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JAN HERCZYŃSKI

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Semiclassical limit for perturbations of non-resonant rotators

by

Jan HERCZYŃSKI

Instytut Matematyki Stosowanej i Mechaniki, Uniwersytet Warszawski,
PKiN 9p., Warszawa 00 901, Polska

ABSTRACT. — We show that the semiclassical limit of the Rayleigh-Schrödinger series for the perturbed multi-dimensional non-resonant rotators coincides with the Birkhoff series for the corresponding classical perturbed Hamiltonian and we calculate quantum corrections to all orders in $\hbar$.

RÉSUMÉ. — On établit que les séries Rayleigh-Schrödinger pour les perturbations d'un rotateur multidimensionnel non resonant tendent, en limite classique, vers les séries de Birkhoff pour l'Hamiltonien classique correspondant, et on calcule toutes les corrections quantiques en $\hbar$.

INTRODUCTION

In this paper we consider the semiclassical limit of the Rayleigh-Schrödinger perturbation series. This study, initiated by Turchetti [13], Graffi and Paul [6], Graffi, Paul, and Silverstone ([7], [8]) (*see* also [1]), complements earlier research which focussed mainly on the semiclassical limit of dynamics, of the spectral density function and of individual eigenvalues (the Bohr formula). Since many problems of non-relativistic quantum mechanics are studied via the perturbation theory it is reasonable to ask what is the semiclassical limit of the successive terms of the Rayleigh-Schrödinger series. The obvious candidate in the non-resonant case is the classical perturbation series, known as the Birkhoff series.

The way to tackle this question worked out in the context of perturbations of non-resonant harmonic oscillators by Graffi and Paul [6] consists in rewriting, in the spirit of WKB approximation ([15], [11]), the Schrödinger equation in the form of the Hamilton-Jacobi equation with quantum corrections and then systematically computing successive terms of the perturbation theory. The non-resonant nature of the problem enters crucially in two places: the eigenvalues in the quantum case are simple and the Birkhoff series can be defined to all orders. The problem of semiclassical limit of the resonant perturbed oscillators requires essentially new ideas, since neither of these two simplifications takes place any more.

The aim of the present note is to extend these methods to the study of perturbations of non-resonant rotators

$$H_0(\hbar) = -\frac{\hbar^2}{2}\sum_{j=0}^{d}\omega_j\frac{\partial^2}{\partial\varphi_j^2}$$

in $L^2(T^d)$, where ω satisfies the diophantine condition to be specified below (we assume that $\omega_j>0$), by a multiplication operator $V(\varphi)$, $H_\varepsilon(\hbar)=H_0(\hbar)+\varepsilon V$. In contrast to [6] the eigenvalues of $H_0(\hbar)$ are not simple. $H_0(\hbar)$ is a model Hamiltonian, describing a system with only rotational degrees of freedom, or alternatively a system of free (uncoupled) one-dimensional rotators, *cf.* [3], [11], [2], [14]. Accordingly, $H_\varepsilon(\hbar)$ may describe a perturbed multidimensional rotator or a system of weakly coupled one-dimensional rotators. Integrability of the corresponding classical system $K_\varepsilon(A, \varphi)=\frac{1}{2}\sum_{j=0}^{d}\omega_j A_j^2+\varepsilon V(\varphi)$ is studied in Kozlov, Treščov [10]. The diophantine condition on ω implies that the multiplicity of the eigenvalue $E_0(\hbar n, \hbar)=\frac{1}{2}\sum_{j=0}^{d}\omega_j\hbar^2 n_j^2$ remains constant when $\hbar\to 0$ with $\hbar n=A$ constant. In this sense we may say that we have here only the trivial multiplicity, easier to deal with, whereas in the resonant case, when the components of ω admit a vanishing non-zero integer combination, the multiplicity of a given eigenvalue grows polynomially as $\hbar\to 0$ (the same difficulty confronts the resonant harmonic oscillator). Fix now $A=\hbar n$ with all components A_j non-zero and consider the 2^d-degenerate eigenvalue $E_0(A, \hbar)$. We show, in the main Theorem 1 and Theorem 1′, that 2^d Rayleigh-Schrödinger series of this eigenvalue converge in the semiclassical limit to 2^d Birkhoff series calculated at 2^d actions $\bar{A}$ for which $K_0(\bar{A})=K_0(A)$, and also that for symmetric potentials these Rayleigh-Schrödinger series are exponentially close as $h \to 0$. In Section 1 we establish the notation and state the main results. The proofs of these results are given in Section 2.

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SECTION 1

Consider the system of d-dimensional rotators defined by the Hamiltonian function K_0 in the angle-action variables $(A, \varphi) \in \mathbf{R}^d \times T^d$, $K_0(A)=\frac{1}{2}\sum_{j=0}^{d} \omega_j A_j^2$. The tori $A = Const.$ are invariant under the Hamiltonian flow associated with K_0. The system is perturbed by the potential V, assumed to be an analytic in the vicinity of T^d, $K_\varepsilon(A, \varphi)=K_0(A)+\varepsilon V(\varphi)$. We assume that V is real on T^d. We assume moreover that $\omega=(\omega_1, \ldots, \omega_d)$ satisfies the following diophantine condition:

$$|\omega m|^{-1} \leqq C |m|^\gamma \tag{1}$$

for any $m \in \mathbf{Z}^d$, $m \neq 0$, with some C, $\gamma > 0$ (here $\omega m = \sum_{j=0}^{d} \omega_j m_j$, we shall also use the notation $\omega AB = \sum_{j=0}^{d} \omega_j A_j B_j$ for any two vectors A, $B \in \mathbf{R}^d$). Thus the motion on the invariant tori is quasi-periodic and not periodic. It will be of importance later that the actions of the form $A=\hbar n$, $n \in \mathbf{Z}^d$, $\hbar > 0$, $n_j \neq 0$, $j=1, \ldots, d$, are non-resonant. As is well known (*cf.* Gallavotti [5]), it is in general impossible to find a completely canonical transformation putting the perturbed Hamiltonian into its action-angle variables, because the KAM theorem guaranties the survival of invariant tori only for a (high) fraction of the phase-space. The insoluble Hamilton-Jacobi equation

$$K_0(\nabla_\varphi W(\varphi, J, \varepsilon)) + \varepsilon V(\varphi) = N(J, \varepsilon) \tag{2}$$

for the generating function $W(\varphi, J, \varepsilon)$ may be solved only in the sense of perturbations, that is by expanding W into a power series in ε,

$$W(\varphi, J, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k W^{(k)}(\varphi, J)$$

which is in general asymptotic and not convergent. It is assumed that for $\varepsilon = 0$ the generating function describes the identity transformation, which imposes $W^{(0)}(\varphi, J) = \varphi J$. The corresponding expansion of $N(J, \varepsilon)$ in the series $N(J, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k N_k(J)$ gives the Birkhoff perturbation series. In particular $N_0(A) = K_0(A)$, $N_1(A) = \hat{V}_0$. Putting for convenience $V^{(1)} = V$, $V^{(k)} = \frac{1}{2} \sum_{s=1}^{k-1} \omega \nabla W^{(s)} \nabla W^{(k-s)}$ we obtain the following equations for $W^{(k)}$ and N_k:

$$\omega A \nabla_\varphi W^{(k)} + V^{(k)} = N_k. \tag{3}$$

It can be seen that for non-resonant actions the above equations can be solved inductively.

