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Resonances of N-body Schrödinger operators with stark effect

by

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ABSTRACT. — In this paper, we apply the analytic distortion technique to study the resonances of atomic type N-body Schrödinger operators with Stark effect. By introducing a suitable Dirichlet problem, we give precise location for resonances generated by eigenvalues below the bottom of the essential spectra and as a consequence, we obtain an upper bound on the widths of resonances which are exponentially small. Our result is almost optimal in some cases.

RÉSUMÉ. — Dans ce travail on étudie les résonances de l'opérateur de Schrödinger à N-corps du type atomique avec l'effet Stark par la technique de distortion analytique. En introduisant un problème de Dirichlet convenable, on donne la localisation précise pour des résonances engendrées par des valeurs propres au-dessous du spectre essentiel. Par conséquent nous obtenons une borne supérieure sur la largeur de résonance qui est exponentiellement petite. Notre résultat est presque optimal dans certain cas.

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1. INTRODUCTION

The resonances in Stark effect have been noticed since the early days of quantum mechanics [20]. Titchmarsh first undertook a systematic mathematical study of such phenomena. In fact he studied only the problem with pure Coulomb potentials [24]. In that case the results are nowadays fairely satisfactory: one knows how to calculate the widths of resonances generated by negative eigenvalues [10]. In general cases, the resonances of Schrödinger operators with Stark effect are defined in [13] and [15] by analytic dilation and in [8] and [9] by parabolic analytic dilation for both one and N-body problems. But as far as the author knows, the question remains open until now how to calculate the widths of resonances for one-body Stark Hamiltonian with general potentials, not to speak of the same problem for N-body Schrödinger operators. In [26], we studied the resonances of one body Schrödinger operators with Stark effect. Making use of the analytic dilation (or more exactly, analytic distortion) in negative x_1-direction, we arrived at obtaining a precise location of resonances generated by a negative eigenvalue λ_0 of $-\Delta+V(x)$. As a corollary, we derive an upper bound on the widths of these resonances of the form:

$$|\operatorname{Im} z(\beta)| \leqq C_\varepsilon e^{-4(|\lambda_0|^{3/2}-\varepsilon)/3\beta}, \qquad \beta \to 0_+ \tag{1.1}$$

for any $\varepsilon>0$. In comparing with the results of [10], we see that this result is almost optimal.

In this paper we shall study the resonances of N-body Schrödinger operator with Stark effect:

$$H(\beta)=\sum_{j=1}^{N}(-\Delta_j+\beta q_j x_j^{(1)}+V_j(x_j))+\sum_{i<j} V_{ij}(x_i-x_j) \tag{1.2}$$

in $L^2(R^{\nu N})$. Here $\beta>0$, $x_j=(x_j^{(1)}, \ldots, x_j^{(\nu)})\in\mathbb{R}^\nu$ and Δ_j is the Laplacian on $\mathbb{R}^\nu_{x_j}$. The charges q_j may be any real number, but it is natural to assume that:

$$Q=\sum_{j=1}^{N}|q_j|>0 \tag{1.3}$$

We shall apply methods of [26] with a little modification to study the resonances of N-body Schrödinger operators (1.2). Let $\Sigma=\inf\sigma_{\text{ess}}(H(0))$. Assume that $\lambda_0<\Sigma$ is an eigenvalue of $H(0)$ with multiplicity m. Choose a smooth real, function χ on R such that for $\eta>0$ sufficiently small,

$$\chi(t)=0 \quad \text{for } t\leqq 1 \qquad \text{and} \qquad \chi(t)=1 \quad \text{for } t\geqq 1+\eta$$

and to simplify the computation of numerical ranges, we assume that $0\leqq\chi(t)\leqq 1$ and $\chi'(t)\geqq 0$ for $1<t<1+\eta$. Then for $\lambda=(\Sigma-\lambda_0)/Q-\varepsilon$ and

$\theta \in \mathrm{R}$ with $|\theta|$ sufficiently small, the map: $x \mapsto e^{\theta \chi(\beta\lambda^{-1}|x|)} x$ is a diffeomorphism on R_x^ν. It induces a unitary operator on $\mathrm{L}^2(\mathrm{R}_x^\nu)$, which we denote $\mathrm{T}(\theta, x)$. For a generic point $x \in \mathrm{R}^{\nu\mathrm{N}}$, we write: $x = (x_1, \ldots, x_\mathrm{N})$ with $x_j \in \mathrm{R}^\nu$. Then the distorted Stark Hamiltonian for θ real is defined by

$$\mathrm{H}(\beta, \theta) = \mathrm{T}(\theta, x_1) \ldots \mathrm{T}(\theta, x_\mathrm{N}) \, \mathrm{H}(\beta) \, \mathrm{T}(\theta, x_\mathrm{N})^{-1} \ldots \mathrm{T}(\theta, x_1)^{-1} \tag{1.3}$$

in $\mathrm{L}^2(\mathrm{R}^{\nu\mathrm{N}})$. To discuss the holomorphic extension of $\mathrm{H}(\beta, \theta)$ in θ, we make reasonable assumptions on V_j and V_{ij}. In particular, our asumptions are verified if $\mathrm{V}_j \in \mathrm{C}_0^\infty(\mathrm{R}^\nu)$ or V_j is a sum of Coulomb type potentials and V_{ij} is a Coulomb or Yukawa potential on R^ν. By the choice of χ, we can check that $\mathrm{H}(\beta, \theta)$ so defined can be extended to a holomorphic family of type (A) for $\theta \in \mathrm{C}$, $|\theta|$ small and $\mathrm{Im}\,\theta > 0$, with constant domain $\mathscr{D} = \mathrm{D}(\Delta) \cap \mathrm{D}(\mathrm{X}_1)$. Here and in the following, $\Delta = \sum \Delta_j$ and $\mathrm{X}_1 = \sum q_j x_j^{(1)}$. For $\beta > 0$ sufficiently small, we prove that the essential spectra of $\mathrm{H}(\beta, \theta)$ are contained in the region: $\{z = \lambda_0 + \varepsilon_0 + e^\theta t + s;\ t \in \mathbb{R},\ s \geqq 0\}$. Therefore the resonances of $\mathrm{H}(\beta)$ near λ_0 are defined. By studying the Dirichlet realization of $\mathrm{H}(\beta)$ on the domain:

$$\mathrm{M} = \{x = (x_1, \ldots, x_\mathrm{N});\ |x_j| < \beta^{-1}\lambda,\ j = 1, \ldots, \mathrm{N}\}$$

we obtain as in [26] the location of the resonances of $\mathrm{H}(\beta)$ near λ_0. In particular we get an upper bound on the width of resonances of the form, for any $\varepsilon > 0$,

$$|\mathrm{Im}\, z(\beta)| \leqq \mathrm{C}_\varepsilon e^{-4(1-\varepsilon)\,\mathrm{S}(\Sigma-\lambda_0)^{3/2}/3\,\mathrm{Q}\,\beta}, \tag{1.4}$$

where $\mathrm{S} \geqq 1$ is some constant depending only on the charges q_j. In case all but one q_j equal zero, we have $\mathrm{S} = 1$. But in general case we have $\mathrm{S} > 1$. *See* Theorem 5.6. In the former case, our result is almost optimal. But in general, one could hope to obtain an upper bound of the form:

$$|\mathrm{Im}\, z(\beta)| \leqq \mathrm{C}_\varepsilon e^{-4(1-\varepsilon)(\Sigma-\lambda_0)^{3/2}/3\,\mathrm{R}\,\beta} \tag{1.5}$$

where $\mathrm{R} = (q_1^2 + \ldots + q_\mathrm{N}^2)^{1/2}$. *See* also Remark 4.4. In comparing with the work [26], we see that the main difference resides in the location of essential spectra for distorted Hamiltonian: In one-body case this follows easily from Weyl's theorem, while in N-body case, we have to use complicated Weinberg-van Winter equations.

When trying to treat the regular N-body Schrödinger operators with Stark effect by our method, one runs into some technical difficulties due to the removal of the centre of mass. In fact because our deformation is not linear in the coordinates, it is difficult to determine the essential spectra by an induction on N. This difficulty might be overcome by the exterior dilation technique introduced as in [5] and [23]. But then the domain question of $\mathrm{H}(\beta, \theta)$ is more delicate.

The plan of this work is as follows. In Section 2, we introduce in detail the analytic distortion machinery and study its action on the free N-body Stark Hamiltonian. In Section 3, we give the definition of resonances. Here the main task is to identify the essential spectra of $H(\beta, \theta)$. In Section 4, we introduce the Dirichlet problem and study the stability of the eigenvalues of $H^D(\beta)$ near λ_0. As in [26], we prove that there are exactly *m* eigenvalues, counted according to their multiplicity, of $H^D(\beta)$ which converges to λ_0 when β tends to zero. In Section 5, we construct a Grushin problem for $H(\beta, \theta)$ and obtain a precise location on the resonances near λ_0. As a corollary, we get (1.4).

This paper is a revised version of preprint [31]. After this work was finished, I learned that I. M. Sigal has also some works on related subjects (*cf.* [29], [30]). In particular in [30], Sigal also studied Stark resonances for N-body operators (1.2) with $q_j<0$ for all j. But both methods and conditions are different from ours. Sigal assumed the interacting potentials V_{ij} to be repulsive and this made it easier to estimate the essential spectrum of distorted Stark Hamiltonian (*see* condition IV in Section 2 [30]).

2. ANALYTIC DISTORTION

In this section, we introduce the analytic distortion used in this paper. For the technical reason in the location of the essential spectra of distorted Hamiltonian, we use a distortion slightly different from that of [26] which depends only on one variable x_1. *See* also [6], [16], [18].

Let χ be a smooth function over R such that:

$$\chi(t)=0 \quad \text{for } t\leqq 1 \qquad \text{and} \qquad 1 \quad \text{for } t\geqq 1+\eta, \quad \eta>0 \tag{2.1}$$

We require χ to satisfy that $0\leqq\chi(t)\leqq 1$ and $\chi'(t)\geqq 0$ for $1<t<1+\eta$. Clearly such functions exist. For $\lambda>0$ and $\theta\in\mathbb{R}$, define the transformation Φ_θ on $\mathbb{R}^{\nu N}$:

$$\Phi_\theta(x)=(x_1\, e^{\theta\chi(\beta\,|x_1|/\lambda)}, \ldots, x_N\, e^{\theta\chi(\beta\,|x_N|/\lambda)}) \tag{2.2}$$

Then if $|\theta|$ is sufficiently small, Φ_θ is a diffeomorphism on $\mathbb{R}^{\nu N}$ and induces a unitary operator (U(θ) (depending on β and λ) on $L^2(\mathbb{R}^{\nu N})$ by:

$$U(\theta) f(x)=(\Phi'_\theta(x))^{1/2} f(\Phi_\theta(x)) \tag{2.3}$$

where $\Phi'_\theta(x)$ denotes the Jacobian of the transformation Φ_θ.

Now let $H_0(\beta)=\sum_{j=1}^{N}(-\Delta_j+\beta q_j x_j^{(1)})$ be the free Hamiltonian with Stark effect. Put:

$$H_0(\beta, \theta)=U(\theta)\, H_0(\beta)\, U(\theta)^{-1} \tag{2.4}$$

Then we can write: $H_0(\beta, \theta) = \sum_{j=1}^{N} H_{0j}(\beta, \theta)$, where

$$H_{0j}(\beta, \theta) = -\sum_{m, k} g^{m, k}(x_j, \theta)\, \partial_m \partial_k + \sum_k a_k(x_j, \theta)\, \partial_k + b(x_j, \theta) + \beta q_j x_j^{(1)} e^{\theta \chi(\beta |x_j|/\lambda)} \quad (2.5)$$

Remark 2.1. – Notice that the explicit calculation of the coefficient functions is not necessary. In the following, we shall use only the properties:

$$g^{mk}(x_j, \theta) = e^{-2\theta\chi(\beta |x_j|/\lambda)} (\delta_{mk} + O(\theta))$$
$$a_k(x_j, \theta) = O(\beta\theta) = b(x_j, \theta)$$

Where $O(\mu)$ is some function with support in $\{|\chi z| < \beta^{-1}\lambda(1+\eta)\}$ and is of the order $O(|\mu|)$ uniformly in x_j.

Let Ω denote the domain: $\Omega = \{\theta \in C;\ 1\theta| \leqq \rho,\ \text{Im}\,\theta > 0\}$. We should keep in mind that in the following, $\rho > 0$ will always be sufficiently small. For $\theta \in \Omega$, $t \in R$, we define:

$$E(\theta, t) = \{e^{\theta} r + s;\ r \in R,\ s \geqq t\}.$$

Then $H_0(\beta, \theta)$ defined by (2.4) has a natural extension in θ into Ω.

PROPOSITION 2.2. – (*i*) *For* $\theta \in \Omega$, $H_0(\beta, \theta)$ *defined on* $\mathcal{D} = D(\Delta) \cap D(X_1)$ *is closed in* $L^2(R^{\nu N})$ *and is a holomorphic family of type* (A), where $X_1 = \sum_{j=1}^{N} q_j x_j^{(1)}$.

(ii) *For any* $\varepsilon > 0$, *let* $\beta_0 = \beta(\varepsilon, \text{Im}\,\theta) > 0$ *be small enough. Then the numerical range of* $H_0(\beta, \theta)$ *is contained in* $E(\theta, -\lambda(1+\eta)Q - \varepsilon)$, *for* $0 < \beta \leqq \beta_0$.

