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On diamagnetism and de Haas-van Alphen effect

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ABSTRACT. — In this work we study the Schrödinger equation with a periodic potential and a constant weak magnetic field. We justify the Peierls substitution without any hypothesis of non overlapping of bands (in the 0 field case), and we apply it to the study of the diamagnetism. Under some additional assumptions we describe asymptotically the singularities of the density of states and the de Haas-van Alphen oscillations.

RÉSUMÉ. — Dans ce travail on étudie l'équation de Schrödinger avec potentiel périodique et champ magnétique faible et constant. On justifie la substitution de Peierls sans hypothèse de non recouvrement des bandes (dans le cas de champ 0), et on l'applique à l'étude du diamagnétisme. Sous quelques hypothèses supplémentaires on décrit asymptotiquement les singularités de la densité d'états et les oscillations de de Haas-van Alphen.

Mots-clés : Schrödinger, magnetic, diamagnetism, de Haas-van Alphen effect.

Classification A.M.S. : 35 S 05, 34 E 20, 35 P 05, 81 C 25, 82 A 99, 81 H 20.

0. INTRODUCTION

The purpose of this article is to clarify the results published since 1930 by the physicists concerning the diamagnetism and the de Haas-van Alphen effect in the weak magnetic field case. We are interested in the study of the spectral properties of the Schrödinger operator with magnetic field:

$$P_{B,V}=\sum_j (D_{x_j}+A_j(x))^2+V(x) \tag{0.1}$$

where V is a C^∞ periodic potential on a lattice Γ, and where $A_j(x)=\frac{1}{2}(\sum_k x_k b_{kj})$, $B=(b_{jk})$ is the antisymmetric matrix defining the constant magnetic field. Our problem will be semi-classical in the sense that we are interested in the asymptotic behavior of different quantities as $\|B\|$ tends to 0. The main object which will be considered in this paper will be the density of states associated to $P_{B,V}$. $P_{B,V}$ is an essentially selfadjoint operator and by the abstract theory it is possible to define for every real valued C_0^∞ function f a selfadjoint operator $f(P_{B,V})$. The restriction of the distribution kernel to the diagonal is a C^∞ periodic function on $\mathbb{R}^n$ and by computing the mean value of this function on a fundamental domain we get some real number denoted by $\tilde{\mathrm{T}}\mathrm{r} f(P_{B,V})$; the density of states is the measure defined by:

$$f\to D_f(B)=\tilde{\mathrm{T}}\mathrm{r} f(P_{B,V})=\rho_{B,V}(f)=\int f(s)\,\rho_{B,V}(ds) \tag{0.2}$$

The different notations correspond to the different points of view taken in the litterature. We can for example study for f fixed, the map $B\to D_f(B)$, or we can also try to give a more explicit formula for the measure $\rho_{B,V}$. Another connected problem is to look to the limit as T (the temperature !) tends to 0 of $(\rho_{B,V}*f_T)(z_0)$ (z_0 is the chemical potential) for a family of functions of the form: $f_T(s)=Tf_1(s/T)$. If the situation is relatively clear in the case of the free electron [*see* the standard books in solid state physics (for example [Ca])], the general case is more obscure and most of the books give some heuristic explanation of the de Haas-van Alphen effect based on the Onsager's rule. One of the reasons for this obscurity (at least from the mathematical point of view) was that the so called Peierls substitution was not rigorously proved until recently. This justification is now obtained after the contributions of [Be 2], [Ne 1, 3] and our paper [He-Sj 4] which will be the starting point of this one. However it was only proved in the case of the single band case and there is a lot to do in the case of overlapping bands.

As we said, this paper is first of all the natural continuation of [He-Sj 4] but one motivation to go further was a recent paper by Guillot-Ralston-Trubowitz [Gu-Ra-Tru] which tries to justify the de Haas-van

Alphen effect by construction of quasi-modes with the help of techniques appearing in the homogeneization theory. With these techniques they can recover the Onsager's rule [On] but this remains in some aspects formal and does not give precise information on the density of states. Moreover these authors are obliged to make technical assumptions on the rationality of the fluxes of the magnetic fields through some basic surfaces related to the lattice (we are here in the 3-dimensional case). The final purpose of this article is to give a precise description of the density of states, but because there are many interesting results connected to this density of states (according to the respective weights of different parameters essentially the chemical potential, the temperature and the magnetic field) we shall rewrite rigorously a lot of other results appearing in the physical litterature. This paper solves completely the case of non crossing Floquet eigenvalues (near the considered Fermi level). We hope to return in a further paper to the case of touching or overlapping bands (case of the bismuth), where also the physical litterature is less clear and where the Onsager's rule is not clearly given.

Let us now more precisely describe the contents of the different sections.

In *section* 1, we consider the free case $(V=0)$ and for fixed f we study the function $B \to D_f(B)$. We show how a recent pseudo-differential calculus (1976) developped in another context by Boutet-Grigis-Helffer [BGH] and a functional calculus inside this class permits to recover (with a proof which is in the same spirit) results given by Peierls in his celebrated paper [Pe] and to prove the C^∞ dependence with respect to B (for B in a neighborhood of 0 in $\Lambda^2(\mathbb{R}^n)$) of $D_f(B)$. We compare with Landau's approach and recall the link with the asymptotic behavior of the free energy per unit volume:

$$(0.3) \qquad \Omega(z_0, B, N, T) = N z_0 - \widetilde{\operatorname{Tr}}\left(T \operatorname{Log}\left(1 + e^{(z_0 - P_{B,0})/T}\right)\right)$$

for fixed T as $\|B\|$ tends to 0.

(Here z_0 is the chemical potential, N is the density of electrons per unit volume). The interesting feature is here to determine the quadratic term of the Taylor expansion with respect to B at $B=0$.

In section 2 we remain in the case $V=0$ and consider the 3D-case where $B_{12}=B>0$, $B_{13}=B_{23}=0$. We shall study the susceptibility in the limit where $T/\|B\|$ and $\|B\|$ are small. The susceptibility is associated to the energy per unit volume by the relation:

$$(0.4) \qquad \chi(z_0, B, N, T) = \left(\frac{1}{B}\frac{d}{dB}\right)(\Omega(z_0, B, N, T)).$$

More precisely we shall consider the limit as $T \to 0$ of this susceptibility. We precise mathematically the proof given by [So-Wi] as presented in Callaway's book [Ca] and give the link with some Riesz means associated to the eigenvalues of the harmonic oscillator. In particular, starting from the formula for the density of states:

$$(0.5)\qquad d\rho_{B,V} = \sum_n (z_0 - ((2n+1)B))_+^{-(1/2)}\, ds,$$

we give the classical formula for the de Haas-Van Alphen effect in the free case and in the limit of 0 temperature with a precise estimate of the remainders. This is in fact just an exercise using the Laplace transform.

Section 3 is a complement to [He-Sj 4]. In the study of the 0-magnetic field but with non trivial periodic electric potential and as a starting point for the study of the spectrum of $P_{B,V}$ in the general case, we have presented in [He-Sj 4] (*cf.* also [Be 2], [Ne 1-3]) a way to justify the Peierls substitution by the introduction of suitable exponentially localized Wannier functions. This was possible under the hypothesis that the band under consideration was simple. We explain here how to define these Wannier functions in the general case where we admit overlapping bands or crossing of the graphs of the Floquet eigenvalues.

Section 4 recalls how to use these Wannier functions (which are less natural than in the simple band case) to construct magnetic Wannier functions which permit to justify the Peierls substitution in the general case. In the place of the reduction to a single pseudodifferential operator whose principal symbol was the Floquet eigenvalue, we find a more general system of pseudo-differential operators which can be in principle studied by techniques used in [He-Sj 2].

Section 5 is devoted to give different formulas for the density of states in the particular case of the odd dimension (in particular $n=3$) under the generic hypothesis that B is of maximal rank. More precisely we consider the family $B(h)=hB_0$ where $h\in]0, h_0]$. We construct a symplectic change of coordinates adapted to the given B_0 permitting to simplify the computation of the density of states.

Section 6 is independent of the preceding ones. This is a parallel section to section 1 in the case where V is not identically 0. We recover old results by [Ad], [Ko], [B 1]... concerning the quadratic term of $D_f(B)$. This gives complementary results to the results given in [He-Sj 4].

Section 7 is the starting section devoted to the study of the de Haas-Van Alphen effect. We suppose in all the remaining sections that the dimension is 3 and that the magnetic field is of the form $B(h)=hB_0$. We give a classification of the nature of the Fermi surface and of their sections by a family of hyperplanes orthogonal to the given magnetic field B_0. We remain here in the case where the Floquet eigenvalue is simple near the chosen energy level (which corresponds to the Fermi level).

Section 8 is devoted to the study of the density of states $\rho_{B,V}$ and we want to determine up to some error which is $O(h^\infty)$ the asymptotic behavior of this measure. Using the Peierls substitution procedure justified in sections 3-5, this computation is related to the computation of explicit quantities associated to a family of $N \times N$ pseudo-differential system depending of the spectral parameter z. In the case where the Floquet eigenvalue is simple, the principal symbol of this $N \times N$ system has an eigenvalue proportional to $(z - \lambda(\theta))$ [where $\lambda(\theta)$ is the Floquet eigenvalue] and the other eigenvalues are non 0.

Section 9 gives the final result which is the natural extension of the formula (0.5) in the case of non zero periodic potential V. Roughly speaking the eigenvalues $(2n+1)$ h of the harmonic oscillator which are the singularities of the density of states are replaced by expansions in power of h given modulo $O(h^2)$ by the eigenvalues of the pseudo-differential operators attached to $\lambda(\theta)$ in the following way. In suitable coordinates adapted to $B(h)$ and constructed in section 5, we can consider the family of p.d.o. $\lambda(x, D_x, t)$ on $L^2(\mathbb{R})$ where $t = t_0$ corresponds to an hyperplane orthogonal to $B(h)$. The rule given by Onsager is the following. Fix some Fermi level z_0. Look at the sections by the hyperplanes $t = \text{Cst}$ of the Fermi surface $\lambda = z_0$. This gives a union of bounded curves. Look at the family of t for which some component of this curve is the exterior boundary of a domain of finite extremal area with respect to t. For each of these values of t_j and for each component α, we get some hamiltonian $\lambda(x, hD_x, t_j)$ defined microlocally whose eigenvalues give approximatively the singularities of the density of states. The precise result is given in theorem 9.1.

Section 10 is devoted to the study of the energy per unit volume, of the susceptibility and of the counting function per unit volume. Starting from the expression for the density of states given in section 9, we can give some analogs of the results obtained in section 2 for the free case. In particular we try to recover in the general case what is usually called the de Haas-Van Alphen effect.

1. DIAMAGNETISM OF THE FREE ELECTRON

1.1. Introduction and results

In this section, we rewrite the 1933 Peierls [Pe] in a modern language. We consider more precisely the section devoted to the diamagnetism of the free electron and we compare it in the particular case of dimension 3 with the more direct approach from Landau [La]. Let us consider the free Hamiltonian with constant magnetic field in $\mathbb{R}^n$:

$$P_B = \sum_j (D_{x_j} + A_j(x))^2 \tag{1.1.1}$$

with the same notation as in [He-Sj 4]:

$$A_j(x) = (1/2) \sum_k b_{kj} x_k \tag{1.1.2}$$

P_B is an essentially selfadjoint operator and for a real function f in $\mathscr{S}(\mathbb{R})$ (or more generally if f is the restriction to $\bar{\mathbb{R}}^+$ of a function in $S(\mathbb{R})$) $f(P_B)$ is well defined by the spectral theorem and we can express $f(P_B)$ as in [He-Sj 4] by the following formula:

$$f(P_B) = (1/2 i \pi) \int (\partial \tilde{f}/\partial \bar{z}) (P_B - z)^{-1} d\bar{z} \wedge dz \tag{1.1.3}$$

where $\tilde{f}$ is an extension of f such that:

$(1.1.4)_a$ $\quad \tilde{f}(z) = f(z) \quad$ for $\quad z \in \mathbb{R}$

$(1.1.4)_b$ $\quad \operatorname{supp}(\tilde{f}) \subset \wedge \quad$ where $\quad \wedge := \{ z \in \mathbb{C}, |\operatorname{Im} z| < 1 \}$

$(1.1.4)_c$ $\quad \tilde{f} \in \mathscr{S}(\wedge)$

$(1.1.4)_d$ The family of functions $x \to (\partial \tilde{f}/\partial \bar{z})(x+iy) . |y|^{-N}$ (with y verifying $0 < |y| < 1$) is bounded in $\mathscr{S}(\mathbb{R})$, $\forall\, N \in \mathbb{N}$.

It is easy to prove that such an extension always exists for a function in $\mathscr{S}(\mathbb{R})$. Then as in [He-Sj 4] we can define the density of states associated to P_B by the following formula:

$$f \to \widetilde{\operatorname{tr}} f(P_B) = \lim_{|L| \to \infty} \frac{\operatorname{Tr}(\chi_L f(P_B))}{(2L)^n} \tag{1.1.5}$$

where χ_L is the characteristic function of the cube Q_L in $\mathbb{R}^n$.

Let us consider now the (pseudo-) differential operator:

$$(l_B^w)_j(x, D) = D_{x_j} + A_j(x) \tag{1.1.6}$$

whose Weyl symbol is the linear form on $\mathbb{R}^{2n}$:

$$(x, \xi) \to l_{B,j}(x, \xi) = \xi_j + A_j(x) \tag{1.1.7}$$

Then, for a in $\mathscr{S}(\mathbb{R}^n)$ [or more generally in $\mathscr{S}'(\mathbb{R}^n)$], we can introduce as in [BGH] the following pseudo-differential operator:

$$(1.1.8)\qquad a^w(l_B(x, D)) = (2\pi)^{-n}.\int_{\mathbb{R}^n} e^{i\tau . l_B^w(x, D)}\,\hat{a}(\tau)\,d\tau$$

If a is in $S^k(\mathbb{R}^n)$ ($k \in \mathbb{R}$), where S^k is defined by:

$$(1.1.9)\quad S^k(\mathbb{R}^n) := \{a \in C^\infty(\mathbb{R}^n), |D_\tau^\alpha a(\tau)| \leqq C_\alpha(1+|\tau|)^{k-|\alpha|}, \forall \alpha \in \mathbb{N}^n\},$$

the associated pseudodifferential operator is continuous from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, and, if $k \leqq 0$, this operator is continuous from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. In particular, $P_B = a^w(l_B(x, D))$ with $a(\tau) = \tau_1^2 + \ldots + \tau_n^2$. More generally, we want to consider the constant magnetic field B as a parameter ($B = b_{jk}$, with $j < k$, B belonging to some open set Ω in $\mathbb{R}^{(n(n-1)/2)}$).

In this case, we shall work with symbols which are C^∞ with respect to B, for example:

$$(1.1.10)\quad \begin{aligned} S^k(\Omega, \mathbb{R}^n) := \{ & a \in C^\infty(\mathbb{R}^n \times \Omega), \forall \alpha \in \mathbb{N}^n, \forall \beta \in \mathbb{N}^{n(n-1)/2}, \\ & \forall K \subset\subset \Omega, \exists C_{\alpha\beta K}\ \text{s.t.}: \\ & |D_\tau^\alpha D_B^\beta a(\tau, B)| \leqq C_{\alpha\beta K}(1+|\tau|)^{k-|\alpha|}, \forall B \in K, \forall \tau \in \mathbb{R}^n\} \end{aligned}$$

A family of operators A_B ($B \in \Omega$) in OPS^k is called uniformly elliptic if we have the following estimate for the symbol $a(B, \tau)$:

$$(1.1.11)\quad \forall K \subset\subset \Omega,\quad \exists C_K > 0, \text{s.t.}$$
$$|a(\tau, B)| \geqq C_K(1+|\tau|)^k, \qquad \forall \tau \in \mathbb{R}^n, \quad \forall B \in \Omega.$$

Let us recall the following theorem in [BGH]:

THEOREM 1.1.1. – *Let* $A_B = a^w(l_B(x, D), B)$ *be a family of uniformly elliptic operators in* $OPS^k(\Omega, \mathbb{R}^n)$, $k > 0$. *Let us assume that, for each* B, A_B *is invertible in the following sense: there exists* $c > 0$ s.t.: $\forall \varphi \in \mathscr{S}(\mathbb{R}^n)$, $\|A_B \varphi\| \geqq c\|\varphi\|$, $\|A_B^* \varphi\| \geqq c\|\varphi\|$, *then* A_B^{-1} *is a pseudo-differential operator* $b^w(l_B(x, D), B)$ *whose symbol is in* $S^{-k}(\mathbb{R}^n \times \Omega)$.

This theorem applies for example, if we consider the family of operators: $P_B - z$ with $\operatorname{Im} z \neq 0$. The symbol is simply: $a(\tau) - z$ with $a(\tau) = \tau_1^2 + \ldots + \tau_n^2$, which is clearly elliptic. However theorem 1.1.1 is not sufficient for our purpose. We shall prove the following theorem:

THEOREM 1.1.2. – *Let* A_B *a family of formally selfadjoint operators* (i.e. *with real symbol a*) *satisfying the hypotheses of theorem* 1.1.1. *Then* A_B *is essentially self-adjoint and if* $f \in \mathscr{S}(\mathbb{R})$, $f(A_B) = b_f(l_B(x, D), B)$, *with:* $b_f(\tau, B) \in S^{-\infty}(\mathbb{R}^n \times \Omega)$. *Moreover, if* $0 \in \Omega$, $b_f(\tau, 0) = f(a(\tau, 0))$ *and it is possible to compute the other terms of the Taylor expansion of* a_f *with respect to* B.

Remark 1.1.3. – This theorem is a variant of results of [He-Ro 1, 2], in the context of the classes of Boutet de Monvel-Grigis-Helffer [BGH].

As in this last paper, the difficulty is that the rank of the matrix (b_{jk}) is not constant when B varies. The theorem will be applied in the case where $A_B = P_B$.

COROLLARY 1.1.4. – *Under the hypotheses of theorem* 1.1.2, *we have:*

$$D_f(B) = \widetilde{\operatorname{tr}} f(P_B) = (2\pi)^{-n} \int b_f(\tau, B)\, d\tau \tag{1.1.13}$$

$D_f(B)$ *is* C^∞ *with respect to* B *in* Ω *and we have, if* $a(\tau, B) = a(\tau)$:

$$D_f(0) = (2\pi)^{-n} \int f(a(\tau))\, d\tau \tag{1.1.14}$$

$$D_f(B) = D_f(0) - (1/48) \sum_{k, k', l, l'} \int f''(a(\tau)) \frac{\partial^2 a}{\partial \tau_k \partial \tau_{k'}} \times B_{kl} B_{k'l'} \frac{\partial^2 a}{\partial \tau_l \partial \tau_{l'}} d\tau + O(B^3). \tag{1.1.15}$$

Application 1.1.5. – If $a(\tau) = \tau_1^2 + \ldots + \tau_n^2$, (1.1.14) et (1.1.15) become:

$$D_f(0) = (2\pi)^{-n} \int f(\tau_1^2 + \ldots + \tau_n^2)\, d\tau \tag{1.1.16}$$

$$D_f(B) = D_f(0) - (1/6) \| B \|^2 . (2\pi)^{-n} . \int f''(\tau^2)\, d\tau + O(B^3) \tag{1.1.17}$$

where $\| B \|^2 = \sum_{j<k} (b_{jk})^2$ which is the expected result (*cf.* [Pe], formula (42)).

1.2. The 3-dimensional case. Landau's approach

Peierls makes this computation in dimension 3 without to analyze the C^∞ dependence with respect to B and his estimation of the remainder term is not rigorous. He takes only h as a parameter: $B_{12} = h$, $B_{13} = 0$, $B_{23} = 0$ and formula (1.1.17) becomes:

$$D_f(B) = D_f(0) - (1/6) h^2 . (2\pi)^{-3} . \int f''(\tau^2)\, d\tau + O(h^3). \tag{1.2.1}$$

Let us observe that:

$$\int_{\mathbb{R}^3} f''(\tau^2)\, d\tau = 2\pi \int_{-\infty}^{\infty} \left(\int_0^\infty f''(\tau_3^2 + r^2)\, r dr \right) d\tau_3 = -\pi \int_{-\infty}^{\infty} f'(\sigma^2)\, d\sigma \tag{1.2.2}$$

At this time, the notion of density of state was not clearly defined but the nature of the "proof" of Peierls is pseudo-differential. In fact, R. Peierls refers for this formula to Landau [La]. The proof of Landau is simpler but uses the explicit knowledge of the spectrum of P_B. Let us sketch is briefly:

Here we have:

$$P_B=(D_{x_1}-(h/2)x_2)^2+(D_{x_2}+(h/2)x_1)^2+D_{x_3}^2 \tag{1.2.3}$$

Using corollary 1.1.4 and the invariance of the pseudo-differential calculus by the metaplectic group, we get from the explicit knowledge of the spectrum of the harmonic oscillator:

$$D_f(B)=|h|.(2\pi)^{-2}.\sum_n\int_{\mathbb{R}} f((2n+1)|h|+\sigma^2)\,d\sigma \tag{1.2.4}$$

which can be written:

$$D_f(B)=|h|.\sum_n g((2n+1)|h|) \tag{1.2.5}$$

with:

$$g(x)=(2\pi)^{-2}.\int f(x+\sigma^2)\,d\sigma \tag{1.2.6}$$

Then we can see (1.2.5) as: $|h|\operatorname{Tr} g(Q_0(h))$ where $Q_0(h)$ is the harmonic oscillator $Q_0(h)=h^2D_x^2+x^2$ in one variable. The trace is taken in the usual sense for operators of trace class. Indeed, if f is in $\mathscr{S}(\mathbb{R})$, g is in $\mathscr{S}(\overline{\mathbb{R}}^+)$ and $g(Q_0)$ has its distribution kernel in $\mathscr{S}(\mathbb{R}^2)$ (*see* for example [He]). The study of the asymptotic behavior of $D_f(B)$ as h tends to 0 results from the general theory developped in [He-Ro] (*see* for example the monograph [Ro]) but there is a more direct approach in Landau's paper [La] by applying a first order version of the well known formula of Euler-Maclaurin (*see* for example [Di], p. 302) (we thank A. Grigis for explaining us the use of this formula in this context). The formula given in Landau's paper [La] was the following approximate relation:

$$\sum_a^b k\left(x+\frac{1}{2}\right)=\int_a^b k(x)\,dx-\frac{1}{24}|k'(x)|_b^a \quad (\text{if } (k(x+1)-k(x))/k(x)\ll 1)$$

The formula of Euler-Maclaurin is the following one:

$$k(m)+k(m+1)+\ldots+k(n)=\int_m^n k(t)\,dt+\frac{1}{2}(k(m)+k(n))$$
$$+\sum_{j=1}^{r}(-1)^{j-1}(B_j/(2j)!)(k^{(2j-1)}(n)-k^{(2j-1)}(m))+R_r \tag{1.2.7}$$

where the B_j are the wellknown Bernouilli coefficients (for example $B_1 = 1/6$, $B_2 = 1/30$, $B_j = (2(2j)!/(2\pi)^j).\left(\sum_{n=1}^{\infty}(1/n^{2j})\right)$ and R_r satisfies the following estimate:

$$(1.2.8) \qquad |R_r| \leqq (2/(2\pi)^{2r}) \int_m^n |k^{(2r+1)}(t)|\,dt,$$

$m < n$ integers, k of class C^{2r+1} in $[m, n]$.
Here, as we shall see later, we have an explicit control of the remainder term and we can justify completely Landau's proof. (Under natural analyticity conditions on f, we can also get that $D_f(B)$ is an analytic symbol in the sense of [Sj]).

Landau's approach

We assume that $k \in \mathscr{S}(\bar{\mathbb{R}}^+)$ and we apply (1.2.7) with $m = 0$, $n = \infty$. Then we get:

$$(1.2.9) \qquad \sum_{p=0}^{\infty} k(p) = \int_0^{\infty} k(t)\,dt + (1/2)\,k(0) + \sum_{j=1}^{r}(-1)^j (B_j/(2j)!)\,k^{(2j-1)}(0) + R_r$$

with: $|R_r| \leqq (2/(2\pi)^{2r}) \int_0^{\infty} |k^{(2r+1)}(t)|\,dt.$

We take: $k(t) = g((2t+1)h)$ (assuming $h > 0$) and we get:

$$\sum_{p=0}^{\infty} hg((2p+1)h) = h \int_0^{\infty} g((2t+1)h)\,dt$$
$$(1/2)\,h.g(h) + \sum_{j=1}^{r}(-1)^j (1/2).(B_j/(2j)!).(2h)^{2j}.g^{(2j-1)}(h) + \tilde{R}_r$$

with:

$$|\tilde{R}_r(h)| \leqq (2/(2\pi)^{2r})(2h)^{2r+2} \int_0^{\infty} |g^{(2r+1)}((2t+1)h)|\,dt$$

After some change of variables, we have:

$$(1.2.10) \qquad \sum_{p=0}^{\infty} hg((2p+1)h) = (1/2) \int_0^{\infty} g(t+h)\,dt$$
$$+ (1/2)\,h.g(h) + \sum_{j=1}^{r}(-1)^j (1/2).(B_j/(2j)!).(2h)^{2j}.g^{(2j-1)}(h) + \tilde{R}_r$$

with:

$$(1.2.11) \qquad |\tilde{R}_r(h)| \leqq (2/(2\pi)^{2r})(2h)^{2r+1} \int_0^\infty |g^{(2r+1)}(t)|\,dt$$

We can write more explicitly the first terms in the right hand side of (1.2.10):

$$(1.2.12) \quad \sum_{p=0}^{\infty} hg((2p+1)h) = (1/2)\int_0^\infty g(t)\,dt + (1/12)g'(0)h^2 + O(h^3)$$

$$\text{with } g(x) = (2\pi)^{-2}.\int f(x+\sigma^2)\,d\sigma$$

which is easy to compare with (1.2.1) according to (1.2.2).

In the applications, the function f which appears in the computation of $\tilde{\mathrm{T}}\mathrm{r} f(\mathrm{P_B})$ is given by:

$$(1.2.13) \qquad f_{z_0, \mathrm{T}}(x) = \mathrm{T}\,\mathrm{Log}\,(1 + e^{((z_0 - x)/\mathrm{T})}).$$

The free energy per unit volume is then defined by:

$$(1.2.14) \qquad \Omega(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T}) = \mathrm{N} z_0 - \tilde{\mathrm{T}}\mathrm{r} f_{z_0, \mathrm{T}}(\mathrm{P_B})$$

We use this function only for $x \geqq 0$ and it is clear that $f_{z_0, \mathrm{T}} \in \mathscr{S}(\bar{\mathbb{R}}^+)$ (we can always modify the function in $\mathbb{R}^-$ to get a function in $\mathscr{S}(\mathbb{R})$. z_0 is called the chemical potential and T is the temperature.

Remark 1.2.1. – As $\mathrm{T} \to 0$, the function $f_{z_0, \mathrm{T}}$ tends to $(z_0 - x)\,1_{x < z_0}$ uniformly on $\mathbb{R}$. As a consequence, $f'_{z_0, \mathrm{T}}$ tends in the distributional sense to $-1_{x < z_0}$ (the same is true for $-\partial f_{z_0, \mathrm{T}}/\partial z_0$) and $f''_{z_0, \mathrm{T}}$ tends to δ_{z_0}. The density of particles is given by:

$$(1.2.15) \quad \mathscr{N}(z_0, \mathrm{B}, \mathrm{T}) = \tilde{\mathrm{T}}\mathrm{r}\,((\partial f_{z_0, \mathrm{T}}/\partial z_0)(\mathrm{A_B})) = -\tilde{\mathrm{T}}\mathrm{r}\,(f'_{z_0, \mathrm{T}})(\mathrm{A_B})).$$

Let us observe the following properties (related to the remark 1.2.1):

(1.2.16) $\mathscr{N}(z_0, 0, \mathrm{T})$ tends as $\mathrm{T} \to 0$ to $(2\pi)^{-3}\left(\int_{a < z_0} d\tau\right)$

(1.2.17) $(\partial\mathscr{N}/\partial z_0)\,(z_0, \mathrm{T}, 0)$ tends as $\mathrm{T} \to 0$ to: $(2\pi)^{-3}\left(\int_{a = z_0} d\tau/da\right)$ if z_0 is not a critical value of a. In particular, if $a(\tau) = \tau^2$, we get:

$$(1.2.18) \qquad \begin{cases} \mathscr{N}(z_0, 0, 0) = c_n z_0^{(n/2)}, \\ (\partial\mathscr{N}/\partial z_0)(z_0, 0, 0) = c_n.(n/2).z_0^{((n-2)/2)} \end{cases}$$

Here $c_n = (2\pi)^{-n}\,\mathrm{Vol}(\mathrm{B}_n)$ where B_n is the unit ball in $\mathbb{R}^n$. In fact, as it is well known and clearly explained for example in [Ad], z_0 depends on B and is determined by the fact that the density of particles N:

$$\mathrm{N}=\mathcal{N}(z_0,\mathrm{B},\mathrm{T}) \tag{1.2.19}$$

which corresponds to:

$$\partial\Omega/\partial z_0=0 \tag{1.2.20}$$

If the temperature T is small enough, then if z_{00} corresponds to a solution of (1.2.19) with $\mathrm{B}=0$, the solution $z_0(\mathrm{B})$ of (1.2.20) [verifying $(\partial\mathcal{N}/\partial z_0)(z_0,\mathrm{T},\mathrm{B})_{/z_0=z_{00},\,\mathrm{B}=0}\neq 0$, because it is true at the limit for $\mathrm{T}\to 0$, if $z_{00}>0$] is C^∞ and has the expansion:

$$z_0=z_{00}+O(\mathrm{B}^2) \tag{1.2.21}$$

which follows from $(\partial z_0/\partial\mathrm{B})(0)=0$ [consequence of $\partial\mathcal{N}(z_0,\mathrm{B})/\partial\mathrm{B}=0$ for $\mathrm{B}=0$] (*see* §2 for a complementary discussion). Now the susceptibility corresponds to the second order term in B of:

$\Omega(z_0(\mathrm{B},\mathrm{N},\mathrm{T}),\,\mathrm{B},\mathrm{N},\mathrm{T})$ and we just observe that:

$$\Omega(z_0(\mathrm{B}))-\Omega(z_{00})=O(\mathrm{B}^4)$$

due to (1.2.20) and (1.2.21). This justifies the computation of the susceptibility assuming that z_0 is independent of B (see also an interesting discussion in [Me]).

