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by

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ABSTRACT. — The problem of characterizing the physical states of unbroken, non-confining, gauge field theories is examined with pertubative methods. It is argued that there does not yet exist a convincing solution for the non-Abelian case. In particular, the Kugo-Ojima criterion does define a space with positive metric in QED, but it is doubtful whether this is also true for Yang-Mills theories. Alternatively one could require the validity, on physical states, of certain algebraic relations between observables, which are induced by the field equations in their gauge invariant form. This criterion is shown to be violated by the vacuum state.

RÉSUMÉ. — Nous examinons par des méthodes perturbatives le problème de la caractérisation des états physiques non confinés pour des théories invariantes de jauge. Nous montrons qu'il existe pas actuellement de solution convaincantes pour le cas non Abélien. En particulier, le critère de Kugo-Ojima définit un espace à métrique positive en Electrodynamique quantique, mais il est douteux que ce soit aussi le cas pour une théorie de Yang-Mills. Un autre critère serait la validité sur les états physiques de certaines relations algébriques entre observables induites par les équations d'évolution sous leur forme invariante de jauge. Nous montrons que ce critère est violé par le vide.

1. INTRODUCTION

The state space $\mathscr{V}$ of a gauge field theory is a complex linear space, on which a non-degenerate, hermitian, but in general indefinite, scalar product $(.,.)$ is defined. Since physical states must have a positive real norm, not all vectors of $\mathscr{V}$ can represent such states. (We assume that we are not working in a "physical gauge", where $(.,.)$ is positive.) Following the lead of the Gupta-Bleuler formalism in QED, we expect to find a suitable subspace $\mathscr{V}_{ph} \subset \mathscr{V}$ on which the scalar product is positive:

$$(V, V) \geqq 0 \qquad \text{for} \quad V \in \mathscr{V}_{ph}, \tag{1.1}$$

so that $\mathscr{V}_{ph}$ is a pre-Hilbert space. The actual physical state space $\mathscr{H}_{ph}$ is then obtained from $\mathscr{V}_{ph}$ by quotienting out the subspace $\mathscr{V}' \subset \mathscr{V}_{ph}$ of zero-norm states, and completing the quotient $\mathscr{V}_{ph}/\mathscr{V}'$ in the Hilbert topology induced by $(.,.)$. Aside from positivity, $\mathscr{V}_{ph}$ must also satisfy such obvious requirements as that it be invariant under the dynamical evolution of the system, i. e. under time translations, and preferably also under space translations, and that it should contain the vacuum state Ω.

These requirements do not determine $\mathscr{V}_{ph}$ uniquely. In a $\mathscr{V}$ with indefinite scalar product there is no unique maximal subspace on which the scalar product is positive. Translational invariance does not change this situation, if the translations preserve the scalar product, as is usually the case. Hence we need an additional requirement for singling out $\mathscr{V}_{ph}$ unambiguously.

For a class of local, covariant, gauges Kugo and Ojima (Refs. [1], henceforth quoted as KO I and II), based on previous work by Curci and Ferrari [2], have proposed the definition

$$\mathscr{V}_{ph} = \{ V \in \mathscr{V} : Q_B V = 0 \}, \tag{1.2}$$

where Q_B is the generator of the BRS-transformation. In a canonical formulation of the theory, also developed in KO, Q_B agrees with the space integral of j^0, when j^μ is the Noether current associated with the BRS invariance of the Lagrangian density. This gives an explicit expression for Q_B in terms of the basic fields. In QED the KO-condition (1.2) is equivalent to the Gupta-Bleuler condition $(\partial_\mu A^\mu)^+ \mathscr{V}_{ph} = 0$. Furthermore this $\mathscr{V}_{ph}$ is translation invariant and can be shown to contain the vacuum at least in perturbation theory. However, the KO-argument for the positivity of the scalar product on $\mathscr{V}_{ph}$ is unconvincing, since it makes essential use of the LSZ asymptotic condition, which has no chance of being satisfied for the charged fields of an interacting gauge theory, due to infrared (IR) problems. This has long been an accepted fact in QED [1],

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[1] For a recent rigorous discussion see Buchholz [3].

and since the IR singularities are worse in the non-Abelian case, we cannot expect the situation to be better there.