Proof. – Let

$$p(x, \xi) = \sum_j \sum_{l, m} g^{lm}(x_j, \theta)\, \xi_j^{(1)} \xi_j^{(m)} + \sum_j \beta q_j x_j^{(1)} e^{\theta\chi(\beta |x_j|/\lambda)}$$

Then for $z \in \mathbb{C}$ with

$$\text{Re}\, z \leqq -\lambda(1+\eta)Q - \varepsilon,$$
$$\text{Im}\, z \geqq 1/2\ \text{Im}\,\theta(\text{Re}\, z + \lambda(1+\eta)Q + \varepsilon),$$

one can prove that for some $C = C(\varepsilon) > 0$, one has:

$$\left.\begin{array}{c} |p(x, \xi) - z| \geqq C\ \text{Im}\,\theta(|\xi|^2 + \beta|X_1| + |\text{Re}\, z|), \\ \theta \in \Omega,\ 0 < \beta \leqq \beta_o. \end{array}\right\} \quad (2.6)$$

This involves an elementary but lengthy calculation. For simplicity, consider only the cas $\text{Re}\,\theta = 0$, $\text{Im}\,\theta = > 0$. Put: $z = z_1 + iz_2$, $z_j \in R$, $\chi_j = \chi(\beta|x_j|/\lambda)$. We need only to prove (2.6) in the region $|\text{Re}(p(x, \xi) - z)| \leqq \delta\varphi(|\xi|^2 + \beta|X_1| + |\text{Re}\, z|)$ for some $\delta > 0$ appropriately small. By the estimates:

$$|\operatorname{Re} \sum_{l,\, m} g^{lm}(x_j, \theta)\, \xi_j^{(l)} \xi_j^{(m)} - |\xi_j|^2 \leqq C\varphi^2 |\xi_j|^2,$$

$$\operatorname{Im} \sum_{l,\, m} g^{lm}(x_j, \theta)\, \xi_j^{(l)} \xi_j^{(m)} \leqq 0,$$

for $|\theta|$ small enough, we obtain in the above region that

$$\left.\begin{aligned} &\left||\xi|^2 - \sum_j \beta q_j x_j^{(1)} \cos \varphi\chi_j - z_1\right| \leqq \delta' \varphi (|\xi|^2 + \beta |X_1| + |z_1|) \\ &\delta' = \delta + O(\varphi). \end{aligned}\right\} \tag{2.7}$$

Let $\sum'$ denote the sum over the j such that $|x_j| > (1+\eta)\lambda/\beta$. Let $\sum^+$ (resp. $\sum^-$) denote the sum over the j such that $q_j x_j^{(1)} > 0$ (resp. $q_j x_j^{(1)} < 0$). Then one has:

$$\begin{aligned} -\sum{}^{',\,-} \beta q_j x_j^{(1)} (\cos 2\varphi \pm \delta' \varphi) &\geqq (1 - \delta' \varphi)(|\xi|^2 + |z_1|) \\ &\quad - \sum{}^- \lambda |q_j| (1 \pm \delta' \varphi)(1+\eta) + \sum{}^+ \beta q_j x_j^{(1)} (\cos 2\varphi\chi_j \pm \delta' \varphi) \end{aligned} \tag{2.8}$$

Here the signs $\pm$ depend on $\mp X_1 \geqq 0$. By the choice of z, we derive from (2.8) for $\beta > 0$ sufficiently small that

$$-\sum{}^{',\,-} q_j x_j^{(1)} \geqq \sum{}^{',\,+} q_j x_j^{(1)}$$

From (2.7) and the above estimate, it follows that:

$$\left.\begin{aligned} &\left||\xi|^2 + \beta X_1 - z_1\right| \leqq \delta'' \varphi (|\xi|^2 + \beta |X_1| + |Z_1|), \\ &\delta'' = \delta + O(\varphi). \end{aligned}\right\} \tag{2.9}$$

For $\delta > 0$ sufficiently small, we have necessarily $X_1 < 0$. From (2.9), we obtain: $\beta X_1 \leqq -\dfrac{1-\delta''\varphi}{1+\delta''\varphi}(|\xi|^2 + |z_1|)$. Therefore in the region in consideration, one has:

$$\begin{aligned} |\operatorname{Im}(p(x, \xi) - z)| &\geqq -\sum \beta q_j x_j^{(1)} \sin \varphi\chi_j + z_2 \\ &\geqq \frac{\sin \varphi}{1 + \delta''\varphi} ((1 - \delta''\varphi)(|\xi|^2 + |z_1|) - \sum{}^- |q_j| \lambda (1+\eta)(1+\delta''\varphi)) \\ &\qquad - \sum{}^+ |q_j| \lambda (1+\eta) \sin \varphi + z_2. \end{aligned}$$

Since $|z_1| \geqq \sum |q_j| (1+\eta)\lambda + \varepsilon$, one obtains for $\varphi > 0$ sufficiently small that $|\operatorname{Im}(p(x, \xi) - z)| \geqq c\varphi(|\xi|^2 + |z_\perp|)$ for some $c > 0$. Now (2.6) follows from (2.9). In the region where (2.6) is valid, the symbol $p - z$ is globally elliptic. Let $B(x, D; z)$ denote the pseudodifferential operator with symbol $(p-z)^{-1}$. Then by the results on composition and continuity of pseudodifferential operators [17], one derives from (2.6) that $(-\Delta + \beta |X_1|)\, B(x, D; z)$ is bounded on L^2 and

$$B(x, D; z)(H_0(\beta, \theta) - z) = I + R_1(z)$$
$$(H_0(\beta, \theta) - z)\, B(x, D; z) = I + R_2(z)$$

where $R_j(z)$, $j=1, 2$, are pseudodifferential operators of order 0 (hence bounded on L^2) and satisfy: $\|R_j(z)\| \leqq C|\mathrm{Re}\, z|^{-1}$, for z in the region such that (2.6) is valid. This enables us to conclude that for $|\mathrm{Re}\, z|$ large enough, $H_0(\beta, \theta) - z$ is invertible and

$$(H_0(\beta, \theta) - z)^{-1} = B(x, D; z)(I + R_2(z))^{-1}$$

It follows that $(-\Delta + \beta |X_1|)(H_0(\beta, \theta) - z)^{-1}$ is bounded and

$$\left.\begin{array}{c} \|(H_0(\beta, \theta) - z) f\| \geqq c(\|\Delta f\| + \beta \|X_1 f\|), \\ \text{for } f \in C_0^\infty(\mathbf{R}^n) \end{array}\right\} \qquad (2.10)$$

By (2.10), we conclude easily that $H_0(\beta, \theta)$ is closed on $\mathscr{D}$. By the expression for $H_0(\beta, \theta)$, one sees clearly that the map: $\Omega \ni \theta \mapsto H_0(\beta, \theta) f$ is holomorphic for each $f \in \mathscr{D}$. This proves (i).

To prove (ii), we write: $H_0(\beta, \theta) = H_0(\theta) + \beta X_1(\theta)$, where $X_1(\theta)$ corresponds to the distorted Stark effect. Remark first that the numerical range of $\beta X_1(\theta)$ is contained in $E(\theta, -\lambda(1+\eta)Q - \varepsilon)$. We compute the numerical range of $H_0(\theta)$ in assuming $N=1$. Denote: $\Phi(x, \theta) = e^{\theta\chi(|x|\beta/\lambda)} x$, $x \in \mathbf{R}^\nu$. Then it is easy to compute:

$$\left.\begin{array}{l} (\Phi')_{ij} = e^{\theta\chi(\beta|x|/\lambda)}(\delta_{ij} + \theta f w_i w_j) \\ (\Phi'^{-1})_{ij} = e^{-\theta\chi(\beta|x|/\lambda)}(\delta_{ij} + F w_i w_j) \end{array}\right\} \qquad (2.11)$$

Here Φ' denotes the Jacobian of Φ and $w = (w_1, w_2, \ldots, w_N) = x/|x|$,

$$f(x) = \chi'(\beta|x|/\lambda)\beta|x|/\lambda \geqq 0 \qquad \text{and} \qquad F = \sum_{j \geqq 1} (-1)^j \theta^j f^j.$$

Since $\partial^\alpha \Phi'(x) = 0(\beta|\theta|)$, for $|\alpha| \geqq 1$, $H_0(\theta)$ can be written as

$$H_0(\theta) = \sum_{l, j, k} -\partial_k (\Phi'^{-1})_{kj} (\Phi'^{-1})_{jl} \partial_l + \text{terms of lower order} \qquad (2.12)$$

where the coefficients in the terms of lower order are of the order $O(\beta|\theta|)$. Let $u \in D(\Delta)$ and $\|u\| = 1$. Making use of (2.11), one has:

$$I \equiv \sum_{j, k, l} \iint (\Phi'^{-1})_{kj} (\Phi'^{-1})_{jl} \partial_k u \overline{\partial_l u}\, dx$$

$$= \iint e^{-2\theta\chi}(|\nabla u|^2 + (2F + F^2)|w.\nabla u|^2)\, dx$$

Since $F = (1 + \theta f)^{-1} - 1$, we deduce from the above expression that:

$$\mathrm{Re}\, I \geqq (1 - C|\theta|) \iint |\nabla u|^2\, dx, \qquad C > 0,$$
$$\mathrm{Im}\, I \leqq 0.$$

Let R denote the remaining terms in (2.12). Then it is easy to estimate:

$$|\langle u, Ru \rangle| \leqq C\beta|\theta,$$
$$(\|\nabla u\| . \|u\| + \|u\|^2) \leqq C'\beta|\theta|(\mathrm{Re}\, I + 1).$$

This shows that for $\beta>0$ sufficiently small,

$$\langle u, H_0(\theta) u\rangle \in E_1(\theta, -\varepsilon) \equiv \{\operatorname{Re} z \geqq -\varepsilon,\ \operatorname{Im} z \leqq \varepsilon \operatorname{Im}\theta \operatorname{Re} z+1)\}.$$

Since

$$E_1(\theta. -\varepsilon)+E(\theta, -\lambda(1+\eta)Q-\varepsilon) \subseteqq E(\theta, -\lambda(1+\eta)Q-3\varepsilon)$$

and $\varepsilon>0$ is arbitrary, we derive that the numerical range of $H_0(\beta, \theta)$ is contained in $E(\theta, -\lambda(1+\eta)Q-\mu)$, $\mu>0$ can be arbitrary, provided $0<\beta\leqq\beta_0$, with $\beta_0=\beta(\operatorname{Im}\theta)$ sufficiently small. ■

From Proposition 2.2, we derive the following.

COROLLARY 2.3. – *For $z\in\mathbb{C}\setminus E(\theta, -\lambda(1+\eta)Q-\varepsilon)$, $\theta\in\Omega$, $H_0(\beta, \theta)-z$ is invertible in L^2 and the resolvent is jointly analytic in z and θ. Moreover, we have*

$$\|(H_0(\beta, \theta)-z)^{-1}\| \leqq \operatorname{dist}(z, E(\theta, -\lambda(1+\eta)Q-\varepsilon)^{-1}$$

for $0<\beta\leqq\beta_0$, with $\beta_0=\beta(\operatorname{Im}\theta)>0$ *small enough.*

Observe that from the proof of Proposition 2.2, it is clear that $\beta_0>0$ can be chosen locally uniformly in $\theta\in\Omega$. The analyticity in $\theta\in\Omega$ in Corollary 2.3 means that for every $K\subset\subset\Omega$ there exists $\beta_K>0$ such that for $0<\beta\leqq\beta_K$, $(H_0(\beta, \theta)-z)^{-1}$ is jointly analytic in $\theta\in K$ and $z\in\mathbb{C}\setminus E(\theta, -\lambda(1+\eta)Q-\varepsilon)$. In the following β_0 will always denote some small positive constant depending on $\operatorname{Im}\theta>0$ locally uniformly in $\theta\in\Omega$.

Let $L_0(\beta, \theta)=ie^{-\theta}H_0(\beta, \theta)$. Then Corollary 2.3 says that

$$\sigma(L_0(\beta, \theta)) \subset \{z;\ \operatorname{Re} z>-\lambda(1+\eta)Q-\varepsilon\}$$

and

$$\|(L_0(\beta, \theta)-z)^{-1}\| \leqq C|\operatorname{Re} z|^{-1},$$

for $\operatorname{Re} z\leqq -2(1+\eta)\lambda Q$.

LEMMA 2.4. – *With the above notations, one has for $\theta\in\Omega$, $0<\beta<\beta_0$:*

(*a*) $\sigma(L_0(\beta, \theta))=\varphi$;

(*b*) *Let* $K=\sum_{j=1}^{N} -\Delta_j$. *Then for any* $E\in R$, *one has:*

$$\sup_{\operatorname{Re} z\leqq E} \|(L_0(\beta, \theta)-z)^{-1}\| + \|K(L_0(\beta, \theta)-z)^{-1}\| < +\infty$$

Proof. – Recall that the results (*a*), (*b*) are true for $\tilde{L}_0(\beta, \theta)=i(e^{-3\theta}K+\beta X_1)$ ([28], [15]). We shall prove the lemma by comparing $L_0(\beta, \theta)$ with $\tilde{L}_0(\beta, \theta)$. Assume first $N=1$. Then $L_0(\beta, \theta)$ coincides with $\tilde{L}_0(\beta, \theta)$ outside some compact set in $\mathbb{R}^\nu$. An easy argument shows that $\sigma_{ess}(L_0(\beta, \theta))=\sigma_{ess}(\tilde{L}_0(\beta, \theta))=\varnothing$. This means that $(L_0(\beta, \theta)-z)^{-1}$ defined for $\operatorname{Re} z<-2(1+\eta)\lambda Q$ can be meromorphically extended to C. Let $A=\{e^{-\alpha|x|^2}P(x);\ \alpha>0, P(.)\text{ polynomial}\}$. For $f\in A$, define;

$f(x, \theta)=e^{-\nu\theta/2} f(e^{-\theta} x)$, $F(\theta)=U(\theta) f(\theta)$. $F(\theta)$ is well defined and belongs to $\mathscr{S}$ for $\theta \in \Omega$. Then for $\operatorname{Re} z \ll -1$, one has:

$$\langle (\tilde{L}_0(\beta, \theta)-z)^{-1} f, g \rangle = \langle (L_0(\beta, 0)-z)^{-1} f(\theta), g(\bar{\theta}) \rangle$$
$$= \langle (L_0(\beta, \theta)-z)^{-1} F(\theta), G(\bar{\theta}) \rangle, \quad (2.14)$$
$$f, g \in A,$$

But the left hand side of (2.14) is holomorphic in $z \in C$. This shows that $\langle (L_0(\beta, \theta)-z)^{-1} F(\theta), G(\bar{\theta}) \rangle$ is entire in $z \in C$. We can check that $\{F(\theta), f \in A\}$ is dense in $L^2(\mathbb{R}^\nu)$. From (2.14) it follows that the poles of $(L_0(\beta, \theta)-z)^{-1}$ are absent, so $\sigma(L_0(\beta, \theta))=\varnothing$.