1.3. Proof of theorems

Proof of Corollary 1.1.4. – According to theorem 1.1.2, the distribution kernel of $f(\mathrm{A_B})$ is given by the following integral:

$$\mathrm{K}(x,y)=(2\pi)^{-n}\int e^{i\langle x-y,\,\xi\rangle}\,b_f(l((x+y)/2,\xi),\mathrm{B})\,d\xi$$

and is a C^∞ function with respect to x, y, B. We observe that

$$\mathrm{K}(x,x)=(2\pi)^{-n}\int b_f(\xi,\mathrm{B})\,d\xi.$$

According to the (classical) definition recalled for example in [He-Sj 4], §7 (*see* (1.1.5)), we get finally:

$$\widetilde{\mathrm{T}}\mathrm{r}\,f(\mathrm{A_B})=(2\pi)^{-n}\int b_f(\tau,\mathrm{B})\,d\tau.$$

Q.E.D.

Proof of theorem 1.1.2. – Let us start from formula (1.1.3) which can be written also:

$$f(\mathrm{A_B})=(1/2\,i\pi)\lim_{\varepsilon\to 0}\int_{|\operatorname{Im} z|\geqq\varepsilon}(\partial\tilde{f}/\partial\bar{z})(\mathrm{A_B}-z)^{-1}\,d\bar{z}\wedge dz \tag{1.3.1}$$

This formula was proved in [He-Sj 4] and the left hand side of (1.3.1) is well defined as an operator in $L^2(\mathbb{R}^n)$ because of $1.1.4_d$ and:

(1.3.2) $$\|(A_B-z)^{-1}\|_{L^2\to L^2}\leqq 1/|\operatorname{Im} z|.$$

To see that this operator is a pseudo-differential operator requires a more careful study of $(A_B-z)^{-1}$ as a pseudo-differential operator (for $\operatorname{Im} z\neq 0$). Let $q(\tau, B, z)$ be the symbol of $(A_B-z)^{-1}$. From theorem 1.1.1, we know that this symbol is in $S^{-k}(\Omega\times(\wedge\setminus\mathbb{R}), \mathbb{R}^n)$ [in the sense of 1.1.10 with Ω replaced by $(\Omega\times(\wedge\setminus\mathbb{R}))$]. Indeed, the hypotheses of this theorem are clearly satisfied because of the essential selfadjointness of P_B. But this result is not sufficiently precise because we have no control with respect to z as $\operatorname{Im} z$ tends to 0 or as $|z|$ tends to ∞. To get this control, we follow the technique used in this context (*see* [Ro], [He-Ro],... for different variants).

Let us introduce the class:

$$\begin{aligned}
&\tilde{S}^l(\Omega\times(\wedge\setminus\mathbb{R}), \mathbb{R}^n)\\
&\quad:=\{a\in C^\infty(\mathbb{R}^n\times\Omega\times(\wedge\setminus\mathbb{R})), \text{ holomorphic with respect to } z, \text{ s.t.}\\
&\qquad\forall\alpha\in\mathbb{N}^n,\ \forall\beta\in\mathbb{N}^{n(n-1)/2},\ \forall p\in\mathbb{N},\ \forall K\subset\subset\Omega,\ \exists C_{\alpha\beta pK},\ \exists N_{\alpha\beta pK}
\end{aligned}$$

s. t.

$$|D_\tau^\alpha D_B^\beta D_z^p a(\tau, B, z)|\leqq C_{\alpha\beta pK}(1+|\tau|)^{l-|\alpha|}((1+|z|)/|\operatorname{Im} z|)N_{\alpha\beta pK},$$
$$\forall B\in K,\ \forall z\in\wedge\setminus\mathbb{R},\ \forall\tau\in\mathbb{R}^n)\}$$

Law of composition

It is not too difficult to see that this class is stable by composition (*see* [BGH]). Let us recall now the definition of the composition of two p.d.o.;

$a^w(l_B(x, D), B)\ b^w(l_B(x, D), B)$ is a p.d.o. whose symbol c is in $S^{k+k'}$ (resp. $\tilde{S}^{k+k'}$) if a is in S^k (resp. $\tilde{S}^k$) and b is in S^k (resp. $\tilde{S}^{k'}$). We denote by $\sharp_B$ the corresponding law of the symbols:

(1.3.3) $$c=a\,\sharp_B\, b=b\,\sharp_{-B}\, a$$

with:

$$\hat{c}(t)=(2\pi)^{-n}\int e^{(i/2)\langle B, y\wedge t\rangle}\hat{a}(y)\hat{b}(t-y)\,dy$$

where $\hat{a}$ (resp. $\hat{b}$) is the Fourier transform of a (resp. b).

Let us also note the following decomposition in formula (1.3.3):

(1.3.4) $$c=\sum_{j=1}^{N}c_j+r_N$$

with:

$$(1.3.5)\qquad \hat{c}_j(t)=(1/j!)(2\pi)^{-n}\int (i/2)^j \langle B, y\wedge t\rangle^j \hat{a}(y)\hat{b}(t-y)\,dy$$

$$(1.3.6)\qquad \hat{r}_N=(2\pi)^{-n}\int \rho_N((i/2)\langle B, y\wedge t\rangle)\hat{a}(y)\hat{b}(t-y)\,dy$$

with:

$$(1.3.7)\qquad \rho_N(s)=\exp(is)-\sum_{j=1}^{N}(is)^j/j!$$

which satisfies:

$$|\rho_N^{(p)}(s)|\leqq |s|^{|N+1-p|_+}$$

We can rewrite the formula for c_j in the following way:

$$(1.3.8)\qquad c_j(\tau)=(1/j!)(1/2\,i)^j((\Sigma\, b_{kl}\,\partial_{\tau_k}\partial_{\sigma_l})^j(a(\tau)\,b(\sigma)))_{/\tau=\sigma}$$

These expansions (which appear also in [BGH]) can be used in two different ways. Let us first observe that: r_N belongs to $S^{k+k'-(2N+2)}$ (resp. $\tilde{S}^{k+k'-(2N+2)}$). This means that the remainder term decreases more rapidly with respect to τ when N increases. On the other hand, these expansions can be used to get asymptotics in powers of B, for B in a neighborhood of 0.

Finally let us mention the following formula on the differentiability with respect to B of the composition law (*see* [BGH] (4.3)); taking the b_{jk} $(j<k)$ as coordinates, we have:

$$(1.3.9)\qquad D_{b_{jk}}(a\,\sharp_B\, b)=(D_{b_{jk}}a)\,\sharp_B\, b+a\,\sharp_B\,(D_{b_{jk}}b)$$
$$\frac{1}{2}(D_{\tau_j}a\,\sharp_B\, D_{\tau_k}b-D_{\tau_k}a\,\sharp_B\, D_{\tau_j}b)$$

This formula appears also naturally in the context of families of C*-algebras (*see* [Be 1, 2]).

We shall prove that $q(\tau, B, z)\in\tilde{S}^{-k}$ $(k>0)$ and more precisely:

LEMMA 1.3.1. — *Under the hypotheses of theorem* 1.1.2 *there exists symbols* $b_j(\tau, B)$ *in* $S^{(k-1)j}$ *s. t. for each* N, *there exists* M(N) *s. t.*:

$$(1.3.10)_N\qquad q(\tau, B, z)-\sum_{j=0}^{M(N)}(a(\tau, B)-z)^{-j-1}\,b_j(\tau, B)\in\tilde{S}^{-N}.$$

Proof of the lemma. — The proof of the lemma follows narrowly the classical proof (*see* for example § 4 in [Ro 1]). First of all, we construct a left parametrix q_N in $\tilde{S}^{-k}$ s. t.:

$$(1.3.11)_N\qquad q_N\,\sharp_B\,(a-z)=1+s_N\quad\text{with } s_N \text{ in } \tilde{S}^{-N}$$

where $q_N = \sum_{j=0}^{M(N)} (a(\tau, B) - z)^{-j-1} d_j(\tau, B)$ (with d_j in $S^{(k-1)j}$).

This is proved using (1.3.4)-(1.3.8) starting from $q_1 = (a-z)^{-1}$ and inverting $(1+s_1)$. The proof that q_1 is in $\tilde{S}^{-k}$ follows from the ellipticity of a and of different estimates corresponding to $|\tau| \geqq C|z|^{(1/k)}$ and $|\tau| \leqq \tilde{C}|z|^{(1/k)}$. We omit some details. In the same way (using the self-adjointness) we construct a right parametrix $\tilde{q}_N$ in $\tilde{S}^{-k}$ s. t.:

$$(1.3.12)_N \qquad (a-z) \#_B \tilde{q}_N = 1 + \tilde{s}_N \quad \text{with } \tilde{s}_N \text{ in } \tilde{S}^{-N}.$$

Then we have for each N the following formula for q:

$$(1.3.13)_N \qquad q = q_N - s_N \#_B \tilde{q}_N + s_N \#_B q \#_B \tilde{s}_N$$

The two first terms of the r.h.s. have the good form; so we replace the problem of the control of q in some class of symbols by the control of $s_N \#_B q \#_B \tilde{s}_N$ where we have the possibility to choose the N arbitrarily.

Let us recall that the problem here is not a problem of existence of the symbol q, neither a problem of C^∞ dependence with respect to the parameters, but a problem of control of the symbol as $|z|$ tends to ∞ or as $|\operatorname{Im} z|$ tends to 0.

For this, we rewrite the formula of composition of the symbols a and b in the following way:

$$(1.3.14) \qquad \hat{c} = (Op(a)\hat{b})$$

Indeed, we have:

$$\begin{aligned}
\operatorname{Op}(a)u(x) &= (2\pi)^{-n} \int e^{i\langle x-y \mid \xi\rangle} a(\xi + A((x+y)/2))\, u(y)\, dy\, d\xi \\
&= (2\pi)^{-n} \int e^{i\langle x-y \mid \xi - A((x+y)/2)\rangle} a(\xi)\, u(y)\, dy\, d\xi \\
&= (2\pi)^{-n} \int e^{-i\langle x-y \mid A((x+y)/2)\rangle} \hat{a}(y-x)\, u(y)\, dy \\
&= (2\pi)^{-n} \int e^{-i\langle x \mid A(y)\rangle} \hat{a}(y-x)\, u(y)\, dy \\
&= (2\pi)^{-n} \int e^{(i/2)\langle B \mid x \wedge y\rangle} \hat{a}(y-x)\, u(y)\, dy.
\end{aligned}$$

We apply (1.3.14) with $b = \tilde{s}_N$ and $a = q$. We then have:

$$(1.3.15)_N \qquad (q \#_B \tilde{s}_N)\hat{}\,(\tau) = ((P_B - z)^{-1}(\hat{\tilde{s}}_N))(\tau)$$

From (1.3.15) and (1.3.2), we get:

$$(1.3.16)_N \qquad \| q \#_B \tilde{s}_N \|_{L^2(\mathbb{R}^n)} = O_N(((1+|z|)/|\operatorname{Im} z|)^{\tilde{N}_0})$$

for some $\tilde{N}_0$ depending on N. In the same way, we get the same estimates for $q \#_B \tilde{s}_N$ in the Sobolev spaces H^s where s can be chosen arbitrarily large if N is large enough. This is just a consequence of (1.3.2), (1.3.15) and the following estimate in weighted L^2 spaces:

$$(1.3.17) \qquad \|u\|_{L^{2,M}} \leqq C((1+|z|)/|\operatorname{Im} z|)^{M+1} \|(A_B - z)u\|_{L^{2,M}}$$

where, for

$$M \in \mathbb{N},\ L^{2,M} = \{u \in L^2,\ x^\alpha u \in L^2\ \forall \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leqq M\}$$

(this is proved by induction starting from (1.3.2) and:

$$\|u\|_{H_B^k} \leqq C(\|(A_B - z)u\|_{L^2} + (|z|+1)\|u\|_{L^2}),$$

where H_B^k is the magnetic Sobolev space introduced in [He-Sj 4] (*see* also section 4) and where C is locally uniform with respect to B.

We can also consider the derivative with respect to B, but this will be made later. We now look at the composed symbol:
$s_N \#_B p_N$ where p_N satisfies for some $t(N)$ tending to ∞ with N and for some $k(N)$:

$$\|p_N\|_{H^{t(N)}} \leqq C_{N,K}((1+|z|)/|\operatorname{Im} z|))^{k(N)}$$

in $K \times \wedge \diagdown \mathbb{R}$ (for each $K \subset\subset \Omega$).

We use now formula (1.3.3) (in the form $p_N \#_{-B} s_N$) to estimate $s_N \#_B p_N$ in the spaces B^l (in fact the Fourier transform of $s_N \#_B p_N$), where, for $l \in \mathbb{N}$, B^l is defined by:

$$B^l(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n),\ x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n),\ \forall \alpha, \beta \text{ s.t. } |\alpha|+|\beta| \leqq l\}.$$

We get finally the following estimate on $s_N \#_B q \#_B \tilde{s}_N$. For each $l \in \mathbb{N}$, there exists N_l, s.t. $\forall N \geqq N_l$, we have, for some $C_{l,K}$ large enough:

$$(1.3.18)_N \quad \|s_N \#_B q \#_B \tilde{s}_N\|_{B^l} \leqq C_l(1+(|z|/|\operatorname{Im} z|))^{C_l} \quad \text{in } K \times \wedge \diagdown \mathbb{R}$$

(for each $K \subset\subset \Omega$).

Using $(1.3.11)_N$, $(1.3.12)_N$, $(1.3.13)_N$, and $(1.3.18)_N$, we get the part of (1.3.10) which corresponds to no derivatives of the symbols with respect to B and z. To finish the proof, we observe again that we can play with the N in formula $(1.3.13)_N$ and that we have quite precise controls of the two first terms of the r.h.s. of this formula.

Let us consider the identity:

$$(1.3.19) \qquad q \#_B (a - z) = 1.$$

Deriving with respect to b_{jk}, we get first from (1.3.9):

$$D_{b_{jk}} q \#_B (a-z) + q \#_B D_{b_{jk}} a + \frac{1}{2}(D_{\tau_j} q \#_B D_{\tau_k} a - D_{\tau_k} q \#_B D_{\tau_j} a) = 0$$

which gives the following formula for $D_{b_{jk}} q$:

$$(1.3.20) \quad D_{b_{jk}} q = -q \#_B D_{b_{jk}} a \#_B q - \frac{1}{2}(D_{\tau_j} q \#_B D_{\tau_k} a - D_{\tau_k} q \#_B D_{\tau_j} a) \#_B q$$

In the same way, we get for $D_z q$:

$$D_z q = q \#_B q \tag{1.3.21}$$

(1.3.20) permit to obtain the good estimates for the first derivatives of q with respect to B or z in the suitable classes of symbols and we can then continue by induction by deriving the formulae (1.3.20) and (1.3.21) with respect to B or z. This finishes the proof of the lemma. □

End of the proof of the theorem

We combine simply formulae (1.3.1) and (1.3.10). The estimates for the different symbols permit, taking account of (1.1.4), to integrate term by term. For each N, we get that modulo some remainder $\mathscr{R}_N$ in S^{-N}, $f(A_B)$ is determined by the following:

$$\begin{aligned}\sigma(f(A_B))(\tau) &= (1/2i\pi)\sum_{j=0}^{M(N)} d_j(\tau, B)\int(\partial\tilde{f}/\partial\bar{z})(a(\tau, B)-z)^{-j-1}\,dz\wedge d\bar{z}+\mathscr{R}_N\\ &= \sum_{j=0}^{M(N)} \tilde{d}_j(\tau, B) f^{(j)}(a(\tau, B))+\mathscr{R}_N.\end{aligned}$$

It is clear that, for each j, $d_j(\tau, B)\, f^{(j)}(a(\tau, B))$ is in $S^{-\infty}$. Playing with N, we get the theorem. □

Explicit computations of $D_f(B)$ *near* $B=0$

Let us first observe that, if:

(1.3.22) $\quad a(\tau, B)$ is real and satisfies: $a(\tau, B)=a(-\tau, -B)$,

we have the following relation:

$$A_B.\Gamma = \Gamma A_{-B} \qquad (\text{with } \Gamma u = \bar{u}) \tag{1.3.23}$$

which implies:

$$D_f(B) = D_f(-B) \tag{1.3.24}$$

Indeed it follows from (1.3.23) that the distribution kernel of $f(A_B)$ satisfies:

$$K_{f,B}(x, y) = \overline{K_{f,-B}(x, y)}$$

and by the selfadjointness of $f(A_B)$, we have:

$$K_{f,B}(y, x) = \overline{K_{f,B}(x, y)}$$

We then get: $K_{f,B}(x, x) = \overline{K_{f,-B}(x, x)} = K_{f,-B}(x, x)$

Let us now compute the first terms in the expansion of $D_f(B)$. The first term is given by taking $B=0$ in the formula (1.3.1). We have:

$$\sigma(f(P_0))(\tau)=(1/2\,i\pi)\lim_{\varepsilon\to 0}\int_{|\operatorname{Im} z|\geqq\varepsilon}(\partial\tilde{f}/\partial\bar{z})(a(\tau,0)-z)^{-1}\,d\bar{z}\wedge dz$$
$$=f(a(\tau,0)).$$

According to (1.3.23), the linear term with respect to B is 0. The quadratic term can be computed explicitly by taking the second derivative with respect to B in formula (1.3.1), by using (1.3.9), (1.3.20) and taking the trace at $B=0$. Another way is to make formal expansions with respect to B taking account of the fact that the law $\#_B$ admits formal expansion with respect to B [*see* formulas (1.3.3)-(1.3.8)]. The result is given in (1.1.15).

2. DE HAAS-VAN ALPHEN EFFECT FOR THE FREE ELECTRON

The study of the de Haas-van Alphen effect for the free electron corresponds to the study of the free energy per unit volume $\Omega(z_0, B, N, T)$ in the limit as $T\to 0$. This means that, in opposite to the preceding section where T was fixed (eventually small) and where we looked to the limit $B\to 0$, we shall here first take the limit as $T\to 0$ and consider afterwards the behavior for B small (this corresponds to the understanding of the situation as (T/B) and B are small). This study is always mentioned in every standard book in solid state physics and is (not far from rigorously) presented in Callaway's book [Ca] who refers to [So-Wi] and [Wi] (*see* also [Me]). We give here a slightly different presentation and refer to [Ca] for some complement. This can be considered as an introduction to the more complicated case which is studied in the last sections. We concentrate in all this section on the case of the dimension 3 and we keep the notations of section 1. The de Haas-van Alphen effect is more apparent in the study of the susceptibility:

$$\chi(z_0, B, N, T)=\frac{1}{B}\left(\frac{d}{dB}(\Omega(z_0(B,T), B, N, T))\right) \tag{2.1}$$

Here $z_0(B,T)$ is the chemical potential which was determined in paragraph 1 by 1.2.20. For $T>0$ fixed, and in the limit as B tends to 0, this corresponds to the study of the coefficient of B^2 in the computation of the free energy per unit volume (this study was made in paragraph 1). As in section 1, we can consider that z_0 is independent of B. Indeed, the computation of (2.1) can, according to: $\partial\Omega/\partial z_0=0$ for $z_0=z_0(B,T)$, be replaced by:

$$(2.1)' \qquad \chi(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T}) = (\partial\Omega/\partial \mathrm{B})/\mathrm{B} \qquad \text{for} \quad z_0 = z_0(\mathrm{B}, \mathrm{T}).$$

At least to start with, we shall consider that $z_0 = \mathrm{Const.}$ [*see* in (2.17) the computation of $z_0(\mathrm{B}, \mathrm{N}, \mathrm{T})$].

We are interested in the study of:

$$(2.2) \qquad \chi(z_0, \mathrm{B}, \mathrm{N}, 0) = \lim_{\mathrm{T} \to 0} \chi(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T})$$

We shall prove the existence of the limit, but a more precise study is possible (expansion with respect to (T/B) along the lines suggested in [Ca]).

Let us start again from the expression defining the density of states:

$$(2.3) \qquad f \to \mathrm{D}_f(\mathrm{B}) = \tilde{\mathrm{T}}\mathrm{r} f(\mathrm{P_B}) = (2\pi)^{-2}\, \mathrm{B} \sum_n \int f((2n+1)\mathrm{B} + \xi_3^2)\, d\xi_3.$$

This defines a measure ρ_B which can be defined by a density in L^1:

$$(2.4) \qquad d\rho_\mathrm{B} = ((2\pi)^{-2}\, \mathrm{B} (\sum_n (s - (2n+1)\mathrm{B})_+^{-(1/2)}))\, ds$$

Let us observe that the support of ρ_B is in $\mathbb{R}^+$ and that if we take:

$$(2.5) \qquad f_\mathrm{T}(s) = \mathrm{T}\,\mathrm{Log}(1 + e^{-s/\mathrm{T}})$$

we have:

$$(2.6) \qquad f_\mathrm{T}(s) = \mathrm{T} f_1(s/\mathrm{T})$$

and

$$(2.7) \qquad \Omega(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T}) = \mathrm{N} z_0 - (\rho_\mathrm{B} * \check{f}_\mathrm{T})(z_0)$$

where $\check{f}_\mathrm{T}(s) = f_\mathrm{T}(-s)$.

The computation of the free energy per unit volume corresponds to some convolution of the density of states μ_B by $\check{f}_\mathrm{T}$ at the point z_0. As observed in section 1, the second derivative of f_T tends as $\mathrm{T} \to 0$ in the distributional sense to the Dirac measure at 0. Let us compute in a different way $\Omega(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T})$; after two integrations by parts we get:

$$(2.8) \qquad \Omega(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T}) = \mathrm{N} z_0 - (2\pi)^{-2}(4/3)\, \mathrm{B} \left(\sum_n \int f_\mathrm{T}''(s - z_0)(s - (2n+1)\mathrm{B})_+^{(3/2)}\, ds \right)$$

Now the susceptibility is given by:

$$(2.9) \qquad \chi(z_0, \mathrm{B}, \mathrm{N}, \mathrm{T}) = - (2\pi)^{-2}(4/3)\, \mathrm{B}^{-1} \left(\sum_n \int f_\mathrm{T}''(s - z_0)(s - (2n+1)\mathrm{B})_+^{(3/2)}\, ds \right) + (2\pi)^{-2}(2) \left(\sum_n \int f_\mathrm{T}''(s - z_0)(2n+1)(s - (2n+1)\mathrm{B})_+^{(1/2)}\, ds \right).$$

Let us look to the limit as T tends to 0. According to the properties of f''_T we get:

$$(2.10)\quad \Omega(z_0, B, N, 0) = N z_0 - (2\pi)^{-2}(4/3)\, B\,(\sum_n (z_0-(2n+1)B)_+^{(3/2)})$$

and

$$(2.11)\quad \chi(z_0, B, N, 0) = -(2\pi)^{-2}(4/3)\, B^{-1}(\sum_n (z_0-(2n+1)B)_+^{(3/2)}) + (2\pi)^{-2}(2)\,B^{-1}(\sum_n ((2n+1)B)(z_0-(2n+1)B)_+^{(1/2)})$$

Let us now introduce the auxiliary function:

$$(2.12)\qquad r_\gamma(h) = \sum_n (1-(2n+1)h)_+^\gamma.$$

This is a well known object related to the Riesz means associated to the eigenvalues of the harmonic oscillator: $h^2 D_x^2 + x^2$ in the semiclassical limit. The following results are not sufficient for our purpose but we think that they will be enlighting. We get from the general semi-classical study of these Riesz means the following results (*see* [He-Ro 3]):

$$(2.13)\qquad r_\gamma(h) = h^{-1}(\sum_{j\leq|\gamma+1|} c_j(\gamma)\,h^j + h^{\gamma+1}\rho(\gamma, h))$$

where $[\gamma+1]$ is the largest integer smaller than $\gamma+1$ (for $\gamma\notin\mathbb{N}$), where c_j can be explicitly computed and $\rho(\gamma, h)$ is an "oscillatory" term which remains bounded for h small.

In particular we have:

$$c_0(\gamma) = (2\pi)^{-1}\left(\int_{\xi^2+x^2\leq 1} (1-(\xi^2+x^2))^\gamma\, dx\, d\xi\right) = (1/(2(\gamma+1)))$$

This corresponds to the Weyl's term.

$$c_1(\gamma) = 0$$
$$c_2(\gamma) = -(\gamma/12).$$

Let us now rewrite (2.10) and (2.11); using (2.12) we get:

$$(2.14)\qquad \Omega(z_0, B, N, 0) = N z_0 - (2\pi)^{-2}(4/3)\, z_0^{(3/2)}\, B\, r_{(3/2)}(B/z_0)$$

and

$$(2.15)\quad \chi(z_0, B, N, 0) = -(2\pi)^{-2}(10/3)\, B^{-1} z_0^{(3/2)}\, r_{(3/2)}(B/z_0) + (2\pi)^{-2}(2 z_0^{(3/2)} B^{-1})\, r_{(1/2)}(B/z_0).$$

For (B/z_0) small, let us use the information given by (2.13):

$$(2.16)\quad \Omega(z_0, B, N, 0) = N z_0 - (2\pi)^{-2}(4/3) \times z_0^{(5/2)}((1/5) + c_2(3/2)(B/z_0)^2 + (B/z_0)^{(5/2)}\rho(3/2, B/z_0)).$$

The diamagnetic term is quite apparent and given by:

$$(2\pi)^{-2}(4/3)\,z_0^{(5/2)}\,c_2(3/2)\,(B/z_0)^2 = -(2\pi)^{-2}(1/6)\,B^2 z_0^{1/2}.$$

The oscillatory term in (2.16) seems to be smaller but we have to be careful because we have to compute in fact $\Omega(z_0(B), B, N, 0)$. Let us determine $z_0(B)$. We first start from the equation:

$$\partial\Omega/\partial z_0 = N - (2\pi)^{-2}(2B)\,z_0^{(1/2)}\,r_{(1/2)}(B/z_0) = 0$$

which permits to determine $z_0(B)$ by the equation:

$$N = (2\pi)^{-2}(2/3)\,z_0^{(3/2)}\,(1 + (B/z_0)^{(3/2)}\,.\,3\,.\,\rho(1/2, B/z_0)).$$

This is possible because the right hand side is strictly monotone and continuous with respect to z_0, as can be seen by derivation of (2.10). Let us remark here that for $T=0$, the function $B \to z_0(B)$ is probably not in C^1 (the chemical potential is in this case equal to the Fermi level).

This gives:

$$(2.17)\qquad z_0(B) = ((2\pi)^2(3/2)\,N)^{(2/3)} + O(B^{(3/2)}) = z_0(0) + O(B^{(3/2)}).$$

It is not difficult so see according to (2.17) and the relation $\partial\Omega/\partial z_0 = 0$ that:

$$(2.18)\qquad \Omega(z_0(B), B, N, 0) = N\,z_0(0)$$
$$-(2\pi)^{-2}(4/3)\,(z_0(0))^{(5/2)}\,((1/5) + c_2(3/2)\,(B/z_0(0))^2 + O((B)^{(5/2)}).$$

This justifies the classical approximation that we can assume that z_0 is constant in the weak magnetic field approximation.