We now turn to the description of the quantum case and consider the operator $H_0(\hbar)$ introduced above. $H_0(\hbar)$ is self-adjoint, its spectrum consists of isolated eigenvalues $E_0(\hbar n, \hbar) = K_0(\hbar n)$ of finite multiplicity, and its resolvent is compact. We see that the Bohr-Sommerfeld quantization $A = \hbar n$ is exact for unperturbed rotators. We formulate separate results for symmetric and non-symmetric potentials, so first we introduce the notation for the symmetry subspaces of $L^2(T^d)$ in the case of symmetric potentials. Let $\mathcal{Q} = \{0, 1\}^d$ and for any $\alpha \in \mathcal{Q}$ let $\mathcal{H}^\alpha$ denote the subspace

$$\mathcal{H}^\alpha = \{ f \in L^2(T^d) : T_j f = (-1)^{\alpha_j} f, j = 1, \ldots, d \}, \tag{4}$$

where T_j is the symmetry operator defined by

$$T_j f(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_d) = f(\varphi_1, \ldots, -\varphi_j, \ldots, \varphi_d), \tag{5}$$

where $\varphi = (\varphi_1, \ldots, \varphi_d)$. We say that V is symmetric if $T_j V = V$ for $j = 1, \ldots, d$. The subspaces $\mathcal{H}^\alpha$ are then invariant for $H_0(\hbar)$ and V, their orthogonal sum is the whole of $L^2(T^d)$. We denote by P^α the orthogonal projection onto $\mathcal{H}^\alpha$. Let $H_0^\alpha(\hbar)$ denote the restriction of $H_0(\hbar)$ onto $\mathcal{H}^\alpha$, $H_0^\alpha(\hbar) = P^\alpha H_0(\hbar)$. It is clear that the spectrum of $H_0^\alpha(\hbar)$ is simple. It is worth noticing that the operators $H_0^\alpha(\hbar)$ are not unitarily equivallent, since they have different spectra. Indeed, for any $A \in \mathbf{R}^d$ let $\Lambda(A) = \{ \alpha \in \mathcal{Q} : \alpha_j = 0$ whenever $A_j = 0 \}$, in particular $\Lambda(A) = \mathcal{Q}$ if all components of A are non-zero. Then we have

$$K_0(\hbar n) \in \sigma(H_0^\alpha(\hbar)) \quad \text{iff} \quad \alpha \in \Lambda(\hbar n).$$

Let the Rayleigh-Schrödinger series (*see* Reed and Simon [12] for the definition) of $H_0^\alpha(\hbar)$ in $\mathcal{H}^\alpha$ be denoted by $\sum_{k=0}^{\infty} \varepsilon^k E_k^\alpha(A, \hbar)$, with $E_0^\alpha(A, \hbar) = E_0(A, \hbar)$ for $\alpha \in \Lambda(\hbar n)$. Then the following theorem holds.

THEOREM 1. — *Suppose* V *is symmetric. Fix* $A \in \mathbf{R} \times \mathbf{Z}^d$, $\Lambda(A) = \mathcal{Q}$, *and let* $\hbar > 0$ *satisfy* $A/\hbar \in \mathbf{Z}^d$.

Then for any $k \geqq 1$ *we have:*

(*a*) *for any* $\alpha, \beta \in \mathcal{Q}$, *and some* $K>0$

$$E_k^{\alpha}(A, \hbar) = E_k^{\beta}(A, \hbar) + O(\exp(-K/\hbar))$$

as $\hbar \to 0$;

(*b*) *there exist independent of* $\hbar$ *functions of* A $N_k^{(j)}(A)$, $j=0, 1, \ldots$, *where* $N_k^{(0)}(A) = N_k(A)$ *is the k-th term of the Birkhoff series, such that for any* $\alpha \in \mathcal{Q}$

$$E_k^{\alpha}(A, \hbar) = \sum_{j=0}^{\infty} \hbar^j N_k^{(j)}(A)$$

as $\hbar \to 0$, *where the series is asymptotic in the sense that for every* $M \in N$

$$E_k^{\alpha}(A, \hbar) - \sum_{j=0}^{M} \hbar^j N_k^{(j)}(A) = O(\hbar^{M+1}) \tag{6}$$

as $\hbar \to 0$.

The statement of analogous result for non-symmetric potentials is weaker, in particular we can only treat trigonometric polynomials. The extension to general non-symmetric potentials V could be attempted by approximating V by trigonometric polynomials, as the functions $N_k^{(j)}(A)$ appearing below are continuous in V. However for degenerate eigenvalues the control of the coefficients of the Rayleigh-Schrödinger series is not simple.

THEOREM 1' — *Suppose* V *is a trigonometric polynomial. Fix* $A \in \mathbf{R} \times \mathbf{Z}^d$, $\Lambda(A) = \mathcal{Q}$, *and let* $\hbar > 0$ *satisfy* $A/\hbar \in \mathbf{Z}^d$. *Then for any* $k \geqq 1$ *there exist independent of* $\hbar$ *functions of* A $N_k^{(j)}(A)$, $j=0, 1, \ldots$, *where* $N_k^{(0)}(A) = N_j(A)$ *is the k-th term of the Birkhoff series, such that*

$$E_k(A, \hbar) = \sum_{j=0}^{\infty} \hbar^j N_k^{(j)}(A),$$

is convergent (with the radius of convergence decreasing to 0 *as* $k \to \infty$), *and the* 2^d *sums* $\sum_{s=0}^{k} \varepsilon^s E_s(\bar{A}, \hbar)$ for $\bar{A} \in \mathbf{R}^d$ *with* $|A_j| = |\bar{A}_j|$, $j=1, \ldots, d$, *are the partial sums of the Rayleigh-Schrödinger series for perturbation of* $E_0(A, \hbar)$.

The above theorems show that the semiclassical limit of the Rayleigh-Schrödinger perturbation series is indeed the Birkhoff perturbation series also for degenerate eigenvalues.

The quantum corrections $N_k^{(j)}(A)$ may be explicitely recursively calculated. It is natural that Theorem 1 holds only for $A = \hbar n$ with $\Lambda(A) = \mathcal{Q}$, that is for eigenvalues of maximal multiplicity 2^d, since for other eigenvalues the corresponding Birkhoff series $\sum \varepsilon^k N_k(A)$ is not defined (because

A is then a resonant action) and, indeed, the perturbed eigenvalue need not have any asymptotics as seen in Example 1 below. For the case $k=1$ we have $N_1^{(0)}(A)=\hat{V}_0$, $N_1^{(j)}(A)=0$ for $j=1, 2, \ldots$ and, as seen from the remark following the proof of Lemma 1 below, an exponential estimate of the error in (*b*) in the following form holds:

$$E_1(A, \hbar)=N_1(A)+O(\exp(-K/\hbar)).$$

For the case $k=2$ in one dimension one can show, using the formulae derived in the proof of Theorem 1, that $N_2^{(j)}(A)=0$ for j odd and

$$N_2^{(j)}(A)=\frac{1}{2\omega A^2}\sum_{m\neq 0}\left(\frac{m}{2A}\right)^j|\hat{V}_m|^2$$

for j even. If V is not a trigonometric polynomial, *i.e.* has infinitely many non-zero components in its Fourier expansion, then the series in $\hbar$ for $E_2^\alpha(A, \hbar)$ is divergent and we cannot improve the asymptotic estimate in (*b*). The estimate of the error in (*a*) is certainly optimal (discounting the problem of determining the constant K), as the following simple example demonstrates.