The question of whether (1.2) indeed defines a positive subspace therefore requires closer inspection. This we shall do in sections 2 and 3. In section 2 we briefly describe the formal mechanism of the KO-argument. In section 3 we discuss how this mechanism can be implemented in perturbative QED, and show, why the methods used for QED do not carry over to the non-Abelian case. While these arguments do not prove that $\mathscr{V}_{\mathrm{ph}}$ is not a positive-metric space, they do raise severe doubts about this point. At the very least they show that the problem is still wide open.

An additional objection to the KO-condition is its lack of a convincing physical motivation. This makes it difficult, if not impossible, to generalize the condition to gauges other than those considered by KO. Also, the equations of motion acquire ghost contributions, even if sandwiched between physical states, and thus lose their gauge invariance. This makes it hard to see what gauge invariance means for such a theory, and seems to knock out the ground from under the original argument of Yang and Mills [4] in favor of gauge theories. For this reason we investigate in section 4 the alternative possibility of characterizing $\mathscr{V}_{\mathrm{ph}}$ by the requirement that on it the original, gauge invariant, equations of motion should hold in a suitable sense. The treatment is again perturbative, and the result is negative.

Since perturbative arguments are hardly convincing for a theory with confinement, we restrict our attention to non-confining theories. We also assume gauge invariance to be unbroken, since in theories with broken invariance — like the Glashow-Salam-Weinberg theory — the problem takes on a wholly new complexion due to the existence of unitary gauges and of gauge invariant fields creating "charged" particles [5]. Even though unbroken, non-confining, non-Abelian, theories do not seem to be realized in nature, their study is clearly of interest, since it may reveal *why* they are not realized. And in this context the main result of the present paper, that for such theories the problem of characterizing physical states is still open, may be of relevance.

When discussing the non-Abelian case we consider only pure YM theories. The inclusion of matter fields — at least if they are massive — is not expected to cause any problems beyond those already encountered in QED. Nor is it expected to cancel the problems originating in the self-interaction of the gauge fields. The basic fields of the theory are, then, the gauge potentials $A_\mu(x)$ and the Faddeev-Popov ghost fields $C(x)$, $\bar{C}(x)$. All these fields take values in the adjoint representation of the Lie algebra of the gauge group. Following KO we treat C, $\bar{C}$, as independent, hermitian, fields. The field strength $F_{\mu\nu}$ is defined, as usual, by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\, A_\mu \times A_\nu, \qquad (1.3)$$

where the cross product is the Lie product and g is a coupling constant. The components of A_μ, C, $\bar{C}$, in an orthonormal basis of the Lie algebra are denoted by A_μ^a, C^a, $\bar{C}^a$, whenever it is convenient to work with components.

As example of an Abelian theory we consider spinorial QED. It includes the Dirac spinors $\Psi(x)$, $\bar{\Psi}(x)$, as additional fields besides A_μ, C, $\bar{C}$. In order to discuss the KO-conjecture in its original form, we retain the ghost fields, even though they are free and decouple from the physically relevant fields A_μ, Ψ. The cross product vanishes in QED.

2. THE KUGO-OJIMA PROPOSAL

In this section the positivity argument of KO will be recapitulated in a form which is suitable for adaptation to pertubation theory. This presentation serves also to introduce our notations. The section does not contain any new results.

For notational convenience we work in the Feynman gauge, this being the formally simplest of the gauges considered in KO.

We start by discussing free ($g=0$) gauge theories, in which case the KO argument is rigorous. The fields of a free YM theory satisfy the equations of motion

$$\Box A^\mu(x) = \Box C(x) = \Box \bar{C}(x) = 0, \qquad (2.1)$$

and their two-point functions are, in p-space:

$$(\Omega, A_\mu^a(p) A_\nu^b(q) \Omega) = -g_{\mu\nu} \delta^{ab} T(p, q)$$
$$(\Omega, C^a(p) \bar{C}^b(q) \Omega) = -(\Omega, \bar{C}^a(p) C^b(q) \Omega) = i \delta^{ab} T(p, q) \qquad (2.2)$$
$$(\Omega, C(p) C(q) \Omega) = (\Omega, \bar{C}(p) \bar{C}(q) \Omega) = 0.$$