Let us consider now the susceptibility. The use of (2.13) gives:

$$(2.19)\qquad \chi(z_0, B, N, 0) = 2(2\pi)^{-2}\,(z_0\,B^{-(1/2)})\,(\rho(1/2,\ B/z_0) + O((B)^{(1/2)}).$$

The oscillatory term appears to be the dominant term and this is what is called the de Haas-van Alphen effect. It remains to compute more explicitly $r_{1/2}(h)$ which is outside the general techniques we mentioned before but can be performed along the lines of Callaway's book [Ca]. It remains also to verify that $\chi(z_0(B), B, N, 0)$ has the same behavior as B tends to 0 as $\chi(z_0(0), B, N, 0)$. Let us recall briefly how $r_{(1/2)}(h)$ can be precisely estimated.

Poisson formula for r_γ:

Let us consider the following expression:

$$(2.20)\qquad f_\gamma(s) = \sum_n (s - (2n+1))_+^\gamma$$

and recall that:

$$(2.21)\qquad r_\gamma(h) = h^\gamma f_\gamma(1/h).$$

Let us start from the following formula valid for $\gamma>0$:

$$(2.22)\quad \sigma_+^\gamma=(1/(2i\pi))\Gamma(\gamma+1)\int_{c-i\infty}^{c+i\infty} e^{t\sigma}t^{-\gamma-1}\,dt \qquad \text{where} \quad c>0.$$

Taking $\sigma=(s-(2n+1))$ and summing with respect to n, we get easily:

$$(2.23)\qquad f_\gamma(s)=(1/2i\pi).(\Gamma(\gamma+1)/2)\int_{c-i\infty}^{c+i\infty} e^{st}(\operatorname{sh} t)^{-1}t^{-\gamma-1}\,dt$$

It is the Cauchy integral of the meromorphic function in $\mathbb{C}\setminus\mathbb{R}^-$:

$$t\to(\Gamma(\gamma+1)/2)e^{st}(\operatorname{sh} t)^{-1}t^{-\gamma-1}$$

along the path: $\mathbb{R}\ni u\to c+iu$.
Applying the residues theorem, we get the following formula for $f_\gamma(s)$:

$$(2.24)\quad f_\gamma(s)=\sum_{n>0}\Gamma(\gamma+1)(-1)^n(\pi n)^{-\gamma-1}\cos\left(n\pi s-\frac{\pi}{2}(\gamma+1)\right)$$
$$+(\Gamma(\gamma+1)/2).\frac{1}{2i\pi}.\int_{\mathscr{C}_1} e^{st}(\operatorname{sh} t)^{-1}t^{-\gamma-1}\,dt$$

where $\mathscr{C}_1$ is some unbounded contour around the real negative axis inside a small neighborhood of this real axis.

In the following, we are interested in computing the asymptotic, as $s\to\infty$, of the second term in the r.h.s. of (2.24): $\mathrm{I}(s,\gamma)$.

Let us now specify the contour we shall consider. For some ε to be chosen later, we consider the contour $\mathscr{C}_1(\varepsilon)$ which is the union $\mathscr{C}'(\varepsilon)\cup\mathscr{C}''(\varepsilon)\cup\mathscr{C}'''(\varepsilon)$ with

$$\mathscr{C}'(\varepsilon)=]-\infty-i0,-\varepsilon-i0],\qquad \mathscr{C}''(\varepsilon)=\varepsilon e^{i\theta}(\theta\in]-\pi,\pi[)$$

and

$$\mathscr{C}'''(\varepsilon)=[-\varepsilon+i0,-\infty+i0[.$$

$\mathrm{I}(s,\gamma)$ is of course independent of ε and we get:

$$(2.25)\quad \mathrm{I}(s,\gamma)=O(\varepsilon^{-\gamma-1})e^{-\varepsilon s}$$
$$+(\Gamma(\gamma+1)/2).\frac{1}{2i\pi}.\int_{\mathscr{C}''(\varepsilon)} e^{st}(\operatorname{sh} t)^{-1}t^{-\gamma-1}\,dt$$

We keep free for the moment the choice of ε, but small enough to have for each M the following expansion of $(t/\operatorname{sh}(t))$:

$$(2.26)\qquad (t/\operatorname{sh} t)=1+\sum_{j=1}^{\mathrm{M}}\gamma_j t^{2j}+O_{\mathrm{M}}(t^{2\mathrm{M}+2})$$

For each M and each ε we get the following decomposition for $\mathrm{I}(s,\gamma)$:

$$\begin{aligned}\mathrm{I}(s,\gamma) &= O(\varepsilon^{-\gamma+1})\,e^{-\varepsilon s}+e^{\varepsilon s}\,O(\varepsilon^{2\mathrm{M}+1-\gamma})\\ &\quad+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j\left(\frac{1}{2i\pi}\,.\int_{\mathscr{C}''(\varepsilon)}e^{st}\,t^{2j-\gamma-2}\,dt\right)\\ &= O(\varepsilon^{-\gamma+1})\,e^{-\varepsilon s}+e^{\varepsilon s}\,O(\varepsilon^{2\mathrm{M}+1-\gamma})+\sum_{j=0}^{2\mathrm{M}}O(\varepsilon^{-\gamma-1})\,.\,O(s^{-2j})\,.\,e^{-(\varepsilon s)/2}\\ &\quad+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j\left(\frac{1}{2i\pi}\,.\int_{\mathscr{C}(\varepsilon)}e^{st}\,t^{2j-\gamma-2}\,dt\right)\\ &= O_{\mathrm{M}}(\varepsilon^{-\gamma+1})\,e^{-(\varepsilon s/2)}+e^{\varepsilon s}\,O_{\mathrm{M}}(\varepsilon^{2\mathrm{M}+1-\gamma})\\ &\quad+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j\left(\frac{1}{2i\pi}\,.\int_{\mathscr{C}(\varepsilon)}e^{st}\,t^{2j-\gamma-2}\,dt\right)\end{aligned}$$

for all $\varepsilon<1$ and all $s>1$.

The integral $\left(\frac{1}{2i\pi}\,.\int_{\mathscr{C}(\varepsilon)}e^{t}\,t^{2j-\gamma-2}\,dt\right)$ is known (*cf.* [Di]) to be equal to $(1/\Gamma(\gamma+2-2j))$, so we have finally $\forall\varepsilon<1$, $\forall s>1$, $\forall\mathrm{M}$:

$$\begin{aligned}\mathrm{I}(s,\gamma) &= O_{\mathrm{M}}(\varepsilon^{-\gamma+1})\,e^{-(\varepsilon s/2)}+e^{\varepsilon s}\,O_{\mathrm{M}}(\varepsilon^{2\mathrm{M}+1-\gamma})\\ &\quad+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j\,.\,(1/\Gamma(\gamma+2-2j))\,.\,s^{\gamma+1-2j}\end{aligned}$$

We determine now $\varepsilon(s)$ by the relation:

$$e^{-(3\varepsilon s/2)}=\varepsilon^{2\mathrm{M}}. \tag{2.27}$$

which is compatible to the preceding conditions for s large enough and we get:

$$\forall\mathrm{M},\forall s>\mathrm{C}_0$$

$$\begin{aligned}\mathrm{I}(s,\gamma) &= O_{\mathrm{M}}(\varepsilon(s)^{((2\mathrm{M}/3)+1-\gamma)}\\ &\quad+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j(1/\Gamma(\gamma+2-2j))\,.\,s^{\gamma+1-2j}\end{aligned}$$

Let us finally observe from (2.27) that we have:

$$\mathrm{I}(s,\gamma)=O_{\mathrm{M}}(s^{-(2\mathrm{M}/3)})+(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{2\mathrm{M}}\gamma_j(1/\Gamma(\gamma+2-2j))\,.\,s^{\gamma+1-2j}$$

We have proved that $\mathrm{I}(s,\gamma)$ admits an asymptotic expansion in powers of $(1/s)$ as $s\to\infty$:

$$\mathrm{I}(s,\gamma)\sim(\Gamma(\gamma+1)/2)\,.\sum_{j=0}^{\infty}\gamma_j(1/\Gamma(\gamma+2-2j))\,.\,s^{\gamma+1-2j}. \tag{2.28}$$

Summing up, we get the following lemma for the asymptotic for the Riesz means of the harmonic oscillator:

LEMMA 2.1:

$$(2.29)\quad r_\gamma(h)-h^\gamma \sum_{n>0} \Gamma(\gamma+1)(-1)^n(\pi n)^{-\gamma-1}\cos(n\pi(1/h)-\frac{\pi}{2}(\gamma+1))$$
$$+(\Gamma(\gamma+1)/2).\sum_{j=0}^{\infty}\gamma_j(1/\Gamma(\gamma+2-2j)).h^{2j-1}.$$

This is of course much more precise than the formula given in (2.13). As a consequence, we recover the result on the de Haas-van Alphen effect for the free electron:

PROPOSITION 2.2 [So-Wi]. — *In the* $z_0=$Const. *approximation, the susceptibility for* $T=0$, *admits as* B *tends to* 0 *the following expansion with respect to* B:

$$(2.30)\quad \chi(z_0,B,N,0)=2(2\pi)^{-2}(z_0 B^{-(1/2)})\left(\sum_{n>0}\Gamma\left(\frac{3}{2}\right)(-1)^n\right.$$
$$\left.\times(\pi n)^{-(3/2)}\cos\left(n\pi(z_0/B)-\frac{3\pi}{4}\right)+O(B)^{(1/2)}\right)$$

Justification of the approximation $z_0(B)=z_0(0)$ *for the computation of the susceptibility*

The only problem is to compute $(\Delta S)(B)=S(z_0(B))-S(z_0(0))$ with:

$$S(z_0)=\sum_{n>0}\Gamma\left(\frac{3}{2}\right)(-1)^n(\pi n)^{-(3/2)}\cos\left(n\pi(z_0/B)-\frac{3\pi}{4}\right)$$

A brutal Taylor expansion at $z_0(0)$ does'nt give the result because $S(z_0)$ is not apparently of class C^1. To estimate $(\Delta S)(B)$, we decompose $S(z_0)$ in two parts by writing:

$$S(z_0)=\sum_{n=0}^{M}\Gamma\left(\frac{3}{2}\right)(-1)^n(\pi n)^{-(3/2)}\cos\left(n\pi(z_0/B)-\frac{3\pi}{4}\right)$$
$$+\sum_{n=M+1}^{\infty}\Gamma\left(\frac{3}{2}\right)(-1)^n(\pi n)^{-(3/2)}\cos\left(n\pi(z_0/B)-\frac{3\pi}{4}\right)$$
$$=S'_M(z_0)+S''_M(z_0)$$

We observe now that we have the following estimates:

$|S''_M(z_0)|\leqq CM^{-(1/2)}$ where C is independent of M and uniform with respect to z_0.

On the other hand, using the Taylor expansion to order 1 for the finite sum S'_M, we get:

$$|S'_M(z_0(B)) - S'_M(z_0(0))| \leqq C.B^{-1}.M^{(1/2)}.|z_0(B) - z_0(0)|$$

with C independent of M and B in a small neighborhood of 0.

We then get that for each M we have the following estimate on $(\Delta S)(B)$:

$$|(\Delta S)(B)| \leqq C(M^{-(1/2)} + M^{(1/2)}(B)^{(1/2)})$$

Taking $B^{-(1/2)} \leqq M \leqq B^{-(1/2)} + 1$, we obtain:

$|\Delta S(B)| \leqq CB^{(1/4)}$ for some constant C independent of B small.

So we have proved the:

PROPOSITION 2.3. – *The susceptibility for* $T=0$, *admits as* B *tends to* 0 *the following expansion with respect to* B:

$$\begin{aligned}(2.31)\quad & \chi(z_0(B), B, N, 0) \\ & = 2(2\pi)^{-2}(z_0(0)B^{-(1/2)})\left(\sum_{n>0}\Gamma\left(\frac{3}{2}\right)(-1)^n(\pi n)^{-(3/2)}\right. \\ & \qquad \left.\times\cos\left(n\pi(z_0(0)/B) - \frac{3\pi}{4}\right)\right) + O((B)^{(1/4)})\end{aligned}$$

As $T \to 0$, the comparison with $T=0$ is explained in Callaway's book [Ca]. No new phenomena appear.

3. THE 0-MAGNETIC FIELD CASE

We now start to consider the case of a non vanishing potential $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$, assumed to be Γ periodic: $V(x+\gamma) = V(x)$, for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$, where Γ is a lattice of the form $\bigoplus_1^n \mathbb{Z} e_j$, for some basis, $e_1, \ldots, e_n$ of $\mathbb{R}^n$. In this section we make some preliminary work in the case when the magnetic field is zero, in other words, we shall study the operator, $P_{0,V} = \sum D_{x_j}^2 + V(x)$. By Floquet theory, $P_{0,V}$ is unitarily equivalent to the direct integral $\int_{\mathbb{R}^n/\Gamma^*}^{\oplus} P_\theta\, d\theta$, where P_θ is the natural selfadjoint realization of $P_{0,V}$ in the space $\mathscr{H}_\theta = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n); u(x+\gamma) = e^{i\gamma\theta}u(x), x \in \mathbb{R}^n, \gamma \in \Gamma\}$. Here $\mathbb{R}^n$ is identified with its own dual, and $\Gamma^* = \{\gamma^*; \gamma\gamma^* \in 2\pi\mathbb{Z}\}$ is the dual lattice. Let $\mathscr{H}_\theta^2 = \mathscr{H}_\theta \cap H^2_{\mathrm{loc}}(\mathbb{R}^n)$, where H^s denotes the standard Sobolev spaces.

We fix an energy level $z_0 \in \mathbb{R}$. The main goal of this section is to show:

THEOREM 3.1. – *There exist* $N \in \mathbb{N}$, *and analytic functions* $\varphi_j^{\pm} : \mathbb{R}^n/\Gamma^* \to \mathcal{H}_\theta$, $1 \leqq j \leqq N$, *such that for every* $\theta \in \mathbb{R}^n/\Gamma^*$ *the Grushin problem,*

$$(3.1) \qquad (P_\theta - z_0) u + R_\theta^- u^- = v, R_\theta^+ u = v^+,$$

has a unique solution $(u, u^-) \in \mathcal{H}_\theta^2 \times \mathbb{C}^N$ *for every* $(v, v^+) \in \mathcal{H}_\theta \times \mathbb{C}^N$. *Here we have put* $R_\theta^+ u(j) = (u \mid \varphi_j^+(\theta))$, $R_\theta^- u^- = \sum_{1 \leqq j \leqq N} u^-(j) \varphi_j^-(\theta)$.

We notice that a necessary condition for (3.1) to be well posed, is that $(\varphi_1^+, \ldots, \varphi_N^+)$ and $(\varphi_1^-, \ldots, \varphi_N^-)$ are linearly independent systems. If these two systems are linearly independent for some fixed θ, then the well posedness of (3.1), only depends on the two vector spaces $\mathcal{I}_+(\theta)$ and $\mathcal{I}_-(\theta)$ generated by the two systems. More explicitly, we have,

PROPOSITION 3.2. – *Fix* θ *and assume that* $(\varphi_1^+, \ldots, \varphi_N^+)$ *and* $(\varphi_1^-, \ldots, \varphi_N^-)$ *are both linearly independent. Then the problem* (3.1) *is well posed if and only if* $(1 - \pi_-)(P_\theta - z_0)$ *as an operator from* $\mathcal{I}_+^\perp \cap \mathcal{H}_\theta^2$ *to* $\mathcal{I}_-^\perp$ *is bijective. Here* π_- *denotes the orthogonal projection onto* $\mathcal{I}_-$.

Proof. – Since θ is fixed we shall suppress the subscript "θ". Although not absolutely necessary, let us first verify that $\mathcal{H}^2 \cap \mathcal{I}_+^\perp$ is a dense subspace of $\mathcal{I}_+^\perp$: Without loss of generality, we may assume that $\varphi_1^+, \ldots, \varphi_N^+$ is an orthonormal system. Let $\tilde{\varphi}_1^+, \ldots, \tilde{\varphi}_N^+ \in \mathcal{H}^2$ have the property that $(\varphi_j^+ \mid \tilde{\varphi}_k^+) = \delta_{j,k}$. (This is easily achieved by approximating the φ_j^+ by $\mathcal{H}^2$-functions ψ_j, and then working in the space generated by the functions ψ_j.) If $u \in \mathcal{I}_+^\perp$ and $\varepsilon > 0$ we first take $v \in \mathcal{H}^2$ with $\| v - u \| \leqq \varepsilon$. Then we replace v by $w = v - \sum_1^N (v \mid \varphi_j^+) \tilde{\varphi}_j^+$. We then have $w \in \mathcal{H}^2 \cap \mathcal{I}_+^\perp$, and $\| w - u \| \leqq (N+1) \varepsilon$.

In the case $v = 0$, $v^+ = 0$, the problem (3.1) takes the form $(P - z_0) u + R^- u^- = 0$, $u \in \mathcal{I}_+^\perp \cap \mathcal{H}^2$. Taking the $\mathcal{I}_-^\perp$ component of this equation gives $(1 - \pi_-)(P - z_0) u = 0$, so the injectivity of the restriction of $(1 - \pi_-)(P - z_0)$ to $\mathcal{I}_+^\perp \cap \mathcal{H}^2$ implies that we have uniqueness for the problem (3.1). If the restriction of $(1 - \pi_-)(P - z_0)$ to $\mathcal{I}_+^\perp \cap \mathcal{H}^2$ is not injective, then there is $u \in \mathcal{I}_+^\perp \cap \mathcal{H}^2$ such that $(P - z_0) u \in \mathcal{I}_- =$ range of R^-, and we see that there is a u^- such that (3.1) holds with $v = 0$, $v^+ = 0$. We have then established the equivalence between uniqueness for (3.1) and injectivity of the restriction of $(1 - \pi_-)(P - z_0)$ to $\mathcal{I}_+^\perp \cap \mathcal{H}^2$.

In order to discuss solvability of (3.1), we notice that since R^+ is surjective, solvability of (3.1) is equivalent to solvability of the first equation of (3.1), with $u \in \mathcal{I}_+^\perp \cap \mathcal{H}^2$. Since the image of R^- is $\mathcal{I}_-$, we see that solvability of (3.1) is equivalent to the surjectivity of the restriction of $(1-\pi_-)$ $(P-z_0)$ to $\mathscr{I}_+^\perp \cap \mathscr{H}^2$. This completes the proof of the proposition. □

It is enough to prove the statement of Theorem 3.1 with $\varphi_j^\pm$ depending continuously on θ. In fact, by the closed graph theorem, the inverse of

$$\mathscr{P}_\theta(z_0)=\begin{pmatrix} P_\theta-z_0 & R_\theta^- \\ R_\theta^+ & 0 \end{pmatrix} \tag{3.2}$$

will then be uniformly bounded, and by the same regularization argument as in section 1 of [He Sj 4], part 2, we can approximate $\varphi_j^\pm(\theta)$ by functions which are analytic in θ.

In our proof of the theorem, we shall take $\varphi_j^+=\varphi_j^-$ so that $R_\theta^-=(R_\theta^+)^*$ for every real θ. Then (3.1) is a self adjoint problem, and the corresponding inverse operator will be self adjoint. The reason why we do not entirely restrict the attention to such self adjoint problems is that it is not excluded that by the use of non-selfadjoint problems one can choose N smaller than for the smaller class of self adjoint ones. (We shall give a result in this direction below.)

PROPOSITION 3.3. — *Let* P *be a second order elliptic self adjoint operator on a compact manifold* M. *Let* $\varphi_1, \ldots, \varphi_N \in L^2(M)$ *be linearly independent functions and assume that there is a constant* $C_0>0$ *such that,*

$$(Pu\,|\,u)\geqq C_0^{-1}\|u\|^2, \qquad u\in H^1(M)\cap[\varphi_1, \ldots, \varphi_N]^\perp, \tag{3.3}$$

where $[\varphi_1, \ldots, \varphi_N]$ *denotes the linear span of the functions* $\varphi_1, \ldots, \varphi_N$, *and* $H^s(M)$ (*for* $s\in\mathbb{R}$) *is the classical Sobolev space on* M *of order* s. *Then, if* $\tilde{P}$ *is another second order self adjoint operator and* $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N\in L^2(M)$, *with* $\|\tilde{P}-P\|_{\mathscr{L}(H^1, H^{-1})}$ *and* $\|\tilde{\varphi}_1-\varphi_1\|, \ldots, \|\tilde{\varphi}_N-\varphi_N\|$ *small enough, there exists a constant* $C_1>0$, *such that,*

$$(\tilde{P}u\,|\,u)\geqq C_1^{-1}\|u\|_1^2-\sum_1^N|(u\,|\,\tilde{\varphi}_j)|^2, \tag{3.4}$$

for all $u\in H^1(M)$.

Proof. — Without loss of generality, we may assume that $\varphi_1, \ldots, \varphi_N$ is an orthonormal system. Choose $\psi_1, \ldots, \psi_N\in H^2$ with $(\psi_j\,|\,\varphi_k)=\delta_{j,k}$. For $u\in\mathscr{H}^1$, we put $u'=u-\sum_1^N(u\,|\,\varphi_j)\,\psi_j\in\mathscr{H}^1\cap[\varphi_1, \ldots, \varphi_N]^\perp$, so we can apply (3.3) to $u':(Pu'\,|\,u')\geqq C_0^{-1}\|u'\|^2$.

Now,

$$(Pu'\,|\,u')\leqq(Pu\,|\,u)+C\|u\|\sum_1^N|(u\,|\,\varphi_j)|+C\sum_1^N|(u\,|\,\varphi_j)|^2,$$

$$\|u\|^2\leqq 2\|u'\|^2+C\sum_1^N|(u\,|\,\varphi_j)|^2,$$

so with a new constant $C>0$, we get,

$$(3.5)\qquad (Pu\,|\,u)\geqq C^{-1}\|u\|^2-C\sum_1^N|(u\,|\,\varphi_j)|^2.$$

On the other hand, we have Garding's inequality,

$$(Pu\,|\,u)\geqq C^{-1}\|u\|_1^2-C\|u\|^2,$$

so after taking a suitable mean value of the two inequalities, we get:

$$(3.6)\qquad (Pu\,|\,u)\geqq C^{-1}\|u\|_1^2-C\sum_1^N|(u\,|\,\varphi_j)|^2,$$

with a new constant C. From this we get (3.4) if $\|\tilde{P}-P\|_{\mathscr{L}(H^1,H^{-1})}$ and $\|\tilde{\varphi}_j-\varphi_j\|$ are sufficiently small. □

For a fixed $\theta_0\in\mathbb{R}^n/\Gamma^*$, we can find $\varphi_1^0,\ldots,\varphi_N^0\in\mathscr{H}_{\theta_0}$, and a constant $C_0>0$ such that,

$$(3.7)\qquad ((P_{\theta_0}-z_0)u\,|\,u)\geqq C_0^{-1}\|u\|^2,\ \text{for all } u \text{ in } \mathscr{H}_{\theta_0}^1\cap[\varphi_1^0,\ldots,\varphi_N^0]^\perp.$$

Let $E\subset\mathbb{R}^n$ be a fundamental domain of Γ. Modifying φ_j^0 by terms with small norm [which will not destroy (3.7)], we may assume that $\operatorname{supp}(\varphi_j^0)\cap\partial E=\emptyset$, so that $\varphi_j^0=\sum_\gamma\Phi_j^0(x-\gamma)e^{i\theta_0\gamma}$, with

$$\Phi_j^0\in L^2(E)\cap\mathscr{E}'(\operatorname{int}(E)).$$

We then put $\varphi_j(x,\theta)=(U\Phi_j^0)(x,\theta)=\sum_\gamma\Phi_j^0(x-\gamma)e^{i\theta\gamma}$. Proposition 3.3 then shows that for θ sufficiently close to θ_0 we have with a new constant $C_0>0$:

$$(3.8)\qquad ((P_\theta-z_0)u\,|\,u)\geqq C_0^{-1}\|u\|^2\quad \text{for all } u \text{ in } \mathscr{H}_\theta^1\cap[\varphi_1(.,\theta),\ldots,\varphi_N(.,\theta)]^\perp.$$

Notice that $\varphi_1(.,\theta),\ldots,\varphi_N(.,\theta)$ are Γ^* periodic in θ and linearly independent for every θ. Clearly, if we add more functions to our system $\varphi_1,\ldots,\varphi_N$, then (3.8) remains valid for θ in the same neighborhood of θ_0 and with the same constant C_0. Varying the point θ_0, and using the compactness of $\mathbb{R}^n/\Gamma^*$, we obtain with a new N a system of functions $\varphi_j(x,\theta)=U(\Phi_j)(x,\theta)$, $\Phi_j\in C_0^\infty(\operatorname{int}(E))$, such that (3.8) holds for all $\theta\in\mathbb{R}^n/\Gamma^*$, with a new constant $C_0>0$ which is independent of θ. From this construction it does not necessarily follow that $\varphi_1,\ldots,\varphi_N$ will become linearly independent for every θ. Without changing (3.8) we may however eliminate successively all the Φ_j's which are linear combinations of the others (and make the corresponding elimination of φ_j). We then obtain (3.8) with $\varphi_1(.,\theta),\ldots,\varphi_N(.,\theta)$ independent for every θ.

(Incidentally, φ_j depend analytically on θ.) It is an easy exercise to show from (3.8) that $(1-\pi_-(\theta))(P_\theta-z_0)$: $\mathscr{H}_\theta^2\cap\mathscr{I}(\theta)^\perp\to\mathscr{I}(\theta)^\perp$ is bijective, and this completes the proof of Theorem 3.1 (with $R_\theta^- = (R_\theta^+)^*$) in view of Proposition 3.2. □

Remark 3.4. — Under the assumptions of Theorem 3.1, for z close to z_0, let

$$\mathscr{E}(z, \theta) = \begin{pmatrix} E(z, \theta) & E_+(z, \theta) \\ E_-(z, \theta) & E_{-+}(z, \theta) \end{pmatrix}$$

denote the inverse of

$$\mathscr{P}(z, \theta) = \begin{pmatrix} P_\theta - z & R_-(\theta) \\ R_+(\theta) & 0 \end{pmatrix}.$$

Using that $R^\pm$ are independent of z, we get

$$\partial_z E_{-+}(z, \theta) = E_-(z, \theta) E_+(z, \theta).$$

In the proof of Theorem 3.1, we constructed $R^\pm$ with $(R^-)^* = R^+$. In that case, $\partial_z E_{-+} = E_+^* E_+$. Since $R^+ E_+ = I$, E_+ is of full rank and we conclude in this case that $\partial_z E_{-+} > 0$ in the sense of self adjoint operators.

We now turn to the problem of finding the smallest possible N in Theorem 3.1. We have already seen that the family $\varphi_1^+(., \theta), \dots, \varphi_N^+(., \theta)$ must be linearly independent for every $\theta \in \mathbb{R}^n/\Gamma^*$, and have the property that $\mathrm{Ker}(P_\theta - z_0) \cap \mathscr{I}_+(\theta)^\perp = \{0\}$, whenever θ belongs to the Fermi surface $\mathscr{I}(z_0) = \{\theta \in \mathbb{R}^n/\Gamma^*; z_0 \in \sigma(P_\theta)\}$. The converse of this is given by,

THEOREM 3.5. — *Suppose that for some* $N \in \mathbb{N}$, *there exist continuous functions* $\varphi_j^+ : \mathscr{F}(z_0) \to \mathscr{H}_\theta$, $1 \leq j \leq N$ *such that*:

1° $\varphi_1^+(., \theta), \dots, \varphi_N^+(., \theta)$ *are linearly independent for every* $\theta \in \mathscr{F}(z_0)$,

2° *Putting* $\mathscr{I}_+(\theta) = \bigoplus_1^N \mathbb{C} \varphi_j^+(., \theta)$, *we have*

$$\mathrm{Ker}(P_\theta - z_0) \cap \mathscr{I}_+(\theta)^\perp = \{0\},$$

for every $\theta \in \mathbb{R}^n/\Gamma^*$.

Then the conclusion of Theorem 3.1 *holds with the same* N (*but with different functions* φ_j^+).