EXAMPLE 1. — Let $d=1$. As is well known, the first order perturbation corrections are given by the eigenvalues of the matrix $P(E_0(A, \hbar))VP(E_0(A, \hbar))$, where $P(E_0(A, \hbar))$ is the orthogonal projection onto the subspace spanned by eigenvectors corresponding to $E_0(A, \hbar)$. For $k=1$, consider the 2×2 matrix

$$\begin{pmatrix}\hat{V}_0 & \hat{V}_{-2n}\\ \hat{V}_{2n} & \hat{V}_0\end{pmatrix},\qquad (7)$$

where $\hat{V}_m$ denotes the Fourier coefficient of V. Assume that $\hbar\to 0$, $n\to\infty$ with $\hbar n=A$ fixed. Due to the analycity of V the off-diagonal entries in (7) are exponentially small as $\hbar\to 0$, so the (two) values of E_k^α are exponentially close to V_0, and hence to each other. However that is all that may be asserted about them in general. Consider now the only eigenvalue of less than maximal multiplicity, that is $E_0(0, \hbar)=0$ with its corresponding eigenvector $\psi_0(\varphi)=1$ (here as in all the paper we assume that the Lebesgue measure on T^d is normalized). $E_0(0, \hbar)$ being a simple eigenvalue the second order perturbation is given by

$$\varepsilon E_1(0, \hbar)+\varepsilon^2 E_2(0, \hbar)\qquad\text{where}\quad E_1(0, \hbar)=\hat{V}_0,$$

and

$$E_2(0, \hbar)=(\psi_0, (V-E_1(0, \hbar))H_0(\hbar)^{-1}(V-E_1(0, \hbar))\psi_0)$$

[here $(.,.)$ denotes the scalar product]. If we suppose that $V(\varphi)=2\cos\varphi$, then $E_1(0, \hbar)=0$, $E_2(0, \hbar)=C\hbar^{-2}$, which diverges as $\hbar\to 0$.

As is well known from the general theory [12], the Rayleigh-Schrödinger perturbation series has a positive radius of convergence whenever, as is the case in Theorem above, the perturbation is relatively bounded. The perturbed eigenvalue $E_\varepsilon(\hbar, n)$ of $H_\varepsilon(\hbar)$ is thus equal to $E_\varepsilon(\hbar, n) = \sum_{k=0}^{\infty} \varepsilon^k E_k(A, \hbar)$. On the other hand, for actions of the form $A = \hbar n$ the KAM theorem [5] may be applied, yielding that in the vicinity of A there exists a completely canonical map putting K_ε into its action-angle variables (A', φ'). More precisely [4], fix a bounded domain $\Omega \in \mathbf{R}^d$, and denote the set of invariant KAM tori of $K_\varepsilon(A, \varphi)$ in $\Omega \times T^d$ by $\Gamma(\varepsilon)$. By results of [4] there exist smooth functions $A'(A, \varphi)$, $\varphi'(A, \varphi)$, defined on $\Omega \times T^d$ such that A' are prime integrals for perturbed motions starting in $\Gamma(\varepsilon)$, are in involution on $\Gamma(\varepsilon)$ and the canonical change of variables $(A, \varphi) \mapsto (A', \varphi')$ on $\Gamma(\varepsilon)$ transforms K_ε to $N(A', \varepsilon)$. Moreover the Birkhoff series is asymptotic to $N(A', \varepsilon)$ in the sense that $N(A, \varepsilon) - \sum_{j=0}^{M} \varepsilon^j N_j(A) = O(\varepsilon^{M+1})$ for any M, for $A \in \Gamma'(\varepsilon) = A'(\Gamma(\varepsilon))$.

In view of Theorem 1 it is reasonable to ask whether the following semiclassical limit holds:

$$E_\varepsilon(\hbar, n) = N(\hbar n, \varepsilon) + O(\hbar), \tag{8}$$

unformly for small ε, as $\hbar \to 0$. The estimates established below in the proof of Theorem 1 are not sufficient to prove (8), we will instead prove the following weaker result, analogous to Theorem 2 of [6].

THEOREM 2. — *Suppose* V *is a trigonometric polynomial. There exists a bounded function* $g(A, \hbar, \varepsilon) \in C^\infty(\Gamma'(\varepsilon) \times (0, \hbar_1) \times (0, \varepsilon_1))$, *with some positive* $\hbar_1$, ε_1 *such that for any* $\hbar$, n *with* $\hbar n \in \Gamma'(\varepsilon)$, $\hbar < \hbar_1$, *we have*

$$E_\varepsilon(\hbar, n) = N(\hbar n, \varepsilon) + \hbar g(\hbar n, \hbar, \varepsilon) + O(\varepsilon^\infty), \tag{9}$$

as $\varepsilon \to 0$, *where* $O(\varepsilon^\infty)$ *is uniform in* $\hbar$.

Theorem 2 shows that the Bohr-Sommerfeld quantization of $K_\varepsilon(A, \varphi) = \frac{1}{2} \sum_{j=0}^{d} \omega_j A_j^2 + \varepsilon V(\varphi)$ in its action-angle variables A', φ' is valid, for $A' = \hbar n \in \Gamma'(\varepsilon)$, in the limit $\hbar \to 0$ to first order in $\hbar$ and to any order in ε. The function g appearing in Theorem 2 is not unique and the major open problem is to construct such function which would yield (8).

SECTION 2

The proof of Theorem 1 and Theorem 1', as mentioned in the Introduction, consists in rewriting the Schrödinger equation in the form of the Hamilton-Jacobi equation with quantum corrections and then applying an inductive solution. Due to the (trivial) degeneracy of eigenvalues of $H_0(\hbar)$ in our case the Hamilton-Jacobi equation cannot be solved exactly and an extra argument is needed to compare its solutions with the Rayleigh-Schrödinger series (such an argument was not necessary in [6]). The estimates needed to solve the Hamilton-Jacobi equation form the proof of Lemma 1 below, while the extra argument is used to prove Theorem 1 and Theorem 1′ from Lemma 1.

