is satisfied, Schwarz inequality implies that $m[(x+y) \circ z - (x \circ z + y \circ z)] = 0$ for any x, y, z, and it is easy to check that $\mathrm{K}_m(a, b) = m(q_a \circ q_b)$ satisfies conditions (i)-(ii) of Theorem 2.1, and hence m is representable by an H-valued state ξ.

(ii) follows directly from (i).

Our next remark concerns some relations between H-valued states and representations of C*-algebras.

Let $(L, \mathscr{S}(L))$ be a sum logic such that L is a projection logic of a C*-algebra $\mathfrak{U}$ (e.g. L is a projection logic of a W*-algebra of operators acting on a complex separable Hilbert space H with no I_2-factor as direct summand, *see* [5], [18], [20], [22]). Then, by GNS-construction, to every state $s \in \mathscr{S}(L)$, there is a cyclic representation π_s of $\mathfrak{U}$ on a Hilbert space H_s, and a unit cyclic vector v_s for π_s such that $s(A) = (\pi(A)v_s, v_s)$ $(A \in \mathfrak{U})$ (*see* [15], p. 278, [4], p. 64). If $a \in L$, $\pi(a)$ is a projection on H_s and hence $s(a) = \|\pi(a)v_s\|^2$. If we put $\eta(a) = \pi(a)v_s$, then it is easy to check that $a \to \pi(a)v_s$, $a \in L$, is an H_s-valued σ-additive state on L.

We note that, using the results in [1], similar results may be obtained for suitable types of JB-algebras.

On the other hand, let $\xi : L \to H$ be a σ-additive H-valued state on L such that $s(a) = \|\xi(a)\|^2$ for some state $s \in \mathscr{S}(L)$. Then there is a lattice σ-morphism $\Phi_\xi : L \to L(H)$ such that $\xi(a) = \Phi_\xi(a)v_\xi$, $a \in L$, for some unit vector $v_\xi \in H$ and Φ_ξ can be extended to a linear and Segal product preserving map from bounded observables on L (*i. e.* Hermitian elements of $\mathfrak{U}$) into the algebra $\mathfrak{B}(H)$ of bounded operators on H. This map Φ_ξ can be in a natural way extended to a linear Jordan morphism $\bar{\Phi}_\xi$ from $\mathfrak{U}$ into $\mathfrak{B}(H)$ (*see* also [16]). By [4], 3.2.2, p. 17), every Jordan morphism is a combination of a morphism and antimorphism. In case that $\bar{\Phi}_\xi$ is a morphism, we put $H_\xi = \{\bar{\Phi}(A)v_\xi \mid A \in \mathfrak{U}\}^-$ (where M^- means the closure of M, $M \subset H$, in H). Then since H_ξ is invariant, the map $A \to P_\xi \bar{\Phi}_\xi(A)$, $A \in \mathfrak{U}$, where P_ξ is the projection in H onto H_ξ, is a cyclic representation of $\mathfrak{U}$ in H_ξ such that

$$s(A) = (P_\xi \bar{\Phi}_\xi(A)v_\xi, v_\xi) = (\bar{\Phi}_\xi(A)v_\xi, v_\xi),$$
$$\xi(a) = P_\xi \Phi_\xi(a)v_\xi = \Phi_\xi(a)v_\xi.$$

By [15], Prop. 4.5.3, this cyclic representation $(H_\xi, P_\xi \bar{\Phi}_\xi, v_\xi)$ is isomorphic to the cyclic representation (H_s, π_s, v_s) produced from s by the GNS construction, in the sense that there is an isomorphism from H_s onto H_ξ such that $v_\xi = U v_s$, $P_\xi \bar{\Phi}_\xi(A) = U \pi_s(A) U^*$ $(A \in \mathfrak{U})$.

Our next results concerns joint distributions of observables. Recall that observables $x_1, x_2, \ldots, x_n$ on a logic L have a joint distribution of type 1 in a state m if there is a probability measure μ_1 on $B(R^n)$ such that

$$\mu_1(E \times E \times \ldots \times E) = m(x_1(E_1) \wedge \ldots \wedge x_n(E_n))$$

for any $E_1, \ldots, E_n \in B(R^n)$, and bounded observables $x_1, \ldots, x_n$ on a sum logic have a joint distribution of type 2 in a state m if there is a measure μ_2 on $B(R^n)$ such that

$$\mu_2\{(t_1, \ldots, t_n) \in R^n \mid \sum_{i \leq n} \alpha_i t_i \in E\} = m((\sum_{i \leq n} \alpha_i x_i)(E))$$

for any $\alpha_1, \ldots, \alpha_n \in R$ and $E \in B(R)$ (*see* [8]).

THEOREM 3.7. — *Let* $(L, \mathscr{S}(L))$ *be a sum logic. Let* $m \in \mathscr{S}(L)$ *and let there be an* H*-valued state* $\xi : L \to H$ *such that* $m(a) = \|\xi(a)\|^2$, $a \in L$. *If for a given set* $x_1, \ldots, x_n$ *of bounded observables joint distribution of type* 1 *in the state m exists, then there exists also joint distribution of type* 2, *and the two joint distributions are identical.*

Proof. — Let us define so-called commutator c of $x_1, \ldots, x_n$ by

$$c = \bigwedge_{E_1 \ldots E_n} \bigvee_{i_1 \ldots i_n = 0}^{1} \bigwedge_{j=0}^{n} x_j(E_j)^{i_j}$$

where $a^1 = a$, $a^0 = a'$, $a \in L$. It is known that c exists and that $x_1, \ldots, x_n$ have a type 1 joint distribution in m iff $m(c) = 1$ (*see* [8]). But $m(c) = 1$ implies $\|\xi(c)\|^2 = \|\Phi(c) v_\xi\|^2 = 1$, hence $\Phi(c) v_\xi = v_\xi$. Since $\Phi : L \to L(H)$ is a lattice σ-morphism, we obtain that $\Phi(c)$ is the commutator of $\Phi(x_1), \ldots, \Phi(x_n)$, and hence the latter observables have a type 1 joint distribution in the vector state m_{v_ξ} on L (H) corresponding to v_ξ. Now by [8], type 2 joint distribution of $\Phi(x_1), \ldots, \Phi(x_n)$ in m_{v_ξ} exists and is equal to the type 1 joint distribution. Let $\alpha_1, \ldots, \alpha_n \in R$, $E \in B(R)$. Then

$$\begin{aligned} m(x_1(E_1) \wedge \ldots \wedge x_n(E_n)) &= (\Phi(x_1(E_1) \wedge \ldots \wedge x_n(E_n)) v_\xi, v_\xi) \\ &= (\Phi(x_1)(E_1) \wedge \ldots \wedge \Phi(x_n)(E_n) v_\xi, v_\xi) \\ &= m_{v_\xi}(\Phi(x_1)(E_1) \wedge \ldots \wedge \Phi(x_n)(E_n)), \\ m((\sum_{i \leq n} \alpha_i x_i)(E)) &= (\Phi(\sum_{i \leq n} \alpha_i x_i)(E) v_\xi, v_\xi) \\ &= ((\sum_{i \leq n} \alpha_i \Phi(x_i))(E) v_\xi, v_\xi) = m_{v_\xi}((\sum_{i \leq n} \alpha_i \Phi(x_i)(E)). \end{aligned}$$

The latter equalities show that there is a type 2 joint distribution of $x_1, \ldots, x_n$ in m, which equals to the type 1 joint distribution.

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