Here $T(p, q) = (2\pi)^{-3} \delta^4(p+q) \delta_+(p)$, $\delta_\pm(p) = \theta(\pm p_0) \delta(p^2)$, is the two-point function of a free, scalar, massless field. In the case of QED there are no internal indices a, b, and the spinor fields satisfy the equations of motion

$$(i\slashed{\partial} - m) \Psi(x) = \bar{\Psi}(x)(i\slashed{\partial} + m) = 0 \qquad (2.3)$$

and have the two-point function

$$(\Omega, \Psi(p) \bar{\Psi}(q) \Omega) = (2\pi)^{-3} \delta^4(p+q)(\slashed{p} + m) \delta_+(p), \qquad (2.4)$$

where now $\delta_\pm(p) = \theta(\pm p_0) \delta(p^2 - m^2)$.

Let $\Phi_{\alpha\beta}$ stand for any of the components of our basic fields. The first index $\alpha = 1, \ldots, 5$, represents the type of field (A, C, etc.) and β runs through all the components of Φ_α. The creation and annihilation operators corresponding to $\Phi_{\alpha\beta}$ are called $\varphi_{\alpha\beta}^+$, $\varphi_{\alpha\beta}^-$. They satisfy the commutation

relations

$$[a_\mu^{a-}(\mathbf{p}), a_\nu^{b+}(\mathbf{q})] = -2\,\omega_p\,\delta^{ab}\,g_{\mu\nu}\,\delta^3(\mathbf{p}-\mathbf{q}) \left.\right\}$$
$$\{c^{a-}(\mathbf{p}), \bar{c}^{b+}(\mathbf{q})\} = -\{\bar{c}^{a-}(\mathbf{p}), c^{b+}(\mathbf{q})\} = 2\,i\,\omega_p\,\delta^{ab}\,\delta^3(\mathbf{p}-\mathbf{q}) \right\} \quad (2.5)$$

$$\{\psi_\rho^-(\mathbf{p}), \bar{\psi}_\sigma^+(\mathbf{q})\} = 2\,\omega_p\,(\slashed{p}+m)_{\rho\sigma}\,\delta^3(\mathbf{p}-\mathbf{q}) \left.\right\}$$
$$\{\bar{\psi}_\rho^-(\mathbf{p}), \psi_\sigma^+(\mathbf{q})\} = 2\,\omega_p\,(\slashed{p}-m)_{\rho\sigma}\,\delta^3(\mathbf{p}-\mathbf{q}), \right\} \quad (2.6)$$

all other (anti-) commutators being zero. In (2.5) we have $\omega_p = |\mathbf{p}|$, in (2.6) $\omega_p = (\mathbf{p}^2 + m^2)^{1/2} = p_0$.

A (non-orthonormal) basis of the state space $\mathscr{V}$ is formed by the kets

$$|\mathbf{p}_1, \alpha_1, \beta_1; \ldots; \mathbf{p}_n, \alpha_n, \beta_n> = \prod_{i=1}^n \varphi_{\alpha_i \beta_i}^+(\mathbf{p}_i)|0>, \quad (2.7)$$

where the factors are ordered such that $\alpha_{i+1} \geq \alpha_i$. The case $n=0$ corresponds to the vacuum $|0> = \Omega$. To (2.7) we associate the bra

$$<\ldots; \mathbf{p}_1, \alpha_i, \gamma_i; \ldots| = <0| \prod_{i=n}^1 \varphi_{\bar{\alpha}_i \gamma_i}^-(\mathbf{p}_i), \quad (2.8)$$

where the order of the factors is reversed relative to (2.7), and $\Phi_{\bar{\alpha}}$ is $\bar{C}$ or $\bar{\Psi}$ if $\Phi_\alpha = C$ or Ψ, and vice versa. The completeness relation for this basis reads

$$\sum_{n=0}^\infty \sum_{\alpha_i, \beta_i, \gamma_i} \int \Pi \frac{d^3 \mathbf{p}_i}{2\,\omega_i} |\ldots; \mathbf{p}_i, \ldots; \ldots>$$
$$\times \Pi_\alpha(n_\alpha!)^{-1} \prod_{i=1}^n K_{\beta_i \gamma_i}^{\alpha_i} <\ldots; \mathbf{p}_i, \ldots; \ldots| = 1. \quad (2.9)$$