LEMMA 1. — *For any* $A \in \mathbf{R} \times \mathbf{Z}^d$ *and* $\hbar > 0$ *with* $A/\hbar \in \mathbf{Z}^d$, $A_j \neq 0$ *for* $j = 1, \ldots, d$, *there exist* $\psi_k(A, \hbar) \in L^2(T^d)$, $S_k(A, \hbar) \in \mathbf{C}$, *and* $N_k^{(j)}(A) \in \mathbf{C}$, k, $j = 0, 1, \ldots$ *with* $\psi_0(A, \hbar) = \exp(iA/\hbar\varphi)$, $S_0(A, \hbar) = E_0(A, \hbar) = K_0(\hbar n)$ *satisfying the following conditions:*

(*a*)

$$S_k(A, \hbar) = \sum_{j=0}^{\infty} \hbar^j N_k^{(j)}(\hbar n),$$

where the series is asymptotic in the sense that for every $M \in \mathbf{N}$ *and any* $0 < \eta < 1$

$$S_k(A, \hbar) - \sum_{j=0}^{M} \hbar^j N_k^{(j)}(\hbar n) = O(\hbar^{M+\eta})$$

as $\hbar \to 0$, *where the error estimate is uniform for potentials* V *with* $\sup_{|\operatorname{Im}\varphi| \leq \varkappa} |V(\varphi)|$ *equibounded;*

(*b*) *for any* $k \geq 1$ $\psi_k(A, \hbar)$ *is in domain of* $H_0(\hbar)$ *and*

$$(H_0(\hbar) - E_0(A, \hbar))\psi_k(A, \hbar) + (V - S_1(A, \hbar))\psi_{k-1}(A, \hbar) \\ - S_2(A, \hbar)\psi_{k-2}(A, \hbar) - \ldots - S_k(A, \hbar)\psi_0(A, \hbar) = O(\exp(-K/\hbar))$$

for $\hbar \to 0$, *for some* $K > 0$, *where the error estimate is in supremum norm over* T^d;

(*c*) *if* V *is a trigonometric polynomial with* $\hat{V}_m = 0$ *for* $|m| > M_V$, *then for any* k

$$(H_0(\hbar) - E_0(A, \hbar))\psi_k(A, \hbar) + (V - S_1(A, \hbar))\psi_{k-1}(A, \hbar) \\ - S_2(A, \hbar)\psi_{k-2}(A, \hbar) - \ldots - S_k(A, \hbar)\psi_0(A, \hbar) = 0,$$

for $\hbar < 2^{-k} \min_{j=1,\ldots,d} |A_j|/M_V$ *and, moreover, for such* $\hbar$ *the series in* (*a*) *is convergent to* $S_k(A, \hbar)$;

(*d*) *for fixed* A *and* k *the norm of* $\psi_k(A, \hbar)$ *in* $L^2(T^d)$ *is polynomially bounded in* $\hbar^{-1}$.

Proof of Lemma 1. As mentioned above we use the WKB approximation. The solution of the eigenequation

$$(H_0(\hbar) + \varepsilon V)\psi_\varepsilon = E_\varepsilon \psi_\varepsilon$$

will be sought in the form $\psi_\varepsilon = \exp\left(\frac{i}{\hbar} W_\varepsilon\right)$, where $W_\varepsilon(\varphi)$ is a periodic function on T^d.

Expanding W_ε and E_ε in the series

$$E_\varepsilon = E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \dots,$$
$$W_\varepsilon(\varphi, \hbar) = W^{(0)}(\varphi, \hbar) + \varepsilon W^{(1)}(\varphi, \hbar) + \varepsilon^2 W^{(2)}(\varphi, \hbar) + \dots,$$

and setting $W^{(0)}(\varphi, \hbar) = \varphi \hbar n$, $E_0 = K_0(\hbar n)$, $A = \hbar n$ we obtain the following equations

$$\omega A \nabla_\varphi W^{(1)} - \frac{1}{2} i\hbar \sum_{j=0}^{d} \omega_j \frac{\partial^2 W^{(1)}}{\partial \varphi_j^2} + V = E_1,$$
$$\omega A \nabla_\varphi W^{(k)} - \frac{1}{2} i\hbar \sum_{j=0}^{d} \omega_j \frac{\partial^2 W^{(k)}}{\partial \varphi_j^2} + \frac{1}{2} \sum_{j=0}^{d} \omega_j \left(\sum_{s=1}^{k-1} \frac{\partial W^{(s)}}{\partial \varphi_j} \frac{\partial W^{(k-s)}}{\partial \varphi_j} \right) = E_k.$$

Here $W^{(k)}$ and E_k depend on A and $\hbar$. The above equations may be called the Hamilton-Jacobi equations with quantum corrections, where the correction is the term with the second derivative which, formally speaking, vanishes in the limit $\hbar \to 0$. We will use the abbreviation $V^{(1)} = V$,

$$V^{(k)} = \frac{1}{2} \sum_{j=0}^{d} \omega_j \left(\sum_{s=1}^{k-1} \frac{\partial W^{(s)}}{\partial \varphi_j} \frac{\partial W^{(k-s)}}{\partial \varphi_j} \right), \tag{10}$$

which permits to put these equations into a more compact form, analogous to (3),

$$\omega A \nabla_\varphi W^{(k)} - \frac{1}{2} i\hbar \sum_{j=0}^{d} \omega_j \frac{\partial^2 W^{(k)}}{\partial \varphi_j^2} + V^{(k)} = E_k. \tag{11}$$

We recall the procedure, well known in classical mechanics ([5], [3]), to solve these equations.

One takes E_k to be the mean over the torus of the left hand side of (11). Then one expands $W^{(k)}$ and $V^{(k)}$ in the Fourier series

$$W^{(k)}(\varphi, \hbar) = \sum_{m \in Z^d} \hat{W}_m^{(k)} \exp(im\varphi),$$
$$V^{(k)}(\varphi, \hbar) = \sum_{m \in Z^d} \hat{V}_m^{(k)} \exp(im\varphi),$$

where the dependence of $\hat{W}_m^{(k)}$ and $\hat{V}_m^{(k)}$ on $\hbar$ and n is supressed, and (11) reduces to

$$\left(i\omega A m + \frac{1}{2} i\omega\hbar m^2 \right) \hat{W}_m^{(k)} + \hat{V}_m^{(k)} = 0, \tag{12}$$

for any $m \neq 0$. We put $\hat{W}_0^{(k)} = 0$. However the equation (12) cannot be solved, because for $m = -2n$, with $A = \hbar n$, the expression in brackets vanishes. This happens for a set of 2^d vectors m with m_j equal either 0 or $-2\,A_j/\hbar$, let us denote this set $U(A, \hbar)$. Note that this difficulty is due precisely to the quantum corrections in (11). Therefore we adopt the following procedure. We define $\hat{W}_m^{(k)}=0$ for $m \in U(A, \hbar)$, and let $S_k(A, \hbar)=\hat{V}_0^{(k)}$. We will show that this procedure defines an analytic function $W^{(k)}$ of φ [*see* (17) below] and allows us to solve inductively the equations

$$\omega A \nabla_\varphi W^{(k)} - \frac{1}{2} i \hbar \sum_{j=0}^{d} \omega_j \frac{\partial^2 W^{(k)}}{\partial \varphi_j^2} + V^{(k)} = S_k(A, \hbar) + O\,[\exp(-K/\hbar)], \quad (13)$$

the estimate of the error in supremum norm over T^d. (*b*) of the lemma follows from (13) after we express $\psi_k(A, \hbar)$ in terms of $i\hbar^{-1} W^{(j)}$, $j=1, \ldots, k$, namely

$$\psi_k(A, \hbar) = \psi_0(A, \hbar) \sum_{\beta \in N^k,\ \|\beta\|=k} \frac{1}{\beta!} \prod_{j=1}^{k} \left(\frac{i}{\hbar} W^{(j)}\right)^{\beta_j},$$

where we use the notation $\|\beta\| = \sum_{j=1}^{k} j\beta_j$. Note that polynomial dependence on h^{-1} is offset by exponential estimate above. We will then prove the estimate (*a*) of the Lemma. Observe also that if V is a trigonometric polynomial, then by (10) and (12) so is $V^{(k)}$ for any k, and hence for any k for sufficiently small h $\hat{V}_m^{(k)}=0$ for any $m \in U(A, \hbar)$, $m \neq 0$. Moreover the finite number of terms in the Fourier series of $V^{(k)}$ for each k enables us to prove inductively that $V^{(k)}$ and hence also $S_k(A, \hbar)$, for $A=\hbar n$ fixed, are well defined and analytic in $\hbar$ for small $\hbar$. This yields (*c*). (*d*) follows from the above formula for $\psi_k(A, \hbar)$ and the estimate (16) below.