Proof. — Since $\mathscr{H}_\theta$ is an infinite dimensional space, we can extend φ_j^+ to continuous functions: $\mathbb{R}^n/\Gamma^* \to \mathscr{H}_\theta$ in such a way that 1° remains valid for every $\theta \in \mathbb{R}^n/\Gamma^*$. (As earlier noticed, it is enough to find φ_j^+, φ_j^- satisfying the conclusions of Theorem 3.1 but with analyticity in θ replaced by continuity.) Outside $\mathscr{F}(z_0)$, the property 2° is trivially satisfied, since $\mathrm{Ker}(P_\theta - z_0)$ is reduced to $\{0\}$ there. For $\varepsilon > 0$, we put:

$$\mathscr{I}_-(\theta, \varepsilon) = \overline{(P_\theta - z_0 + i\varepsilon)^{-1}(\mathscr{I}_+(\theta))}. \tag{3.9}$$

We claim that,

$$(3.10)\qquad (P_\theta - z_0 - i\varepsilon)(\mathscr{I}_+(\theta)^\perp \cap \mathscr{H}^2_\theta) = \mathscr{I}_-(\theta,\varepsilon)^\perp.$$

The inclusion "$\subset$" is immediate from (3.9), and the opposite inclusion is proved by the following chain of implications:

$$(3.9) \Rightarrow \mathscr{I}_- \supset (P_\theta - z_0 + i\varepsilon)^{-1}(\mathscr{I}_+)$$
$$\Rightarrow (P_\theta - z_0 - i\varepsilon)^{-1}(\mathscr{I}_-^\perp) \subset \mathscr{I}_+^\perp \cap \mathscr{H}^2$$
$$\Rightarrow \mathscr{I}_-^\perp \subset (P_\theta - z_0 - i\varepsilon)(\mathscr{I}_+^\perp \cap \mathscr{H}^2).$$

Recall from [He Sj 6] that if F, G are closed subspaces of a Hilbert space $\mathscr{H}$, then we can define a non-necessarily symmetric "distance":

$$\vec{d}(F,G) = \sup_{x\in F,\ \|x\|=1} \operatorname{dist}(x,G) = \|\pi_F - \pi_G\pi_F\| = \|\pi_F - \pi_F\pi_G\|,$$

where π_F, π_G are the orthogonal projections onto F, G respectively. If H is a third closed subspace of H, we have $\vec{d}(F,H) \leqq \vec{d}(F,G) + \vec{d}(G,H)$. It was proved in [HS 6] that if $\vec{d}(F,G)$ and $\vec{d}(G,F)$ are both strictly smaller than 1, then they are equal. We get a distance on the set of all closed subspaces of $\mathscr{H}$ by putting $d(F,G) = \max(\vec{d}(F,G), \vec{d}(G,F))$. Changing the norm into an equivalent one, changes d into an equivalent distance.

LEMMA 3.6. — *Let* F, G *be closed subspaces of a Hilbert space* $\mathscr{H}$. *Then* $\vec{d}(F,G) = \vec{d}(G^\perp, F^\perp)$.

Proof. — The orthogonal projections onto $F^\perp$ and $G^\perp$ are equal to $1-\pi_F$ and $1-\pi_G$ respectively. Using the formula above for $\vec{d}$ in terms of projections, we get,

$$\vec{d}(G^\perp, F^\perp) = \|(1-\pi_G) - (1-\pi_F)(1-\pi_G)\| = \|\pi_F - \pi_F\pi_G\| = \vec{d}(F,G). \quad \square$$

In the present situation, we also want to compare subspaces of different $\mathscr{H}_\theta$. Let us introduce the distance $d(\theta,\theta') = \min_{\gamma\in\Gamma^*} |\theta + \gamma - \theta'|$ on $\mathbb{R}^n/\Gamma^*$. If $E \subset \mathbb{R}^n$ is a bounded fundamental domain for Γ, and $u \in \mathscr{H}_\theta$, $v \in \mathscr{H}_{\theta'}$, we consider the quantity

$$(3.11)\qquad d_{E,\theta,\theta'}(u,v) = \|u-v\|_{L^2(E)} + d(\theta,\theta')\min(\|u\|, \|v\|).$$

Here $\|u\| = \|u\|_{L^2(E)}$, $\|v\| = \|v\|_{L^2(E)}$ are the norms (independent of the choice of E) in $\mathscr{H}_\theta$ and $\mathscr{H}_{\theta'}$ respectively. If $\gamma \in \Gamma$, then

$$\|u-v\|_{L^2(E+\gamma)} = \|e^{i\gamma(\theta-\theta')}u - v\|_{L^2(E)} \leqq \|u-v\|_{L^2(E)} + |e^{i\gamma(\theta-\theta')} - 1|\,\|u\|,$$

and in the last term, we may replace u by v. Since

$$|e^{i\gamma(\theta-\theta')} - 1| \leqq |\gamma|\, d(\theta,\theta'),$$

we obtain,

$$(3.12)\qquad \|u-v\|_{L^2(E+\gamma)} \leqq \|u-v\|_{L^2(E)} + |\gamma|\, d(\theta,\theta')\min(\|u\|, \|v\|).$$

From this we conclude that if Ω is a bounded domain (and hence coverable by a finite number of translates $E+\gamma$), we have,

$$\|u-v\|_{L^2(\Omega)} \leqq C(\Omega, E)\, d_{E,\theta,\theta'}(u,v). \tag{3.13}$$

In particular, if $\Omega=\tilde{E}$ is another bounded fundamental domain, we obtain:

$$C(E,\tilde{E})^{-1} d_{E,\theta,\theta'} \leqq d_{\tilde{E},\theta,\theta'} \leqq C(E,\tilde{E})\, d_{E,\theta,\theta'}. \tag{3.14}$$

LEMMA 3.7. — *Let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis for Γ and let Ω be bounded with the property that for every j, the set $\Omega \cap (\Omega+e_j)$ contains a bounded fundamental domain for Γ. Then for all $\theta, \theta' \in \mathbb{R}^n/\Gamma^*$ and all $u \in \mathscr{H}_\theta$, $v \in \mathscr{H}_{\theta'}$, we have*:

$$C(\Omega, E)^{-1} d_{E,\theta,\theta'}(u,v) \leqq \|u-v\|_{L^2(\Omega)} \leqq C(\Omega,E)\, d_{E,\theta,\theta'}(u,v). \tag{3.15}$$

Here E is some fixed bounded fundamental domain and $C(\Omega, E)$ is independent of u, v, θ, θ'.

Proof. — The right inequality in (3.15) has already been established. For the left inequality, we may assume that the norm used in the definition of $d(\theta,\theta')$ is $|x| = \max_{1\leqq j\leqq n} |e_j x|$. Then

$$d(\theta,\theta') = \min_{\gamma\in\Gamma^*} \max_{1\leqq j\leqq n} |e_j(\theta-\theta'-\gamma)| = \max_{1\leqq j\leqq n} \min_{\upsilon\in\mathbb{Z}} |e_j(\theta-\theta')-2\pi\upsilon|.$$

Since,

$$C_0^{-1} \min_{\upsilon\in\mathbb{Z}} |t-2\pi\upsilon| \leqq |e^{it}-1| \leqq \min_{\upsilon\in\mathbb{Z}} |t-2\pi\upsilon|,$$

for some universal constant $C_0>0$, we get,

$$C_0^{-1} d(\theta,\theta') \leq \max_{1\leqq j\leqq n} |e^{ie_j(\theta-\theta')}-1| \leqq d(\theta,\theta'). \tag{3.16}$$

For given θ, θ', we choose j such that the maximum above is attained. Then let E_j be the Γ-fundamental domain contained in $\Omega\cap(\Omega+e_j)$, so that E_j and E_j-e_j are contained in Ω. Then,

$$\|u-v\|_{L^2(\Omega)} \geqq \frac{1}{2}(\|u-v\|_{L^2(E_j)} + \|e^{-ie_j(\theta-\theta')}u-v\|_{L^2(E_j)}). \tag{3.17}$$

Treating separately the cases when $\|u-v\|_{L^2(E_j)}$ is larger or smaller than $|e^{ie_j(\theta-\theta')}-1|\,\|u\|$, we get,

$$\|u-v\|_{L^2(E_j)} + \|e^{-ie_j(\theta-\theta')}u-v\|_{L^2(E_j)} \geqq \frac{1}{2}(\|u-v\|_{L^2(E_j)} + |e^{ie_j(\theta-\theta')}-1|\,\|u\|). \tag{3.18}$$

Combining this with the left inequality in (3.16) and (3.14) with $\tilde{E}=E_j$, we get the left inequality in (3.15), which completes the proof of the lemma. $\square$

It is now clear how we shall measure distances between elements or subspaces of different $\mathscr{H}_\theta$ spaces: We take $\Omega \subset \mathbb{R}^n$ with the properties of Lemma 3.7, and we consider the various $\mathscr{H}_\theta$ spaces as subspaces of $L^2(\Omega)$. According to Lemma 3.7 the choice of Ω will only affect $\|u-v\|_{L^2(\Omega)}$ with $u \in \mathscr{H}_\theta$, $v \in \mathscr{H}_{\theta'}$ up to equivalence. With the following special choice of Ω, we can arrange so that the norm in $L^2(\Omega)$ reduces to a constant multiple of the norm of $\mathscr{H}_\theta$ for every θ. Indeed, it suffices to fix a fundamental domain E, and choose $\Omega = E \cup \left(\bigcup_1^N (E+e_j) \right)$. If E, F are closed subspaces of $\mathscr{H}_\theta$, $\mathscr{H}_{\theta'}$, we can then define $\vec{d}(E, F)$, $\vec{d}(F, E)$, $d(E, F)$ in the sense of $L^2(\Omega)$. It is easy to check that,

$$d(\mathscr{H}_\theta, \mathscr{H}_{\theta'}) \leqq C\, d(\theta, \theta'). \tag{3.19}$$

LEMMA 3.8. – *With* $F \subset \mathscr{H}_\theta$, $G \subset \mathscr{H}_{\theta'}$ *as above, we have*

$$\vec{d}(G^\perp, F^\perp) \leqq \vec{d}(F, G) + 5\, d(\mathscr{H}_\theta, \mathscr{H}_{\theta'}),$$

where $F^\perp$, $G^\perp$ *denote the orthogonal complements in* $\mathscr{H}_\theta$ *and* $\mathscr{H}_{\theta'}$ *respectively.*

Proof. – Let π_F, π_G, $\pi_{F^\perp}$, $\pi_{G^\perp}$, π_θ, $\pi_{\theta'}$ denote the orthogonal projections in $L^2(\Omega)$ onto F, G, $F^\perp$, $G^\perp$, $\mathscr{H}_\theta$, $\mathscr{H}_{\theta'}$ respectively. Then

$$\pi_F = \pi_F \pi_\theta = \pi_\theta \pi_F, \qquad \pi_G = \pi_G \pi_{\theta'} = \pi_{\theta'} \pi_G$$

and

$$\pi_{F^\perp} = (1-\pi_F)\pi_\theta = \pi_\theta - \pi_F, \qquad \pi_{G^\perp} = \pi_{\theta'} - \pi_G,$$

so we get

$$\begin{aligned}
\vec{d}(G^\perp, F^\perp) &= \| \pi_{G^\perp} - \pi_{F^\perp} \pi_{G^\perp} \| \\
&= \| \pi_{\theta'} - \pi_G - (\pi_\theta - \pi_F)(\pi_{\theta'} - \pi_G) \| \\
&= \| \pi_{\theta'} - \pi_\theta \pi_{\theta'} - \pi_G + \pi_\theta \pi_G + \pi_F \pi_{\theta'} - \pi_F \pi_G \| \\
&= \| \pi_{\theta'} - \pi_\theta \pi_{\theta'} + (\pi_\theta - \pi_{\theta'}) \pi_G + \pi_F (\pi_{\theta'} - \pi_\theta) + \pi_F - \pi_F \pi_G \| \\
&\leqq \| \pi_{\theta'} - \pi_\theta \pi_{\theta'} \| + 2 \| \pi_\theta - \pi_{\theta'} \| + \| \pi_F - \pi_F \pi_G \| \leqq 5\, d(\mathscr{H}_\theta, \mathscr{H}_{\theta'}) + \vec{d}(F, G). \qquad \square
\end{aligned}$$

Returning to (3.10), we define for $\theta \in \mathbb{R}^n/\Gamma^*$, $1 > \varepsilon \geqq 0$:

$$N(\theta, \varepsilon) = (P_\theta - z_0 - i\varepsilon)(\mathscr{I}_+(\theta)^\perp \cap \mathscr{H}_\theta^2) \subset \mathscr{H}_\theta. \tag{3.20}$$

We shall prove that $N(\theta, \varepsilon)$ varies continuously with θ, ε, in the sense that,

$$d(N(\theta, \varepsilon), N(\theta', \varepsilon')) \leqq f(\theta - \theta', \varepsilon - \varepsilon'), \tag{3.21}$$

with $f(\theta, \varepsilon) \to 0$, when $(\theta, \varepsilon) \to (0, 0)$. To do this, we start by recalling that since $\varphi_j^+(\theta)$ vary continuously with θ, we have the property analogous to (3.21) for $\mathscr{I}_+(\theta)$. By regularizing the φ_j^+ and passing to suitable linear recombinations, we can find $\tilde{\varphi}_j \in \mathscr{H}_\theta^2$ depending continuously on θ [in

$\mathscr{H}^2_\theta(\Omega)$, with Ω as above] such that,

$$(3.22) \qquad (\tilde{\varphi}_j \mid \varphi_k^+) = \delta_{j,k}.$$

For $u \in \mathscr{H}_\theta$, we put $\tilde{\pi}_\theta u = u - \sum (u \mid \varphi_j^+) \tilde{\varphi}_j$. Then $\tilde{\pi}_\theta : \mathscr{H}_\theta \to \mathscr{I}_+(\theta)^\perp$ is a bounded projection onto $\mathscr{I}_+(\theta)^\perp$, with a bounded extension: $\mathscr{H}^2_\theta \to \mathscr{H}^2_\theta \cap \mathscr{I}_+(\theta)^\perp$. We notice that

$$\begin{aligned}\| u - \tilde{\pi}_\theta u \|_2 &= \| \sum (u \mid \varphi_j^+) \tilde{\varphi}_k \|_2 \\ &\leqq (\sum | (u \mid \varphi_j^+) |^2)^{1/2} (\sum \| \tilde{\varphi}_k \|_2^2)^{1/2} \leqq C \| u - \pi_\theta u \|.\end{aligned}$$

Here $\pi_\theta : \mathscr{H}_\theta \to \mathscr{I}_+(\theta)^\perp$ is the orthogonal projection. We can then verify that $\mathscr{I}_+(\theta)^\perp \cap \mathscr{H}^2_\theta$ depends continuously on θ [in $\mathscr{H}^2(\Omega)$]: If $u \in \mathscr{I}_+(\theta')^\perp \cap \mathscr{H}^2_{\theta'}$ and θ' is close to θ, we consider $e^{i(\theta-\theta')(.)} u \in \mathscr{H}^2_\theta$ which has the property that,

$$(3.23) \qquad \| e^{i(\theta-\theta')(.)} u - u \|_{\mathscr{H}^2(\Omega)} \leqq C\, d(\theta, \theta').$$

By the continuity of $\mathscr{I}_+(\theta)^\perp$ in $\mathscr{H}_\theta$, we know that

$$\| e^{i(\theta-\theta')(.)} u - \tilde{\pi}_\theta e^{i(\theta-\theta')(.)} u \| \leqq g(\theta - \theta') \| u \|.$$

Combining this with (3.23), we get

$$\| u - \tilde{\pi}_\theta e^{i(\theta-\theta')(.)} u \|_{\mathscr{H}^2(\Omega)} \leqq C\, h(\theta - \theta') \| u \|,$$

where $h(\theta) \to 0$, $\theta \to 0$. This implies that

$$(3.24) \qquad \vec{d}(\mathscr{I}_+(\theta')^\perp \cap \mathscr{H}^2_{\theta'}, \mathscr{I}_+(\theta)^\perp \cap \mathscr{H}^2_\theta) \leqq C\, h(\theta - \theta'),$$

where the distance is taken in the sense of $\mathscr{H}^2(\Omega)$.

We can now prove (3.21). Let $u \in N(\theta, \varepsilon)$ with $\| u \| \leqq 1$. Then

$$u = (P_\theta - z_0 - i\varepsilon) v, \quad v \in \mathscr{I}_+(\theta)^\perp \cap \mathscr{H}^2_\theta, \quad \| v \|_{\mathscr{H}^2(\Omega)} \leqq C.$$

(Here we use the assumption that $\mathrm{Ker}(P_\theta - z_0) \cap \mathscr{I}_+(\theta)^\perp = 0$.) According to (3.24) (with θ, θ' interchanged) there is a $w \in \mathscr{I}_+(\theta')^\perp \cap \mathscr{H}^2$, with

$$\| v - w \|_{\mathscr{H}^2(\Omega)} \leqq C^2 h(\theta - \theta').$$

Put $\tilde{u} = (P_\theta - z_0 - i\varepsilon') w \in N(\theta', \varepsilon')$. Then [as elements in $L^2(\Omega)$] we get $u - \tilde{u} = (P - z_0 - i\varepsilon)(v - u) + i(\varepsilon - \varepsilon') w$. Since $\| w \|_{L^2(\Omega)} \leqq$ Const., we get with a new constant C:

$$\| u - \tilde{u} \| \leqq C (h(\theta - \theta') + | \varepsilon - \varepsilon' |).$$

This gives (3.21) with d replaced by $\vec{d}$. By symmetry in θ, θ', we then get (3.21).

In view of (3.10), (3.20), we can now extend the definition of $\mathscr{I}_-(\theta, \varepsilon)$ to the case $\varepsilon = 0$, by putting $\mathscr{I}_-(\theta, \varepsilon) = N(\theta, \varepsilon)^\perp$, $0 \leqq \varepsilon < 1$. (Here the orthogonal complement is taken in $\mathscr{H}_\theta$.) Combining (3.21), Lemma 3.8 and (3.19), we then get

$$(3.25) \qquad d(\mathscr{I}_-(\theta, \varepsilon), \mathscr{I}_-(\theta', \varepsilon')) \leqq f(\theta - \theta', \varepsilon - \varepsilon'),$$

where $f(\theta,\varepsilon)\to 0$, when $(\theta,\varepsilon)\to(0,0)$. In particular, the dimension of $\mathscr{I}_-(\theta,\varepsilon)$ is constant and equal to $N=\dim\mathscr{I}_+(\theta)$. For $\varepsilon>0$, (3.9) tells us that $(P_\theta-z_0+i\varepsilon)(\mathscr{I}_-(\theta,\varepsilon))=\mathscr{I}_+(\theta)$. From (3.20) we get when $\varepsilon=0$:

$$(P_\theta-z_0)(\mathscr{I}_-(\theta))\subset\mathscr{I}_+(\theta), \tag{3.26}$$

where $\mathscr{I}_-(\theta)=_{\text{def.}}\mathscr{I}_-(\theta,0)$.

If we recall that $\mathscr{I}_+(\theta)$ is a trivial bundle over $\mathbb{R}^n/\Gamma^*$, we see from (3.9) that the same is true for $\mathscr{I}_-(\theta,\varepsilon)$ for every fixed $\varepsilon>0$ and a globally defined basis is given by $\varphi_j^-(\theta,\varepsilon)=(P_\theta-z_0+i\varepsilon)^{-1}\varphi_j^+(\theta)$. From this fact, and the continuity property (3.25), we deduce that $\mathscr{I}_-(\theta)$ is also a trivial bundle, although we can no more give the basis explicitly. (Orthonormalizing the basis $\varphi_j^-(\theta,\varepsilon)$ for a small fixed $\varepsilon>0$, and then projecting this basis to $\mathscr{I}_-(\theta)$, gives a global basis in the latter bundle.) The property (3.20) implies that $P_\theta-z_0$ maps $\mathscr{I}_+(\theta)^\perp\cap\mathscr{H}_\theta^2$ bijectively onto $\mathscr{I}_-(\theta)^\perp$, se we get a globally well posed Grushin problem if we define R_θ^+ and R_θ^- by using global bases in $\mathscr{I}_+(\theta)$ and $\mathscr{I}_-(\theta)$ respectively. (Here $R_\theta^\pm$ will perhaps only depend continuously on θ, but as mentioned earlier we can make an analytic regularization.) This completes the proof of Theorem 3.5. □

Recall (as in [He Sj 4], part II) that $Uu(x,\theta)=\sum_{\gamma\in\Gamma}u(x-\gamma)e^{i\gamma\theta}$ defines a unitary operator $U: L^2(\mathbb{R}^n)\to\mathscr{H}=\int^\oplus\mathscr{H}_\theta\,d\theta$ and that the inverse is given by:

$$U^{-1}v(x)=(\mathrm{Vol}(\mathbb{R}^n/\Gamma^*))^{-1}\int_{\mathbb{R}^n/\Gamma^*}v(x,\theta)\,d\theta.$$

By means of U we can identify P with $\int^\oplus P_\theta\,d\theta$, and we obtain the well posed Grushin problem:

$$(P-z_0)u+R_-u^-=v,\qquad R_+u=v^+, \tag{3.27}$$

for $u\in\mathscr{H}^2(\mathbb{R}^n)$, $v\in L^2(\mathbb{R}^n)$, u^-, $v^+\in L^2(\mathbb{R}^n/\Gamma^*;\mathbb{C}^N)$. Here

$$R_+u(\theta)=R_\theta^+(Uu(.,\theta)), R_-u^-(x)=(U^{-1}R_{..}^-v(.,..))(x).$$

In the case when $\varphi_j^+=\varphi_j^-$, we had $R_\theta^-=(R_\theta^+)^*$, which here gives $R_-=R_+^*$. In order to transform (3.27) further, we identify $L^2(\mathbb{R}^n/\Gamma^*;\mathbb{C}^N)$ with $l^2(\Gamma;\mathbb{C}^N)$ by means of Fourier series. We then obtain the well posed Grushin problem,

$$(P-z_0)u+R_-^0u^-=v,\qquad R_+^0u=v^+, \tag{3.28}$$

$u\in H^2(\mathbb{R}^n)$, $v\in L^2(\mathbb{R}^n)$, u^-, $v^+\in l^2(\Gamma;\mathbb{C}^N)$, where

$$R_+^0u(\gamma)=\mathrm{Vol}(\mathbb{R}^n/\Gamma^*)^{-1}\int_{\mathbb{R}^n/\Gamma^*}R_+u(\theta)e^{i\theta\gamma}\,d\theta=\varsigma\mathscr{F}R_+u(\gamma).$$

Here,

$$\mathscr{F} w(\gamma)=\operatorname{Vol}(\mathbb{R}^n/\Gamma^*)^{-1}\int_{\mathbb{R}^n/\Gamma^*} w(\theta)\, e^{-i\theta\gamma}\, d\theta$$

associates to w the corresponding Fourier coefficients, and $\varsigma u(\gamma)=u(-\gamma)$. The operator R_-^0 is given by $R_-^0=R_-\mathscr{F}^{-1}\varsigma$.

Let us write explicitly the j:th component of $R_+^0 u(\gamma)$ (with E, E* denoting some fundamental domains for Γ and Γ^*):

(3.29)
$$\begin{aligned}
(R_+^0 u(\gamma))_j &= \operatorname{Vol}(E^*)^{-1}\int_{E^*}\sum_{\Gamma}\int_{E} u(x-\tilde{\gamma})\,(e^{-i\tilde{\gamma}\theta}\varphi_j^+(x,\theta))^*\,dx\, e^{i\theta\gamma}\,d\theta\\
&= \operatorname{Vol}(E^*)^{-1}\int_{E^*}\sum_{\Gamma}\int_{E} u(x-\tilde{\gamma})\,(\varphi_j^+(x-\tilde{\gamma},\theta))^*\,dx\, e^{i\theta\gamma}\,d\theta\\
&= \operatorname{Vol}(E^*)^{-1}\int_{E^*}\int_{\mathbb{R}^n} u(x)\,(\varphi_j^+(x,\theta))^*\,e^{i\theta\gamma}\,dx\,d\theta\\
&= \operatorname{Vol}(E^*)^{-1}\int_{E^*}\int_{\mathbb{R}^n} u(x)\,(\varphi_j^+(x-\gamma,\theta))^*\,dx\,d\theta=(u\,|\,\Phi_j^+(\,.\,-\gamma)),
\end{aligned}$$

where $\Phi_j^+=U^{-1}\varphi_j^+(x)=\operatorname{Vol}(E^*)^{-1}\int\varphi_j^+(x,\theta)\,d\theta$, and a^* denotes the complex conjugate of a. A similar but easier computation shows that,

$$R_-^0 u^-(x)=\sum_{\Gamma}\sum_{j=1}^{N} u_j^-(\gamma)\,\Phi_j^-(x-\gamma). \tag{3.30}$$

where $\Phi_j^-(x)=U^{-1}\varphi_j^{-1}(x)$.

We can now proceed more or less as in section 1 of [HeSj 4], part II. Since $\varphi_j^{\pm}(x,\theta)$ are analytic in θ, we see that the problem (3.1) remains well posed for θ in a small complex neighborhood of $\mathbb{R}^n/\Gamma^*$ and for z_0 replaced by z in a small complex neighborhood of z_0. Let,

$$\mathscr{E}(\theta,z)=\begin{pmatrix} E(\theta,z) & E_+(\theta,z)\\ E_-(\theta,z) & E_{-+}(\theta,z)\end{pmatrix}:\mathscr{H}_\theta\times\mathbb{C}^N\to\mathscr{H}_\theta^2\times\mathbb{C}^N, \tag{3.31}$$

denote the inverse operator. Similarly (3.28) remains well posed for z_0 replaced by z, varying in a small neighborhood of z_0. Let,

$$\mathscr{E}^0(z)=\begin{pmatrix} E^0(z) & E_+^0(z)\\ E_-^0(z) & E_{-+}^0(z)\end{pmatrix} \tag{3.32}$$

denote the inverse operator. Then $E_{-+}^0(z): l^2(\Gamma;\mathbb{C}^N)\to l^2(\Gamma;\mathbb{C}^N)$ is given by a matrix $E_{-+}^0(z;\alpha,\beta)$, and it follows from the constructions above that,

$$E^0_{-+}(z;\alpha,\beta)=\hat{E}_{-+}(.,z)(\beta-\alpha). \tag{3.33}$$

In other words, $E^0_{-+}(z)$ is convolution by $\varsigma\hat{E}_{-+}$.

As in [HeSj 4], part II, section 1, we also see that if $f:\mathbb{R}^n\to\mathbb{R}$ is of class C^2 with $\|f'\|_\infty+\|f''\|_\infty$ sufficiently small, then

$$\|e^f\mathscr{E}^0(z)v\|_{\mathscr{H}^2\times l^2}\leqq C\|e^f v\|_{\mathscr{H}^0\times l^2}, \tag{3.34}$$

for every $v\in\mathscr{H}^2\times l^2$ with compact support. [For $v=(v_1,v_2)\in\mathscr{H}^0\times l^2$, we write $e^f v=(e^f v_1,(e^{f/\Gamma})v_2)$]. In the proof of this, one uses the fact that there is a constant $C>0$ such that for all $\alpha\in\mathbb{N}^n$, there is a constant C_α such that,

$$|\partial_x^\alpha\Phi_j^\pm(x)|\leqq C_\alpha e^{-|x|/C}. \tag{3.35}$$

(Here we assume for simplicity that $\varphi_j^\pm(x,\theta)$ is C^∞ in x, which can easily be achieved by regularization.)

4. A GRUSHIN PROBLEM IN THE CASE OF A WEAK MAGNETIC FIELD

What follows will just be an easy adaptation of the sections 2-4 in [HeSj 4], part II. Let $P=P_0$ be the operator of the preceeding section and put,

$$P=P_A=\sum_1^n(D_{x_j}+A_j(x))^2+V(x), \tag{4.1}$$

$$A=\sum_1^n A_j(x)\,dx_j, \tag{4.2}$$

where $A_j\in C^\infty(\mathbb{R}^n;\mathbb{R})$. Writing

$$dA=\frac{1}{2}\sum\sum b_{j,k}\,dx_j\wedge dx_k,\ \text{with } b_{j,k}=-b_{k,j}, \tag{4.3}$$

we assume that $b_{j,k}$ are constant, and by gauge invariance, we may assume that,

$$A_k(x)=\frac{1}{2}\sum_j x_j b_{j,k}. \tag{4.4}$$

From now on we write P_B instead of P_A and we may realize P_B by means of the Friedrichs extension. We define a family of unitary "magnetic translation" operators T^B_α, $\alpha\in\mathbb{R}^n$ by,

$$T^B_\alpha u(x)=e^{i\langle B,x\wedge\alpha\rangle/2}u(x-\alpha), \tag{4.5}$$

and we have the commutation relations,

$$(4.6)\qquad T^{B}_{\alpha} T^{B}_{\beta} = e^{-i\langle B, \alpha\wedge\beta\rangle/2} T^{B}_{\alpha+\beta} = e^{-i\langle B, \alpha\wedge\beta\rangle} T^{B}_{\beta} T^{B}_{\alpha}.$$

For $k \in \mathbb{N}$, we put

$$H^{k}_{B}(\mathbb{R}^{n}) = \{u \in \mathscr{D}'(\mathbb{R}^{n});\ (D_{j_1} + A_{j_1}(x)) \circ \ldots \circ (D_{j_p} + A_{j_p}) u \in L^{2}(\mathbb{R}^{n}),$$

for all $p \leqq k$ and all sequences $j_1, \ldots, j_p \in \{1, \ldots, n\}\}$. Then:

1° $H^{k}_{B}(\mathbb{R}^{n})$ is a Hilbert space (with the natural norm).