Here n_α is the number of fields of type α in the left-hand ket, and

$$K_{\beta\gamma}^\alpha = \begin{cases} -g_{\mu\nu}\,\delta^{ab} & \text{for the } p\text{-pair } a_\mu^{a+} - a_\nu^{b-} \\ i\,\delta^{ab} & \text{for the } p\text{-pair } c^{a+} - \bar{c}^{b-} \\ -i\,\delta^{ab} & \text{for the } p\text{-pair } \bar{c}^{a+} - c^{b-} \\ (2m)^{-1}\,\delta_{\sigma\rho} & \text{for the } p\text{-pair } \bar{\psi}_\sigma^+ - \psi_\rho^- \\ -(2m)^{-1}\,\delta_{\sigma\rho} & \text{for the } p\text{-pair } \psi_\sigma^+ - \bar{\psi}_\rho^-. \end{cases} \quad (2.10)$$

The field $A_\mu(p)$ is split into its transversal, longitudinal, and scalar parts:

$$A_\mu = \mathscr{A}_\mu + i\,p_\mu\,S + \omega_p^{-2}(p_\mu - p_0\,\eta_\mu)\,B, \quad (2.11)$$

where $\eta = (1, 0, 0, 0)$, and

$$\mathscr{A}_0 = 0, \quad \mathscr{A}_i = A_i - \omega_p^{-2}\,p_i\,p_j\,A_j \quad \text{for} \quad i = 1, 2, 3, \quad (2.12)$$
$$B = i\,p^\mu\,A_\mu, \quad S = -i\,p_0\,\omega_p^{-2}\,A_0. \quad (2.13)$$

The corresponding creation and annihilation operators are again denoted by lower-case letters. Their commutation relations or, equivalently, their

two-point functions can be easily derived from (2.2) and (2.5). We find, in obvious notation [see (2.2)]

$$(\Omega, \ \mathscr{A}_i \mathscr{A}_j \Omega) = \left(\delta_{ij} - \frac{p_i p_j}{\omega_p^2} \right) T$$

$$(\Omega, \ \mathscr{A}_i \, \mathrm{B}\,\Omega) = (\Omega, \ \mathscr{A}_i \, \mathrm{S}\,\Omega) = (\Omega, \ \mathrm{B}\,\mathrm{B}\,\Omega) = 0 \qquad (2.14)$$

$$(\Omega, \ \mathrm{S}\,\mathrm{S}\,\Omega) = \omega_p^{-2}\,\mathrm{T}, \qquad (\Omega, \ \mathrm{S}\,\mathrm{B}\,\Omega) = (\Omega, \ \mathrm{B}\,\mathrm{S}\,\Omega) = \mathrm{T}.$$

Note that $(\Omega, \ \mathscr{A}_i \mathscr{A}_j \Omega)$ is a positive matrix and that the states created by $\mathfrak{a}_i^+$ are orthogonal to those created by b^+ and s^+.

Particles created by $\mathfrak{a}^+$, ψ^+, $\bar{\psi}^+$, are called "physical", those created by b^+, s^+, c^+, $\bar{c}^+$, "unphysical". Let $\mathscr{V}^n \subset \mathscr{V}$ be the space of states with n unphysical and any number of physical particles. The $\mathscr{V}^n$ are mutually orthogonal. The projector P^n onto $\mathscr{V}^n$ is obtained from the decomposition of identity (2.9) by splitting $a_\mu^\pm$ according to (2.11) and retaining the terms with n pairs of unphysical and any number of pairs of physical operators. Mixed pairs do not occur. We have

$$\mathrm{P}^n\,\mathrm{P}^m = \delta^{nm}\,\mathrm{P}^n, \qquad \mathrm{P}^{n*} = \mathrm{P}^n, \qquad (2.15)$$

$$\sum_{n=0}^{\infty} \mathrm{P}^n = \mathbf{1}. \qquad (2.16)$$

P^n can be written recursively as [see KO II, eq. (3.29)]

$$\mathrm{P}^n = \frac{1}{n} \int \frac{d^3\,\mathbf{p}}{2\,\omega_p} \left[b^+(\mathbf{p})\,\mathrm{P}^{n-1}\,s^-(\mathbf{p}) + s^+(\mathbf{p})\,\mathrm{P}^{n-1}\,b^-(\mathbf{p}) \right.$$