We assume that $\mathscr{A} \in \mathbf{R}_+ \times \mathbf{Z}^d$, that is that A has the form $A=\hbar n$, $n \in \mathbf{Z}^d$, $\hbar>0$ (this representation is not unique, of course). Let also $n_j \neq 0$ for $j=1, \ldots, d$. Using (1) one verifies that the following estimates hold:

$$|\omega A m|^{-1} \leqq C_1 |m|^\gamma, \quad (14)$$

for any $m \neq 0$, and

$$\left|\bar{\omega} A m + \frac{1}{2} \hbar \omega m^2\right|^{-1} \leqq C_1 \frac{1}{\hbar^{\gamma+1}} |m|^{2\gamma}, \quad (15)$$

for any $\hbar$ with $A \in \hbar \mathbf{Z}^d$, $0<\hbar<1$ and any $m \in \mathbf{Z}^d$ with $m \notin U(A, \hbar)$. Here C_1 depends only on A, not on $\hbar$ or n (this will be our standard convention about the constants appearing below). To see (14) observe that by (1)

$$|\omega A m|^{-1} = \hbar^{-1} |\omega n m|^{-1} \leqq \hbar^{-1} C \|n\|_\infty^\gamma |m|^\gamma,$$

where $\|n\|_\infty = \sup_{j=1, \ldots, d} |n_j|$ and $A=\hbar n$. If we now choose the representation $A=\hbar_1 n_1$ with the maximal $\hbar_1$ (such $\hbar_1$ exists and depends only on A, as does the corresponding n_1), we obtain (14) with appropriate C_1. To establish (15) note that with m, n and $\hbar$ as above we have

$$\left|\omega A m+\frac{1}{2}\hbar\omega m^2\right|^{-1}=\hbar^{-1}\left|\omega\left(n+\frac{1}{2}m\right)m\right|^{-1}\leqq \hbar^{-1}C\left\|n+\frac{1}{2}m\right\|_\infty^\gamma |m|^\gamma.$$

Furthermore,

$$\left\|n+\frac{1}{2}m\right\|_\infty \leqq \hbar^{-1}\|A\|_\infty+\|m\|_\infty \leqq \hbar^{-1}(\|A\|_\infty+1)|m|,$$

since $\hbar<1$ and $|m|\geqq 1$. The estimate (15) follows immediately.

The analicity of V implies that $|\hat{V}_m|\leqq D\exp(-\kappa|m|)$, for some positive D, κ. Let $\kappa_k=\kappa\left(\frac{1}{2}+\frac{1}{2^k}\right)$ (the choice of κ_k is to a large extend arbitrary). We will prove inductively that

$$|\hat{V}_m^{(k)}|\leqq D_k \exp(-\kappa_k|m|), \tag{16}$$

where D_k satisfy $D_k=\hbar^{-2(\gamma+1)(k-1)}B_k$, which shows that the inductive solution of (13) is possible (B_k is a positive constant which does not depend on $\hbar$). Moreover (16) implies (13) and hence (*b*) of the Lemma. Indeed, for any k the error in (13) will be of order of $\sum_{m\in U(A,\hbar),\, m\neq 0}|\hat{V}_m^{(k)}|^2$, that is of the order $2^d D_k\exp(-|A|_0\kappa_k/\hbar)$, where $|A|_0=\inf_{j=1,\ldots,d}|A_j|>0$. Since D_k grows polynomially with $\hbar^{-1}$, (16) implies that this error will be exponentially bounded, with any K lesser that $|A|_0\kappa_k$. We may put therefore $K=\frac{1}{2}|A|_0\kappa$. Note that although sufficient for our purposes, (16) is a crude estimate, indeed (22) below establishes that D_k may be chosen independent of $\hbar$.

For $k=1$ (16) follows from analycity of V. Assume that (16) holds for $k=1,\ldots,N$.

Using (15) and (12) we have

$$|\hat{W}_m^{(s)}|\leqq \frac{1}{\hbar^{\gamma+1}}C_1 D_s|m|^{2\gamma}\exp(-\kappa_s|m|), \tag{17}$$

for $m\notin U(A,\hbar)$, for $s=1,\ldots,N$. This will be used to estimate $V^{(N+1)}$, given by (10).

Observe that $\nabla W^{(s)} \sim \sum_{m \in Z^d} im\, \hat{W}_m^{(s)} \exp(im\,\varphi)$, so for $|\operatorname{Im} \varphi| \leqq \kappa_{N+1}$ we have, using (17),

$$\|\nabla W^{(s)}\|_{\infty,\, |\operatorname{Im} \varphi| \leqq \varkappa_{N+1}} \leqq \frac{1}{\hbar^{\gamma+1}} C_1 D_s \sum_{m \in Z^d} |m|^{2\gamma+1} \exp(-\kappa_s |m|) \exp(\kappa_{N+1} |m|)$$

$$\leqq \frac{1}{\hbar^{\gamma+1}} C_1 D_s \sum_{m \in Z^d} |m|^{2\gamma+1} \exp\left(-\frac{\kappa}{2^{s+1}} |m|\right)$$

by our choice of κ_k. Summing the above series we obtain

$$\|\nabla W^{(s)}\|_{\infty,\, |\operatorname{Im} j| \leqq \varkappa_{N+1}} \leqq \frac{1}{\hbar^{\gamma+1}} C_1 D_s C^1(s), \tag{18}$$

with some constant $C'(s)$, depending on κ, γ, d but not on $\hbar$. Given the definition (10), this estimate implies

$$\|V^{(N+1)}\|_{\infty,\, |\operatorname{Im} \varphi| \leqq \varkappa_{N+1}} \leqq \frac{1}{\hbar^{2\gamma+2}} C_1^2 \|\omega\|_\infty \times \sum_{s=1}^{N} D_s C'(s) D_{N+1-s} C'(N+1-s)$$

$$= \frac{1}{\hbar^{2(\gamma+1)N}} C_1^2 \|\omega\|_\infty \sum_{s=1}^{N} B_s C'(s) B_{N+1-s} C'(N+1-s),$$

where the inductive hypothesis was used. Since clearly

$$|\hat{V}_m^{(N+1)}| \leqq \|V^{(N+1)}\|_{\infty,\, |\operatorname{Im} \varphi| \leqq \varkappa_{N+1}} \exp(-\kappa_{N+1} |m|),$$

in order to establish (16) for $k = N+1$ we have to set

$$B_{N+1} = C_1^2 \|\omega\|_\infty \sum_{s=1}^{N} B_s C'(s) B_{N+1-s} C'(N+1-s).$$

The proof of (16) is complete.