Next we construct approximation for $(L_0(\beta, \theta)-z)^{-1}$. Let F, w be given in (2.11). Define:

$$l_0(\beta, \theta)=e^{-\theta} i (e^{-2\theta\chi}(\xi^2+(2F+F^2)|w.\xi|^2+\beta e^{\theta\chi} q_1 x_1),$$
$$q_1 \neq 0, \qquad x=(x_1, \ldots, x_\nu) \in \mathbb{R}^\nu.$$

Then for any $E>0$, one has

$$|l_0(\beta, \theta)-z| \geqq \operatorname{Re}(l_0(\beta, \theta)-z) \geqq c_1 \operatorname{Im} \theta \xi^2 - C_2 - E,$$
$$\text{for} \quad \operatorname{Re} z < E.$$

Here c_1, $c_2>0$ depends only the choice of χ. Take $\mathscr{S} \in C^\infty(\mathbb{R}^\nu)$ such that

$$\mathscr{S}(\xi)=0 \quad \text{if } |\xi| \leqq 2; \qquad \mathscr{S}(\xi)=1, \qquad \text{if } |\xi| \geqq 3.$$

Put $\psi(\xi)=\mathscr{S}\left(\left(\dfrac{c_1 \operatorname{Im} \theta}{c_2+E}\right)^{1/2} \xi\right)$, $a(\beta, \theta; z)=\psi(l_0(\beta, \theta)-z)^{-1}$. Then

$$|\partial_\chi^\alpha \partial_\xi^\gamma a(\beta, \theta; z)| \leqq c_{\alpha\gamma} \beta^{|\alpha|} < \xi >^{-2}, \quad \text{unformly in } \operatorname{Re} z < E. \quad (2.15)$$

Here $C_{\alpha\gamma}$ may depend on $\operatorname{Im} \theta > 0$, but is independent of $E>0$. Let $A(\beta, \theta; z)$ denote the pseudo-differential operator with symbol $a(\beta, \theta; z)$ and $B=(1-\psi(D))(\tilde{L}_0(\beta, \theta)-z)^{-1}$. Then we can write;

$$(L_0(\beta, \theta)-z) A(\beta, \theta; z)=\psi(D)+R_1(\beta, \theta; z)$$
$$(L_0(\beta, \theta)-z) B(\beta, \theta; z)=(1-\psi(D))+R_2(\beta, \theta; z)$$

By (2.15), one has; $\|R_1(\beta, \theta; z)\| \leqq C(\operatorname{Im} \theta) \beta$, uniformly in $E>0$ and $\operatorname{Re} z < E$. Note that

$$R_2(\beta, \theta; z)=[L_0(\beta, \theta), 1-\psi((D))](\tilde{L}_0(\beta, \theta)-z)^{-1}+(1-\psi(D))$$
$$\times ((L_0(\beta, \theta)-\tilde{L}_0(\beta, \theta))(\tilde{L}_0(\beta, \theta)-z)^{-1}.$$

Since

$$[L_0(\beta, \theta), 1-\psi(D)]$$

and

$$(1-\psi(D))(L_0(\beta, \theta)-\tilde{L}_0(\beta, \theta))$$

are smorthing in both x and ξ variables, we obtain:

$$\|R_2(\beta, \theta; z)\| \leqq c \|(\tilde{L}_0(\beta, \theta)+1)^{-1}(\tilde{L}_0(\beta, \theta)-z)^{-1}\| \leqq c/|z|.$$

Let $\beta_0=\beta\,(\operatorname{Im}\theta)$ be sufficiently small and $R_0>0$ sufficiently large. Then the above estimates shows that $\|R_1(\beta,\theta;z)+R_2(\beta,\theta;z)\|<\frac{1}{2}$, for $0<\beta\leqq\beta_0$, $\operatorname{Re}z<E$, $|z|\geqq R_0$. Therefore in the same region, we have the expression:

$$\left.\begin{aligned}(L_0(\beta,\theta)-z)^{-1}&=(A(\beta,\theta;z)+B(\beta,\theta;z))\,R(\beta,\theta;z),\\ \|R(\beta,\theta;z)\|&\leqq 2.\end{aligned}\right\}\tag{2.16}$$

By (2.15), one sees that $A(\beta,\theta;z)$ is uniformly bounded in $\operatorname{Re}z\leqq E$. Consequently,

$$\sup_{\substack{\operatorname{Re}z<E\\ |z|>R_0}}\|K(L_0(\beta,\theta)-z)^{-1}\|<+\infty.$$

Now (*b*) follows from the local boundedness of the resolvent. Lemma 2.4 for $N=1$ is proved. For $N\geqq 2$, remark that:

$$H_0(\beta,\theta)=H_{0,1}(\beta,\theta)\otimes I\ldots\otimes I+I\otimes H_{0,2}(\beta,\theta)\otimes\ldots\\ \otimes I+\ldots+I\otimes\ldots\otimes I\otimes H_{0,N}(\beta,\theta).$$

For $q_j\neq 0$, $L_{0,j}(\beta,\theta)\equiv ie^{-\theta}H_{0,j}(\beta,\theta)$ verifies (*a*)-(*b*). For $q_j=0$, by estimating the numerical range of $H_{0,j}(\beta,\theta)$, one sees that

$$\sigma(L_{0,j}(\beta,\theta))\subset W\equiv\left\{z;\ \operatorname{Re}z>-\varepsilon,\ |\arg z|<\frac{\pi}{2}-\varepsilon\operatorname{Im}\theta\right\}$$

and

$$\|L_{0,j}(\beta,\theta)-z)^{-1}\|\leqq cd(z,W)^{-1},\qquad z\notin W.$$

Therefore, all $L_{0,j}(\beta,\theta)$ statisfy the condition (P) of Theorem 3.1 in [14]. In addition, they generate exponentially bounded semigroups. By repeatedly applying Theorem 3.1 in [14], we obtain

$$\sigma(L_0(\beta,\theta))=\sigma(L_{0,1}(\beta,\theta))+\ldots+\sigma(L_{0,N}(\beta,\theta))=\varnothing$$

since we always assume that $Q=\sum|q_j|\neq 0$. (*a*) is proved for all N. (*b*) for $N\geqq 2$ can be proved in the same way as in the step for $N=1$. The details are omitted. ■

COROLLARY 2.5. – *Let* $U_0(t;\beta,\theta)$, $t\geqq 0$, *denote the semigroup generated by* $L_0(\beta,\theta)$. *Then for any* $E>0$ *there exists* $C>0$ *such that:*

$$\|U_0(t;\beta,\theta)\|\leqq Ce^{-Et},\qquad\text{for}\quad t\geqq 0.$$

Proff. – It suffices to apply Lemma 2.4 and Proposition 3.4 in [28] which says that if A generates exponentially bounded semigroup and if

$$\sigma(A)\cap\{\operatorname{Re}z\leqq\mu\}=\varphi\qquad\text{and}\qquad\sup_{\operatorname{Re}z\leqq\mu}\|(A-z)^{-1}\|<+\infty,$$

then $\|e^{-tA}\|\leqq Ce^{-\mu t}$, for $t\geqq 0$. ■

3. RESONANCES

Let $H(\beta)$ denote the operator defined by (1.2). In this section, we want to define the resonances of $H(\beta)$ near a given value $\lambda_0 \in \mathbb{R}$. To ensure the self-adjointness of $H(\beta)$ in $L^2(\mathbb{R}^{\nu N})$, we assume initially that V_j's and V_{ij}'s are real valued functions and as multiplication in $L^2(\mathbb{R}_x^\nu)$, they are compact relative to Δ_X. In the later we shall frequently utilize the following conditions. Assume that there exists $R>0$ such that

$$|\partial_x^\alpha V_{ij}(x)| \leqq C_\alpha \langle x \rangle^{-\varepsilon-|\alpha|}, \qquad \text{for} \quad |\alpha| \leqq 2 \tag{3.1}$$

and for $x \in R^\nu$ with $|x|>R$, $i<j$, i, $j=0, 1, \ldots, N$. Here and in the following, we often write: $V_j = V_{0j}$. For $\lambda>0$, let $U(\theta)$ denote the distortion operator introduced in section 2. Put:

$$H(\beta, \theta) = H_0(\beta, \theta) + V(\theta)$$

where $V(\theta) = U(\theta) V U(\theta)^{-1}$. For the notational convenience, we have omitted the dependence of $V(\theta)$ on β and λ. In order to discuss the holomorphic extension of $H(\beta, \theta)$ in θ, we make the following distorsion analyticity assumption on V_{ij}, $0 \leqq i<j \leqq N$.

> There exists $\beta_0>0$ such that for $0<\beta<\beta_0$, as multiplication operator from $D(\Delta)$ to $L^2(R^{\nu N})$, $V_{ij}^{(\theta)}$ defined for θ real as above extends to a holomorphic family of bounded operators for θ in Ω with relative bound 0. For simplicity we shall assume that (3.1) is satisfied by $V_{ij}(\theta)$, $\theta \in \Omega$, with x replaced by $x_i - x_j$, $x_0=0$. (3.2)

If all V_{ij}'s satisfy the assumption (3.2), then we conclude from Proposition 2.2 that $H(\beta, \theta)$ defined on $\mathscr{D}$ is closed for $\theta \in \Omega$ and is a holomorphic family of type (A). Observe also that by our choice of χ, we can show that if V_{0j} has holomorphic extension into the domain: $\{z \in C^n; |\operatorname{Re} z| \geqq R, |\operatorname{Im} z| \leqq \varepsilon |\operatorname{Re} z|\}$ for some R, $\varepsilon>0$, then it satisfies the assumption (3.2). For the two body potentials, we assume:

> For $1 \leqq i<j \leqq N$, V_{ij} is dilation analytic
> in the sense of Combes (*see* [2], [4], [15]). (3.3)

Remark 3.1. — Our assumptions on V_{ij} are verified if (*a*) $V_{0j} \in C_0^2(R^\nu)$ or V_{0j} is of the form $\sum z_k/|x-r_k|$, where the sum is finite and $z_k \in R$, $r_k \in R^\nu$;

(*b*) V_{ij}, $i \geqq 1$, is a Coulomb potential or Yukawa potential.

(*a*) is obvious and (*b*) can be proved by the method of Hunziker [18], § 5. Note that the distortion utilized by Hunziker corresponds to a linearization of the present one. The details are omitted.

In order to define the resonances of $H(\beta)$, we have to identify the essential spectra of $H(\beta, \theta)$, making use of Weinberg-van Winter equations. Introduce first some notations needed later on. *See* [21]. Let $D=(C_0, C_1, \ldots, C_k)$ be a cluster decomposition of the set $(0, 1, \ldots, N)$. Let $|D|$ denote the number of clusters in D. We define (assume always that $0 \in C_0$),

$$H^{C_0}(\beta, \theta) = \sum_{j \in C_0 \setminus 0} (H_{0j}(\beta, \theta) + V_j(\theta)) + \sum_{\substack{i, j \in C_0 \\ i \neq 0}} V_{ij}(\theta)$$

$$H^{C_k}(\beta, \theta) = \sum_{j \in C_k} H_{0j}(\beta, \theta) + \sum_{i, j \in C_k} V_{ij}(\theta),$$

for $k \geqq 1$. Put:

$$H_D(\beta, \theta) = H^{C_0}(\beta, \theta) \otimes I \otimes \ldots \otimes I + \ldots + I \otimes I \otimes \ldots \otimes H^{C_k}(\beta, \theta),$$

which acts on the space

$$L^2(R^{(n_0-1)\nu}) \otimes L^2(R^{n_1 \nu}) \otimes \ldots \otimes L^2(R^{n_k \nu}),$$

where $n_j = |C_j|$. In the case $|D|=1$, we have $H_D(\beta, \theta) = H(\beta, \theta)$ and in the case $|D| = N+1$, we have: $H_D(\beta, \theta) = H_0(\beta, \theta)$. If $S=(D_{N+1}, \ldots, D_k)$, is a string of cluster decomposition with $D_{j+1} \succ D_j$ and $|D_j| = j$, we define:

$$V_{S, k} = \Sigma' V_{ij}(\theta)$$

where the sum $\sum'$ is taken over all indices i, j which belong to different clusters in D_{k+1}, but belong to the same cluster in D_k. For $S=(D_{N+1}, \ldots, D_k)$, we define $i(S)=k$. For $z \notin \bigcup_{|D| \geqq 2} \sigma(H_D(\beta, \theta))$, we define:

$$D(\beta, \theta; z) = \sum_{i(S) \geqq 2} (H_{D_k} - z)^{-1} V_{S, k} \times \ldots \times$$
$$\times (H_{D_N} - z)^{-1} V_{S, N} (H_0 - z)^{-1} \tag{3.4}$$
$$I(\beta, \theta; z) = \sum_{i(S)=1} V_{S, 1} (H_{D_2} - z)^{-1} \ldots V_{S, N} (H_0 - z)^{-1}$$

Then we have the Weinberg-van Winter equation:

$$(H(\beta, \theta) - z) D(\beta, \theta; z) = I + I(\beta, \theta; z) \tag{3.5}$$

By the standard technique used in the proof of HVZ Theorem [23] we can show that:

$$\sigma_{ess}(H(\beta, \theta)) = \bigcup_{|D| \geqq 2} \sigma(H_D(\beta, \theta)) \tag{3.6}$$

This result is also true for $\beta=0$ and $\theta=0$. In this case, we have:

$$\sigma_{ess}(H(0, 0)) = [\Sigma, +\infty[\tag{3.7}$$

where

$$\Sigma = \inf_{|D| \geqq 2} \{ \Sigma_0 + \Sigma_1 + \ldots + \Sigma_k; \Sigma_j = \inf \sigma (H^{C_j}(0, 0), j \geqq 0 \}$$

We want to clarify the relationship between $\sigma_{\text{ess}}(H(\beta, \theta))$ and $\sigma_{\text{ess}}(H(0, 0))$. Our main result in this section is the following.