2° $C^{\infty}_{0}(\mathbb{R}^{n})$ is dense in $H^{k}_{B}(\mathbb{R}^{n})$.

3° There is a constant $C > 0$ such that $\|u\|_{2, B} \leqq C(\|P_{B} u\| + \|u\|)$ for every $u \in C^{\infty}_{0}(\mathbb{R}^{n})$. Here $\|.\|_{k, B}$ denotes the norm in H^{k}_{B}.

4° P_{B} is essentially selfadjoint with domain H^{2}_{B}.

5° $P_{B} T^{B}_{\alpha} = T^{B}_{\alpha} P_{B}$ for all $\alpha \in \Gamma$.

Put

$$(4.7)\qquad \varphi^{\pm}_{j, \gamma} = \Phi^{\pm}_{j}(x - \gamma) = T^{0}_{\gamma} \Phi^{\pm}_{j},$$

$$(4.8)\qquad \varphi^{B}_{\pm, j, \gamma} = T^{B}_{\gamma} \varphi^{\pm}_{j, 0}, \qquad \gamma \in \Gamma,$$

$$(4.9)\qquad R^{B}_{+} u(\gamma, j) = (u \mid \varphi^{B}_{+, j, \gamma}),$$

$$(4.10)\qquad R^{B}_{-} u^{-} = \sum u^{-}(\gamma, j)\, \varphi^{B}_{-, j, \gamma}.$$

Again we have $R^{B}_{-} = R^{B*}_{+}$ in the case when $\varphi^{+}_{j} = \varphi^{-}_{j}$ in the conclusion of Theorem 3.1. We have

$$R^{B}_{+} \in \mathscr{L}(L^{2}(\mathbb{R}^{n}); l^{2}(\Gamma; \mathbb{C}^{N})), \qquad R^{B}_{-} \in \mathscr{L}(l^{2}(\Gamma; \mathbb{C}^{N}), L^{2}(\mathbb{R}^{n}))$$

uniformly with respect to B. Put

$$(4.11)\qquad \mathscr{P}_{B} = \begin{pmatrix} P_{B} - z & R^{B}_{-} \\ R^{B}_{+} & 0 \end{pmatrix} : H^{2}_{B} \times l^{2} \to L^{2} \times l^{2},$$

considered sometimes as an unbounded operator on $L^{2} \times l^{2}$. When $z \in \mathbb{R}$ and $R^{B}_{-} = R^{B*}_{+}$, this operator is selfadjoint. As in [HeSj 4], part II, we then obtain,

THEOREM 4.1. – *For* (z, B) *in a neighborhood of* $z_0 \times \{0\}$ *in*

$$\mathbb{C} \times \mathbb{R}^{n(n-1)/2}_{B}$$

we have:

1. $\mathscr{P}_{B}(z): H^{2}_{B} \times l^{2} \to L^{2} \times l^{2}$ *is bijective with an inverse* $\mathscr{E}_{B}(z)$ *depending holomorphically on* z *and bounded in norm by a constant independent of* (B, z).

2. z *belongs to* $\sigma(P_{B})$ *iff* $0 \in \sigma(E_{-+}(B, z))$. *Here we write,*

$$\mathscr{E}_{\mathscr{B}}(z) = \begin{pmatrix} E(B, z) & E_{+}(B, z) \\ E_{-}(B, z) & E_{-+}(B, z) \end{pmatrix}.$$

3. *There exists a function* $f(\mathrm{B}, z, \alpha)$, $\alpha \in \Gamma$ *with values in the* $\mathrm{N} \times \mathrm{N}$ *matrices, such that the matrix of* $\mathrm{E}_{-+}(\mathrm{B}, z)$ *is given by*

$$\mathrm{E}_{-+}(\mathrm{B}, z;\ \alpha, \beta) = e^{i \langle \mathrm{B}, \alpha \wedge \beta \rangle / 2} f(\mathrm{B}, z;\ \alpha - \beta).$$

4. *The function* f *is* C^{∞} *in* B, *holomorphic in* z, *and there exists an* $\eta > 0$ *such that for every* $\gamma \in \mathbb{N}^{n(n-1)/2}$, *we have*:

$$|\partial_{\mathrm{B}}^{\gamma} f(\mathrm{B}, z;\ \alpha)| \leqq \mathrm{C}_{\gamma} e^{-\eta \langle \alpha \rangle}, \tag{3.12}$$

where $\langle \alpha \rangle = (1 + |\alpha|^2)^{1/2}$.

5. $f(0, z; \alpha) = (\mathrm{E}_{-+}(., z))^{\wedge}$, *where* $\mathrm{E}_{-+}(\theta, z)$ *is defined in* (3.31), *and* $\mathscr{E}(\theta, z)$ *denotes the inverse of the problem* (3.1).

In the case when the problem (3.1) is selfadjoint, the same is true for $\mathscr{P}_{\mathrm{B}}$ when z is real and $\mathrm{E}_{-+}(\mathrm{B}, z;\ \alpha, \beta)$ is selfadjoint. In terms of $f(\mathrm{B}, z;\ \alpha)$, this is reflected by the property,

$$f(\mathrm{B}, z;\ -\alpha) = f(\mathrm{B}, z;\ \alpha)^{*}, \tag{4.13}$$

for real z.

In the general case when the problem (3.1) is not necessarily selfadjoint, we can still keep track of the selfadjointness of the operator P_{B}, by imitating the arguments of [HeSj 3].

5. REDUCTION OF THE STUDY OF THE DENSITY OF STATES

Recall (*cf.* § 1.1) that if P is self adjoint and $\mathrm{F} \in \mathrm{C}_0^{\infty}(\mathbb{R})$, then

$$\mathrm{F}(\mathrm{P}) = -\pi^{-1} \int \bar{\partial} \tilde{\mathrm{F}}(z) (z - \mathrm{P})^{-1} \mathrm{L}(dz), \tag{5.1}$$

where $\tilde{\mathrm{F}} \in \mathrm{C}_0^{\infty}(\mathbb{C})$, $\tilde{\mathrm{F}}|_{\mathbb{R}} = \mathrm{F}$, $\bar{\partial} \tilde{\mathrm{F}} = \mathcal{O}(|\operatorname{Im} z|)$. If $\mathrm{F} \geqq 0$, then $\widetilde{\operatorname{tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B}, \mathrm{V}}) \geqq 0$, and the density of states measure $\rho_{\mathrm{B}, \mathrm{V}}$ is defined as the unique Radon measure on $\mathbb{R}$ such that,

$$\widetilde{\operatorname{tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B}, \mathrm{V}})) = \int \mathrm{F}(t)\, \rho_{\mathrm{B}, \mathrm{V}}(dt), \quad \text{for every } \mathrm{F} \in \mathrm{C}_0^{\infty}(\mathbb{R}). \tag{5.2}$$

We now fix $z_0 \in \mathbb{R}$, and assume that we have a well posed Grushin problem for $\mathrm{P} - z_0$ as in section 3. (Here we write P instead of $\mathrm{P}_{\mathrm{B}, \mathrm{V}}$ for short.)

If $\begin{pmatrix} \mathrm{E} & \mathrm{E}_{+} \\ \mathrm{E}_{-} & \mathrm{E}_{-+} \end{pmatrix}$ is the corresponding inverse of $\begin{pmatrix} \mathrm{P} - z & \mathrm{R}_{-} \\ \mathrm{R}_{+} & 0 \end{pmatrix}$ we have:

$$(z - \mathrm{P})^{-1} = -\mathrm{E}(z) + \mathrm{E}_{+}(z) \mathrm{E}_{-+}(z)^{-1} \mathrm{E}_{-}(z). \tag{5.3}$$

Since $E(z)$ is holomorphic in z, we obtain for $F \in C_0^\infty(\mathbb{R})$ with support in a small fixed neighborhood of z_0:

$$(5.4)\quad \widetilde{\operatorname{tr}} F(P) = -\pi^{-1} \int \bar{\partial} \tilde{F} \,\widehat{\operatorname{tr}}(E_{-+}^{-1}(z)\, \partial E_{-+}(z))\, L(dz)$$
$$= -\pi^{-1} \int \bar{\partial} \tilde{F} \,\widehat{\operatorname{tr}}((\partial E_{-+}(z))\, E_{-+}^{-1}(z))\, L(dz).$$

Here we recall that $\widehat{\operatorname{tr}}(\mathcal{M}_{B,\Gamma}(f)) = \operatorname{Vol}(\mathbb{R}^n/\Gamma)^{-1} \operatorname{tr}(f(0))$, with the notations of [HeSj 4]. Let $l: T^* \mathbb{R}^m \to \mathbb{R}^{n*}$ be linear and surjective with the property:

$$(5.5)\qquad \{\langle t, l(.)\rangle, \langle s, l(.)\rangle\} = \langle B, t \wedge s\rangle, \quad \text{for all } t, s \in \mathbb{R}^n.$$

We then have the isomorphism of algebras: $\mathcal{M}_{B,\Gamma}(f) \mapsto \operatorname{Op}^w(g \circ l)$, where $g \in C^\infty(\mathbb{R}^{n*}/\Gamma^*)$ is the function whose Fourier coefficients are the $f(\alpha)$. (Here Γ^* is the dual lattice. *See* [HeSj 4], part II, for more details.) Here the regularity assumptions are that f is exponentially decreasing and that g is analytic.

If $Q = Q(B, z, \theta)$ denotes the symbol associated to $E_{-+}(B, z)$ in this manner, we get

$$\widetilde{\operatorname{tr}} F(P) = -(\pi \operatorname{Vol}(\mathbb{R}^n/\Gamma))^{-1} \int (\bar{\partial}_z \tilde{F})\, \widetilde{\operatorname{tr}}((\operatorname{Op} \partial_z Q \circ l)(\operatorname{Op} Q \circ l)^{-1})\, L(dz)$$
$$= -(\pi \operatorname{Vol}(\mathbb{R}^n/\Gamma)^{-1} \int (\bar{\partial}_z \tilde{F})\, \widetilde{\operatorname{tr}}((\operatorname{Op} Q \circ l)^{-1} (\operatorname{Op} \partial_z Q \circ l))\, L(dz).$$

Here we recall that $\widetilde{\operatorname{tr}}(A)$ is defined as the mean value of the symbol of A, when A is a Weyl pseudodifferential operator. We can interpret $\rho_{B,V}$ as $\pi^{-1} \bar{\partial}_z (\widetilde{\operatorname{tr}}((\operatorname{Op} Q \circ l)^{-1} \operatorname{Op} \partial_z Q \circ l)$ in the sense of distributions.

If $e_1, \ldots, e_n$ is a basis for $\mathbb{R}^n$, we put $l_j = \langle e_j, l(.)\rangle$. Then the matrix $\mathcal{L} = (\{l_j, l_k\})$ of Poisson brackets is equal to the matrix $\tilde{B} = (b_{j,k})$ of B for the basis just chosen. If $x_1, \ldots, x_n$ are the corresponding coordinates, then $B = \frac{1}{2} \sum\sum b_{j,k}\, dx_j \wedge dx_k$, $b_{j,k} = -b_{k,j}$. If n is even and B is of maximal rank, we can choose $e_1, \ldots, e_n$ such that

$$\tilde{B} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The functions $l_1, \ldots, l_n$ then form a partial system of symplectic coordinates in the following sense: Writing $x_j = l_j$, $\xi_j = l_{j+n/2}$ for $1 \leqq j \leqq n/2$, we can complete the x_j, ξ_j into a system of linear symplectic coordinates:

$x_j, \xi_j, 1 \leqq j \leqq m$ on $T^* \mathbb{R}^m$. Then

$$g \circ l = \tilde{g}(x_1, \ldots, x_{n/2}, \xi_1, \ldots, \xi_{n/2}),$$

where $\tilde{g}$ is the function g expressed in the coordinates $\theta_j = \langle e_j, \theta \rangle$.

The case $n=3$ is the one that we are the most interested in, but before specializing to that case we repeat the discussion above for a general odd n. We then assume that B is of maximal rank: $n-1$. (In the case $n=3$ this only amounts to assuming that B is not the 0 form.) We can then choose a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ such that

$$\tilde{B} = \begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Viewing B as an antisymmetric mapping: $\mathbb{R}^n \to \mathbb{R}^{n*}$, we have $\operatorname{Ker}(B) = \mathbb{R} e_n$. We put,

$$(5.6) \quad x_j = l_j, \qquad \xi_j = l_{j+n'}, \quad 1 \leqq j \leqq n', \qquad x_{n'+1} = l_n, \qquad n' = (n-1)/2,$$

and obtain

$$(5.7) \qquad g \circ l = \tilde{g}(x_1, \ldots, x_{n'}, \xi_1, \ldots, \xi_{n'}, x_{n'+1}),$$

where again $\tilde{g}$ is the function g expressed in the coordinates $\theta_j = \langle e_j, \theta \rangle$. To fix $x_{n'+1} = C_0$ means to restrict g to the affine hyperplane $\theta_n = C_0$. Notice that this affine hyperplane is a translate of $\operatorname{Im}(B) = (\operatorname{Ker}(B))^{\perp}$.

We know that symbols of the form $g \circ l$ form on algebra for the Weyl composition (*cf.* [BGH] and § 1.3): $(g_1 \circ l) \# (g_2 \circ l) = g_3 \circ l$, where # denotes Weyl composition of symbols. Using the formula (5.7) we see that the Weyl composition reduces to the Weyl composition in the variables $x_1, \ldots, x_{n'}, \xi_1, \ldots, \xi_{n'}$, so that

$$(5.8) \qquad \tilde{g}_3(\,.\,, t) = \tilde{g}_1(\,.\,, t) \# \tilde{g}_2(\,.\,, t),$$

for every t.

Write $Q \circ l = \tilde{Q}(x_1, \ldots, x_{n'}, \xi_1, \ldots, \xi_{n'}, x_{n'+1})$, where $Q = Q(B, z, \theta)$ is the symbol corresponding to $E_{-+}(B, z)$. Then

$$(5.9) \quad \tilde{\operatorname{tr}} F(P_{B,V}) = -(\pi \operatorname{Vol}(\mathbb{R}^n/\Gamma))^{-1} \int \bar{\partial}_z \tilde{F}(z) \tilde{\operatorname{tr}}((\partial_z \tilde{Q}) \tilde{Q}^{-1}) L(dz),$$

where by abuse of notations, we write $\tilde{Q}$ and $\tilde{Q}^{-1}$ instead of $\operatorname{Op}(\tilde{Q})$ and $\operatorname{Op}(\tilde{Q})^{-1}$.

Let $\Omega^* \subset \mathbb{R}^{n*}$ be a fundamental domain for Γ^*, and let $\tilde{\Omega} \subset \mathbb{R}^n_{x'', \xi'', x_{n'+1}}$ be the inverse image by l. Here we put $x'' = (x_1, \ldots, x_{n'})$,

$\xi=(\xi_1,\ldots,\xi_{n'})$. In view of (5.8), we get:

$$(5.10)\quad \tilde{\mathrm{tr}}\,\mathrm{F}(\mathrm{P}_{\mathrm{B},\mathrm{V}})=$$

$$-\frac{\displaystyle\int_{-\infty}^{+\infty}\int_{\tilde\Omega_t}\int_t \bar\partial_z\tilde{\mathrm{F}}(z)\,\mathrm{tr}\,[(\partial_z\tilde{\mathrm{Q}}(\mathrm{B},z,x'',\xi'',t))\#\tilde{\mathrm{Q}}^{-1}(\mathrm{B},z,x'',\xi'',t)]\,\mathrm{L}(dz)\,dx''\,d\xi''\,dt}{\displaystyle\pi\,\mathrm{Vol}(\mathbb{R}^n/\Gamma)\iint_{\tilde\Omega}dx''\,d\xi''\,dt}.$$

Here $\#$ denotes Weyl composition in the variables x'', ξ'' and $\tilde{\mathrm{Q}}^{-1}$ denotes the symbol of $\mathrm{Op}(\tilde{\mathrm{Q}})^{-1}$. $\tilde\Omega_t=\{(x'',\xi'');(x'',\xi'',t)\in\tilde\Omega\}$.

If $\chi^*\in\mathrm{C}_0^\infty(\mathbb{R}^{n*})$ has the property that $\sum_{\gamma\in\Gamma^*}\chi^*(\theta-\gamma)=1$, and $\chi=\chi^*\circ l$, then we have the following variant of (5.10):

$$(5.11)\quad -\tilde{\mathrm{tr}}\,\mathrm{F}(\mathrm{P}_{\mathrm{B},\mathrm{V}})=$$

$$\frac{\displaystyle\iiint\chi(x'',\xi'',t)\,\bar\partial_z\tilde{\mathrm{F}}(z)\,\mathrm{tr}\,[(\partial_z\tilde{\mathrm{Q}}(\mathrm{B},z,x'',\xi'',t))\#\tilde{\mathrm{Q}}^{-1}(\mathrm{B},z,x'',\xi'',t)]\,\mathrm{L}(dz)\,dx''\,d\xi''\,dt}{\displaystyle\pi\,\mathrm{Vol}(\mathbb{R}^n/\Gamma)\iint\chi\,dx''\,d\xi''\,dt}.$$

Remark 5.1. — If we replace B by hB, then without changing the basis $e_1,\ldots,e_n$, we see that we can use the algebra isomorphism:

$$\mathscr{M}_{h\mathrm{B},\Gamma}(f)\mapsto\mathrm{Op}^w(\tilde g(x'',h\xi'',x_{n'+1}))=\mathrm{Op}_h^w(\tilde g(x'',\xi'',x_{n'+1})),$$

where g and $\tilde g$ are defined as above. The formula (5.11) becomes:

$$(5.12)\quad -\tilde{\mathrm{tr}}\,\mathrm{F}(\mathrm{P}_{h\mathrm{B},\mathrm{V}})=$$

$$\frac{\displaystyle\iiint\chi\,\bar\partial_z\tilde{\mathrm{F}}(z)\,\mathrm{tr}\,[(\partial_z\tilde{\mathrm{Q}}(h\,\mathrm{B},z,x'',\xi'',t))\#_h\tilde{\mathrm{Q}}^{-1}(h\,\mathrm{B},z,x'',\xi'',t)]\,\mathrm{L}(dz)\,dx''\,d\xi''\,dt}{\displaystyle\pi\,\mathrm{Vol}(\mathbb{R}^n/\Gamma)\iint\chi\,dx''\,d\xi''\,dt}.$$

6. COMPUTATIONS ON THE DENSITY OF STATES: COMPLEMENTS

We continue the study initiated in section 1, but we consider now the case when the electron is submitted to a periodic electric C^∞ potential V. In this context we have explained in [He-Sj 4] how to reduce the study of the localization of the spectrum near one band [or many of them (*see* also section 3 in this paper)] to the study of a reduced system $\mathrm{E}^{\mathrm{B}}_{-+}(z)$ attached to the bands. We want to give here some complements and explain how to compute concretely $\widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ knowing the symbol of $\mathrm{E}^{\mathrm{B}}_{-+}(z)$. All the computation can theoretically be made to arbitrary order in powers of B but we shall emphasize on the computation modulo $O(\mathrm{B}^4)$. Let us recall that $\mathrm{E}^{\mathrm{B}}_{-+}(z)$ is not intrinsic (*see* remarks 6.3 and 6.4 in [He-Sj 4]) but by definition $\widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ is intrinsic. F is here a C^∞ function with compact support and the choice of our Grushin problem depends of the choice of the support for F.

At least to begin with (*see* remark 6.5) we start with the case of the single isolated band and we assume that F has its support contained in a small neighborhood of the band. The article [He-Sj 4] was devoted to the case where F was equal to one on a part of the spectrum of $\mathrm{P}_{\mathrm{B,V}}$ and 0 on the complementary and in this case we gave a new way of determining the possible values which could be obtained when computing the integrated density of states: $\widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$. Let us recall some notations:

$$\mathrm{P}_{\mathrm{B,V}} = \sum_j (\mathrm{D}_{x_j} + \mathrm{A}_j(x))^2 + \mathrm{V} \tag{6.1}$$

with:

$$\mathrm{A}_j(x) = (1/2) \sum_k b_{kj} x_k \tag{6.2}$$

$\mathrm{P}_{\mathrm{B,V}}$ is an essentially selfadjoint operator and for a real function F in $\mathscr{C}_0^\infty(\mathbb{R}^n)$, $\mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ is well defined by the spectral theorem and we can express $\mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ as in [He-Sj 4] by the following formula:

$$\mathrm{F}(\mathrm{P}_{\mathrm{B,V}}) = (1/2i\pi) \int (\partial \tilde{\mathrm{F}}/\partial \bar{z})(\mathrm{P}_{\mathrm{B,V}} - z)^{-1}\, d\bar{z} \wedge dz \tag{6.3}$$

where $\tilde{\mathrm{F}}$ is an extension of F such that:

$$\tilde{\mathrm{F}}(z) = \mathrm{F}(z) \quad \text{for} \quad z \in \mathbb{R} \tag*{$(6.4)_a$}$$

$$\operatorname{supp}(\tilde{\mathrm{F}}) \subset \wedge \quad \text{where} \quad \wedge := \{ z \in \mathbb{C}, |\operatorname{Im} z| < 1 \} \tag*{$(6.4)_b$}$$

$$\tilde{\mathrm{F}} \in \mathrm{C}_0^\infty(\wedge) \tag*{$(6.4)_c$}$$

$$\partial \tilde{\mathrm{F}}/\partial \bar{z} = O_{\mathrm{N}}((\operatorname{Im} z)^{\mathrm{N}}) \quad \text{in } \mathrm{C}_0^\infty(\wedge), \quad \forall \mathrm{N} \in \mathbb{N}. \tag*{$(6.4)_d$}$$

The first result used in a simple special case in [He-Sj 4] is the following:

THEOREM 6.1. — *Assume that* F *has its support in a small neighborhood of a single isolated band, then:* $\mathrm{B} \to \widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ *is a* C^∞ *function.*

Proof. — The function $\mathrm{B} \to \mathrm{D}_{\mathrm{F,V}}(\mathrm{B}) = \widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}})$ was introduced in [He−Sj 4] (7.14). We start from the following formula:

$$\widetilde{\mathrm{Tr}}\, \mathrm{F}(\mathrm{P}_{\mathrm{B,V}}) = \frac{1}{\pi} \int (\partial \tilde{\mathrm{F}}/\partial \bar{z})(z)\, \widehat{\mathrm{tr}}\, (\mathrm{E}_{-+}(z)^{-1} (\partial \mathrm{E}_{-+}/\partial z)(z))\, \mathrm{L}(dz) \tag{6.5}$$

and recall that by 7.15 in [HeSj 4] we have the following estimate:

$$|\mathrm{E}_{-+}^{-1}(z, \alpha, \beta; \mathrm{B})| \leqq (\mathrm{C}_0/|\operatorname{Im} z|)\, e^{-|\alpha - \beta|\,|\operatorname{Im} z|/\mathrm{C}_0} \tag{6.6}$$

then we shall prove that we have the same property for the derivative with respect to B (this result is used in a particular case in paragraph 8 of [He-Sj 4]). We start from an element f_B in the algebra introduced in [He-Sj 4] (here we have in mind E_{-+}) and let us first observe the following property:

LEMMA 6.2. – *Let f_B be in the class of exponentially decreasing symbols* (see [He-Sj 4]). *If f_{B_0} is invertible in the algebra, then f_B is invertible for* B *in the neighborhood of* B_0 *and the inverse g_B depends continuously of* B.

Proof. – According to the hypotheses we have:

$$|f_B(\alpha)| \leqq C \exp(-|\alpha|/C) \text{ for C large enough.}$$

Let us also recall that the law of composition is a distorted convolution:

$$(f \#_B g)(\alpha) = \sum_{\beta+\gamma=\alpha} e^{i\langle B \mid \beta \wedge \gamma\rangle} f(\beta) g(\gamma). \tag{6.7}$$

Then to invert f_B for B in a neighborhood of B_0 we use the formula:

$$f_B \#_B g_{B_0} = 1 + (f_B - f_{B_0}) \#_B g_{B_0} + (f_{B_0} \#_B g_{B_0} - f_{B_0} \#_{B_0} g_{B_0}) \tag{6.8}$$

and the r. h. s. is easily seen to be invertible due to the exponential decay of the symbols and to (6.7). We get also the continuity of g_B with respect to B. □

In a second step, we get the derivability by using the formula [*see* formula (1.3.9)]:

$$\begin{aligned} D_{b_{jk}}(f_B \#_B g_B) = (D_{b_{jk}} f_B) \#_B g_B + f_B \#_B (D_{b_{jk}} g_B) \\ + \frac{1}{2}((\alpha_j f_B) \#_B (\alpha_k g_B) - (\alpha_k f_B) \#_B (\alpha_j g_B)) \end{aligned} \tag{6.9}$$

We get for the derivative of g_B the following formula:

$$\begin{aligned} (D_{b_{jk}} g_B) = -(g_B \#_B (D_{b_{jk}} f_B) \#_B g_B) \\ - \left(g_B \#_B \left(\frac{1}{2}((\alpha_j f_B) \#_B (\alpha_k g_B) - (\alpha_k f_B) \#_B (\alpha_j g_B))\right)\right) \end{aligned} \tag{6.10}$$

which can be easily verified.

(6.10) and (6.6) give (with $\mathscr{M}_B(f_B) = E^B_{-+}(z)$, $\mathscr{M}_B(g_B) = E^B_{-+}(z)^{-1}$) the following estimate:

$$|(D_{b_{jk}} g_B)(\alpha)| \leqq (C_1/|\operatorname{Im} z|^5) e^{-|\alpha||\operatorname{Im} z|/C_1}$$

and we get finally the following estimate for $h(\alpha, B, z) := g_B \#_B \partial f_B/\partial z$:

$$(6.11) \qquad |\partial_B^\gamma h_B(\alpha)| \leqq (C_\gamma/|\operatorname{Im} z|^{C_\gamma}).\exp(-|\alpha|\,|\operatorname{Im} z|/C_0)$$

for sufficiently large constants C_γ, C_0, locally uniform with respect to B, and uniform with respect to z, for z in the support of $\tilde{F}$, and $|\operatorname{Im} z| \neq 0$.

The proof of theorem 6.1 is then easy using (6.11) and (6.4_d). In particular, we get:

$$\partial_B^\gamma D_{F,V}(B) = (1/2 i\pi) \int (\partial\tilde{F}/\partial\bar{z})(\partial_B^\gamma h_B)(0)\, d\bar{z} \wedge dz$$

and we can theoretically compute all the different terms using formulas like (6.10) and (6.9). In particular, we can write the Taylor expansion with respect to B at the point 0. Let us write the result which can be obtained in this case. By the standard argument it is easy to see that the linear term with respect to B vanishes (because $P_{B,V}\Gamma = \Gamma P_{-B,V}$ where Γ is the operator $u \to \Gamma u = \bar{u}$). To make this computation, it is easier to use the pseudo-differential representation of our magnetic algebra. And in fact it is another variant of the algebra we studied in section 1: the algebra of the periodic analytic symbols. If $E_{-+}(z, B)$ is written as $\mathscr{M}_B(f_B)$, we write the corresponding symbol [which is defined by $p(\tau, B, z) = \sum_\alpha e^{-i\tau\alpha} f_B(\alpha, z)$]:

$$(6.12) \quad p(\tau, B, z) = p_0(\tau, z) + \sum_{|\gamma|=1} p_1^\gamma(\tau, z)\, b^\gamma + \sum_{|\gamma|=2} p_2^\gamma(\tau, z)\, b^\gamma + o(B^3)$$

This symbol was computed in [He-Sj 4] (in particular, we have $p_0(\tau, z) = p_0(\tau) - z$ where p_0 is the Floquet eigenvalue). We gave also in that paper, conditions under which p_1 was equal to 0. The computation of $D_F(B)$ is quite analogous to the computation in section 1. The only difference is that the dependence with respect to z is different, so we have to replace the computation of the symbol of $(P_B - z)^{-1}$ (formally in powers of B) by the computation (in the class of p. d. o.) of $E_{-+}(z)^{-1}.\,\partial E_{-+}(z)/\partial z$. This is obtained by formal computation of the symbol of:

$$\operatorname{Op}(p(\,.\,, B, z))^{-1}.\operatorname{Op}((\partial p/\partial z)(\,.\,, B, z)).$$

What is used here is the following law of composition for two symbols a and b:

$$(6.13) \quad a \#_B b = a.b + (1/2i) \sum_{k,l} b_{k,l}\, \partial a/\partial\tau_k . \partial b/\partial\tau_l$$
$$- 1/8 \sum_{k,l,k',l'} b_{k,l}.b_{k',l'}.\partial^2 a/\partial\tau_k \partial\tau_{k'}.\partial^2 b/\partial\tau_l \partial\tau_{l'} + O(B^3)$$

After some computations we get the following result:

$$(6.14)\quad (2\pi)^n D_F(B) = \int F(p_0(\tau))\,d\tau$$
$$-(1/48)\sum_{k,k',l,l'} b_{k,l}b_{k',l'}\int F''(p_0)(\partial^2_{kk'}p_0)(\partial^2_{ll'}p_0)\,d\tau$$
$$+(1/2)\sum_{|\gamma|=2} b^\gamma \int F''(p_0(\tau))\,p_1^\gamma(\tau, p_0(\tau))\,d\tau$$
$$+\sum_{|\gamma|=2} b^\gamma \int F'(p_0)\,p_2^\gamma(\tau, p_0(\tau))\,d\tau + O(B^4).$$

These formulas were given in an approximative way in Peierls' paper and were also given in a different way in the articles by Adams [Ad], Kohn [Ko], Blount [Bl] (*see* this proof in [Ca]), Nenciu [Ne]. For example the last three terms in the r.h.s. of (6.14) correspond to 3.17, 3.18, 3.19 in Adams' paper. Here let us recall that $p_0(\tau)$ is the Floquet eigenvalue and that p_1^γ for $\gamma=(\gamma_1,\gamma_2)$ is simply

$$p_1^{\gamma_1}\cdot p_1^{\gamma_2},$$

In the case where p_1 is zero we have the simplified formula:

$$(6.15)\quad (2\pi)^n D_F(B) = \int F(p_0(\tau))\,d\tau$$
$$-(1/48)\sum_{k,k',l,l'} b_{k,l}b_{k',l'}\int F''(p_0)(\partial^2_{kk'}p_0)(\partial^2_{ll'}p_0)\,d\tau$$
$$+\sum_{|\gamma|=2} b^\gamma \int F'(p_0)\,p_2^\gamma(\tau, p_0(\tau))\,d\tau + O(B^4).$$

The second term of the r.h.s. corresponds to the Landau-Peierls susceptibility. The third term contains the contribution of the other bands.