We now pass to the proof of (*a*) of Lemma 1. In order to avoid confusion with the notation, the functions $W^{(k)}$ and $V^{(k)}$ appearing in (3), that is in the classical limit, will be now distinguished from $W^{(k)}$, $V^{(k)}$ above by the subscript *cl*. Expanding $W_{cl}^{(k)}$ and $V_{cl}^{(k)}$ in the Fourier series we see that (3) reduces to

$$i\omega A m\, \hat{W}_{cl,\,m}^{(k)} + \hat{V}_{cl,\,m}^{(k)} = 0 \quad \text{for } m \neq 0, \qquad \hat{V}_{cl,\,0}^{(k)} = N_k. \tag{19}$$

Thus $W_{cl}^{(k)}(\varphi)$ is defined by $\hat{W}_{cl,\,m}^{(k)} = -\dfrac{\hat{V}_{cl,\,m}^{(k)}}{i\omega A m}$ for $m \neq 0$, where the dependence on A is suppressed. Furthermore we define inductively functions $W_{cl}^{(k,\ j)}(\varphi)$, $V_{cl}^{(k,\ j)}(\varphi)$, $j=0, 1, \ldots$, also depending on A, by setting $V_{cl}^{(1,\ 0)}=V$, $V_{cl}^{(1,\ j)}=0$ for $j \geqq 1$, and

$$\hat{W}_{cl,\ m}^{(k,\ j)}=i \sum_{r=0}^{i} \frac{\hat{V}_{cl,\ m}^{(k,\ j-r)}}{\omega A m}\left(\frac{-\omega m^2}{2 \omega A m}\right)^r \quad \text{for } m \neq 0, \qquad \hat{W}_{cl,\ 0}^{(k,\ j)}=0,$$

$$V_{cl}^{(k+1,\ j)}(\varphi)=\frac{1}{2} \sum_{r=0}^{j} \sum_{s=1}^{k} \omega \nabla W_{cl}^{(s,\ r)}(\varphi) \nabla W_{cl}^{(k+1-s,\ j-r)}(\varphi),$$

for $k \geqq 1$. We obtain $V_{cl}^{(k,\ 0)}=V_{cl}^{(k)}$. Recall that $N_k(A)=\hat{V}_{cl,\ 0}^{(k)}$, we also define $N_k^{(j)}(A)=\hat{V}_{cl,\ 0}^{(k,\ j)}$. $N_k^{(j)}$ are the classically defined, independent of $\hbar$ functions appearing in the semi-classical limit of S_k. An estimate analogous to (16) holds for $\hat{V}_{cl,\ m}^{(k,\ j)}$, namely

$$|\hat{V}_{cl,\ m}^{k,\ j}| \leqq C_{k,\ j} \exp(-\kappa_k |m|), \tag{20}$$

where $C_{k,\ j}$ are constants depending on A, common for potentials V with $\sup_{|\operatorname{Im} \varphi| \leqq \varkappa} |V(\varphi)|$ equibounded. The estimates (20) imply, among others, that the recursive definition of $W_{cl}^{(k,\ j)}$, $V_{cl}^{(k,\ j)}$ is correct. Fix now arbitrary $M \in \mathbf{N}$. Our aim is to prove that for every k

$$\left(\hat{V}_m^{(k)}-\sum_{j=0}^{M} \hbar^j \hat{V}_{cl,\ m}^{(k,\ j)}\right) \exp(\kappa_k |m|)=O(\hbar^{M+\eta}) \tag{21}$$

as $\hbar \rightarrow 0$ in such a way that $\hbar n=A$ is fixed, uniformly in $m \in \mathbf{Z}^d$, for any $0<\eta<1$. By the definition of $S_k(A, \hbar)$ and $N_k^{(j)}(A)$, $j=0, 1, \ldots$, and by arbitrariness of M, (*a*) of Lemma 1 follows upon putting $m=0$ in (21).

The proof of (21) is by induction. For $k=1$ we have $V^{(1)}=V_{cl}^{(1)}=V$, so we have equality (without the error term). Suppose we have (21) for $k=1, \ldots, N$. It is enough to show that this implies

$$\nabla W^{(N)}(\varphi, \hbar)-\sum_{j=0}^{M} \hbar^j \nabla W_{cl}^{(N,\ j)}(\varphi)=O(\hbar^{M+\eta}), \tag{22}$$

uniformly in $|\operatorname{Im} \varphi| \leqq \kappa_{N+1}$. Indeed, if that implication is proved then by construction we know that

$$V^{(N+1)}(\varphi, \hbar)-\sum_{j=0}^{M} \hbar^j V_{cl}^{(N+1,\ j)}(\varphi)=O(\hbar^{M+\eta}), \tag{23}$$

uniformly in $|\operatorname{Im} \varphi| \leqq \kappa_{N+1}$, whence the estimate (21) of the Fourier coefficients of the left hand side of (23) follows immediately.

In order to prove (22) we set

$$R(\hbar)=-2^{N+2}(2\gamma+1)(M+1)(N+1)\kappa^{-1} \ln \hbar.$$

Note that for sufficiently small $\hbar m \in U(A, \hbar)$, $m \neq 0$ implies $|m| > R(\hbar)$. Using (17) we have

$$\sum_{|m|>R(\hbar)} im\, \hat{W}_m^{(N)} \exp(im\,\varphi) = O(\hbar^{M+1}),$$

as $\hbar \to 0$, and analogously we estimate $W_{cl}^{(N,\,j)}$ using (20). Therefore

$$\begin{aligned}
&\nabla W^{(N)}(\varphi, \hbar) - \sum_{j=0}^{M} \hbar^j \nabla W_{cl}^{(N,\,j)}(\varphi) \\
&= \sum_{|m| \leqq R(\hbar)} im \left(\hat{W}_m^{(N)} - \sum_{j=0}^{M} \hbar^j \hat{W}_{cl,\,m}^{(N,\,j)} \right) \exp(im\,\varphi) + r(N, M, \hbar) \\
&= - \sum_{|m| \leqq R(\hbar)} m \left(\frac{\hat{V}_m^{(N)}}{\omega A m + (1/2)\hbar\omega m^2} \right. \\
&\quad \left. - \sum_{j=0}^{M} \hbar^j \sum_{r=0}^{j} \frac{\hat{V}_{cl,\,m}^{(N,\,j-r)}}{\omega A m} \left(\frac{-\omega m^2}{2\omega A m} \right)^r \right) \exp(im\,\varphi) + r(N, M, \hbar) \\
&= - \sum_{|m| \leqq R(\hbar)} m \left(\left(\hat{V}_m^{(N)} - \sum_{j=0}^{M} \hbar^j \hat{V}_{cl,\,m}^{(N,\,j)} \right) \frac{1}{\omega A m + (1/2)\,\hbar\omega m^2} \right. \\
&\quad + \sum_{j=0}^{M} \hbar^j \hat{V}_{cl,\,m}^{(N,\,j)} \left(\frac{1}{\omega A m + (1/2)\hbar\omega m^2} \right. \\
&\quad \left. \left. - \sum_{r=0}^{M-j} \frac{\hbar^r}{\omega A m} \left(\frac{-\omega m^2}{2\omega A m} \right)^r \right) \right) \exp(im\,\varphi) + r(N, M, \hbar) \\
&= - \sum_{|m| \leqq R(h)} \left(\hat{V}_m^{(N)} - \sum_{j=0}^{M} \hbar^j \hat{V}_{cl,\,m}^{(N,\,j)} \right) \frac{m}{\omega A m + (1/2)\hbar\omega m^2} \exp(im\,\varphi) \\
&\quad - \hbar^{M+1} \sum_{|m| \leqq R(h)} \sum_{j=0}^{M} \frac{m}{\omega A m + (1/2)\hbar\omega m^2} \\
&\quad \times \hat{V}_{cl,\,m}^{(N,\,j)} \left(\frac{-\omega m^2}{2\omega A m} \right)^{M+1-j} \exp(im\,\varphi) + r(N, M, \hbar). \qquad (24)
\end{aligned}$$