THEOREM 3.2. – *Put*: $\Lambda = \inf \sigma (H(0, 0)$. *Under the assumptions* (3.1)-(3.3), *for every* $\varepsilon > 0$, $\theta \in \Omega$, *there exists* $\beta_0 > 0$ *such that for* $0 < \beta \leqq \beta_0$, *one has:*

(i) $\sigma_{\text{ess}} (H(\beta, \theta)) \subset E(\theta, \Sigma - \lambda(1+\eta) Q - \varepsilon)$;

(ii) $\sigma(H(\beta, \theta)) \subset E(\theta, \Lambda - \lambda(1+\eta) Q - \varepsilon)$;

(iii) *There exists* $C = C(\theta) > 0$, *such that* $\| (H(\beta, \theta) - z)^{-1} \| \leqq C$, *uniformly in* $z \notin \mathbb{C} \setminus E(\theta, \Lambda - \lambda(1+\eta) Q - \varepsilon)$ and $0 < \beta \leqq \beta_0$.

The proof of Theorem 3.2 is broken into several steps. We first compute the numerical range of $H(\beta, \theta)$ and $H(\theta)$.

LEMMA 3.3. – *Let* $N(\beta, \theta)$ [*resp.* $N(\theta)$] *denote the numerical range of* $H(\beta, \theta)$ [*resp.* $H(\theta) \equiv H_0(\theta) + V(\theta)$]. *Then one has:*

(i) *There exists* $C > 0$ *such that* $N(\beta, \theta) \subseteqq E(\theta, -C)$. *Assume* $N = 1$. *Then for any* $\varepsilon > 0$, *one has* $N(\beta, \theta) \subseteqq E(\theta, \Lambda - \lambda(1+\eta) Q - \varepsilon)$ *for* $\beta > 0$ *sufficiently small*;

(ii) *For any* $\varepsilon > 0$, *there exists* $C > 0$ *such that*

$$N(\theta) \subseteqq \{ z \in \mathbb{C}; \operatorname{Re} z \geqq \Lambda - \varepsilon, \operatorname{Im} z \leqq \varepsilon \operatorname{Im} \theta (\operatorname{Re} z + C) \},$$

for all $\theta \in \Omega$ *and* $\beta > 0$ *sufficiently small.*

Proof. – For $u \in \mathcal{D}$ with $\| u \| = 1$, we want to estimate $\langle H(\beta, \theta) u, u \rangle$. We write $H(\beta, \theta)$ as $H(\beta, \theta) = H(\theta) + \beta X_1(\theta)$, where the last term corresponds to the Stark effect. The numerical range of $\beta X_1(\theta)$ is easy to compute. By the choice of χ, it is contained in the domain $E(\theta, -\lambda(1+\eta) Q - \varepsilon)$. By the definition of Λ, we have: $\langle H(0) u, u \rangle \geqq \Lambda$. We estimate:

$$\begin{aligned} \operatorname{Re} \langle H(\theta) u, u \rangle &= 1/2 \langle (H(\theta) + H(\theta)^*) u, u \rangle \\ &= \langle H u, u \rangle + 1/2 \langle (H(\theta)^* - 2H) u, u \rangle \end{aligned} \tag{3.8}$$

Making use of (3.1)-(3.3) and the Cauchy equation, we can show that the second term on the right side of (3.8) can be estimated by:

$$|\langle (1/2 (H(\theta) + H(\theta)^*) - H) u, u \rangle| \leqq C |\theta| (\langle H_0 u, u \rangle + \langle u, u \rangle)$$

Noticing that V is H_0-bounded with relative bound zero, we can derive that $|\langle Hu, u\rangle| \leqq a\langle H_0 u, u\rangle + b\|u\|^2$, $\forall u \in D(H_0)$, with a, b independent of u. Then (3.8) gives:

$$\operatorname{Re}\langle H(\theta)u, u\rangle \geqq (1-C|\theta|)\langle Hu, u\rangle - C'|\theta| \tag{3.9}$$

for some constants C, $C' > 0$.

From the proof of Proposition 2.2, we see that:

$$|\operatorname{Im}\langle H_0(\theta)u, u\rangle| \leqq \varepsilon \operatorname{Im}\theta(\langle H_0 u, u\rangle + 1) \tag{3.10}$$

for $\beta > 0$ sufficiently small. By the assumptions on V, we can write: $\operatorname{Im} V(\theta) = \operatorname{Im}\theta V_1(\theta)$, where $V_1(\theta)$ is H_0-bounded with relative bound 0. Therefore for any $\varepsilon > 0$, there exists $C > 0$ such that

$$|\operatorname{Im}\langle V(\theta)u, u\rangle| \leqq \varepsilon \operatorname{Im}\theta(\langle H_0 u, u\rangle + C)$$

for any $u \in D(\theta)$ with $\|u\| = 1$. This implies that for $\beta > 0$ sufficiently small, one has:

$$\operatorname{Im}\langle H(\theta)u, u\rangle \leqq \varepsilon \operatorname{Im}\theta(\langle H_0 u, u\rangle + C') \leqq \varepsilon \operatorname{Im}\theta(\operatorname{Re}\langle H(\theta)u, u\rangle + C'')$$

Taking $\varepsilon = 1/2$, we obtain from the above estimates that for $N \geqq 1$, $N(\beta, \theta) \subseteqq E(\theta. -C)$, for some $C > 0$ sufficiently large but independent of θ and $\beta > 0$. If $N = 1$, from (3.2) and the choice of χ, it follows that $V_1(\theta) = O(\beta^b)$, for some $b > 0$. Then one has:

$$\operatorname{Im}\langle H(\theta)u, u\rangle \leqq \varepsilon \operatorname{Im}\theta(\operatorname{Re}\langle H(\theta)u, u\rangle + C\beta^b)$$

This proves the assertion in (i) for $N = 1$. (ii) follows from (3.9) and the estimate on $\operatorname{Im}\langle H(\theta)u, u\rangle$. ■

Remark that if V_{ij}, $1 \leqq i < j \leqq N$ are repulsive, then the second part of (i) in Lemma 3.3 holds for any N and Theorem 3.2 can be proved much more easily.

Since $H(\beta, \theta) - z$ is invertible for some z outside $N(\beta, \theta)$, Lemma 3.3 tells us that the domain $\mathbb{C} \setminus E(\theta, -C)$ is contained in the resolvent set of $H(\beta, \theta)$ (*see* Kato [19]).

We want to apply the method of geometrical spectral analysis to locate the essential spectra of $H(\beta, \theta)$. For this purpose, recall from [7] some notations.

Definition 3.4. — Let A be a closed operator in $L^2(R^m)$ with C_0^∞ as its core. We define $N_{ess}(A)$ and $N_\infty(A)$ respectively by:

$N_{ess}(A) = \{\lambda \in C$; there exists a sequence $u_n \in D(A)$ with $\|u_n\| = 1$ such that (u_n) converges weakly to 0 and $\|(A-\lambda)u_n\| \mapsto 0$, when n tends to $+\infty\}$.

$N_\infty(A) = \{\lambda \in C$; there exists $u_n \in C_0^\infty(R^m)$ with $\|u_n\| = 1$ such that for any compact K, there is N_K such that $\operatorname{supp} u_n \cap K = \varnothing$ for $n > N_K$ and $\|(A-\lambda)u_n\| \mapsto 0$ when n tends to $+\infty\}$.

In order to apply these notions to the proof of Theorem 3.2, we introduce the following partition of unity on $R^{\nu N}$. Let $D=(C_0, C_1)$ be any two cluster decomposition of $\{0, 1, \ldots, N\}$ with $0 \in C_0$. Let Q_D denote the set:

$$Q_D = \{(x_1, \ldots, x_N) \in R^{\nu N};\ |x_j| > 1 \text{ for } j \in C_1 \text{ and } |x_i - x_j| > 1 \text{ for } i \in C_0 \text{ and } j \in C_1\}$$

$$Q_0 = \{(x_1, \ldots, x_N);\ |x_j| < 2N, \text{ for } j = 1, 2, \ldots, N\}$$

Notice that $Q_0 \cup \bigcup_{|D|=2} Q_D = R^{\nu N}$. Let $\chi_0 + \sum \chi_D = 1$ be a partition of unity subordinate to this covering. We define:

$$N_\infty^D(A) = \{\lambda \in C;\ \text{there exists a Weyl sequence } (u_n) \text{ for } A \text{ and } \lambda \text{ such that supp } u_n \subset \text{supp } \chi_D^n, \text{ where } \chi_D^n(.) = \chi_D(./n)\}.$$

For any two cluster decomposition D, let $H_D(\beta, \theta)$ denote the corresponding cluster operator as before. Then applying the result of [7] to $H(\beta, \theta)$, we obtain:

$$N_{ess}(H(\beta, \theta)) = \bigcup_D N_\infty^D(H_D(\beta, \theta)) \tag{3.11}$$

LEMMA 3.5. — *Assume the conditions* (3.1)-(3.3). *For any cluster decomposition* $D=(C_0, C_1)$, *set* $\tilde{H}_D(\beta, \theta) = H^{C_0}(\beta, \theta) \otimes I + I \otimes \tilde{H}^{C_1}(\beta, \theta)$ *where* $\tilde{H}^{C_1}(\beta, \theta)$ *is the operator obtained from* $H^{C_1}(\beta, 0)$ *by analytic dilation. Then we have:*

$$N_{ess}(H(\beta, \theta)) = \bigcup_{|D|=2} N_\infty^D(\tilde{H}_D(\beta, \theta))$$

Proof. — Note that $\tilde{H}^{C_1}(\beta, \theta)$ is well defined by the assumption (3.3). The desired result follows easily from (3.11) and the fact that $\tilde{H}_D(\beta, \theta) = H_D(\beta, \theta)$ on the support of χ_D^n and therefore

$$N_\infty^D(\tilde{H}_D(\beta, \theta)) = N_\infty^D(H_D(\beta, \theta)). \quad \blacksquare$$

LEMMA 3.6. — *Under the assumptions* (3.1)-(3.3), *for* $\theta \in \Omega$, $\sigma_{disc}(H(\beta, \theta))$ *is contained in the lower half complex plane* $\{\operatorname{Im} z \leqq 0\}$.

The above lemma can be proved by constructing a suitable space of distortion analytic functions and making use of the analytic continuation of $(H(\beta) - z)^{-1}$ from the upper half complex plane. Since the proof is by now standard, we omit it here (*cf.* [2], [4], [15] and [18]).

PROPOSITION 3.7. — *Let* $L(\beta, \theta) = i.\ e^{-\theta} H(\beta, \theta)$. *Then under assumptions* (3.1) *and* (3.2), $L(\beta, \theta)$ *generates an exponentially bounded strongly continuous semi-group and satisfies condition* (P) *in* [14]: *For each* $E \in R$ *there* *exists* $R_E>0$ *such that for* $W_E=\{z;\ \mathrm{Re}\,z<E,\ |\mathrm{Im}\,z|>R_E\}$, *one has:*

$$\begin{aligned} &\sigma(L(\beta,\theta))\cap W_E=\varnothing \\ \text{and}\quad &\sup_{z\in W_E}\|(L(\beta,\theta)-z)^{-1}\|<+\infty \end{aligned} \tag{3.12}$$

Proof. – Seeing Lemma 3.3, we need only to prove (3.12) for W_E replaced by $W'_E=\{-C\leqq \mathrm{Re}\,z\leqq E;\ |\mathrm{Im}\,z|>R_E\}$, for some $C>0$ large engouh. We use an induction on N. For $N=1$, we need only to prove:

$$\lim_{|\mathrm{Im}\,z|\to\infty}\|V(\theta)(L_0(\beta,\theta)-z)^{-1}\|=0.$$

Then the desired result follows from Lemma 2.4. By approximating $V(\theta)$ by C_0^∞ – potentials, we can assume without loss that $V(\theta)$ is in $C_0^\infty(R^\nu)$. Put $g(t;z)=V(\theta)U_0(t;\beta,\theta)e^{tz}$. Then Corollary 2.5 gives for $\mu>E$: $\|g(t;z)\|\leqq Ce^{-t(\mu-E)}$, for $\mathrm{Re}\,Z\leqq E$ and $t>0$. Clearly $g(t;z)$ is compact and measurable for $t>0$. We can apply Riemann-Lebesgue lemma and by the arguments already used in [15] we conclude that $V(\theta)(L_0(\beta,\theta)-z)^{-1}\to 0$, when $|\mathrm{Im}\,z|\to\infty$ and $-C\leqq\mathrm{Re}\,z\leqq E$. This shows that (3.12) is true for $N=1$. Assume now (3.12) is proved for $1\leqq N\leqq m$. Let $N=m+1$. For an arbitary cluster decomposition $D=\{C_0, C_1, \dots, C_k\}$, $k\geqq 1$, put $L_D(\beta,\theta)=ie^{-\theta}H_D(\beta,\theta)$ and $L_{C_j}(\beta,\theta)=ie^{-\theta}H^{C_j}(\beta,\theta)$. Since $|C_j|\leqq m$ and $L_D(\beta,\theta)$ and $L^{C_j}(\beta,\theta)$ generate exponentially bounded semigroups, it follows from the induction hypothesis and Theorem 3.1 in [14] $L_D(\beta,\theta)$ also satifies (3.12). Let $D(\beta,\theta;z)$ and $I(\beta,\theta;z)$ be defined by (3.4) with H_D replaced by L_D. Then in W_E with R_E large enough $D(\beta,\theta;z)$ and $I(\beta,\theta;z)$ are well defines and uniformly bounded and

$$(L(\beta,\theta)-z)D(\beta,\theta;z)=I+I(\beta,\theta;z).$$

If $\mathrm{Re}\,z=-C$ with $C>0$ large enough $I(\beta,\theta;z)$ can be expanded in a convergent series whose general terms are of the form:

$$I_\Gamma(z)=V_{i_1j_1}(L_0(\beta,\theta)-z)^{-1}\dots V_{i_Nj_N}(L_0(\beta,\theta)-z)^{-1}$$

where $\Gamma=\{i_1j_1, i_2j_2, \dots, i_Nj_N\}$ forms a connected diagram [21]. $I_\Gamma(z)$ is compact and uniformly bounded in $z\in W_E$. By Corollary 2.5 and the arguments used in step $N=1$, we derive that: $\lim\limits_{|s|\to\infty}\|I_\Gamma(-C+is)\|=0$ and so $\lim\limits_{|s|\to\infty}\|I(\beta,\theta;\ -C+is)\|=0$ Applying Lemma B.5 in [5], we obtain:

$$\lim_{|s|\to\infty}\|I(\beta,\theta;r+is)\|=0,$$

uniformly in $-C\leqq r\leqq E$. Now (3.12) results from (3.13) with $R_E>0$ large enough. Proposition 3.7 is proved by induction. ■