Remark 6.3: *The case of the Pauli equation.* – In the study of the Pauli equation (*cf.* [Ca] p. 245), we have to consider the expression:

$$D_F^P(B) := D_{F+}(B) + D_{F-}(B)$$

where:

$$F_\pm(s) = F(s \pm \|B\|), \qquad \|B\|^2 = \sum_{j<k} b_{jk}^2.$$

Let us observe the following relation:

$$(6.16)\quad (2\pi)^n D_F^P(B) = 2(2\pi)^n D_F(B) + \|B\|^2 \int F''(p_0)\,d\tau + O(B^3).$$

The second term of the r.h.s. of (6.16) is called the Pauli susceptibility. Let us recall the classical comparison between the two susceptibilities in the case of dimension 2. We denote B_{12} simply by B. The Landau-Peierls susceptibility χ_{LP} is then given by:

$$-(1/48).(2).B^2 \times ((2\pi)^{-2}\int F''(p_0)(\partial^2_{xx}p_0 . \partial^2_{\xi\xi}p_0 - (\partial^2_{x\xi}p_0)^2)\, d\tau$$

(This formula appears to be, in the case where $F = f_{\omega, T}$ (*see* § 1.1.1) and in the limit when T tends to 0, equal to:

$$-(1/24)\, B^2 \times ((2\pi)^{-2}\int_{p_0=\omega} (\partial^2_{xx}p_0 . \partial^2_{\xi\xi}p_0 - (\partial^2_{x\xi}p_0)^2)\, d\tau/dp_0)$$

and is mentioned in [Ad]).

If we assume that $p_0(\tau) = (\tau_1^2 + \tau_2^2)/m^*$, we get finally the formula:

$$(6.17) \qquad \chi_{LP}(B) = -(2\pi)^{-2}.(1/6).B^2.m^{*-2}.\int F''(p_0)\, d\tau$$

which is a diamagnetic term (due to the $-$sign) to compare with the Pauli susceptibility:

$$(6.18) \qquad \chi_S(B) = (2\pi)^{-2}.B^2.\int F''(p_0)\, d\tau$$

which is a paramagnetic term (due to the $+$sign).

The sign of $2\chi_{LP}(B) + \chi_S(B)$ which is the sign of $(1 - (1/3\, m^{*2})$ plays the important role for the physical properties in the case when the third term in (6.15) can be neglected. The case of the free electron corresponds to $m^* = 1$.

Remark 6.4: *Taylor expansions near a rational.* — It is possible to write down similar expansions near a rational B in the sense of [He-Sj 2]. We have first to reduce the problem to a similar problem but for a system (as in [He-Sj 2,4]) and follow the same ideas.

Remark 6.5: *The general case.* — The theorem 6.1 is true for F in C_0^∞ (or more generally for F with a right bounded symbol) *without any hypotheses on the Floquet eigenvalues.* Moreover $B \to \tilde{\mathrm{T}}\mathrm{r}\, F(P_{B,V})$ is an even function with respect to B according to the property: $P_{B,V}\Gamma = \Gamma P_{-B,V}$. The proof is in fact almost the same as in the particular case of the isolated single band. Indeed, if F belongs to C_0^∞, we can after some partition of unity assume that the support of F is contained in some small neighborhood of z_0 given by theorem 4.1 (the restriction with respect to B can be assumed independent of z_0). Using the Grushin problem defined in section 4, we then follow the proof of theorem 6.1 line by line just thinking that now $E_{-+}(z)$ is a $N \times N$ matrix (where N depends of z_0);

one has of course to replace in (6.6) and in other places $|\ |$ by $\|\ \|_{\mathscr{L}(\mathbb{C}^N, \mathbb{C}^N)}$. Of course, (relatively) simple formulas like (6.14) do not exist in general unless in the case when some Floquet eigenvalue is simple in the support of F.

7. ASSUMPTIONS AND GEOMETRIC PRELIMINARIES FOR THE STUDY OF THE DE HAAS VAN ALPHEN EFFECT

From now on we assume that $n=3$. We fix $z_0 \in \mathbb{R}$ and put $\mathscr{F}(z_0)=\{\theta \in \mathbb{R}^{3*}/\Gamma^*;\ z_0 \in \sigma(P_\theta)\}$. Here P_θ is the operator $P_{0, V}$, acting on θ-Floquet periodic functions. Our first assumption is:

(H.1) For every $\theta \in \mathscr{F}(z_0)$, z_0 is a simple eigenvalue of P_θ.

Recall that generically, a self adjoint complex matrix, depending smoothly on three real parameters, will have multiple eigenvalues only at isolated points, so "morally" (H.1) will be satisfied if z_0 avoids some isolated values. Notice also that (H.1) is a much weaker assumption than the single band hypothesis, used in [HeSj 4] to justify the Peierls substitution.

In a small neighnorhood of $\mathscr{F}(z_0)$, we let $\lambda(\theta)$ be the simple eigenvalue which is close to z_0. $\lambda(\theta)$ depends analytically on θ and is equal to z_0 precisely when θ belongs to $\mathscr{F}(z_0)$. Our second assumption is that:

(H.2) $d\lambda(\theta) \neq 0$ for all $\theta \in \mathscr{F}(z_0)$.

This implies that $\mathscr{F}(z_0)$ is a closed analytic hypersurface, and that any compact set in $\mathbb{R}^{3*}$ can intersect only finitely many connected components of $\mathscr{F}(z_0)$. In the remainder of this section the whole discussion will take place in $\mathbb{R}^{3*}$ and the lattice will be Γ^*. To simplify notations we shall drop all the stars and write $\mathbb{R}^3$, Γ instead of $\mathbb{R}^{3*}$, Γ^*.

LEMMA 7.1. — *There exists* $N \in \mathbb{N}$ *such that if* $K_1, \ldots, K_M$ *are connected components of* $\mathscr{F}(z_0)$ *with* $(K_j+\Gamma) \cap K_k = \varnothing$ *for* $j \neq k$, *then* $M \leq N$.

Proof. — Let $K_1, \ldots, K_M$ be as in the lemma. Let $d_0 = \sup_{x \in \mathbb{R}^3} \operatorname{dist}(x, \Gamma)$. For every j, there is a $\gamma_j \in \Gamma$ with $\operatorname{dist}(\gamma_j, K_j) \leq d_0$. Then $K_j - \gamma_j$, $1 \leq j \leq M$, are disjoint components, all intersecting the ball $B(0, d_0)$, so $M \leq N$ for some fixed number N [by the remark following (H.2)]. $\square$

For each component K, we put $J(K)=\{\gamma \in \Gamma;\ K+\gamma=K\}$. $J(K)$ is a subgroup of Γ.

LEMMA 7.2. — *Let* K *be a component of* $\mathscr{F}(z_0)$. *Then* (i) *and* (ii) *are equivalent*:

(i) K *is compact*,

(ii) $J(K)=0$.

Proof. – It is obvious that (i) $\Rightarrow$ (ii). In order to prove the opposite implication, let us assume that K is non compact but that $J(K)=0$. Let $\gamma_j \in \Gamma$, $j=1, 2, \ldots$ satisfy: $\gamma_j \neq \gamma_k$ when $j \neq k$, and: $\operatorname{dist}(\gamma_j, K) \leqq d_0$ (where d_0 is the number defined above). Since (ii) is assumed to hold, the sets $\underline{K-\gamma_j}$, $j=1, 2, \ldots$ form an infinite family of components, all intersecting $\overline{B(0, d_0)}$. As in the proof of Lemma 7.1 this is impossible. □

Now fix a hyperplane $\mathscr{H} \subset \mathbb{R}^3$ [which later will be $\operatorname{Im}(B)=\operatorname{Ker}(B)^\perp$] and introduce the following additional assumptions:

(H 3) If $x_0 \in \mathscr{F}(z_0)$, then either $d(\lambda|_{x_0+\mathscr{H}}) \neq 0$ at x_0 or $\operatorname{Hess}(\lambda|_{x_0+\mathscr{H}})$ is definite (positive or negative).

Geometrically, (H.3) means that if $x_0+\mathscr{H}$ does not intersect $\mathscr{F}(z_0)$ transversally at x_0, then x_0 is an isolated point of intersection and $x_0+\mathscr{H}$ is tangent to $\mathscr{F}(z_0)$ to precisely the second order.

Our last assumption in this section is:

(H.4) For every $x_0 \in \mathbb{R}^3$ and every component K of $\mathscr{F}(z_0)$, $(x_0+\mathscr{H}) \cap K$ is compact.

If we fix $e_0 \in \Gamma \setminus \mathscr{H}$, then $\mathscr{H}_t = te_0 + \mathscr{H}$, $t \in \mathbb{R}$ constitute the set of affine hyperplanes parallel to $\mathscr{H}$. We shall sometimes identify: $\mathbb{R}^3 = \mathbb{R}_t \times \mathscr{H}$ and write: $x = te_0 + h = (t, h)$, $t \in \mathbb{R}$, $h \in \mathscr{H}$.

LEMMA 7.3. – *Let* K *be a component of* $\mathscr{F}(z_0)$. *Then there are only two possibilities:*

(*a*) K is compact: $K_t = \mathscr{H}_t \cap K$ is non-empty precisely for $a \leqq t \leqq b$, where $-\infty < a < b < +\infty$. Moreover, K_a, K_b are points, while K_t for $a < t < b$ is a simple closed smooth analytic curve.

(*b*) K is non-compact: For every $t \in \mathbb{R}$, K_t is a simple closed smooth analytic curve. Moreover $J(K) = \mathbb{Z} f_0$, for some $f_0 \in \Gamma \setminus \mathscr{H}$. Writing $f_0 = t_0 e_0 + h_0$, $t_0 \in \mathbb{R} \setminus \{0\}$, $h_0 \in \mathscr{H}$, and identifying K_t with its projection in $\mathscr{H}$, we have $K_{t+t_0} = K_t + h_0$.

Proof. – For every $t \in \mathbb{R}$, the components of $\mathscr{F}_t(z_0) = \mathscr{H}_t \cap \mathscr{F}(z_0)$ are either points or simple closed analytic curves. Moreover, there is an $\varepsilon_0 > 0$, independent of t, such that the distance between different components of $\mathscr{F}_t(z_0)$ is $\geqq \varepsilon_0$. (Using the periodicity, the proof of this statement can be reduced to the case when t belongs to a compact set, and we consider two components whose distance is realized by points in a fixed compact set.) A component of $\mathscr{F}_{t_0}(z_0)$ which is reduced to a point: (t_0, h_0) is the limit of components given for neighboring values of t by the equation $g(t, h) =_{\text{def.}} \lambda(t, h) - z_0 = 0$, where g is analytic, $\partial_h g(t_0, h_0) = 0$, $g''_{hh}(t_0, h_0)$ is definite, $\partial_t g(t_0, h_0) \neq 0$. Depending on the sign of $\partial_t g/(\text{sign of } g''_{hh})$, $\mathscr{F}_t(z_0)$ has a component close to h_0 (roughly an ellipse of diameter $\sim |t-t_0|^{1/2}$) either when $t-t_0$ is positive or negative. (For the opposite sign of $t-t_0$, there is no component of $\mathscr{F}_t(z_0)$ close to h_0.) Since there is a minimal separation between components of $\mathscr{F}_t(z_0)$, we see that if K is a component of $\mathscr{F}(z_0)$, then for every t, the set $K_t = K \cap \mathscr{H}_t$ has a t most one component. Moreover, $I = \{ t \in \mathbb{R}; K_t \neq \varnothing \}$ is a closed interval with non-empty interior.

In the case when K is compact, the remarks above imply that (*a*) holds. If K is unbounded, then we know that $J(K) \neq 0$, and we must have $J(K) \cap \mathscr{H} = 0$ since otherwise K_t would not be compact for some t in contradiction with (H.4). From this it follows that I is invariant under translations by $\pm t_1$, where $f_1 = t_1 e_0 + h_1$ is some element in $J(K) \setminus 0$. Hence $I = \mathbb{R}$, and for every $t \in \mathbb{R}$, we see that K_t is a simple closed curve. Moreover (identifying K_t with it's projection in $\mathscr{H}$) we have $K_{t+t_1} = K_t + h_1$. Let $f_2 = t_2 e_0 + h_2$ be another non-vanishing element of $J(K)$, and take a sequence $(n_j, m_j) \in \mathbb{Z}^2$ such that $|n_j| + |m_j| \to \infty$ and $|n_j t_1 - m_j t_2| \leqq \text{Const}$. (This implies that $m_j/n_j \to t_1/t_2$, $j \to \infty$.) Then by the t_1-periodicity property of $K_t - (t/t_1) h_1$, we see that

$$\operatorname{dist}(K_{n_j t_1}, K_{m_j t_2}) \leqq \text{Const}.$$

On the other hand,

$$\begin{aligned}\operatorname{dist}(K_{n_j t_1}, K_{m_j t_2}) &= \operatorname{dist}(K_0 + n_j h_1, K_0 + m_j h_2)\\ &= \operatorname{dist}(K_0, K_0 + m_j h_2 - n_j h_1).\end{aligned}$$

It follows that $\| m_j h_2 - n_j h_1 \| \leqq \text{Const.}$, so $\| h_1 - (m_j/n_j) h_2 \| \to 0$ and then $h_1 = (t_1/t_2) h_2$, $h_2 = (t_2/t_1) h_1$. Hence $f_2 = (t_2/t_1) f_1$, so we have proved that $J(K) \subset \mathscr{R} f_1$.. If follows that $J(K) = \mathbb{Z} f_0$ for some $f_0 \in \Gamma$. Writing $f_0 = t_0 e_0 + h_0$, we then have $K_{t+t_0} = K_t + h_0$. $\square$

Let now $K_1, \ldots, K_M$ be a maximal family as in Lemma 7.1, so that the family of all components is given by $K_{j,\gamma} = K_j + \gamma$, $\gamma \in \Gamma / J(K_j)$. Put:

$$(7.1) \qquad \hat{\Omega}_j = \{ x \in \mathbb{R}^3;\ \operatorname{dist}(x, K_j) = \operatorname{dist}(x, \mathscr{F}(z_0)) \}.$$

If K_j is unbounded, we let $J(K) = \mathbb{Z} f_j$, $f_j = t_j e_0 + h_j$, $t_j > 0$. Then $E_j = \{ t e_0 + h;\ 0 \leqq t < t_j,\ h \in \mathscr{H} \}$ is a fundamental domain for $J(K_j)$. If K_j is bounded, then $E_j = \mathbb{R}^3$ is a fundamental domain for $J(K_j) = 0$. In both cases, we put:

$$(7.2) \qquad \Omega_j = \hat{\Omega}_j \cap E_j.$$

LEMMA 7.4. – *Put* $\Omega^* = \bigcup_1^M \Omega_j$. *Then* Ω^* *is a fundamental domain up to a set of measure* 0, *in the following sense:*

$$(7.3) \qquad (\Omega^* + \gamma) \cap \Omega^* \textit{ is of measure } 0 \textit{ for every } \gamma \in \Gamma \setminus 0.$$

$$(7.4) \qquad \bigcup_{\gamma \in \Gamma} (\Omega^* + \gamma) = \mathbb{R}^3.$$

Proof. — Since

$$\bigcup_{\gamma \in J(K_j)} (\Omega_j+\gamma)=\hat{\Omega}_j,$$

we have

$$\bigcup_{\gamma \in \Gamma} (\Omega_j+\gamma)= \bigcup_{\gamma \in \Gamma/J(K_j)} (\hat{\Omega}_j+\gamma).$$

If $x\in\mathbb{R}^3$, we have $\operatorname{dist}(x, K_{j,\gamma})=\operatorname{dist}_{M}(x, \mathscr{F}(z_0))$ for some $K_{j,\gamma}$. But this means that x belongs to $\hat{\Omega}_{j,\gamma}=_{\text{def.}}\hat{\Omega}_j+\gamma$. Hence $\mathbb{R}^3=\bigcup_{l}\bigcup_{\gamma}(\Omega_j+\gamma)$, and we have proved (7.4). To prove (7.3) it is enough to prove that

(7.5) $\quad (\Omega_j+\gamma)\cap\Omega_k$ is of measure 0 whenever $(j,\gamma)\neq(k,0)$.

Notice that $(\Omega_j+\gamma)\cap\Omega_k\subset(\hat{\Omega}_j+\gamma)\cap\hat{\Omega}_k$. If $K_j+\gamma\neq K_k$, then $(\hat{\Omega}_j+\gamma)\cap\hat{\Omega}_k$ is of measure 0. If $K_j+\gamma=K_k$, then $j=k$ and $\gamma\in J(K_j)$, $\gamma\neq 0$. Then $(E_j+\gamma)\cap E_k=\varnothing$, so $(\Omega_j+\gamma)\cap\Omega_k=\varnothing$. This completes the proof of (7.3). □

Remark 7.5. — For z real and close to z_0, we put

$$\mathscr{F}(z)=\{\theta\in\mathbb{R}^{3*}/\Gamma^*;\ z\in\sigma(P_\theta)\}.$$

Then $\mathscr{F}(z)$ is contained in a neighborhood of $\mathscr{F}(z_0)$ and is given by $z=\lambda(\theta)$. In view of (H.2), the $\mathscr{F}(z)$ for $|z-z_0|\leqq\varepsilon_0$ form a fibration of a neighborhood of $\mathscr{F}(z_0)$.

8. STUDY OF THE TRACE INTEGRALS

To simplify notations, we shall write $Q(h,z;x,\xi,t)$ instead of $\tilde{Q}(hB,z;x'',\xi'',t)$ introduced in section 5, and in the Remark 5.1, and identify functions and domains in θ-space with their pullbacks with respect to l. (We are now restricted to the 3-dimensional case and make all the assumptions of section 7.) Q has the following properties:

1° Q is hermitian (when all the arguments are real).

2° $\operatorname{Det}(Q(0,z;x,\xi,t)$ vanishes on $\mathscr{F}(z)$, when z is real (close to z_0) and the dimension of the kernel is 1, when $(x,\xi,t)\in\mathscr{F}(z)$.

3° In a neighborhood of a component of $\mathscr{F}_{t_0}(z_0)$, we have (microlocally):

$$(8.1)\quad Q(h,z;x,hD_x,t) = A(h,z;x,hD_x,t)\begin{pmatrix} P(h,z;x,hD_x,t) & 0 \\ 0 & B(h,z;x,hD_x,t)\end{pmatrix}A^{-1},$$

where A, P, B are classical symbols of order zero, and $A(h,z;x,hD_x,t)$ denotes the Weyl quantization of $A(h,z;x,h\xi,t)$, etc.. Moreover, A and

B are elliptic of size $N \times N$ and $(N-1) \times (N-1)$ respectively, while P is scalar. We can also choose A to be (formally) unitary. We have the factorization:

$$\mathrm{P}(0, z; x, \xi, t) = c(z, x, \xi, t)(z - \lambda(x, \xi, t)), \tag{8.2}$$

where c is a smooth non-vanishing function of all it's arguments.

When we vary t_0 or the component of $\mathscr{F}_{t_0}(z_0)$, the above result is uniform in the sense that it holds in an ε-neighborhood of the component, with $\varepsilon > 0$ independent of all other parameters, and that we have uniform bounds on all seminorms of A, P, B, c, B^{-1}, c^{-1}.

Combining the arguments of [HeSj 1], section 4, [HeSj 2], section 3.1, 3.3, and [HeSj 3], section 3, we arrive at the following result valid uniformly for: z_1 in a small neighborhood of z_0, $t_1 \in \mathbb{R}$, $|z - z_1| \leqq \varepsilon_0 h$, $|t - t_1| \leqq \varepsilon_0 h$, for some fixed but sufficiently small $\varepsilon_0 > 0$:

THEOREM 8.1. — *For $\delta_0 > 0$ sufficiently small and fixed, we let $\mathscr{C}_{t_1, \delta_0}$ be the set of components of $\mathscr{F}(z_0) \cap \{|t - t_1| \leqq \delta_0\}$. Then there is a constant $h_0 > 0$ such that for $|z - z_1| \leqq \varepsilon_0 h$, $|t - t_1| \leqq \varepsilon_0 h$, $h < h_0$, there is a subset $\Gamma \subset \mathscr{C}_{t_1, \delta_0}$ (depending on z_1, t_1, h) such that the operator:*

$$\mathscr{P}(t, z) = \begin{pmatrix} \mathrm{Q}_t & \mathrm{R}_- \\ \mathrm{R}_+ & 0 \end{pmatrix} : \mathrm{L}^2(\mathbb{R}) \times l^2(\Gamma) \to \mathrm{L}^2(\mathbb{R}) \times l^2(\Gamma) \tag{8.3}$$

is bijective with a bounded inverse of norm $O(h^{-1})$.

Here $\mathrm{Q}_t = \mathrm{Q}(h, z, x, h\mathrm{D}_x, t)$, $\mathrm{R}_- = \mathrm{R}_+^*$, and R_+ is independent of t and z and given by $\mathrm{R}_+ u(\gamma) = (u \mid \varphi_\gamma)$, where φ_γ is normalized in $\mathrm{L}^2(\mathbb{R})$ and localized to γ in the sense that

$$\|\chi\varphi_\gamma\| \leqq \mathrm{C}_\mathrm{N} h^\mathrm{N} \operatorname{dist}(\operatorname{supp}\chi, \gamma)^{-\mathrm{N}},$$

for $\chi = \chi(x, h\mathrm{D}_x, h)$ in a fixed bounded family in $\mathrm{S}^0(\mathbb{R}^2)$ with

$$\operatorname{dist}(\operatorname{supp}\chi, \gamma) \geqq \mathrm{Const} > 0.$$

Moreover, the inverse (depending holomorphically on z):

$$\mathscr{E}(t, z) = \begin{pmatrix} \mathrm{E} & \mathrm{E}_+ \\ \mathrm{E}_- & \mathrm{E}_{-+} \end{pmatrix} \tag{8.4}$$

has the following properties:

(8.5) $\|\mathscr{E}\|_{\mathscr{L}(\mathrm{L}^2 \times l^2, \mathrm{L}^2 \times l^2)} \leqq \mathrm{C}/h$, *and more precisely,*

$$\|\mathrm{E}\| \leqq \mathrm{C}/h, \|\mathrm{E}_\pm\| \leqq \mathrm{C}, \|\mathrm{E}_{-+}\| \leqq \mathrm{C}h.$$

(8.6) $\mathrm{E}_-^* u = \mathrm{E}_+ u = \sum_{\gamma \in \Gamma} \mathrm{E}_+(x, \gamma) u(\gamma)$, *where $\mathrm{E}_+(., \gamma)$ is of norm bounded by a constant and localized to γ in the sense explained above.*

(8.7) *Writing $\mathrm{E}_{-+} = (\mathrm{E}_{-+}(z, t, h; \alpha, \beta))_{\alpha, \beta \in \Gamma}$, we have*

$$|\mathrm{E}_{-+}(\alpha, \beta)| \leqq \mathrm{C}_\mathrm{N} h^\mathrm{N} \operatorname{dist}(\alpha, \beta)^{-\mathrm{N}}$$

for every N *and all* α, β *with* $\alpha \neq \beta$.

(8.8) *The function* $z \mapsto E_{-+}(z, t, h; \alpha, \alpha)$ *has at most one zero for* $|z - z_1| \leqq \varepsilon_0 h$. *This zero is real (if it exists) and is given by a Bohr-Sommerfeld condition:* $f_\alpha(z, t, h) = kh + O(h^\infty)$, $k \in \mathbb{Z}$.

Here f_α *is a classical real valued symbol of order* 0, *independent of the choice of* z_1, *and belonging to a bounded family when* α *varies. Moreover,*

(8.9) $f_\alpha(z, t, 0)$ *is the symplectic area of the domain in* $\mathbb{R}^2_{x,\xi}$, *encercled by the component of* $\mathscr{F}_t(z)$ *which is close to* α [*which is a component of* $\lambda(x, \xi) = z$].

(Actually, it may happen that the component of $\mathscr{F}_t(z)$ that we speak about in (8.9), is empty, but then there is a neighboring value t', for which the component is non-empty, and the symplectic area extends in a C^∞ fashion as a function of t.)

Indications on the proof:

– The set Γ corresponds to the resonant wells for which the quantization condition in (8.8) gives a level in $|z - z_1| \leqq \varepsilon_0 h$ with $t = t_1$. The φ_γ are then quasi-modes corresponding to those wells.

– For the proof of (8.5) we refer to the discussion between (3.11) and (3.21) in [HeSj 3].

– To prove the first part of (8.8), we observe that by Remark 3.4, it follows that $\partial_z Q_t > 0$ is an elliptic operator, and repeating the argument of that remark for the Grushin problem associated to Q_t, we see that:

$$\partial_z E_{-+}(z, t, h; \alpha, \alpha) \leqq \text{Const.} < 0. \tag{8.10}$$

[*Cf.* (8.33).]

From the above properties, we deduce that $\mp \operatorname{Im} E_{-+} \geqq C_0^{-1} |\operatorname{Im} z| I$ for $\operatorname{Im} z \gtrless 0$. Hence for z non-real, E_{-+}^{-1} exists and satisfies,

$$\| E_{-+}^{-1} \| \leqq C_0 / |\operatorname{Im} z|. \tag{8.11}$$

If we further restrict to a region $|\operatorname{Im} z| \geqq h^{N_0}$ for some fixed N_0, we can treat the off diagonal part of E_{-+} as a small perturbation, and we obtain in that region:

$$\| E_{-+}^{-1}(z, t; \alpha, \alpha) - \dot{E}_{-+}(z, t; \alpha, \alpha)^{-1} \| \leqq C_N h^N, \quad \text{for every } N > 0. \tag{8.12}$$

We now turn to the study of the numerator in the right hand side of (5.12). We choose $\tilde{\Omega}$ (identified with Ω^* as stated in the beginning of this section) as in section 7. Letting Ω_j be the domains constructed in the same section, and $\Omega_{j,t} = \{(x, \xi); (x, \xi, t) \in \Omega_j\}$, we are then reduced to the problem of studying

$$\int_{-\infty}^{+\infty} \int \bar{\partial}_z \tilde{F}(z) \iint \chi_{j,t} \operatorname{tr} \sigma(\partial_z Q \circ Q^{-1})(x, \xi, t) \, dx \, d\xi \, L(dz) \, dt \tag{8.13}$$

where Q^{-1}, $\partial_z Q$ denote h-pseudodifferential operators, $\sigma(A)$ is defined as the h-Weyl symbol of A, and $\chi_{j,t}$ is the convolution of $1_{\Omega_{j,t}}$ with an appropriate approximation of δ. [This is not quite the formula (5.12) but rather the version with χ equal to a regularization in the x'', ξ''-variables of the characteristic function of $\tilde{\Omega}$.] Notice that if K_j is unbounded, then the t-integration reduces to an integration over the interval $[0, t_j]$, but that the boundary of this interval will not be "singular" since the x, ξ-integral is periodic with respect to t.