Here $r(N, M, \hbar) = O(\hbar^{M+1})$. We will separately estimate the two finite sums in (24). For $0 < |m| \leqq R(\hbar)$ we have, using (14),

$$\begin{aligned}
\left| \omega A m + \frac{1}{2} \hbar\omega m^2 \right| &\geqq |\omega A m| - \frac{1}{2} \hbar \|\omega\|_\infty |m|^2 \\
&\geqq \frac{1}{C_1} |m|^{-\gamma} - \frac{1}{2} \hbar \|\omega\|_\infty R(\hbar)^2 \\
&\geqq \frac{1}{C_2} |\ln \hbar|^{-\gamma} (1 - C_3 \hbar |\ln \hbar|^{2+\gamma}),
\end{aligned}$$

where C_2 and C_3 are constants depending only on N, M, γ, ω and C_1. Thus for sufficiently small $\hbar$ (depending on N, etc.) we find

$$\left|\omega A m+\frac{1}{2}\hbar\omega m^2\right| \geqq \frac{1}{2C_2}|\ln \hbar|^{-\gamma}, \tag{25}$$

which together with the inductive hypothesis (21) implies that

$$\sum_{|m|\leqq R(\hbar)} \left(\hat{V}_m^{(N)} - \sum_{j=0}^{M} \hbar^j \hat{V}_{cl,\,m}^{(N,\,j)}\right) \frac{m}{\omega A m+(1/2)\hbar\omega m^2} \exp(im\,\varphi) = O(\hbar^{M+\eta'}),$$

for any $0<\eta'<\eta$, uniformly in $|\operatorname{Im}\varphi|\leqq\kappa_{N+1}$. To deal with the second sum in (24), observe that (14), (25) and the definition of $R(\hbar)$ imply that

$$\left|\frac{m}{\omega A m+(1/2)\hbar\omega m^2}\left(\frac{-\omega m^2}{2\omega A m}\right)^{M+1-j}\right| \leqq C_4 |\ln \hbar|^{\gamma} |m|^{(2+\gamma)(M+1-j)+1}$$
$$\leqq C_5 |\ln \hbar|^{(2+\gamma)(M+2)} = O(\hbar^{\eta'-1}), \tag{26}$$

for any $0<\eta'<1$. Now (26) implies

$$\sum_{|m|\leqq R()} \sum_{j=0}^{M} \hat{V}_{cl,\,m}^{(N,\,j)} \frac{m}{\omega A m+(1/2)\hbar\omega m^2} \times \left(\frac{-\omega m^2}{2\omega A m}\right)^{M+1-j} \exp(im\,\varphi) = O(\hbar^{\eta'-1}),$$

uniformly in $|\operatorname{Im}\varphi|\leqq\kappa_{N+1}$, because $\sum_{m}\sum_{j=0}^{M}|\hat{V}_{cl,\,m}^{(N,\,j)}|$ may be summed and its sum is independent of $\hbar$. An inductive argument shows that the error estimate in the above formula is also uniform for potentials V with $\sup_{|\operatorname{Im}\varphi|\leqq\varkappa}|V(\varphi)|$ equibounded. We have thus estimated both sums in (24), so we obtain (22) with the left hand side replaced by $O(\hbar^{M+\eta'})$. However as in (21) η is an arbitrary positive number less than 1, so is η' in (22). The proof of (21) and hence of Lemma 1 is complete.

We remark that for $k=1$ the above proof implies that the series in $\hbar$ reduces to zeroth term only and we have strict equality (without the error term) in Lemma 1 (*a*), and hence an exponential estimate in Theorem 1 (*b*). Indeed, in the above notation, $V^{(1)}(\varphi, \hbar)=V_{cl}^{(1)}(\varphi)=V(\varphi)$, and $V_{cl}^{(1,\,j)}(\varphi)=0$, $N_1^{(j)}=0$ for $j=1, 2, \ldots$, therefore $S_1(A, \hbar)=N_1(\hbar n)=\hat{V}_0$ (this argument is essentially contained in Example 1 in the Introduction). Thus first order quantum perturbation is exponentially close to first order classical perturbation in the semiclassical limit. This implies that (*a*) of Theorem 1 for $k=1$ is true also for non-symmetric potentials.

Proof of Theorem 1. Fix $A\in\mathbf{R}^d$, $A_j\neq 0$ for $j=1, \ldots, d$, and consider $\hbar>0$, $n\in\mathbf{Z}^d$ such that $A=\hbar n$. For any $\alpha\in\Lambda(A)$ let $\psi_k^\alpha(A, \hbar)=P^\alpha\psi_k(A, \hbar)$. We may assume that ψ_0^α has norm 1. Since both

$H_0(\hbar)$ and V are invariant with respect to P^α, Lemma 1 (*b*) implies that

$$(H_0^\alpha(\hbar)-E_0(A,\hbar))\psi_k^\alpha(A,\hbar)+(V-S_1(A,\hbar))\psi_{k-1}^\alpha(A,\hbar)$$
$$-S_2(A,\hbar)\psi_{k-2}^\alpha(A,\hbar)-\ldots-S_k(A,\hbar)\psi_0^\alpha(A,\hbar)=O(\exp(-K/\hbar)) \quad (27)$$

as $\hbar\to 0$. It is clear that (27) is very close to the equations satisfied by the Rayleigh-Schrödinger series, which read

$$(H_0^\alpha(\hbar)-E_0^\alpha(A,\hbar))p_k^\alpha(A,\hbar)+(V-E_1^\alpha(A,\hbar))p_{k-1}^\alpha(A,\hbar)$$
$$-E_2^\alpha(A,\hbar)p_{k-2}^\alpha(A,\hbar)-\ldots-E_k^\alpha(A,\hbar)p_0^\alpha(A,\hbar)=0, \quad (28)$$

where $\sum_{k=0}^{\infty}\varepsilon^k p_k^\alpha(A,\hbar)$ is the Rayleigh-Schrödinger in $\mathscr{H}^\alpha$ for the perturbed eigenvector of $H_0^\alpha(\hbar)$. Obverve that without influencing the $S_k(A,\hbar)$ and $E_k^\alpha(A,\hbar)$ we may assume in (27), (28) that $p_k^\alpha(A,\hbar)$ and $\psi_k^\alpha(A,\hbar)$ are orthogonal to $\psi_0^\alpha(A,\hbar)=p_0^\alpha(A,\hbar)$ [here we use Lemma 1 (*d*)]. We will now show that (27) and (28) imply that

$$S_k(A,\hbar)=E_k^\alpha(A,\hbar)+O(\exp(-K/\hbar)), \quad (29)$$
$$\psi_k^\alpha(A,\hbar)=p_k^\alpha(A,\hbar)+O(\exp(-K'/\hbar)), \quad (30)$$