Proof of Theorem 3.2. – We use an induction on N. For $N=1$, we deduce from lemma 3.5 that

$$N_{ess}(H(\beta, \theta)) = N_\infty(\tilde{H}_0(\beta, \theta)) \subset \sigma_{ess}(\tilde{H}_0(\beta, \theta)) \tag{3.14}$$

where $\tilde{H}_0(\beta, \theta)$ is obtained from $H_0(\beta)$ by analytic dilation. According to the results of [13], one has:

$$\sigma_{ess}(\tilde{H}_0(\beta, \theta)) = \varnothing, \quad \text{if} \quad q_1 \neq 0,$$
$$\sigma_{ess}(H_0(\beta, \theta)) = \{e^{-\theta}\lambda;\ \lambda \geqq 0\}, \quad \text{if} \quad q_1 = 0.$$

By (ii) of Theorem 3.1 in [7], if $N_{ess}(A) \subset W$ and $C \setminus W$ is connected with $(\mathbb{C} \setminus W) \cap \rho(A) \neq \varnothing$, then $\sigma_{ess}(A) \subset W$. Therefore,

$$\sigma_{ess}(H(\beta, \theta)) \subset \sigma_{ess}(\tilde{H}_0(\beta, \theta))$$

which proves (i) for $N=1$, (ii) and (iii) for $N=1$ follows from (i) of Lemma 3.3. Now assume that the results are ture for $N=m$. When $N=m+1$, let $D=\{C_0, C_1\}$ be a two-cluster decomposition of $\{0, 1, \ldots, N\}$. Put:

$$\tilde{H}_D(\beta, \theta) = H^{C_0}(\beta, \theta) \otimes I + I \otimes \tilde{H}^{C_1}(\beta, \theta)$$

where $\tilde{H}^{C_1'}(\beta, \theta)$ is obtained from $H^{C_1}(\beta, 0)$ by analytic dilation. By the results of [15],

$$\sigma(\tilde{H}^{C_1}(\beta, \theta)) = \sigma_{ess}(H^{C_1}(\beta, \theta)) \subset E(\theta, \Sigma_1 - \varepsilon),$$

$\Sigma_1 = \inf \sigma(H^{C_1}(0, 0))$. Since the number of particles in C_0 is less than or equal to m, by the induction assumption,

$$\sigma(H_0^{C'}(\beta, \theta)) \subset E(\theta, \Sigma_0 - (1+\eta)\lambda Q_{C_0} - \varepsilon)$$

with $\Sigma_0 = \inf \sigma(H^{C_0}(0, 0))$ and $Q_{C_0} = \sum_{j \in C_0 \ 0} |q_j|$. Proposition 3.7 implies that $ie^{-\theta} H^{C_0}(\beta, \theta)$ has the properties of contained spectrum (condition (P) in [14]). It is known that $ie^{-\theta} \tilde{H}^{C_1}(\beta, \theta)$ has the some properties (*cf.* [15]). From Theorem 3.1 in [14], we obtain;

$$N_\infty^D(\tilde{H}_D(\beta, \theta)) \subset \sigma(\tilde{H}_D(\beta, \theta))$$
$$= \sigma(H^{C_0}(\beta, \theta)) + \sigma(\tilde{H}^{C_1}(\beta, \theta) \subset E(\theta, \Sigma_0 + \Sigma_1 - (1+\eta)\lambda G - \varepsilon)$$

Since $\Sigma \leqq \Sigma_0 + \Sigma_1$, $N_{ess}(H(\beta, \theta)) \subset E(\theta, \Sigma\text{-}(1+\eta)\lambda Q - \varepsilon)$, so by the argument already used in the step $N=1$, (i) for $N=m+1$ is poved. To prove (ii) for $N=m+1$, we use the Weiberg van Winter equation:

$$(H(\beta, \theta) - z)^{-1} = D(\beta, \theta; z)(I + I(\beta, \theta; z))^{-1}. \tag{3.15}$$

According to (i) of lemma 3.3, this equation is valid in $\mathbb{C} \setminus E(\theta; -C)$, for some $C>0$ large enough. We want to prove that it is also valid for $z \in \mathbb{C} \setminus E(\theta, \Lambda - (1+\eta)\lambda\theta - \varepsilon)$.

For any two-cluster decomposition $D=(C_0, C_1)$, by proposition 3.7 and Theorem 3.1 in [14], one has:

$$\sigma(H_D((\beta, \theta))=\sigma(H^{C_0}(\beta, \theta))+\sigma(H^{C_1}(\beta, \theta))$$

By the induction hypothesis,

$$\sigma(H^{c_j}(\beta, \theta)) \subset E(\theta, \Sigma_j-(1+\eta)\lambda Q_{c_j}-\varepsilon),$$

$\Sigma_j=\operatorname{Inf}\sigma(H^{c_j}(0, 0))$ and $Q_{c_j}-\sum_{k \in c_j} |q_k|$, and

$$\|(H^{c_j}(\beta, \theta)-z)^{-1}\| \leqq M, \qquad \text{for } j=0.1,$$

uniformly in $z \notin E(\theta, \Sigma_j-(1+\eta)\lambda Q_{c_j}-\varepsilon)$ and $0<\beta \leqq \beta_0$. Since $L_j(\beta, \theta)=ie^{-\theta} H^{c_j}(\beta, \theta)$ generates exponentially bounded semigroup, we derive from Proposition 3.4 in [28] that there exists $C>0$, independent of $\beta>0$, such that

$$\|e^{-tL_j(\beta, \theta)}\| \leqq ce^{-t(\Sigma_j-(1+\eta)\lambda Q_{c_j}-\varepsilon)},$$
$$j=0, 1, t \geqq 0.$$

Therefore

$$\|e^{-tie^{-\theta} H_D(\beta, \theta)}\| \leqq c^2 e^{-t(\Sigma'-(1+\eta)\lambda Q-2\varepsilon)},$$
$$\Sigma'=\Sigma_1+\Sigma_2 \geqq \Sigma.$$

This shows that $\|(H_D(\beta, \theta)-z)^{-1}\|$ is bounded uniformly in $z \in \mathbb{C} \setminus E(\theta, \Sigma-(1+\eta\}\lambda Q-2\varepsilon)$ and $0<\beta \leqq \beta_0$. By an easy induction, we can show that the same estimates for $H_D(\beta, \theta)$ are true when $|D| \geqq 2$. Let $H(\theta)$ denote the distorted schrödinger operator without homogeneous field. Then the corresponding Weinberg-van Winter equation:

$$(H(\theta)-z)^{-1}=D(\theta; z)(I+I(\theta, z))^{-1}$$

is valid in $\mathbb{C} \setminus E(\theta, \Lambda-\varepsilon)$. *See* (ii) of lemma 3.3 and Theorem 4 in [18]. (The proof of Theorem 4 in [18] implies that $\sigma(H(\theta)) \subset \{\operatorname{Im} z \leqq 0\}$.) From (ii) of lemma 3.3 for $H_D(\theta)$, we derive that $I(\theta; z)$ and $D(\theta; z)$ are bounded uniformly in $C \setminus E(\theta, \Lambda-\varepsilon)$. Since

$$I+I(\beta, \theta; z))(I+I(\theta; z))^{-1}=I+(I(\beta, \theta; z)-I(\theta; z))(I+I(\theta; z))^{-1},$$

in order to show the invertibility of $I+I(\beta, \theta; z)$ for $z \in \mathbb{C} \setminus E(\theta, \Lambda-(1+\eta)\lambda Q-\varepsilon)$ and $0<\beta \leqslant \beta_0$, which implies that (3.14) is valid in $\mathbb{C} \setminus E(\theta, \Lambda-(1+\eta)\lambda Q-\varepsilon)$, it suffices to prove:

$$\lim_{\beta \to 0+} \|I(\beta, \theta; z)-I(\theta; z)\|=0. \tag{3.16}$$

uniformly in $z \in \mathbb{C} \setminus E(\theta, \Lambda-\lambda(1+\eta)Q-\varepsilon)$. The proof for (3.16) follows the argument of Proposition 3.3 in [28]. Remark first that general terms in $I(\beta, \theta; z)$ can be written as:

$$A(\beta, \theta; z) \equiv (H_{D_1}(\beta, \theta)-z)^{-1} V_{i_1 j_1} \times \ldots (H_{D_k}(\beta, \theta)-z)^{-1} V_{i_k j_k} \tag{3.17}$$

where $|D_j| \geqq 2$ and $(i_1, j_1), \ldots, (i_k, j_k))$ forms a connected diagram (*see* [21]). Let $A(\theta, z)$ denote the same operator as (3.17) with $H_{D_j}(\beta, \theta)$ replaced by $H_{D_j}(\theta)$. We need only to prove

$$\lim_{\beta \to 0_+} \| A(\beta, \theta; z) - A(\theta, z) \| = 0 \tag{3.18}$$

which implies (3.16). By the assumptions (3.1), (3.2), we can approximate V_{ij} by functions in $C_0^\infty(R^\nu)$. Thus without loss, we consider only the case $V_{ij} \in C_0^\infty(R^\nu)$. (*See* also the proof of Lemma 2.8 in [28].) Since the diagram associated to $A(\beta, \theta; z)$ is connected, we can write: $X_1(\theta) = \sum q_j x_j^{(1)}(\theta) = \sum c_1 (x_{i_1}^{(1)}(\theta))$ (with $x_0 = 0$). Commute $x_{i_1}^{(1)}(\theta) - x_{j_1}^{(1)}(\theta)$ with $(H_{D_p} - z)^{-1}$ until it is absorbed by $V_{i_1 j_1}$. Since all the resulting commutators are uniformly bounded, we derive from the boundedness of $(x_i^{(1)} - x_j^{(1)}) V_{ij}(x_i - x_j)$ that $A(\beta, \theta; z) - A(\theta; z) = O(\beta)$ in the norm of operators. Here $O(\beta)$ depends on $\operatorname{Im}\theta > 0$, but is uniform in $z \in \mathbb{C} \setminus E(\theta, \Lambda - \lambda(1+\eta)Q - \varepsilon)$. This proves (3.16), so (3.15) is valid in the above domain. (ii) for $N = m+1$ is proved. (iii) follows from (3.14) and the uniform boundedness of $(I + I(\beta, \theta; z)^{-1})$. This finishes the proof of Theorem 3.2 by induction. ■

From the proof of Theorem 3.2, it is clear that if all charges are different from zero and have the same sign, say, $q_j > 0$, $j = 1, \ldots, N$, then $\sigma_{\text{ess}}(H(\beta, \theta)) = \varnothing$. This should not be surprising, seeing the result of [15].

Let S_θ be an open connected domain strictly contained in the complement of $E(\theta, \Sigma - \lambda(1+\eta)Q - \varepsilon)$. Then Theorem 3.2 says that the spectra of $H(\beta, \theta)$ in S_θ are discrete.

THEOREM 3.8. – *Under the assumptions* (3.1)-(3.3), *the spectra of* $H(\beta, \theta)$ *in* S_θ *are independent of* $\theta \in \Omega$. *More precisely, if* $z \in S_{\theta_1} \cap S_{\theta_2}$ *and* $z \in \sigma(H(\beta, \theta_1))$, *then* $z \in \sigma(H(\beta, \theta_2))$ *and the algebraic multiplicity of* z *relative to* $H(\beta, \theta_1)$ *and* $H(\beta, \theta_2)$ *is the same.*

Proof. – Let A denote the set of functions: $\{e^{-\alpha x^2} P(x);\ \alpha > 0$ and $P(.)$ polynomial$\}$. If $f \in A$, $U(\theta) f$ is holomorphic in θ in a whole small complex neighbourhood of 0. We can prove as in [18] and [26] that for f, g in A, $\langle f, (H(\beta) - z)^{-1} g \rangle$ defined for $\operatorname{Im} z > 0$ has a meromorphic extension in z into S_θ and that

$$\sigma(H(\beta, \theta)) \cap S_\theta = \bigcup_{f, g \in A} \{\text{poles of} \langle f, (H(\beta) - z)^{-1} g \rangle \text{in } S_\theta\}$$

This proves that $\sigma(H(\beta, \theta)) \cap S_\theta$ is independent of θ as point sets. Now let $z \in S_{\theta_1} \cap S_{\theta_2}$ be an eigenvalue of $H(\beta, \theta_j)$, $j = 1, 2$. To prove that the multiplicity of z relative to $H(\beta, \theta_1)$ and $H(\beta, \theta_2)$ is the same, we remark that for any compact $K \subset\subset \Omega$, we define the spectral projector of $H(\beta, \theta)$, $\theta \in K$, relative to z. Let $\pi(\theta)$ denote this projector. Then since $\{H(\beta, \theta), \theta \in K\}$ is *a* holomorphic family of type (A), $\pi(\theta)$ is holomorphic in $\theta \in K$. In particular, if K is connex, the multiplicity of z relative to $H(\beta, \theta)$, $\theta \in K$, is constant. Then the desired result follows by suitably choosing K. See the proof of Theorem 4 in [18] and Theorem 2.2 in [26] for more details. ■

The proof of Theorem 3.8 shows also that the spectra of $H(\beta, \theta)$ in S_θ are independent of the choice of χ. We define $R = \bigcup_\theta (S_\theta \cap \sigma(H(\beta, \theta)))$ as the resonances of $H(\beta)$. Observe that by Remark 3.1, we have included a slightly larger class of potentials than usual (*cf.* [9]). But the main interest of this work is to estimate the width of resonances which is exponentially small. The remainder of this paper is devoted to the study of resonances generated by the eigenvalues of $H(0, 0)$ below the bottom of the essential spectra.