F is assumed to have it's support in $]z_0-\varepsilon_0, z_0+\varepsilon_0[$ for some small fixed ε_0, and satisfy:

$$F(z), F'(z), F''(z) = O(h^{-N_0}), \tag{8.14}$$

for some fixed N_0. Since we can cover $]z_0-\varepsilon_0, z_0+\varepsilon_0[$ by small intervals of size $h/\text{Const.}$, and apply a partition of unity, it will be no loss of generality to assume that F has it's support in one of those intervals, the center of which, we denote by z_1. The extension $\tilde{F}$ can then be taken with support in $|z-z_1| \leq h/\text{Const.}$, satisfying,

$$\tilde{F}, \tilde{F}' = O(h^{-N_0}), \qquad \bar{\partial}\tilde{F} = O(h^{-N_0}|\operatorname{Im} z|). \tag{8.15}$$

For simplicity, we shall write χ instead of $\chi_{j,t}$ and not indicate the t-variable, which for the moment is not of great interest. Using that,

$$Q^{-1} = E - E_+ E_{-+}^{-1} E_-, \tag{8.16}$$

and that E depends holomorphically on z, we see that the integral (8.13) is equal to,

$$-\int J(t)\, dt, \tag{8.17}$$

where $J(t)$ is given by,

$$J = \int \bar{\partial}\tilde{F} \iint \chi \operatorname{tr} \sigma((\partial_z Q) E_+ E_{-+}^{-1} E_-)(x, \xi)\, dx\, d\xi\, L(dz). \tag{8.18}$$

Write, $\partial_z Q E_+ u = \Sigma_\alpha\ ((\partial_z Q) E_+)(x, \alpha) u(\alpha)$. Identifying operators with their kernels, we can then write:

$$J = \sum_\alpha \sum_\beta \int \bar{\partial}\tilde{F}(z) \iint \chi \operatorname{tr} \sigma(((\partial_z Q) E_+)(x, \alpha) \times E_{-+}^{-1}(\alpha, \beta) E_+(y, \beta)^*)(x, \xi)\, dx\, d\xi\, L(dz). \tag{8.19}$$

Here we recall that the h-Weyl symbol, $\sigma(K)$ of an operator K with distribution kernel also denoted by K, is given by:

$$\sigma(K)(x, \xi) = h^{-n} (\mathscr{F}_{h, v \to \xi})(K(x + v/2, x - v/2))(\xi), \tag{8.20}$$

where,

$$\mathscr{F}_h u(\xi) = \mathscr{F} u(\xi/h) = \int e^{-ix\xi/h} u(x)\, dx. \tag{8.21}$$

In the case when,

$$\begin{aligned} \varphi(x) &= \mathrm{C}h^{-1/4} e^{(i/h)((x-x_0)\xi_0 + i(x-x_0)^2/2)}, \\ \psi(x) &= \mathrm{C}h^{-1/4} e^{(i/h)((x-y_0)\eta_0 + i(x-y_0)^2/2)}, \end{aligned} \tag{8.22}$$

are normalized in L^2, and $\mathrm{K}(x, y) = \varphi(x)\overline{\psi(y)}$, we get,

$$\begin{aligned} \sigma(\mathrm{K})(x, \xi) &= \sigma(|\varphi\rangle\langle\psi|)(x, \xi) \\ &= e^{(i/h)}\Bigg[\left(x - \frac{1}{2}(x_0 - y_0)\right)(\xi_0 - \eta_0) - (x_0 - y_0)\xi \\ &\qquad + i\left(x - \frac{1}{2}(x_0 + y_0)\right)^2 + i\left(\xi - \frac{1}{2}(\xi_0 + \eta_0)\right)^2\Bigg] \end{aligned} \tag{8.23}$$

Notice that this symbol is exponentially small outside a neighborhood of $\left(\frac{1}{2}(x_0 + y_0), \frac{1}{2}(\xi_0 + \eta_0)\right)$ and that it is rapidly oscillating near that point, unless $(x_0, \xi_0) = (y_0, \eta_0)$. This leads to the estimate,

$$\begin{aligned} &\iint \chi(x, \xi)\, \sigma(|\varphi\rangle\langle\psi|)(x, \xi)\, dx d\xi \\ &\quad = O(h^{\mathrm{N}})\Bigg(|x_0 - y_0| + |\xi_0 - \eta_0| + \\ &\qquad\qquad \operatorname{dist}\left(\left(\frac{1}{2}(x_0 + y_0), \frac{1}{2}(\xi_0 + \eta_0)\right), \operatorname{supp}\chi\right)\Bigg)^{-\mathrm{N}}, \end{aligned} \tag{8.24}$$

for $\chi \in \mathrm{C}_0^\infty$ and every N, provided that the expression inside the parenthesis is bounded from below by some fixed constant > 0.

Let $\alpha_0 \in \mathscr{F}_{t_1, z_0}$ be the element associated to $\chi = \chi_{j, t}$ (so that α_0 may or may not belong to Γ). The functions $((\partial \mathrm{Q})\mathrm{E}_+)(., \alpha)$ and $\mathrm{E}_+(., \beta)$ are microlocally concentrated to α and β respectively, in the sense explained in Theorem 8.1. Representing these functions as superpositions of Gaussians of the type $\varphi_{(x_0, \xi_0)}$ given by (8.20), we see that

$$((\partial \mathrm{Q})\mathrm{E}_+)(., \alpha) = \iint f(x, \xi)\, \varphi_{(x, \xi)}(.)\, dx\, d\xi + O(h^\infty) \quad \text{in } \mathscr{S}, \tag{8.25}$$

where $|f| \leqq \mathrm{C}h^{-\mathrm{N}_0}$ for some fixed N_0, and

$$|f(x, \xi)| \leqq \mathrm{C}_{\mathrm{N}} h^{\mathrm{N}} \operatorname{dist}((x, \xi), \alpha)^{-\mathrm{N}}, \tag{8.26}$$

for every N, when $\operatorname{dist}((x,\xi),\alpha)\geqq \text{Const.}>0$. The analogous result holds for $E_+(.,\beta)$. Combining this with (8.19), (8.24), (8.11), we obtain

$$J=\sum_{\alpha,\beta\in\Gamma}\sum J_{\alpha,\beta},$$

where

$$(8.27)\qquad |J_{\alpha,\beta}|\leqq C_N h^N(\operatorname{dist}(\alpha,\alpha_0)+\operatorname{dist}(\beta,\alpha_0))^{-N},$$

for every N, when $(\alpha,\beta)\neq(\alpha_0,\alpha_0)$. From this, we conclude that if $\alpha_0\notin\Gamma$, then

$$(8.28)\qquad J=O(h^N),\ \text{for all N},$$

and if $\alpha_0\in\Gamma$, then

$$(8.29)\qquad J=\int\bar\partial\tilde F(z)\iint\chi(x,\xi)\operatorname{tr}\sigma(((\partial_z Q)E_+)(x,\alpha_0)\\ \times E_{-+}^{-1}(z;\alpha_0,\alpha_0)E_+(y,\alpha_0)^*)(x,\xi)\,dx\,d\xi\,L(dz)+O(h^N)$$

for every N. Again, by the fact that $(\partial_z Q)E_+(.,\alpha_0)$ and $E_+(.,\alpha_0)$ are concentrated close to α_0, we can drop χ:

$$(8.30)\qquad J=\int\bar\partial\tilde F(z)\iint\operatorname{tr}\sigma(((\partial Q)E_+)(x,\alpha_0)E_{-+}(z;\alpha_0,\alpha_0)^{-1}\\ \times E_+(y,\alpha_0)^*)(x,\xi)\,dx\,d\xi\,L(dz)+O(h^N).$$

Here we also used (8.15), (8.11), (8.12).

Since the integral of the Weyl symbol of an operator is $2\pi h$ times the trace of the operator, we get modulo $O(h^\infty)$:

$$(8.32)\qquad J\equiv\int\bar\partial\tilde F(z)2\pi h\int\operatorname{tr}[((\partial_z Q)E_+)(x,\alpha_0)\\ \times E_{-+}(z;\alpha_0,\alpha_0)^{-1}E_+(x,\alpha_0)^*]\,dx\,L(dz)\\ \equiv\int\bar\partial\tilde F(z)2\pi h\int E_+(x,\alpha_0)^*\\ \times((\partial_z Q)E_+)(x,\alpha_0)E_{-+}(z;\alpha_0,\alpha_0)^{-1}]\,dx\,L(dz)\\ \equiv\int\bar\partial\tilde F(z)2\pi h(E_+^*(\partial_z Q)E_+)(z;\alpha_0,\alpha_0)E_{-+}(z;\alpha_0,\alpha_0)^{-1}L(dz)\\ \equiv 2\pi h\int\bar\partial\tilde F(z)(E_-(\partial_z Q)E_+)(z;\alpha_0,\alpha_0)E_{-+}(z;\alpha_0,\alpha_0)^{-1}L(dz).$$

Exploiting the fact that the auxiliary operators R_+, R_- are independent of z, we obtain,

$$(8.33)\qquad E_-(\partial_z Q)E_+=-\partial_z E_{-+},$$

which gives with (8.30):

$$(8.34)\quad J=-2\pi h\int \bar{\partial}\tilde{F}(z)\,(\partial_z E_{-+}(z;\alpha_0,\alpha_0))\times E_{-+}(z;\alpha_0,\alpha_0)^{-1}\,L(dz)+O(h^{\infty}).$$

If $z\mapsto E_{-+}(z;\alpha_0,\alpha_0)$ has no zeros in the support of F, then the integral vanishes. If not, let z_2 be the (unique, simple and real) zero of E_{-+} in this region. Then (8.34) simplifies to

$$(8.35)\qquad J=2\pi^2 hF(z_2)+O(h^{\infty}).$$

If we let $g=g_{\alpha_0}$ be the inverse of the function f_{α_0} given in (8.8), (8.9) in Theorem 8.1, we get the following result:

THEOREM 8.2. – *Let* $\mu_t{}^j=(g_{\alpha_0})_*\left(\sum_{k=0}^{\infty} h\delta(.-kh)\right)$ *be the direct image under the map* $z\mapsto g_{\alpha_0}(z,t,h)$. *Then for every* $F\in C_0^{\infty}(]z_0-\varepsilon_0,z_0+\varepsilon_0[)$, *satisfying* (8.14), *we have*

$$(8.36)\quad -\int\bar{\partial}_z\tilde{F}(z)\iint\chi_{j,t}(x,\xi)\,\mathrm{tr}\,\sigma((\partial_z Q_t)\circ Q_t^{-1})(x,\xi,t)\,dx\,d\xi\,L(dz)=2\pi^2\int F(z)\,\mu_t^j(dz)+O(h^{\infty}).$$

9. CONCLUSION CONCERNING THE DENSITY OF STATES

We consider the contribution to the density of states from one component of $\mathscr{F}(z_0)$ (or a part of such a component in the unbounded case). For z in a neighborhood of z_0, we then have a real valued classical symbol of order $0: f=f(t,z;h)\geqq 0$, such that in the case when we are looking at a bounded component then f is defined for $a\leqq t\leqq b$ and vanishing at $t=a$ and b (where $a=a(z,h), b=b(z,h)$) and satisfying

$$(9.1)\quad \partial_t f(a,z;h)>0,\qquad \partial_t f(b,z;h)<0,\qquad \text{and}\qquad f>0\ \text{ for } a<t<b.$$

In the case when we consider the contribution from a part of an unbounded component of $\mathscr{F}(z_0)$, then $f>0$ is a periodic function of t and we let the period be $b-a$. In both cases we have:

(9.2) $f(\tilde{t},z;0)$ is the symplectic area of the bounded domain in the plane $t=\tilde{t}$, whose boundary is the intersection of that plane and $\mathscr{F}(z)$.

From our earlier hypotheses (H.1) and (H.2) in section 7, we conclude that,

$$(9.3)\qquad \partial_z f\neq 0,$$

and in order to fix the ideas, we shall assume most of the time that

$$(9.3)_+ \qquad \partial_z f > 0.$$

We then have the measure μ_t, $a \leqq t \leqq b$, defined in a neighborhood of z_0 by:

$$(9.4) \qquad \int u(z)\,\mu_t(dz) = \sum_{k=0}^{\infty} h\,u(z_k(t,h))$$

for $u \in C_0(\,]z_0-\varepsilon_0, z_0+\varepsilon_0[\,)$. Here $z_k(t,h)$ is defined by,

$$(9.5) \qquad f(t, z_k(t,h); h) = kh.$$

The object of study of this section will be the measure, μ, given by,

$$(9.6) \qquad \int u(z)\,\mu(dz) = \int_a^b \int u(z)\,\mu_t(dz)\,dt = \sum_{k=0}^{\infty} h \int_a^b u(z_k(t;h))\,dt.$$

To see in a simpler case the corresponding formula, it can be useful to compare with the corresponding situation in the free case (see formula (2.3) and (2.4), with $B=h$, $f(z,t) = \frac{1}{2}(z-t)^2 - h$, $a = -b = (2)^{1/2}$).

Let us introduce an additional hypothesis:

(9.7) The critical points of $[a,b] \ni t \mapsto f(t, z_0; 0)$ are all non-degenerate.

We then have only finitely many critical points, $T_1, \ldots, T_m$, which are local non-degenerate maxima or minima. In the case when $t \mapsto f$ is periodic we may assume for simplicity that a, b are non-critical, and in the other case we have the same fact, in view of (9.1). Let $\chi_j \in C_0^\infty(\mathbb{R}; [0,1])$ have its support in a small neighborhood of T_j and be equal to 1 near T_j. Put $\chi = \sum_1^m \chi_j$. We shall first study

$$(9.8) \qquad \int_a^b \int u(z)\,\mu_t(dz)\,(1-\chi(t))\,dt = \sum_0^{\infty} h \int_a^b (1-\chi(t))\,u(z_k(t,h))\,dt.$$

In the region $z_k(t,h) \approx z_0$, we deduce from (9.5) that

$$(9.9) \qquad z_k(t,h) = z(t, kh; h),$$

where $z(t,s;h)$ is a classical symbol of order 0, defined for (t,s) in a neighborhood of the graph of $f(.\,, z_0; 0)$. If we further restrict to $t \in \operatorname{Supp}(1-\chi)$, we see that,

$$(9.10) \qquad \partial_t z_k = -(\partial_t f / \partial_z f)(t, z(t, kh; h); h) \neq 0.$$

Restricting further to $[T_j, T_{j+1}]$, $j = 0, \ldots, M$, with the convention that $T_0 = a - \varepsilon$, $T_{m+1} = b + \varepsilon$, we can introduce the inverse function $t = t_k(z,h)$, given by.

$$f(t_k(z,h),z;h)=kh, \qquad t_k(z,h)=t(z,kh;h), \tag{9.5}$$

and we get,

$$\int (1-\chi(t))\,u(z_k(t,h))\,dt=\int u(z)\,(1-\chi(t(z,kh;h))\,(|\partial_t z_k|)^{-1}\,dz. \tag{9.11}$$

Here,

$$|\partial_z^\upsilon[(1-\chi(t(z,kh;h))/|\partial_t z_k|]\leqq C_\upsilon, \tag{9.12}$$

for $\upsilon=0,1,2,\ldots$, so we deduce that [*cf.* (9.8)],

$$\sum h\int_a^b (1-\chi(t))\,u(z_k(t,h))\,dt=\int u(z)\,m_0(z,h)\,dz \tag{9.13}$$

where $|\partial_z^\upsilon m_0(z,h)|\leqq C_\upsilon$ for every $\upsilon\geqq 0$.

We now look at

$$\sum_0^\infty h\int u(z_k(t,h))\,\chi_l(t)\,dt, \tag{9.14}$$

for some fixed $l\in\{1,\ldots,M\}$. For simplicity, we shall write χ instead of χ_l and we may assume that the corresponding T_l is equal to 0. To fix the ideas, let us assume that 0 is a (non-degenerate) local maximum for the function $t\mapsto f(t,z_0;0)$. This together with $(9.3)_+$ will constitute the case 1. The other three cases (that we will merely comment on) are:

Case 2: $\partial_z f>0$, $t\mapsto f$ has a minimum,

Case 3: $\partial_z f<0$, $t\mapsto f$ has a maximum,

Case 4: $\partial_z f<0$, $t\mapsto f$ has a minimum.

We again consider the equation (9.5), now with t in a small neighborhood of 0, and for k such that $z_k(t,h)=z(t,kh;h)$ is close to z_0. Since $\partial_z f>0$, the function $t\mapsto z_k(t,h)$ will have a critical point precisely when $(\partial_t f)(t,z_k(t,h);h)=0$. Using that $\partial_t^2 f>0$, we see that the equation,

$$\partial_t f(t,z;h)=0, \tag{9.15}$$

has a unique solution $t=\tau(z;h)$ close to 0, and that τ is a classical symbol of order 0, satisfying $\tau(z_0;0)=0$. The possible critical values $\zeta_k(h)$ are then given by substitution into (9.5):

$$f(\tau(\zeta_k,h),\zeta_k;h)=kh. \tag{9.16}$$

Since $\partial_\zeta(f(\tau(\zeta;h),\zeta;h)=(\partial_z f)\,(\tau(\zeta;h),\,\zeta;h)>0$, we have a (locally) unique solution of (9.16):

$$\zeta_k(h)=\zeta(kh;h), \tag{9.17}$$

where ζ is a classical symbol of order 0 with $\partial_s\zeta(s;h)>0$. Summing up, we have found the critical value, $\zeta(kh;h)$ and the corresponding critical point $t=\tau(\zeta(kh;h);h)=t(kh;h)$ for the function $t\mapsto z_k(t;h)$. Differentiating (9.5) twice and restricting to the critical point just determined, we get,

$$(9.18)\quad \partial_t^2 z(t(kh;h),kh;h)=-[(\partial_t^2 f)/(\partial_z f)](t(kh;h),z(t(kh;h),h);h)>0,$$

since $\partial_t^2 f<0$, $\partial_z f>0$. (More generally this quantity is negative in the cases 2 and 3 and positive in the case 4.)

For $z\geqq\zeta_k(h)$, we can define two solutions $t_k^\pm(z,h)$ with

$$t_k^-(z,h)\leqq t(kh,h)\leqq t_k^+(z,h),$$

of the equation $(9.\tilde{5})$, that we write in the form,

$$(9.19)\qquad z(t_k^\pm(z,h),kh;h)=z.$$

Thanks to (9.18), we have

$$(9.20)\qquad z(t,kh;h)-\zeta_k(h)=g(t-t(kh,h),kh;h)(t-t(kh,h))^2,$$

where $g>0$ is a classical symbol of order 0, satisfying

$$(9.21)\qquad g(0,kh;h)=-\frac{1}{2}[(\partial_t^2 f)/(\partial_z f)](t(kh;h),\zeta(kh;h);h).$$

We can then rewrite (9.19) as

$$(9.22)\quad (g(t_k^\pm(z;h)-t(kh;h),kh;h))^{1/2}(t_k^\pm-t(kh;h))=\pm(z-\zeta_k(h))^{1/2},$$

from which we deduce that:

$$(9.23)\quad t_k^\pm(z;h)-t(kh;h)=\pm(z-\zeta_k(h))^{1/2}j(\pm(z-\zeta_k(h))^{1/2},kh;h),$$

where $j>0$ is a classical symbol of order 0, satisfying

$$(9.24)\quad j(0,kh;h)=\left[-\frac{1}{2}[(\partial_t^2 f)/(\partial_z f)](t(kh;h),\zeta(kh;h);h)\right]^{-1/2}.$$

Differentiating (9.23), we obtain

$$(9.25)\qquad \partial_z t_k^\pm(z;h)=\pm(z-\zeta_k(h))^{-1/2}l(\pm(z-\zeta_k(h))^{1/2},kh;h),$$

where $l>0$ is a classical symbol of order 0 and $l(0,kh;h)=\frac{1}{2}j(0,kh;h)$.

The k:th integral appearing in (9.14) can be rewritten as

$$(9.26)\quad \int u(z_k(t,h))\chi(t)\,dt$$
$$=\int_{\zeta(kh;h)}^{+\infty} u(z)\,|\partial_z t_k^+(z;h)|\,\chi(t_k^+(z;h))\,dz$$
$$+\int_{\zeta(kh;h)}^{+\infty} u(z)\,|\partial_z t_k^-(z;h)|\,\chi(t_k^-(z;h))\,dz$$
$$=\int_{\zeta(kh;h)}^{+\infty} u(z)\,m(z,kh;h)\,|z-\zeta_k(h)|^{-1/2}\,dz.$$

where

$$m(z,kh;h)=\chi(t_k^+(z;h))\,l((z-\zeta_k(h))^{1/2},kh;h)\\ +\chi(t_k^-(z;h))\,l(-(z-\zeta_k(h))^{1/2},kh;h)$$

is a non-negative symbol of order 0 satisfying $m(\zeta_k(h),kh;0)>0$ and vanishing for z in a neighborhood of z_0 when $\zeta_k(h)\leqq z_0-\text{Const}$. (The last property is due to the presence of the cutoff function.) When $\zeta_k(h)$ is close to z_0, we have

$$(9.27)\quad m(\zeta(kh;h),kh;h)\\ =j(0,kh;h)=\left[-\frac{1}{2}[(\partial_t^2 f)/(\partial_z f)](t(kh;h),\zeta(kh;h);h)\right]^{-1/2}.$$

When u has its support in a small but fixed neighborhood of z_0, we can rewrite the sum (9.14) as

$$(9.28)\qquad \int u(z)\,b(z,h)\,dz,$$

with

$$(9.29)\qquad b(z,h)=\sum hm(z,kh;h)\,\mathrm{H}(z-\zeta_k(h))\,|z-\zeta_k(h)|^{-1/2}.$$

Here $\mathrm{H}=1_{[0,+\infty[}$ is the standard Heaviside function. In the case 4 we get the same result, and in the cases 2 and 3 we get (9.29) but with $\mathrm{H}(z-\zeta_k)$ replaced by $\mathrm{H}(-(z-\zeta_k))$.

We are now ready to summarize the results of the last three sections into the following rather long theorem:

THEOREM 9.1. – *We take* $n=3$, *we fix* $\mathrm{B}_0\neq 0$ *and put* $\mathrm{B}=h\mathrm{B}_0$. *We make the assumptions* (H.1-4) *of section* 7. *Let* $\mathscr{H}=\mathrm{Im}(\mathrm{B}_0)\subset\mathbb{R}^{3*}$, *and choose* $e_0\in\Gamma\setminus\mathscr{H}$, *so that* $\mathbb{R}^{3*}\cong\mathbb{R}\times\mathscr{H}$, *via* $x=te_0+h$, $t\in\mathbb{R}$, $h\in\mathscr{H}$. *We define* Ω_j, $j=1,\dots,\mathrm{M}$, *as in the end of section* 7. *Let* $\Sigma_j(z)=\Omega_j\cap\mathscr{F}(z)$ *and let* $[a_j,b_j]$ *be the t-projection of* $\Sigma_j(z)$. *Let* $f_j(t,z;0)$, *for* $a_j(z)\leqq t\leqq b_j(z)$ *be the area with respect to the dual form*, B_0^*, *of* B_0, *of the bounded domain in* $te_0+\mathscr{H}$ *with boundary* $\Sigma_j(z)\cap(te_0+\mathscr{H})$. *For every j, we make the assumption* (9.7) *about the function* $t\mapsto f_h(t,z_0;0)$. *Then there exist classical symbols* $f_j(t,z;h)$, *extending* $f_j(t,z;0)$ *in the natural way, defined for z close to* z_0 *and t close to* $[a_j(z_0),b_j(z_0)]$, *so that the following holds:*

For $\mathrm{F}\in\mathrm{C}_0^\infty(]z_0-\varepsilon_0,z_0+\varepsilon_0[)$ (*where* $\varepsilon_0>0$ *is small but fixed*), *satisfying* (8.14) *for some fixed* N, *we have,*

$$(9.30)\quad \widetilde{\mathrm{tr}}\,\mathrm{F}(\mathrm{P}_{h\mathrm{B}_0,\mathrm{V}})\\ =(\pi\,\mathrm{Vol}(\mathbb{R}^3/\Gamma)\times\mathrm{Vol}_{\mathrm{B}_0^*\wedge dt}(\mathbb{R}^{3*}/\Gamma^*))^{-1}\int\mathrm{F}(z)\,b(z)\,dz+O(h^\infty),$$

where $b \in L^1$ is independent of F *and can be written as a finite sum:*

$$b_0(z) + \sum_{j=1}^{M} \sum_{l=1}^{m(j)} b_{j,l}(z).$$

Here $|\partial_z^\upsilon b_0(z)| \leqq C_\upsilon$ *(independent of* h*) and* $b_{j,l}$ *is the contribution from the critical point,* $T_{j,l}$ *of the function* $t \mapsto f_j(t, z_0; 0)$*. To describe this contribution, we drop the subscript* j *and assume for simplicity that* $T_{j,l} = 0$*. (*j*,* l *are now fixed.) Let* $\tau(\upsilon h; h)$*,* $\zeta(\upsilon h; h)$ *be the solution of the system,*

$$(9.31)\quad \partial_t f(\tau(\upsilon h; h), \zeta(\upsilon h; h); h) = 0, \qquad f(\tau(\upsilon h; h), \zeta(\upsilon h; h); h) = \upsilon h,$$

for $\upsilon \in \mathbb{N}$ *with* υh *close to* $f((0, z_0; 0)$*, and with* (τ, ζ) *close to* $(0, z_0)$*. Then,* τ*,* ζ *are classical symbols of order* 0*, and we have:*

$$(9.32)\quad b(z,h) = \sum_{\{\upsilon \in \mathbb{Z};\, |\upsilon h - f(0, z_0; 0)| \leqq \delta_0\}} hm(z, \upsilon h; h)\, |z - \zeta(\upsilon h; h)|^{-1/2}\, H(\pm(z - \zeta(\upsilon h; h))),$$

where $\delta_0 > 0$ *is small and fixed, and* $m(z, s; h)$ *is a classical symbol of order* 0 *such that,*

$$(9.33)\quad m(\zeta(\upsilon h; h), \upsilon h; h) = \left| \frac{1}{2} [(\partial_t^2 f)/(\partial_z f)](\tau(\upsilon h; h), \zeta(\upsilon h; h); h) \right|^{-1/2}.$$

The $\pm$ *sign in* (9.32) *is that of* $-(\partial_t^2 f)/(\partial_z f)$*. Notice that a modification of* δ_0 *will only lead to a modification of the term* b_0 *discussed above.*

Remark 9.2. – The relation (9.31) is given by Onsager [On].

Remark 9.3. – By an approximation argument we can replace the assumption (8.14) by the assumption that the Hölder norm of order α is of temperate growth when $h \to 0$, for some $\alpha \in]0, 1[$. *See* Proposition 10.4.

10. DE HAAS-VAN ALPHEN EFFECT IN THE GENERAL CASE

We now want to present the analogue of formula (2.16), (2.19) in section 2. Let us recall the definition of the energy per unit volume in the case of temperature 0:

$$(10.1)\qquad \Omega(z_0, B, N, 0) = N z_0 - \widetilde{\mathrm{Tr}}\, f_{z_0, 0}(P_{hB_0, V})$$

with $f_{z_0, 0}(s) = (z_0 - s)\, 1_{s < z_0} = (z_0 - s)_+$.

In the limit T equal to 0, we get immediately a problem in the definition of the susceptibility whose tentative definition would be:

$$(10.2)\qquad \chi(z_0, B, N, 0) = -(1/h)(d/dh)(\widetilde{\mathrm{Tr}}\, f_{z_0, 0}(P_{hB_0, V}))$$

because we don't know if the energy per unit volume is differentiable with respect to h.

In the same way, a tentative definition for the Fermi level z_0 would be given by the equation:

$$N = -(\widetilde{\mathrm{Tr}}\, f'_{z_0,\,0}(P_{hB_0,\,V}))$$

More precisely, because the left hand side is monotone but not necessarily continuous or strictly monotone, it is better to take the following definition which will have a sense in any case; let us define the counting function per unit volume by:

$$\mathcal{N}(z, B) = 1/2\,(\widetilde{\mathrm{tr}}\,(1_{]-\infty,\,z[} + 1_{]-\infty,\,z]})\,(P_{B,\,V})) \tag{10.3}$$

Then for N given, the Fermi level is obtained by:

$$z_0 = \inf_z \{ \mathcal{N}(z, B) = N \}. \tag{10.4}$$

We are interested in the behavior as h tends to 0 of these 3 expressions. We refer to the section 2 for the results in the free case which will be the inspiring model for the general case and for which the tentative definition appears to be correct if one replaces (10.2) by the limit as T tends to 0 of the expression (10.2) (with $f_{z_0,\,T} = T.\mathrm{Log}(1 + e^{-(z_0 - s)/T})$ instead of $f_{z_0,\,0}$).