as $\hbar\to 0$, the second estimate in the sense of L^2 norm, with any $K'<K$. We will proceed by induction on k. For $k=0$ we have equalities (without the error terms). Suppose we have (29) and (30) for $k=0,\ldots,N-1$. Applying to both (27) and (28) the orthogonal projection $P_0^\alpha(\hbar,n)$ onto the one-dimensional subspace of $\mathscr{H}^\alpha$ spanned by the eigenvector $\psi_0^\alpha(A,\hbar)$ we obtain

$$S_k(A,\hbar)=-(V\psi_{k-1}^\alpha(A,\hbar),\psi_0^\alpha(A,\hbar))+O(\exp(-K/\hbar)),$$
$$E_k^\alpha(A,\hbar)=-(Vp_{k-1}^\alpha(A,\hbar),p_0^\alpha(A,\hbar)),$$

whence we get (29) for $k=N$ (here $(.,.)$ denotes the scalar product). Observe now that $\psi_k^\alpha(A,\hbar)$ and $p_k^\alpha(A,\hbar)$ in (27) and (28) are obtained by applying $(H_0^\alpha(\hbar)-E_0^\alpha(A,\hbar))^{-1}$ to vectors whose difference has norm exponentially small as $\hbar\to 0$, namely

$$p_N^\alpha(A,\hbar)=-(H_0^\alpha(\hbar)-E_0^\alpha(A,\hbar))^{-1}(1-P_0^\alpha(\hbar,n))$$
$$((V-E_1^\alpha(A,\hbar))p_{N-1}^\alpha(A,\hbar)-E_2^\alpha(A,\hbar)p_{N-2}^\alpha(A,\hbar)-\ldots$$
$$-E_N^\alpha(A,\hbar)p_0^\alpha(A,\hbar))+O(\exp(-K/\hbar)),$$
$$\psi_N^\alpha(A,\hbar)=-(H_0^\alpha(\hbar)-E_0^\alpha(A,\hbar))^{-1}(1-P_0^\alpha(\hbar,n))$$
$$((V-E_1(A,\hbar))\psi_{N-1}^\alpha(A,\hbar)-S_2(A,\hbar)\psi_{N-2}^\alpha(A,\hbar)-\ldots$$
$$-S_N(A,\hbar)\psi_0^\alpha(A,\hbar)).$$

But clearly the norm of $(H_0^\alpha(\hbar)-E_0^\alpha(A,\hbar))^{-1}$ on the orthogonal complement of $\psi_0^\alpha(A,\hbar)$ is bounded by a polynom in $\hbar^{-1}$, the constant depending on A, α and ω. Hence we obtain (30) for $k=N$ with possibly slightly decreased K, which ends the inductive argument. Observe now that (*a*) and (*b*) of Theorem 1 follow from (*a*) of Lemma 1 and from (29). The proof for the symmetric case is complete.

Note that in the above proof the exponential estimate of Lemma 1 is crucial, for instance a polynomial estimate in Lemma 1 (*b*) would not be sufficient to prove the polynomial estimate in Theorem 1 (*b*). So Lemma 1 is the least we need, while Example 1 above shows that it is also the most we can hope for.

The proof of Theorem 1′ follows immediately from Lemma 1 (*c*) and from the following simple Lemma.

LEMMA 2. — *Let* A_0 *be self-adjoint and* B *symmetric and* A_0*-bounded, put* $A_\varepsilon = A_0 + \varepsilon B$. *Let* E_0 *be an isolated eigenvalue of* A_0 *of finite multiplicity* N. *Suppose there exist approximate eigenvalue* $E_M(\varepsilon) = \sum_{j=0}^{M} \varepsilon^j E_j$ *of* A_ε *and approximate eigenvector* $\psi_M(\varepsilon) = \sum_{j=0}^{M} \varepsilon^j \psi_j$, such that

$$(A_\varepsilon - E_M(\varepsilon))\, \psi_M(\varepsilon) = O(\varepsilon^{M+1})$$

as $\varepsilon \to 0$, *and with* $\psi_1, \ldots, \psi_M$ *orthogonal to* ψ_0. *Then* $E_M(\varepsilon)$ *and* $\psi_M(\varepsilon)$ *are partial sums of the Rayleigh-Schrödinger series for perturbed eigenvalue and eigenvector of* A_ε. *Moreover if there exist* N *approximate eigenvalues* $E_M^{(k)}(\varepsilon) = \sum_{j=0}^{M} \varepsilon^j E_j^{(k)}$ *and eigenvectors* $\psi_M^{(k)}(\varepsilon) = \sum_{j=0}^{M} \varepsilon^j \psi_j^{(k)}$ of A_ε, $k = 1, \ldots, N$, *satisfying the above conditions and with* $\psi_0^{(k)}$ *mutually orthogonal, then these approximate eigenvalues and eigenvectors are partials sums of all* N *Rayleigh-Schrödinger series associated with* A_ε *and* E_0.

Proof of Lemma 2. Suppose M is fixed. Together with A_ε we consider another analytic family of operators $\tilde{A}_\varepsilon$, defined as

$$\tilde{A}_\varepsilon = A_0 + \varepsilon B + \varepsilon^M \sum_{j=1}^{M} \varepsilon^j C_j,$$

where C_j are defined to be zero on the orthogonal complement of ψ_0 and by

$$C_1 \psi_0 = -B \psi_M + \sum_{k=1}^{M} E_k \psi_{M+1-k},$$

$$C_j \psi_0 = \sum_{k=j}^{M} E_k \psi_{M+j-k},$$

for $j = 2, \ldots, M$. Using orthogonality of $\psi_2, \ldots, \psi_M$ to ψ_0 we conclude that

$$(\tilde{A}_\varepsilon - E_M(\varepsilon))\, \psi_M(\varepsilon) = 0,$$

that is $E_M(\varepsilon)$ and $\psi_M(\varepsilon)$ are exact eigenvalues and eigenvectors of $\tilde{A}_\varepsilon$. The result of the lemma now follows when we note that perturbation series for eigenvalues and eigenvectors of A_ε and $\tilde{A}_\varepsilon$ coincide up to order M.

Proof of Theorem 2. Fix any $M \in N$, $A = \hbar n \in \Gamma'(\varepsilon)$. Using the results of Theorem 1 we note that

$$E_\varepsilon(\hbar, n) = \sum_{j=0}^{M} \varepsilon^j E_j(A, \hbar) + O(\varepsilon^{M+1}), \tag{31}$$

where $O(\varepsilon^{M+1})$ is uniform for $\hbar \in (0, 1)$. We also have

$$N(A, \varepsilon) = \sum_{j=0}^{M} \varepsilon^j N_j(A) + O(\varepsilon^{M+1}), \tag{32}$$

independent of course of $\hbar$. Observe now that for $j = 0, 1, \ldots$ we have

$$E_j(A, \hbar) = N_j(A) + \hbar g_j(A, \hbar),$$

where g_j is a C^∞ function obtained as a sum of convergent power series in $\hbar$. Using Theorem 2.6.1 of [9] we conclude that there exists a smooth function $g(A, \hbar, \varepsilon)$ such that

$$g(A, \hbar, \varepsilon) = \sum_{j=0}^{M} \varepsilon^j g_j(A, \hbar) + O(\varepsilon^{M+1}), \tag{33}$$

as $\varepsilon \to 0$, uniformly in $\hbar$, for any M. Now (31), (32) and (33) together imply (9).

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