4. STABILITY OF EIGENVALUES

We keep the assumption (3.1) on the potentials V_{ij} and denote: $H = H(0)$. Let $\lambda_0 < \Sigma \equiv \inf \sigma_{ess}(H)$ be an eigenvalue of H with multiplicity m. In this section, we shall introduce a suitable Dirichlet problem and study the stability of λ_0 under the perturbation of Stark effect. Since the proofs are often almost the same as in [26], we will only sketch the details.

Let λ be a fixed number in the interval $]0, (\Sigma - \lambda_0)/Q[$. Let M be the domain depending on $\beta < 0$ with piecewise smooth boundary defined by:

$$M = \{x = (x_1, \ldots, x_N) \in R^{vN};\ |x_j| < \lambda/\beta, j = 1, \ldots, N\} \qquad (4.1)$$

Let $P(\beta)$ denote the Dirichlet realization of $H(\beta)$ on M with zero boundary condition on ∂M. $P(\beta)$ is selfadjoint with domain $D(\Delta_D)$. Let $c(\beta) > \beta^{1/2}$ be a positive function of $\beta > 0$ such that $c(\beta) \to 0$ when $\beta \to 0$. Put: $I(\beta) = [\lambda_0 - c(\beta), \lambda_0 + c(\beta)]$. Then,

$$\operatorname{dist}(\sigma(H) \setminus \{\lambda_0\}, \quad I(\beta)) \geqq C > 0$$

for $\beta > 0$ sufficiently small. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $P(\beta)$ in $I(\beta)$, repeated according to their multiplicity. To study the relationship between the μ_j's and λ_0, we recall first Agmon's result on the exponential decay of eigenfunctions of N-body schrödinger operators.

THEOREM 4.1 [1]. – *For a unit vector* $\omega \in (\omega_1, \ldots, \omega_N)$ *in* R^{vN}, *we define the operator* H_ω *by:*

$$H_\omega = -\Delta + \sum_{\omega_j = 0} V_j(x_j) + \sum_{\omega_i = \omega_j} V_{ij}(x_i - x_j)$$

Set: $\Sigma_\omega = \inf \sigma(H_\omega)$. *Let* f *be an eigenfunction of* H *with eigenvalue* λ_0. *Then for any* $\varepsilon > 0$, *we have*

$$\| e^{(1-\varepsilon) d(x)} f \|_{H^1(R^{vN})} < +\infty \qquad (4.2)$$

where $d(x)$ denotes the geodesic distance from x to some fixed point x_0 in $\mathbf{R}^{\nu N}$ *in the metric $c(x)\,dx^2$. Here $c(x)=\Sigma_{x/|x|}-\lambda_0$, for $x\neq 0$ and $c(0)=-\lambda_0$.*

Notice that Σ_ω takes only a finite number of values and

$$\min \Sigma_\omega = \Sigma. \tag{4.3}$$

To estimate the decay of the eigenfunctions of $P(\beta)$, we observe that for every $\varepsilon>0$, there exists $R>0$ such that:

$$\langle H-\lambda_0 u, u\rangle \geqq \langle (c(x)-\varepsilon)u, u\rangle$$

for any $u\in C_0^\infty(\Omega_R)$, $\Omega_R=\{|x|>R\}$. For the operator $P(\beta)$ on M, we get:

$$\langle P(\beta)-\lambda_0 u, u\rangle \geqq \langle (c(x)+\beta X_1-\varepsilon)u, u\rangle \tag{4.4}$$

for any $u\in C_0^\infty(M\cap\Omega_R)$. By the definition of M, we can choose $\varepsilon>0$ sufficiently small so that $c(x)+\beta X_1-\varepsilon>0$ on M. From (4.4) we can prove the following result (*see* also [26]).

LEMMA 4.2. – *Let $f_j(\beta)$ be the orthonormalized eigenfunctions of* $P(\beta)$ *associated with the eigenvalue $\mu_j(\beta)$ in $I(\beta)$. Then for every $\varepsilon>0$, there exists $\beta_0>0$ such that:*

$$\sup_{0<\beta<\beta_0} \|e^{(1-\varepsilon)d_\beta} f_j\|_{H^1(M)} < +\infty,$$

$$j=1, \ldots, m \tag{4.5}$$

where $d_\beta(x)$ denotes the distance from x to zero in the metric $(c(x)+\beta X_1)_+\,dx^2$ and dx^2 is the usual Euclidean metric on $\mathbf{R}^{\nu N}$.

In order to obtain an upper bound on the widths of resonances, it is important to know the behavior of f_j's near the boundary ∂M. For this purpose, we give a lower bound of $d_\beta(x)$ on M.

LEMMA 4.3. – *Let $\bar{d}_\beta(x)$ denote the distance from x to 0 in the metric $(\Sigma+\beta X_1-\lambda_0)_+\,dx_2$. Then we have:*

$$d_\beta(x)\geqq \bar{d}_\beta(x)\geqq \frac{2}{3}|x|(2(\Sigma-\lambda_0)+\beta X_1)^{1/2}, \qquad x\in M. \tag{4.6}$$

Recall that $X_1=\sum q_j x_j^{(1)}$.

The proof of (4.6) is the same as in [26]. We omit it here.

COROLLARY 4.4. – *With the above notations, we have:*

$$d_\beta(x)\geqq 2\lambda S(\Sigma-\lambda_0)^{1/2}/3\beta, \qquad \textit{for} \quad x\in\partial M \tag{4:7}$$

Here the constants S depending only on the charges q_j is defined as follows. Set: $a_k=\sum_{j\neq k} q_j^2$, $b_k=2(\Sigma-\lambda_0)-|q_k|\lambda$ and $c_k=b_k+(b_k^2-3\lambda^2 a_k)^{1/2}$. Then

$$S=\min_k\,[(1+\lambda^2 a_k/c_k^2)(2-(\lambda|q_K|+\lambda^2 a_k)/(\Sigma-\lambda_0)c_k)]^{1/2}$$

Proof. – It follows from (4.6) and an easy computation. In fact, let $f(x)$ denote the right hand side of (4.6) and $f_k(x)$ be the restriction of f to the part of boundary where $|x_k| = \lambda/\beta$. We want to find the minimum of f_k for $|x_j| \leqq \lambda/\beta$, $j \neq k$. To be definitive, consider only the case $k=1$. Thus let

$$f_1(x) = ((\lambda/\beta)^2 + x_2^2 + \ldots + x_N^2)(b_1 + \beta \sum_{j \geqq 2} c_j x_j^{(1)}),$$

for $|x_j| \leqq \lambda/\beta$, $j \geqq 2$. We can check that the minimum of this function is attained at the point:

$$x_j^{(1)} = -q_j \lambda^2/\beta c_1, \quad x_j^{(k)} = 0, \qquad \text{for} \quad k=2, \ldots, \nu.$$

and the minimum equals: $\left(\dfrac{\lambda}{\beta}\right)^2 (1 + \lambda^2 a_1/c_1^2)(b_1 - \lambda^2 a_1/c_1)$. This proves (4.7). ■

Remark 4.5. – Note that $S \geqq 1$ in general and the equality holds if and only if there exists some $1 \leqq j_0 \leqq N$, such that $q_j = 0$ for $j \neq j_0$. Observe also that the metric $(c(x) + \beta X_1) dx^2$ is non-degenerate on the domain $M' = \{\beta X_1 > -\lambda/\beta\}$ and $d_\beta(x) \geqq 2\lambda Q(2(\Sigma - \lambda_0) - \lambda Q)^{1/2}/3\beta(\Sigma q_j^2)^{1/2}$ for $x \in \partial M'$. This suggests that one might hope to have an upper bound on the widths of resonances of the form (1.5). ■

THEOREM 4.6. – *Under the assumption* (3.1), *assume that* $\lambda_0 < \Sigma = \inf \sigma_{ess}(H)$ *is an eigenvalue of* H *with multiplicity m. Then for* $\beta > 0$ *sufficiently small, there exist exactly m eigenvalues, counted with their multiplicity, of* $P(\beta)$ *in* $I(\beta)$. *Let* $\mu_1(\beta), \mu_2(\beta), \ldots, \mu_m(\beta)$ *denote these eigenvalues. Then,*

$$|\mu_j - \lambda_0| \leqq C\beta, \qquad \textit{for} \quad 0 < \beta < \beta_0, \ j = 1, \ldots, m.$$

The proof of Theorem 4.6 is similar to that of Theorem 3.5 in [26] and we omit it here. Remark that as in the proof of Theorem 3.5 there, we can show that when β tends to zero, the spectra of $P(\beta)$ below $\lambda_0 + \varepsilon$ ($\varepsilon > 0$ sufficiently small) tend to cluster around those of H. This enables us to choose an interval $I = [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ independent of $\beta > 0$ such that:

$$\# \sigma(P(\beta)) \cap I = m \tag{4.8}$$

and

$$\operatorname{dist}(\sigma(P(\beta)), [\lambda_0 - 3\varepsilon, \lambda_0 + 3\varepsilon] \setminus \{\mu_1, \ldots, \mu_m\}) \geqq c > 0$$

uniformly in $0 < \beta < \beta_0$. From Lemma 4.2, we can prove the following result. *See* also [11].

$$\mu(\beta) \sim \lambda_0 + \sum_{k \geqq 1} \lambda_k \beta^k, \qquad \beta \to 0_+.$$

5. LOCATION OF RESONANCES

In this section, we shall give precise information on the location of the resonances generated by the eigenvalues of $H(\beta)$ below the bottom of essential spectra and as a corollary, we derive an upper bound on the widths of these resonances. For the sake of coherence, we precise the data utilized in the previous sections.

Let $\lambda_0<\Sigma$ be an eigenvalue of $H(0)$ with multiplicity m. For $\eta>0$ sufficiently small, let χ be a function chosen as in section 2: $\operatorname{supp}\chi\leqq\{t>1\}$ and $\chi=1$ for $t\geqq 1+\eta$. Set: $\lambda=(1-\eta)(\Sigma-\lambda_0)/Q$. Define the analytic distortion induced by the map Φ_θ as in section 2. Let $H(\beta,\theta)$ be the distorted N-body Schrödinger operator. Then the restriction of $H(\beta,\theta)$ on the domain M [*cf.* (4.1)] coincides with that of $H(\beta)$. By the result of section 3, the resonances of $H(\beta)$ are defined in the region S'_θ:

$$S'_\theta=\{z\in\mathbf{C};\ \operatorname{Re}z\leqq\lambda_0+3\mu,\ \pi/2\leqq\arg(z-\lambda_0-3\mu)\leqq\pi+\varepsilon\operatorname{Im}\theta\}$$

where $\mu=(\Sigma-\lambda_0)\eta^2/6$, provided $\beta>0$ is sufficiently small. We want to study the resonances of $H(\beta)$ near λ_0. For $\beta>0$ sufficiently small, there are exactly m eigenvalues of $P(\rho)$ in $[\lambda_0-\rho^{1/2},\lambda_0+\rho^{1/2}]$. Set:

$$S(\theta)=\{z\in S'_\theta;\ |\operatorname{Re}z-\lambda_0|\leqq\beta^{1/2}\}\tag{5.1}$$

To study the resonant problem:

$$H(\beta,\theta)f=zf,\qquad z\in S(\theta)\quad\text{and}\quad f\in\mathscr{D}$$

we shall approximate f in M by the eigenfunctions of $P(\beta)$ and outside M, we shall construct an approximate inverse of $H(\beta,\theta)-z$. *See* also [26].

Since the eigenvalues of $H(\beta,\theta)$ in $S(\theta)$ are essentially independent of $\theta\in\Omega$ (*see* Theorem 3.7), θ will be fixed in the following discussions. For $R\geqq 1$, define the operator $\tilde H(\beta,\theta)$ by:

$$\tilde H(\beta,\theta)=H_0(\beta,\theta)+\sum_{0\leqq i<j\leqq N}\tilde V_{ij}(\theta)\tag{5.2}$$

where $\tilde V_{ij}(\theta)=V_{ij}(\theta)$ for $i\geqq 1$ and $\tilde V_{0j}(x,\theta)=\xi(x/R)V_{0j}(x,\theta)$, ξ is a smooth function on $\mathbf{R}^\nu$ which equals 0 in some neighborhood of 0 and equals 1 outside a slightly bigger neighborhood.

PROPOSITION 5.1. — *For any* $\theta\in\Omega$ *and* $\varepsilon>0$, *there exist* R_0, $\beta_0>0$ *such that for* $z\in\mathbf{C}\setminus E(\theta,\Sigma-(1-)\lambda Q-\varepsilon)$, $\tilde H(\beta,\theta)-z$ *is invertible for* $0<\beta\leqq\beta_0$ *and* $R\geqq R_0$. *Moreover, we have:*

$$\|(\tilde H(\beta,\theta)-z)^{-1}\|_{\mathscr{L}(L^2)}\leqq C,\tag{5.3}$$

uniformly in $R\geqq R_0$ *and* $0<\beta\leqq\beta_0$.