In the case when the function F is C^∞ (and independent of h) we have seen in section 6 that all the quantities described in (10.1) and (10.2) (with $f_{z_0,\,0}$ replaced by F, for example $f_{z_0,\,T}$) are C^∞ with respect to h. The new phenomenon occurs because $f_{z_0,\,0}$ is not a C^∞ function but only in the Lipschitz class $C^{0,\,1}$. Of course, this is only a limit case (which can be seen as non-physical) but it is a good model when the temperature decreases more rapidly than a sufficiently large power of h.

In theorem (9.1) we have described quite precisely the measure:

$$F \to \widetilde{\mathrm{Tr}}\, F\,(P_{hB_0,\,V})$$

but modulo an error term which is $O(h^\infty)$ for F satisfying to (8.14) (*see* also remark 9.3).

It seems that this remainder term can hide not only technical problems but also deep problems relative to the nature of the spectrum. In particular, we are unable to give the exact equivalent of the results in the free case concerning the susceptibility and the Fermi level.

However we can prove the following results:

THEOREM 10.1: asymptotics of the energy per unit volume. — *We keep the same hypotheses and notations as in theorem* 9.1. *Then we have the following expansion for* $\Omega(z_0, h B_0, N, 0)$ *with respect to h; there exists classical symbols in h:* $c_0(h)$, $d^{jl}_\gamma(h)$, $s_{jl}(z, h)$ $\left(\gamma \in \wedge := \left\{\frac{3}{2} + \mathbb{N}\right\},\ j = 1\right.$

to M, $l=1$ *to* $m(j)\Big)$ *of order* 0 *such that:*

$$(10.5)\quad \Omega(z_0, h\mathrm{B}_0, \mathrm{N}, 0) = c_0(h) + \sum_{j=1}^{\mathrm{M}} \sum_{l=1}^{m(j)} \sum_{\gamma \in \wedge} h^{\gamma+1} d_\gamma^{jl}(h) \,.\, \tilde{\rho}(\gamma, s_{jl}(z_0) h^{-1} + y_{jl}(h)) + O(h^\infty)$$

with

(*a*) $s_{jl}(z_0) = s_{jl}(z_0, 0) = f_j(\mathrm{T}_{jl}(z_0); 0)$

(*b*) $y_{jl}(h) = (s_{jl}(z_0; h) - s_{jl}(z_0; 0))/h$

(*c*) $c_0(h) = c_{00} + c_{02} h^2 + O(h^3)$

(*d*) $c_{00} = \mathrm{N} z_0 - c'_{00}(z_0)$

where:

$$(e)\quad c'_{00}(z_0) = (\mathrm{Vol}_{\mathrm{B}_0^* \wedge dt}(\mathbb{R}^3/\Gamma^*))^{-1} \sum_p \int_{(\lambda_p(\theta) < z_0) \cap (\mathbb{R}^3/\Gamma^*)} (z_0 - \lambda_p(\theta))\, d\theta$$

where the $\lambda_p(\theta)$ *are the floquet eigenvalues.*

$$(f)\quad d_{(3/2)}^{jl}(0) = -(4/3)(\pi \,\mathrm{vol}(\mathbb{R}^3/\Gamma))^{-1} \times (\mathrm{Vol}_{\mathrm{B}_0^* \wedge dt}(\mathbb{R}^3/\Gamma^*))^{-1} \times |(\partial_z f_j)(\mathrm{T}_{jl}(z_0), z_0; 0)|^{(-1/2)} \times \left| \frac{1}{2} (\partial_t^2 f_j)(\mathrm{T}_{jl}(z_0), z_0; 0) \right|^{-1/2}$$

$$(g)\quad \tilde{\rho}(\gamma, \sigma) = 2^{-\gamma} . \Gamma(\gamma+1) . \left(\sum_{n>0} (\pi n)^{-\gamma-1} \cos\left(2\pi n \sigma - \frac{\pi}{2}(\gamma+1) \right) \right)$$

The principal terms in this formula are given in many standard books in solid state physics (*see* for example formula 3.75 in White [Wh]). The dominant perturbation induced by B on Ω is given by the diamagnetic term $c_{02} h^2$; consequently the de Haas-van Alphen effect is difficult to see; this is not the same thing for the asymptotic expansion we get by differentiating formally with respect to h (we do not know if the remainder can be differentiated); we shall call this expression the formal susceptibility χ^f [which is defined modulo $O(h^\infty)$]:

THEOREM 10.2: asymptotics of the formal susceptibility. – *We keep the same hypotheses and notations as in theorem* 9.1. *Then we have the following expansion for* $\chi^f(z_0, h\mathrm{B}_0, \mathrm{N}, 0)$ *with respect to h; there exists classical symbols in* $h: e_0(h), f_\delta^{jl}(h)\left(\delta \in \Delta := \left\{ -\frac{1}{2} + \mathbb{N} \right\}, j=1 \text{ to } \mathrm{M}, l=1 \text{ to } m(j)\right)$ *of order* 0 *such that:*

$$(10.6)\quad \chi^f(z_0, h\mathrm{B}_0, \mathrm{N}, 0) = e_0(h) + \sum_{j=1}^{\mathrm{M}} \sum_{l=1}^{m(j)} \sum_{\delta \in \Delta} h^\delta f_\delta^{jl}(h) \times \tilde{\rho}(\delta+1, s_{jl}(z_0) h^{-1} + y_{jl}(h)) + O(h^\infty)$$

(*a*) $$e_0(h)=e_0(0)+O(h)$$

(*b*) $$e_0(0)=2c_{02}$$

(*c*) $$f^{jl}_{-(1/2)}(0)=-4(\mathrm{vol}(\mathbb{R}^3/\Gamma))^{-1}\times(\mathrm{Vol}_{\mathrm{B}_0^*\wedge dt}(\mathbb{R}^3/\Gamma^*))^{-1}$$
$$\times f_j(\mathrm{T}_{jl}(z_0);0)\times|(\partial_z f_j)(\mathrm{T}_{jl}(z_0),z_0;0)|^{(-1/2)}\times\left|\frac{1}{2}(\partial_t^2 f_j)(\mathrm{T}_{jl}(z_0),z_0;0)\right|^{-1/2}$$

In particular, the dominant term in the expansion is given modulo $O(1)$ *by:*

$$\sum_{j=1}^{\mathrm{M}}\sum_{l=1}^{m(j)}h^{-1/2}f^{jl}_{-(1/2)}(0)\,\tilde{\rho}(1/2,s_{jl}(z_0)h^{-1}+y_{jl}(h)). \tag{10.7}$$

This formal susceptibility will probably give the behavior of the susceptibility for small temperature [in the sense that we have: $(1/\mathrm{C}_0)\ h^{\mathrm{M}_0}\leqq\mathrm{T}\leqq\mathrm{C}_0 h^{\mathrm{M}_1}$ where M_0 and M_1 are real numbers s.t.: $2\leqq\mathrm{M}_1\leqq\mathrm{M}_0$, and C_0 is a fixed constant independent of h]. Because we have no satisfactory theorem in this direction, we omit to developp this idea.

Another problem was that the Fermi level $z_0(h)$ depends in fact of h and was in principle determined by the condition (10.3) and (10.4). So we need at least the corresponding asymptotic formula for $\mathcal{N}(z,\mathrm{B})$ as B tends to 0. This will permit to have an approximation of the Fermi level.

THEOREM 10.3: asymptotics of the counting function per unit volume. — *We keep the same hypotheses and notations as in theorem* 9.1. *Then we have the following expansion for* $\mathcal{N}(z_0,h\mathrm{B}_0)$ *with respect to h; there exists classical symbols in* h: $b_0(h)$, $c^{jl}_\delta(h)$ ($\delta\in\wedge$, $j=1$ *to* M, $l=1$ *to* $m(j)$) *such that:*

$$\mathcal{N}(z_0,h\mathrm{B}_0)=b_0(h)+\sum_{j=1}^{\mathrm{M}}\sum_{l=1}^{m(j)}\sum_{\delta\in\wedge}h^{\delta}f^{jl}_{\delta}(h)\times\tilde{\rho}(\delta,s_{jl}(z_0).h^{-1}+y_{jl}(h))+O(h^{\infty}) \tag{10.8}$$

(*a*) $$b_0(h)=b_{00}(z_0)+O(h^2)$$

(*b*) $$b_{00}(z_0)=(\mathrm{Vol}_{\mathrm{B}_0^*\wedge dt}(\mathbb{R}^3/\Gamma^*))^{-1}\sum_p\int_{(\lambda_p(\theta)<z_0)\cap(\mathbb{R}^3/\Gamma^*)}d\theta.$$

The proof of theorems 10.1, 10.2, 10.3 are similar, so we shall make some emphasis on the first of these theorems and we shall just explain the points which are different for the two other theorems.

First reduction. — Let us also recall that, according to the results of section 6 (particularly remark 6.5), we have for each χ with right compact support in $]-\infty,z_0[$ the following property:

(10.9) $\tilde{\mathrm{T}}\mathrm{r}\,\chi f_{z_0,0}(\mathrm{P}_h)$ is a classical symbol with respect to h with no even terms.

If we are interested in the singularities (here we mean by "singularities" the terms in the expansion which can not be written as a classical symbol with respect to h) of the energy per unit volume, we get that it is sufficient to analyze: $\tilde{\operatorname{Tr}}\, \theta_{z_0} . f_{z_0, 0}(P_h)$ for some C^∞ function θ_{z_0} equal to 1 near z_0 and with compact support in a neighborhood of z_0 such that we can apply the results of section 9. The problem is that: $F = \theta_{z_0} f_{z_0, 0}$ is not a C^∞ function so we can not apply directly the results of theorem 9.1. To circumvent this problem we shall prove the following improvement of theorem 9.1 (mentioned in remark 9.3).

PROPOSITION 10.4. – *The conclusions of theorem* 9.1 *are true under the weaker assumptions* (*instead of* 8.14) *on* F:

$$F = O(h^{-N_0}) \tag{10.10}$$

(10.11) *There exists* $\alpha > 0$, *some* C *and some* N_0 *s. t.* $\forall x, y$:

$$|F(x+y) - F(x)| \leqq C h^{-N_0} |y|^\alpha.$$

Proof. – Let us introduce, for $M \in \mathbb{N}$, $\Psi_{h, M}(x) = h^{-M} \Psi(h^{-M} x)$ where Ψ is a C^∞ positive function with compact support in $\mathbb{R}$ s. t. $\int \Psi(x)\, dx = 1$.

If F satisfies (10.10) and (10.11), we introduce $F_{h, M} = F * \Psi_{h, M}$. Let us observe that we can apply theorem 9.1 with $F_{h, M}$. To recover the result we have just to observe the following facts:

$$\sup_x |F(x) - F_{h, M}(x)| \leqq C h^{\alpha M - N_0}. \tag{10.12}$$

(10.13) The measure associated to b is temperate with respect to h^{-1}

(10.14) The measure $\rho_{hB_0, V}$ is temperate with respect to h^{-1}

Using (9.30) for $F_{h, M}$ and (10.11)-(10.14) and playing with M we get easily (9.30) for F.

According to (9.30), we are now reduced modulo $O(h^\infty)$ to the study of:

$$\int F(z)\, b(z)\, dz \tag{10.15}$$

where $F(z) = \theta(z - z_0) f_{z_0, 0}(z)$ with support of θ small enough in a neighborhood of 0 in order to apply theorem 9.1.

Preliminaries. — Let us collect some technical results which will be useful in the proof:

LEMMA 10.5 (see [He-Ro 2] and §1). — *If f is C^∞ with right compact support then:*

(10.16) $h(\sum_{j \in \mathbb{N}} f(jh))$ *is a classical symbol with respect to* h.

The value at 0 is given of course by:

$$\int_0^\infty f(t)\,dt. \tag{10.17}$$

Because we have to work with a non regular function f, it is natural to imagine that we want to get (10.6) modulo $O(h^N)$, then we have to assume that f is in some class $C^{k(N)}$ for $k(N)$ sufficiently large. This is probably easy to prove by following the proof of the C^∞ case and will be sufficient for what we need. But it was proved for other purpose in [He-Ro 3] the following lemma [*see* (2.12)]:

LEMMA 10.6. — *Let* $\gamma \in \mathbb{R}^+$, $s_0 \in \mathbb{R}$, g *a* C^∞ *function on* $\mathbb{R}$. *Then:* $h(\sum_{j \in \mathbb{N}} g(jh)(s_0 - jh)_+^\gamma)$ *is a classical symbol of order* 0 *modulo* $O(h^{\gamma+1})$

LEMMA 10.7. — *Let* $\gamma \in \mathbb{R}^+$, $\zeta(s, h)$ *a* C^∞ *function in*

$$[s_0 - \varepsilon_0, s_0 + \varepsilon_0] \times [0, h_0],$$

such that $(\partial_s \zeta)(s, h) > 0$, $\zeta(s_0; 0) = z_0$, f *a* C^∞ *function with compact support in* $]s_0 - \varepsilon_0, s_0 + \varepsilon_0[$, *then:*

$$I_{\gamma, z_0, \zeta, f}(h)\,(= h(\sum_{j \in \mathbb{N}} (z_0 - \zeta(jh; h))_+^\gamma f(jh))) \text{ is } O(h^{1+\gamma}) \tag{10.18}$$

modulo a classical symbol.

More precisely:

$$\begin{aligned} I_{\gamma, z_0, \zeta, f}(h) = c_\gamma(h) + h^{\gamma+1} \\ \times (\sum_{\delta \in \mathbb{N}} h^\delta . a_{\gamma, \delta}(h)\, \tilde{\rho}(\gamma + \delta, s(z_0; h) . h^{-1})) + O(h^\infty) \end{aligned} \tag{10.19}$$

where:

$c_\gamma(h)$ *and* $a_{\gamma, \delta}$ *are classical symbols of order* 0,

$s(z_0; h)$ *is determined by the implicit equation:*

$$\zeta(s, h) = z_0, \qquad s(z_0, 0) = s_0, \qquad a_{\gamma, 0}(0) = |(\partial_s \zeta)(s_0; 0)|^\gamma . f(s_0),$$

the summation with respect to δ *is asymptotic.*

Proof of lemma 10.7. — This lemma is an extension of lemma 10.6 and was proved in some particular cases in section 2. Let us start from the results we got:

[*a*] If $\zeta(s,h)=s$, then we have proved that if $f=1$:

$$(10.20)\quad \mathrm{I}_{\gamma, z_0, s, f}(h) = h^{\gamma+1}\,\tilde{\rho}(\gamma, z_0 h^{-1}) \quad \text{(modulo a classical symbol of order 0)}$$

with:

$$(10.21)\quad \tilde{\rho}(\gamma,\sigma)=\Gamma(\gamma+1).2^{-\gamma} \times\left(\sum_{n>0}(\pi n)^{-\gamma-1}\cos\left(2\pi n\sigma-\left(\frac{\pi}{2}\right)(\gamma+1)\right)\right)$$

[*b*] Let us now consider the case where $\zeta(s,h)=s$, but only $f=1$ in a neighborhood of z_0. Then (10.20) is still true because the function: $s\to(1-f)(s)(z_0-s)_+^{\gamma}$ is a C^∞ function and we can apply lemma (10.5).

[*c*] When f is not equal 1 in a neighborhood of z_0, with compact support in a small neighborhood of z_0, we can perform a Taylor expansion of f at z_0 and write:

$$f(z)=\sum_{0\leq k\leq \mathrm{N}}(z-z_0)^k f^{(k)}(z_0)/k!+(z-z_0)^{\mathrm{N}+1}f_{\mathrm{N}}(z,z_0)$$

and we get:

$$\begin{aligned}\mathrm{I}_{\gamma, z_0, \zeta, f}(h) &= h\Big(\sum_{j\in\mathbb{N}}[(z_0-jh)_+^{\gamma}\Big(\sum_{0\leq k\leq \mathrm{N}}(-1)^k(z_0-jh)^k f^{(k)}(z_0)/k!\Big)\tilde{f}(jh)]\\ &\quad + h\Big(\sum_{j\in\mathbb{N}}[(z_0-jh)_+^{\gamma}((z_0-jh)^{\mathrm{N}+1}f_{\mathrm{N}}(jh,z_0)f(jh)]\\ &= h\Big(\sum_{0\leq k\leq \mathrm{N}}(f^{(k)}(z_0)/k!)\sum_{j\in\mathbb{N}}(z_0-jh)_+^{\gamma+k}\tilde{f}(jh))\Big)+r_{\mathrm{N}}(h)\end{aligned}$$

where: $\tilde{f}$ is equal to 1 in a neighborhood of z_0 and satisfies $\tilde{f}f=f$

$$r_{\mathrm{N}}=h\Big(\sum_{j\in\mathbb{N}}[(z_0-jh)_+^{\gamma+\mathrm{N}+1}f_{\mathrm{N}}(jh,z_0)f(jh)].$$

Now observe that r_{N} is a classical symbol modulo $O(h^{\mathrm{N}+1+\gamma})$ according to lemma (10.6). This finishes the proof in this case.

[*d*] Let us now consider the general case. As we have observed before we can always suppose that the support of f is very small around s_0 and it is also clear that the proof of case [*c*] extends to the case where $f=f(.,h)$ is a symbol with respect to h. Let us consider:

$$h\Big(\sum_{j\in\mathbb{N}}(z_0-\zeta(jh;h))_+^{\gamma}f(jh)\Big)$$

and observing that there exists a symbol $s(z,h)$ s.t. $\zeta(s(z,h),h)=z$, we rewrite this expression as:

$$h\Big(\sum_{j\in\mathbb{N}}(s_0-jh)_+^{\gamma}\,\tilde{g}(jh,h,s_0)\Big)$$

where $\tilde{g}(z, h, s_0)$ is a symbol with respect to h for which we can apply the extension of [c] mentioned above. The explicitation of the computation gives the lemma.

Proof of theorem 10.1. — We have to consider the term we mentioned in 10.15:

$$(10.22)\qquad (\pi \operatorname{vol}(\mathbb{R}^3/\Gamma))^{-1} \times (\operatorname{Vol}_{B_0^* \wedge dt}(\mathbb{R}^3/\Gamma^*))^{-1} \int F(z)\, b(z)\, dz$$

with:

$$(10.23)\qquad F(z) = \theta(z - z_0)(z_0 - z)_+$$

$$(10.24)\qquad b(z) = b_0 + \sum_{j=1}^{M} \sum_{l=1}^{m(j)} b_{j,l}$$

Let us denote by I_0, I_{jl} the integrals:

$$(10.25)\qquad I_0 = \int F(z)\, b_0(z)\, dz, \qquad I_{jl} = \int F(z)\, b_{j,l}(z)\, dz.$$

We shall study these different quantities successively.

Study of I_0. — b_0 appears in 9.11 and is a sum (over the different components) of terms of the form:

$$h \sum_k (1 - \chi(t(z, kh, h)))(|(\partial_t z_k)(t(z, kh, h), kh; h)|)^{-1}$$

that we can rewrite as: $\Psi(z, h) = h \sum_k \varphi(z, kh, h)$ where

$\varphi(z, s, h)$ is a classical symbol with respect to h regular with respect to z and s with compact support in s (*see* between 9.9 and 9.10).

We have:

$I_0 = h \sum_k \varphi_1(z_0, kh, h)$

where

$$\varphi_1(z_0, s, h) = h \int_{-\infty}^{z_0} \theta(z - z_0)(z_0 - z)\, \varphi(z, s, h)\, dz.$$

It is then clear that:

(10.26) I_0 is a classical symbol of order 0.

Study of I_{jl}. — We return to formulas (9.31) and (9.32) and forgetting the reference to j and l, recall that:

$$(10.27)\qquad b(z, h) = \sum hm(z, kh; h)\, |z - \zeta(kh; h)|^{-(1/2)}\, H(\pm(z - \zeta(kh; h)))$$

where we have the sign + (resp. −) if $(-(\partial_t^2 f)/\partial_z f))$ is >0 (resp. <0), where $m(z, s; h)$ is a classical symbol such that (*cf.* 9.33):

$$(10.28)\quad m(\zeta(kh; h), kh; h) = \left|\frac{1}{2}[(\partial_t^2 f)/\partial_z f)](\tau(kh; h), \zeta(kh; h); h)\right|^{-1/2}$$

and where $\tau(s; h)$, $\zeta(s; h)$ are classical symbols defined for s near $f_j(\mathrm{T}_{jl}, z_0; 0)$ written more simply $f(0, z_0; 0)$.

All these properties are true for z in a neighborhood of z_0.

So let us compute I_{jl} in the case where $(-(\partial_t^2 f)/\partial_z f))$ is >0:

$$\mathrm{I}_{jl} = \int \mathrm{F}(z)\, b_{j,l}(z)\, dz$$
$$= h\left(\sum_{\zeta(kh;h) < z_0} \int_{\zeta(kh;h)}^{z_0} \theta(z - z_0)(z_0 - z)\, m(z, kh; h)\, |z - \zeta(kh; h)|^{-(1/2)}\, dz\right)$$

Let us take the Taylor expansion of $\theta(z - z_0)(z_0 - z)\, m(z, s; h)$ at the point $\zeta(s; h)$:

$$\theta(z_0 - z)(z_0 - z)\, m(z, s; h) = \theta(z_0 - \zeta(s; h))$$
$$\times (z_0 - \zeta(s; h))\, m(\zeta(s; h), s; h) + \sum_{i \geqq 1} \varphi_i(s; h)(z - \zeta(s; h))^i$$

where $\varphi_i(s; h)$ is a classical symbol with respect to h and C^∞ in s. We get the following asymptotic expansion for I_{jl}:

$$\mathrm{I}_{jl} = 2h\Big(\sum_{\zeta(kh;h) < z_0} \theta(z_0 - \zeta(kh; h))\, m(\zeta(kh; h), kh; h)(z_0 - \zeta(kh; h))^{3/2}\Big)$$
$$+ h\Big(\sum_{i \geqq 1} \sum_{\zeta(kh;h) < z_0} \tilde{\varphi}_i(kh; h)(z_0 - \zeta(kh; h))^{i + (1/2)}\Big) + O(h^\infty)$$
$$= h\Big(\sum_{\zeta(kh;h) < z_0} 2\,\theta(z_0 - \zeta(kh; h))\, m(\zeta(kh; h), kh; h)(z_0 - \zeta(kh; h))^{3/2}\Big)$$
$$+ h\Big(\sum_{\zeta(kh;h) < z_0} (2/3)\,\varphi_1(kh; h)(z_0 - \zeta(kh; h))^{3/2}\Big)$$
$$+ h\Big(\sum_{i > 1} \sum_{\zeta(kh;h) < z_0} \tilde{\varphi}_i(kh; h)(z_0 - \zeta(kh; h))^{i + (1/2)}\Big) + O(h^\infty)$$

Let us observe now that:

$$2\,\theta(z_0 - \zeta(kh; h))\, m(\zeta(kh; h), kh; h) + (2/3)\,\varphi_1(kh; h)$$
$$= (4/3)\, m(\zeta(kh; h), kh; h) + O((z_0 - \zeta));$$

we can then rewrite the expansion of I_{jl} in the form:

$$(10.29)\quad \mathrm{I}_{jl} = (4/3)\, h\Big(\sum_{\zeta(kh;h) < z_0} m(\zeta(kh; h), kh; h)(z_0 - \zeta(kh; h))^{3/2}\Big)$$
$$+ h\Big(\sum_{i > 1} \sum_{\zeta(kh;h) < z_0} \tilde{\Psi}_i(kh; h)(z_0 - \zeta(kh; h))^{i + (1/2)}\Big) + O(h^\infty)$$

applying lemma 10.7, we get the general form announced for theorem 10.1 and in particular, the dominant terms are given by:

$$\mathrm{I}_{jl}=c_{jl}(0)+h^2 c_{jl}^2(0)+h^{(5/2)}\Big((4/3)\,|\,\partial_s\zeta(s_{jl}(z_0;0);0)\,|^{(3/2)}$$
$$\times\left|\frac{1}{2}[(\partial_t^2 f_j)/\partial_z f_j)](\mathrm{T}_{jl}(z_0),z_0;0)\right|^{-1/2}$$
$$\times\,\tilde{\rho}(3/2,s_{jl}(z_0;h)/h)\Big)+O(h^3)$$

Using the relation: $\partial_s\zeta(s_{jl}(z_0;0);0)=(\partial_z f_j)^{-1}$ deduced from (9.31), we get finally:

$$\text{(10.30)}\quad \mathrm{I}_{jl}=c_{jl}(0)+h^2 c_{jl}^2(0)$$
$$+h^{(5/2)}\Big((4/3)\,|\,(\partial_z f_j)(\mathrm{T}_{jl}(z_0),z_0;0)\,|^{(-1/2)}$$
$$\times\left|\frac{1}{2}(\partial_t^2 f_j)(\mathrm{T}_{jl}(z_0),z_0;0)\right|^{-1/2}$$
$$\times\,\rho(3/2,s_{jl}(z_0;h)/h)\Big)+O(h^3)$$

Recall here that f_j and T_{jl} are defined in theorem 9.1 and that $s_{jl}(z_0)$ is given by $f_j(\mathrm{T}_{jl}(z_0),z_0;0)$.

It is interesting to compare with the formula (2.16) (with $\mathrm{B}=h$, $f(t,z;h)=(1/2)((z-t^2)-h)$, $\mathrm{T}=0$, $s(z_0,h)=(z_0-h)/2)$).

This finishes the proof of theorem 10.1.

Proof of theorem 10.2. – We want now to prove theorem 10.2. The proof is quite similar because by definition we have decided to differentiate term by term. Let us now remark the following relation for the function $\tilde{\rho}(\gamma,\sigma)$ introduced in (10.21) [or after (10.4)]:

$$\text{(10.31)}\qquad \partial_\sigma\tilde{\rho}(\gamma,\sigma)=-(\gamma)\,\tilde{\rho}(\gamma-1,\sigma).$$

Using this formula, we get immediatley the formula (10.5).

Proof of theorem 10.3 Modulo a classical symbol with respect to h, we have now to compute $\int_{-\infty}^{z_0}\theta(z_0-z)\,b(z)\,dz$, with b given in (10.24) modulo an error term, we hope to be $O(h^\infty)$. We can not apply the same lemma as in the proof of theorem 10.1 because the function $z\to\theta(z_0-z)\,1_{z<z_0}$ is not Hölder near z_0. But let us prove the following lemma:

LEMMA 10.8. – *The measure $\rho_{h\mathrm{B}_0,\mathrm{V}}$ has the following property:*

If $\mathrm{I}_{h,\mathrm{M}}$ is an interval of length less than $\mathrm{C}\,h^{\mathrm{M}}$ contained in a small neighborhood of z_0, then:

$$\text{(10.32)}\qquad \rho_{h\mathrm{B}_0,\mathrm{V}}(\mathrm{I}_{h,\mathrm{M}})=O(h^{(\mathrm{M}/3)}).$$

In particular the mass of a point (for $\rho_{hB_0,V}$*) is* $O(h^\infty)$.

Proof. – Because ρ is a positive measure, we can majorize $\rho(I_{h,M})$ by computing $\rho(\varphi_{h,M})$ where $\varphi_{h,M}$ is a positive C^∞ function equal to 1 on $I_{h,M}$ and with compact support in an intervall $I'_{h,M}$ of the same type. $\varphi_{h,M}$ satisfies the hypotheses of theorem 9.1 and in particular (8.15). Then it is sufficient to study the integral $\int \varphi_{h,M}(z)\, b(z)\, dz$ which is easy according to theorem 9.1.

We can now prove theorem 10.3. If we want to prove theorem 10.3 modulo $O(h^N)$, we choose $M=3N$, and compare $\mathcal{N}(z_0, hB_0)$ with $\tilde{T}r(F_{h,M})$ as in the proof after proposition 10.4 but now with $F(z)=\theta(z_0-z).1_{z<z_0}$. We observe here that near z_0 the support of $(F(z)-F_{h,M})$ is in $I_{h,M}$ and that this function, is bounded by 2. The lemma 10.8 permits to control the error.

Remark 10.9. – An interval of measure 0 inside the domain of validity of theorem 9.1 is of length $O(h^\infty)$. This can be deduced from theorem 9.1 and from the positivity of the measure. In particular, the Fermi level is determined by the expansion given in theorem 10.3 modulo $O(h^\infty)$. We see as in the free case that the Fermi level is essentially constant as h tends to 0 modulo an error of $O(h^{3/2})$:

$$z_0(h)=z_{00}+O(h^{3/2}). \tag{10.33}$$

As in the free case we can then insert $z_0(h)$ in the formulas (10.5) or (10.6) and we get the same principal terms in the expansion as if z_0 was constant and equal to z_{00} [*see* the discussion in (2.17) and (2.18)].

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