Proof. — We use an induction on N. For $N=1$, this result is proved in [26]. Suppose that the result is true for all $1\leqq N\leqq m$. For $N=m+1$, we shall apply the Weinberg-van Winter equation (3.5) for $\tilde H(\beta,\theta)$. Let

$D=(C_0, C_1, \ldots, C_k)$ be a cluster decomposition of the set $\{0, 1, \ldots, N\}$. Let $\tilde{H}_D(\beta, \theta)$ denote the corresponding operator of $\tilde{H}(\beta, \theta)$. Then,

$$\begin{aligned}\tilde{H}_D(\beta, \theta) &= \tilde{H}^{C_0}(\beta, \theta) \otimes I \otimes \ldots \otimes I \\ &+ I \otimes H^{C_1}(\beta, \theta) \otimes I \otimes \ldots \otimes I + \ldots \end{aligned} \tag{5.4}$$

where $H^{C_j}(\beta, \theta)$, $j \geqq 1$, is the same as in (3.5). By the proof of Theorem 3.2, we see that the spectra of $H^{C_j}(\beta, \theta)$ are contained in the region $E(\theta, \Sigma_j - \lambda(1+\eta) Q_{C_j} - \varepsilon)$. By the induction hypothesis, we see that the spectra of $\tilde{H}^{C_0}(\beta, \theta)$ are contained in $E(\theta, \Sigma_0 - \lambda(1+\eta) Q_{C_0} - \varepsilon)$, provided $R > R_0$ is sufficiently large. This means that for any D with $|D| \geqq 2$, we have:

$$\sigma(\tilde{H}_D(\beta, \theta)) \leqq E(\theta, \Sigma - (1+\eta)\lambda Q - \varepsilon)$$

Now define $\tilde{I}(\beta, \theta)$ and $\tilde{D}(\beta, \theta)$ as in section 3 with $H_D(\beta, \theta)$ replaced by $\tilde{H}_D(\beta, \theta)$ and $V_{ij}(\theta)$ by $\tilde{V}_{ij}(\theta)$. Then for z in the complementary of the domain $E(\theta, \Sigma - (1+\eta)\lambda Q - \varepsilon)$, we have the Weinberg-van Winter equation:

$$(\tilde{H}(\beta, \theta) - z)\tilde{D}(\beta, \theta; z) = I + \tilde{I}(\beta, \theta; z) \tag{5.5}$$

Observe that $\tilde{I}(\beta, \theta; z)$ is a finite sum of the operators of the form:

$$\tilde{V}_{i_1 j_1}(\tilde{H}_{D_2} - z)^{-1} \tilde{V}_{i_2 j_2} \ldots \tilde{V}_{i_N j_N}(\tilde{H}_{D_{N+1}} - z)^{-1}. \tag{5.6}$$

Since the diagrams associated to $\tilde{I}(\beta, \theta; z)$ are all connected (*see* [21]), the $\tilde{V}_{i_k j_k}$'s contain at least one of the potentials $\tilde{V}_{0j}$. We approximate the $\tilde{V}_{ij}$'s appeared in (5.6) by smooth potentials (uniformly in $R > 1$). From the choice of $\xi(.)$, we see that for such smooth potentials, we have:

$$|\tilde{V}_{i_1 j_1} \tilde{V}_{i_2 j_2} \ldots \tilde{V}_{i_N j_N}| \leqq C R^{-s}, \quad \text{for some } s > 0$$

uniformly in $x \in \mathbb{R}^N$ and $R > 1$. From the proof of Theorem 3.2, we see that $(H_{D_j} - z)^{-1}$ is uniformly bounded in $z \in \mathbb{C} \setminus E - (1+\eta)\lambda Q - \varepsilon)$. By a limiting argument, we derive for general potentials satisfying (3.1)–(3.3) that:

$$\|\tilde{I}(\beta, \theta; z)\| \langle 1/2, \quad \text{for } R \rangle 1 \text{ large enough}$$

uniformly in $z \in \mathbb{C} \setminus E(\theta, \Sigma - (1+\eta)\lambda Q - \varepsilon)$. This proves that $\tilde{H}(\beta, \theta) - z$ has a right inverse. In the same way we can prove that its left inverse exists, so $\tilde{H}(\beta, \theta) - z$ is invertible. (5.3) follows easily. ■

Remark that $\Sigma - (1+\eta)\lambda Q = \lambda_0 + (\Sigma - \lambda_0)\eta^2 > \lambda_0$. For $\beta > 0$ sufficiently small, one has: $S(\theta) \subseteq \mathbb{C} \setminus E(\theta, \Sigma - (1+\eta)\lambda Q - \varepsilon)$. In the latter applications, $R > R_0$ will be fixed so that (5.3) holds for $z \in S(\theta)$. To study the local behavior of the resolvent, it is convenient to use the notations of Helffer-Sjöstrand [12]. Let A_β, $0 < \beta < \beta_0$, be a family of bounded operators. Let $K_{A_\beta}(.,.)$ be its integral kernel. For a positive continuous function f on $U \times U$ we shall write: $K_{A_\beta}(x, y) = \tilde{O}(\exp(-f_\beta(x, y)))$ on $U \times U$, if for any $\varepsilon>0$ and for any $(x_0, \ldots, y_0)\in U\times U$, there exist two neighborhood $W_1\ni x_0$, $W_2\ni y_0$ such that for some $C>0$, independent of (x_0, y_0) and $\beta\in]0, \beta_0]$, one has:

$$\| A_\beta \psi \|_{L^2(W_1)} \leqq C e^{-f_\beta(x_0, y_0)+\varepsilon/\beta} \| \psi \|_{L^2(W_2)} \tag{5.7}$$

for any $\psi\in C_0^\infty(W_2)$. Similarly in β-independent case, we write $K_A(x, y)=O(e^{-f(x, y)})$, if (5.7) is satisfied by A and f with $\beta=1$.

LEMMA 5.2. — *Let Ω be a domain in $\mathbf{R}^n$ and $\Gamma\subset\mathbf{C}$. Let P denote the Dirichlet realization of $-\Delta+V(x)$ (not necessarily selfadjoint). Suppose that there exists some positive continuous function $\lambda(x)$ on Ω such that:*

$$\operatorname{Re}\langle (P-z)u, u\rangle \geqq \langle \lambda u, u\rangle, \qquad u\in C_0^\infty(\Omega) \text{ and } z\in\Gamma \tag{5.8}$$

Assume that $P-z$ is invertible for $z\in\Gamma$ and the resolvent is uniformly bounded in z. Then for any $\varepsilon>0$, we have:

$$K_{(P-z)^{-1}}(x, y)=O(e^{-(1-\varepsilon)\rho_\lambda(x, y)})$$

uniformly in $(x, y)\in\Omega\times\Omega$ and $z\in\Gamma$ Here $\rho_\lambda(x, y)$ denotes the geodesic distance from x to y in the metric $ds^2=\lambda(x)\,dx^2$.

Proof. — For given $(x_0, y_0)\in\Omega\times\Omega$, $x_0\neq y_0$, we choose two disjoint neighborhood $U\ni x_0$ and $V\ni y_0$. Let ψ be a function equal to 1 on U and to 0 on V and $0\leqq\psi\leqq 1$ in general. For $v\in C_0^\infty(V)$, set: $u=(P-z)^{-1}v\in H_0^1(\Omega)$. Then we can verify the following identity (*see* Agmon [1]):

$$\iint_\Omega (|\nabla(fu)|^2-|u|^2|\nabla f|^2-\operatorname{Re}(V-z)|fu|^2)\,dx=\operatorname{Re}\iint_\Omega f^2 v\bar{u}\,dx$$

where $f=e^h$ and $h(x)=(1-\varepsilon)\psi(x)\rho_\lambda(x, y_0)$. From the above identity, we obtain from (5.8) that (*see* also Agmon [1])

$$\iint_\Omega e^{2h}|u|^2(\lambda(x)-|\nabla h|^2)dx\leqq \operatorname{Re}\iint_\Omega e^{2h}|v|^2(\lambda(x)-|\nabla h|^2)^{-1}\,dx. \tag{5.9}$$

By the choice of ψ, $|\nabla h|^2\leqq(1-\varepsilon)\lambda(x)$ on U and $|h|^2=0$ on V. We derive from (5.9) that:

$$\| e^{(1-\varepsilon)\rho_\lambda(x, y_0)} u \|_{L^2(U)} \leqq C \| v \|_{L^2(V)}.$$

This proves the result in the off-diagonal case. When (x_0, y_0) is on the diagonal of $\Omega\times\Omega$, the desired result follows from the assumption on the uniform boundedness of the resolvent. ■

From Lemma 5.2, we obtain the following corollary.

COROLLARY 5.3. — *Let $\tilde{P}(\beta)$ denote the Dirichlet realization of $\tilde{H}(\beta, \theta)$ on M, which is independent of θ. Then for $z\in S(\theta)$ and $0<\beta<\beta_0\,\tilde{P}(\beta)-z$*

is invertible and

$$K_{(\tilde{P}(\beta)-z)^{-1}}(x, y)=\tilde{O}(e^{-d_\beta(x, y)}) \quad \textit{on } M\times M \tag{5.10}$$

uniformly in $z\in S(\theta)$. *Here* $d_\beta(.,.)$ *is the distance function in the metric* $(c(x)+\beta X_1)_+\,dx^2$.

Proof. – The inversibility of $\tilde{P}(\beta)-z$ for $z\in S(\theta)$ can be proved by an induction on N as in Proposition 5.1. To prove (5.10), we notice that for any $\varepsilon>0$,

$$\operatorname{Re}\langle (\tilde{P}(\beta)-z)u, u\rangle \geqq \langle (c+\beta X_1-\varepsilon)u, u\rangle, \qquad u\in C_0^\infty(M),$$

provided $\beta>0$ is sufficiently small and $R>1$ is sufficiently large. The function $c(x)+\beta X_1$ is positive, lower semicontinuous on M. We can approximate it arbitarily well by positive continuous functions. Therefore we can apply Lemma 5.2 to obtain the desired result. ■

PROPOSITION 5.4. – *Let* $\tilde{d}_\beta(x, y)$ *be defined by:*

$$\tilde{d}_\beta(x, y)=\min(d_\beta(x, y),\ d_\beta(x, \partial M)+d_\beta(\partial M, y)) \quad \textit{on } M\times M$$

Then for fixed $\theta\in\Omega$, *we have:*

$$K_{(\tilde{H}(\beta, \theta)-z)^{-1}}(x, y)=\tilde{O}(e^{-\tilde{d}_\beta(x, y)}) \quad \textit{on } M\times M$$

for $\beta>0$ *sufficiently small and uniformly in* $z\,S(\theta)$.

Seeing Corollary 5.3, Proposition 5.4 follows from the same arguments as those used in the proof of Proposition 4.2 in [26]. *See* also [12]. We omit its proof here. Observe in particular that if U and V are two compacts in M, the same arguments give the estimate:

$$K_{(\tilde{H}(\beta, \theta)-z)^{-1}}(x, y)=\tilde{O}(e^{-\tilde{d}_\beta(U, V)}) \qquad \text{for } x\in U,\ y\in V \tag{5.11}$$

in the sense of (5.7).

Now we turn to construct a Grushin problem for $H(\beta, \theta)-z$. This procedure is the same as in [12] and [26]. But for the reader's convenience, we give still a brief description. Let $\mu_1(\beta), \dots, \mu_m(\beta)$ be the eigenvalues of $P(\beta)$ in the interval $I(\beta)$, with associated eigenfunction $u_1(\beta), \dots, u_m(\beta)$. Define the maps:

$$R_0^+:\ L^2(M)\to C^m \qquad \text{by } \ (R_0^+u)_j=\langle u, u_j\rangle \quad \text{for } u\in L^2(M)$$

$$R_0^-:\ C^m\to L^2(M) \qquad \text{by } \ R_0^- v=\sum_1^m v_j u_j$$

for $v=(v_1, \dots, v_m)\in C^m$. For $z\in S(\theta)$, consider the Grushin problem for $P(\beta)$ in matrix form:

$$\begin{pmatrix} P(\beta)-z & R_0^- \\ R_0^+ & 0\end{pmatrix}\cdot\begin{pmatrix} u \\ u^-\end{pmatrix}=\begin{pmatrix} v \\ v^+\end{pmatrix} \tag{5.12}$$

with $(v, v^+)\in L^2(M)\times C^m$. We decompose $L^2(M)$ as the orthogonal sum: $L^2(M)=E'+E''$, where $E''=\{u_1, \dots, u_m\}$. For $f\in L^2(M)$, we shall write the corresponding decomposition as $f=f'+f''$. Let $P'(\beta)$ denote the restriction of $P(\beta)$ on E'. Since there are no spectra of $P(\beta)$ in $S(\theta)$ other than $\mu_1, \ldots, \mu_m$, $P'(\beta)-z$ is invertible for $z\in S(\theta)$ with uniformly bounded inverse. For any (v, v^+) the problem (5.12) has a unique solution given by

$$\begin{pmatrix} u \\ u^- \end{pmatrix} = \mathscr{R}_0(z)\begin{pmatrix} v \\ v^+ \end{pmatrix},$$

where

$$\mathscr{R}_0(z) = \begin{pmatrix} (P'(\beta)-z)^{-1} & R_0^- \\ R_0^+ & \operatorname{Diag}(z-\mu_j) \end{pmatrix} \equiv \begin{pmatrix} E_0(z) & E_0^+(z) \\ E_0^-(z) & E_0^-+(z) \end{pmatrix} \quad (5.13)$$

As in [26], we have the following.

LEMMA 5.5. — *With the notations of Proposition* 5.4, *we have:*

$$K_{E_0(z)}(x, y) = \tilde{O}(e^{d_\beta(x, y)}) \quad \textit{on } M\times M \quad (5.14)_1$$

$$K_{E_0^+(z)}(x) = \tilde{O}(e^{-d_\beta(x)}) \quad (5.14)_2$$

and

$$K_{E_0^-(z)}(y) = \tilde{O}(e^{-d_\beta(y)}) \quad \textit{on } M \quad (5.14)_3$$

where $(5.14)_2$ *and* $(5.14)_3$ *should be interpreted in suitable spaces* (cf. [12], [26]).

Consider now the operator:

$$\mathscr{P}(\beta, z) = \begin{pmatrix} H(\beta,\theta)-z & R^- \\ R^+ & 0 \end{pmatrix} : \mathscr{D}\times C^m \to L^2(R^{\nu N})\times C^m$$

where $R^- = \rho R_0^-$, $R_0^+\rho$ and ρ is a smooth function with compact support in M and $\rho=1$ on $\{|x_j|\leqq(\lambda-\varepsilon/2)/\beta$ for all $j\}$. Let ψ be a smooth function with support contained in $\{|x_j|<(c\lambda+\varepsilon)/\beta\}$ and $\psi=1$ on $\{|x_j|\leqq(c\lambda-\varepsilon/2)/\beta\}$, where $o<c<1$ is chosen so that

$$d_\beta(x, y) \geqq (1-\varepsilon)\lambda S(\Sigma-\lambda_0)^{1/2}/3\beta, \qquad 0<\beta\leqq\beta_0 \quad (5.15)$$

for $x\in\operatorname{supp}(1-\rho)$ and $y\in\operatorname{supp}\psi$ or for $x\in\operatorname{supp}(1-\psi)$ and $y\in\operatorname{supp}(1-\xi(./R))$. By Corollary 4.4, such c exists and in the case $N=1$, we can take $c=1-2^{-3/2}$ [26]. Now for $z\in S(\theta)$, define $F(z)$ by:

$$F(z) = \begin{pmatrix} \rho E_0(z)\psi + (\tilde{H}(\beta,\theta)-z)^{-1}(1-\psi) & \rho E_0^+(z) \\ E_0^-(z)\psi & E_0^{-+}(z) \end{pmatrix}$$

Then a direct computation gives: $\mathscr{P}(z)\,\mathrm{F}(z)=\mathrm{I}+\mathrm{K}(z)$, where $\mathrm{K}(z)=(\mathrm{K}_{ij}(z))_{1\leqq i,\,j\leqq 2}$ with $\mathrm{K}_{ij}(z)$ given by:

$$\mathrm{K}_{11}=[\mathrm{H}(\beta,\theta),\rho]\,\mathrm{E}_0(z)\,\psi-\left(1-\xi\left(\frac{x}{\mathrm{R}}\right)\right)\sum \mathrm{V}_j(\theta)\,(\tilde{\mathrm{H}}(\beta,\theta)-z)^{-1}(1-\psi)$$

$$\mathrm{K}_{12}=[\mathrm{H}(\beta,\theta),\rho]\,\mathrm{E}_0^+(z)$$

$$\mathrm{K}_{21}=\mathrm{R}_0^+(\rho^2-1)\,\mathrm{E}_0(z)\,\psi+\mathrm{R}_0^+\,\rho\,(\tilde{\mathrm{H}}(\beta,\theta)-z)^{-1}(1-\psi)$$

$$\mathrm{K}_{22}=\mathrm{R}_0^+(\rho^2-1)\,\mathrm{E}_0^+(z)$$

Making use of Proposition 5.4 and Lemma 5.5, we can derive from (5.15) as in [26] that in the norm of bounded operators on $\mathrm{L}^2(\mathrm{R}^{\nu\mathrm{N}})\times\mathrm{C}^m$,

$$\mathrm{K}(z)=\tilde{O}\,(e^{-\lambda\mathrm{S}\,(\Sigma-\lambda_0)^{1/2}/3\beta}) \tag{5.16}$$

uniformly in $z\in\mathrm{S}(\theta)$. As a consequence of (5.16), $\mathscr{P}(z)$ is invertible for $\beta>0$ sufficiently small. We write this inverse, $\mathscr{R}(z)$, as:

$$\mathscr{R}(z)=\begin{pmatrix}\mathrm{E}(z) & \mathrm{E}^+(z)\\ \mathrm{E}^-(z) & \mathrm{E}^{-+}(z)\end{pmatrix}$$

From (5.16) and the expression $\mathscr{R}(z)=\mathrm{F}(z)\,(\mathrm{I}-\mathrm{K}(z)+\mathrm{K}(z)^2-\ldots)$, we can prove that

$$\mathrm{E}^{-+}(z)=\mathrm{E}_0^{-+}(z)+\tilde{O}\,(e^{-4\mathrm{S}_0/3\beta}) \tag{5.17}$$

where $\mathrm{S}_0=\lambda\,\mathrm{S}\,(\Sigma-\lambda_0)^{1/2}>0$. In fact, by a direct computation, one has:

$$\begin{aligned}&\mathrm{E}^{-+}(z)-\mathrm{E}_0^{-+}(z)\\ &\quad=\mathrm{E}_0^-(z)\,\mathrm{K}_{11}\mathrm{K}_{12}+\mathrm{E}_0^{-+}(-\mathrm{K}_{22}+\mathrm{K}_{21}\mathrm{K}_{12})+\tilde{O}\,(e^{-5\mathrm{S}_0/3\beta})\end{aligned} \tag{5.18}$$

Making use Proposition 5.4 and Lemma 5.5, we can prove as in [12] section 9 that the first two terms on the right hand side of (5.18) are of the order $\tilde{O}\,(e^{-4\mathrm{S}_0/3\beta})$. This justifies (5.17).

THEOREM 5.6. – *Under the assumptions* (3.1)-(3.3), *suppose that* $\lambda_0<\Sigma=\inf\sigma_{\mathrm{ess}}(\mathrm{H}(0))$ *is an eigenvalue of* $H(0)$ *with multiplicity* m. *Let* $\mu_1(\beta),\ldots,\mu_m(\beta)$ *be the eigenvalues of* $P(\beta)$ *in the interval* $\mathrm{I}(\beta)$. *Let* $\Gamma(\beta)$ *denote the resonances of* $\mathrm{H}(\beta)$ *in* $\mathrm{S}(\theta)$. *Then there exists a bijection* $b:\{\mu_1(\beta),\ldots,\mu_m(\beta)\}\to\Gamma(\beta)$ *such that:*

$$|b(\mu)-\mu|=\tilde{O}\,(e^{-4\lambda\mathrm{S}\,(\Sigma-\lambda_0)^{1/2}/3\beta}),\qquad \beta\to 0_+ \tag{5.19}$$

Here the constant $\mathrm{S}\geqq 1$ *is defined in Corollary* 4.4.

Proof. – We sketch only the proof since it is the same as in [12] and [26]. The basic point is that $\mathrm{H}(\beta,\theta)-z$ is invertible if and only if $\mathrm{E}^{-+}(z)$ is bijective on C^m. Then we have the formula:

$$(\mathrm{H}(\beta,\theta)-z)^{-1}=\mathrm{E}(z)-\mathrm{E}^+(z)\,\mathrm{E}^{-+}(z)^{-1}\,\mathrm{E}^-(z) \tag{5.20}$$

See [12]. By (5.20), we can show that the spectrum of $\mathrm{H}(\beta,\theta)$ in $\mathrm{S}(\theta)$ is in one-one correspondence with the zero of det $\mathrm{E}^{-+}(z)$, even if we count the multiplicity of these elements. Since the spectrum of $H(\beta, \theta)$ in $S(\theta)$ is essentially independent of $\theta \in \Omega$ (theorem 3.8), the desired result follows from (5.17). ■

Remark that $\mu_1(\beta), \ldots, \mu_m(\beta)$ are all real. For given $\varepsilon > 0$, choose $\eta > 0$ sufficiently small and study the Dirichlet problem of $H(\beta, \theta)$ on M defined with $\lambda = (1-\eta)(\Sigma - \lambda_0)/Q$. Then we conclude from Theorem 5.6 the following result on the widths of resonances.

COROLLARY 5.7. — *Under the assumptions of Theorem* 5.6, *for any* $\varepsilon > 0$ *there exists* $\beta_0 > 0$ *such that for* $z(\beta) \in \Gamma(\beta)$, *we have:*

$$|\operatorname{Im} z(\beta)| \leq C_\varepsilon e^{-(1-\varepsilon) 4 (\Sigma - \lambda_0)^{3/2} S/3Q\beta} \tag{5.21}$$

for $0 < \beta < \beta_0$. *Here* $Q = \sum |q_j|$ *and* S *is given in Corollary* 4.4

Notice that in Theorem 5.6, the error estimate contains a factor depending on $\eta > 0$, the parameter used in the definition of the Dirichlet problem, while in (5.21), the estimate is independent of η. Finally we indicate that the resonances in $S(\theta)$ defined here are the same as those defined by analytic dilation ([9], [15]). In fact let A denote the space of functions of the form: $e^{-c|x|^2} P(x)$, $c > 0$ and $P(.)$ is polynomial on $R^{\nu N}$. These functions are both distortion analytic and dilation analytic. If V_j's are also dilation analytic, we can show as in [18] that the resonances defined by these two approaches both coincides with the set:

$$\bigcup_{\psi, \varphi \in A} \{\text{poles of} \langle \psi, (H(\beta) - z)^{-1} \varphi \rangle \text{in } S(\theta)\}$$

Thus they are the same as point set. But in both cases one knows there are exactly m resonances, we conclude that they are the same, even if we take into account the multiplicity of each element. As in [12], we can show that the eigenfunctions of $P(\beta)$ are good approximations of the resonant states. The details are omitted.

REFERENCES

[1] S. AGMON, Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations, *Math. Notes,* No. **29**, Princeton Univ. Press, 1982.

[2] J. AGUILAR and J. M. COMBES, A Class of analytic Perturbations for One-body Schrödinger Hamiltonians, *Comm. Math. Phys.,* Vol. **22**, 1971, pp. 269-279.

[3] P. AVENTINI and R. SEILER, On the Electronic Spectrum of the Diatomic Molecular Iron, *Comm. Math. Phys.,* Vol. **41**, 1975, pp. 119-134.

[4] E. BALSLEV and J. M. COMBES, Spectral Properties of Many-Body Schrödinger Operators with Dilation Analytic Interactions, *Comm. Math. Phys.,* Vol. **22**, 1971, pp. 180-194.

[5] J. M. COMBES, P. DUCLOS, M. KLEIN and R. SEILER, The Shape Resonances, *Comm. Math. Phys.,* Vol. **110**, 1987, pp. 215-236.

[6] H. L. CYCON, Resonances Defined by Modified Dilations, *Helv. Phys. Acta,* Vol. **58**, 1985, pp. 969-981.

[7] P. DEIFT, W. HUNZIKER, B. SIMON and E. VOCK, Pointwise Bounds on Eigenfunctions and Wave Packets in N-Body Quantum Systems. IV., *Comm. Math. Phys.*, Vol. **64**, 1978, pp. 1-34.

[8] S. GRAFFI and V. GRECCHI, Resonances in Stark Effect and Perturbation Theory, *Comm. Math. Phys.*, Vol. **62**, 1979, pp. 63-96.

[9] S. GRAFFI and V. GRECCHI, Resonances in the Stark Effect of Atomic Systems, *Comm. Math. Phys.*, Vol. **79**, 1981, pp. 91-109.

[10] E. M. HARRELL and B. SIMON, The Mathematical Theory of Resonances Whose Widths are Exponentially small, *Duke Math. J.*, Vol. **47**, 1980, pp. 845-902.

[11] B. HELFFER and J. SJÖSTRAND, Multiple Wells in the Semiclassical Limit. I, *Comm. in P.D.E.*, Vol. **9**, 1984, pp. 337-408.

[12] B. HELFFER and J. SJÖSTRAND, Résonances en limite semi-classique, *Bull. Soc. Math. France*, Mémoire No. **24/25**, 1986.

[13] I. HERBST, Dilation Analyticity in Constant Electric Field, I: the Two-Body Problem, *Comm. Math. Phys.*, Vol. **64**, 1979, pp. 279-298.

[14] I. HERBST, Contraction Semigroups and the Spectrum of $A_1 \otimes I + I \otimes A_2$, *J. Operator Theory*, Vol. **7**, 1982, pp. 61-78.

[15] I. HERBST, B. SIMON, Dilation Analyticity in Constant Electric Field, II: N-Body Problem, Borel Summability, *Comm. Math. Phys.*, Vol. **80**, 1981, pp. 181-216.

[16] P. D. HISLOP and I. M. SIGAL, Sphape Resonances in Quantum Mechanics, *Memoirs of A.M.S.* (to appear).

[17] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, III, Springer-Verlag, Berlin, Heidelberg, 1985.

[18] W. HUNZIKER, Distortion Analyticity and Molecular Resonance curves, *Ann. Inst. Henri Poincaré*, sect. A, Vol. **45**, 1986, pp. 339-358.

[19] T. KATO, *Perturbation Theory for Linear Operators*, Berlin Heidelberg, Springer-Verlag, 1966.

[20] L. D. LANDAU and E. M. LIFSHITZ, *Quantum Mechanics*, New York, Pergamon Press, 1977.

[21] M. REED and B. SIMON, *Methods of Modern Mathematical Physics*, IV., New York, Academic Press, 1978.

[22] D. ROBERT and X. P. WANG, Time-Delay and Spectral Density for Stark Hamiltonians, I. Existence of Time-Delay Operator, *Commun. in P.D.E.*, Vol. **14**, 1989.

[23] B. SIMON, The Definition of Molecular Resonance Curves by the Method of Exterior Complex Scaling, *Phys. Lett.*, Vol. **71 A**, 1979, pp. 211-214.

[24] E. C. TITCHMARSH, *Eigenfunction Expansions Associated with Second Order Differential Equations*, II., Oxford Univ. Press, 1958.

[25] X. P. WANG, Low Energy Resolvent Estimates and Continuity of time-Delay Operators, *Proc. R. Soc. Edinburgh*, Vol. **105A**, 1987, pp. 229-242.

[26] X. P. WANG, Bound on Width of Resonances for Stark Hamiltonians, *Acta Math. Sinica* (in press).

[27] X. P. WANG, Asymptotics on Width of Resonances for Stark Hamiltonians (to appear).

[28] I. W. HERBST, Schrödinger Operators with External Homogeneous Electric and Magnetic Fields, *Proc. of Int. School Math. Phys., Erice*, 1980, pp. 131-183.

[29] I. M. SIGAL, Sharp Exponential bounds on Resonances States and Width of Resonances, *Adv. in Appl. Math.*, 1988.

[30] I. M. SIGNAL, Geometric Theory of Stark Resonances in Multi-electron Systems, *Comm. in Math. Phys.*, Vol. **119**, 1988, p. 287-314.

[31] X. P. WANG, *Resonances of N-Body Schrödinger Operators with Stark Effect, in Annual Report*, Inst. Math., Academia Sinica, 1987, pp. 108-126.

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