$$\left. + \frac{1}{\omega_p^2}\,b^+(\mathbf{p})\,\mathrm{P}^{n-1}\,b^-(\mathbf{p}) + i\,c^+(\mathbf{p})\,\mathrm{P}^{n-1}\,\bar{c}^-(\mathbf{p}) - i\,\tau^+(\mathbf{p})\,\mathrm{P}^{n-1}\,c^-(\mathbf{p}) \right] \qquad (2.17)$$

for $n > 0$. Note that the unphysical pairs $s^+ - s^-$, $c^+ - c^-$, $\bar{c}^+ - \bar{c}^-$, do not occur.

The fermion part of P^0 has the same form as in (2.9). The $\mathscr{A}$-part of P^0 is

$$\mathrm{P}_{\mathscr{A}}^0 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_\alpha=1}^{3} \int \prod_{\alpha=1}^{n} \frac{d^3\,\mathbf{p}_\alpha}{2\,\omega_\alpha} \Pi_1^n\,\mathfrak{a}_{i_\alpha}^+(\mathbf{p}_\alpha)\,|\,0> <0\,|\,\Pi_1^n\,\mathfrak{a}_{i_\alpha}^-(\mathbf{p}_\alpha). \qquad (2.18)$$

P^0 is positive, i. e. $(\mathrm{V}, \mathrm{P}^0 \mathrm{V}) \geq 0$ for all $\mathrm{V} \in \mathscr{V} = \sum \mathscr{V}^n$.

The BRS charge Q_B acts on our fields as follows [see KO II, eqs. (2.16) and (2.21)]:

$$[Q_B, A_\mu] = -p_\mu C, \qquad \{Q_B, C\} = \{Q_B, \Psi\} = \{Q_B, \bar{\Psi}\} = 0, \qquad (2.19)$$

$$\{Q_B, \bar{C}\} = B,$$

hence

$$[Q_B, \mathscr{A}_\mu] = [Q_B, B] = 0, \qquad [Q_b, S] = i\,C. \qquad (2.20)$$

Also, it has been shown in KO II, Chap. 3, that

$$[Q_B, P^n] = 0 \tag{2.21}$$

and

$$P^n = \{Q_B, R^n\} \qquad \text{for} \quad n \geq 1 \tag{2.22}$$

with

$$R^n = \frac{1}{n} \int \frac{d^3 \mathbf{p}}{2 \omega_p} \left[\bar{c}^+ (\mathbf{p}) \, P^{n-1} s^- (\mathbf{p}) + s^+ (\mathbf{p}) \, P^{n-1} \bar{c}^- (\mathbf{p}) \right.$$
$$\left. + \omega_p^{-2} b^+ (\mathbf{p}) \, P^{n-1} \bar{c}^- (\mathbf{p}) \right]. \tag{2.23}$$

This form of P^n can be verified by inserting (2.23) into (2.22) and calculating the intervening commutators from (2.19) and (2.20). The result is the recursive formula (2.17).

With these results the positivity proof is now elementary: let $V \in \mathscr{V}_{ph}$, *i. e.* $Q_B V = 0$. Then:

$$(V, V) = \sum_n (V, P^n V) = (V, P^0 V)$$

$$+ \sum_{n=1}^{\infty} [(Q_B V, R^n V) + (V, R^n Q_B V)] = (V, P^0 V) \geq 0. \tag{2.24}$$

For the interacting case KO argue as follows. Assume that for $t \to \infty$ the fields A, C, ..., converge in the LSZ sense towards free fields A^{out}, C^{out}, ..., and that asymptotic completeness holds. Then the foregoing proof applies, with the out-fields being the free fields of the proof.

As mentioned in the introduction, the first of these assumptions is not acceptable. The proof might be salvaged if it were possible to make sense of the sums in (2.9) and (2.18) written for the out-fields, despite the non-existence of the individual terms, in such a way that the KO-argument can be adapted to the new situation. An obvious idea is to look for an IR regularization which makes the individual terms exist, which disturbs the KO-argument only in a controllable way, and which is such that on removal of the regularization the relevant sums have a finite limit. One might hope to achieve this by a generalization of asymptotic completeness, *i. e.* by writing the out-states in (2.9) formally as limits $t \to \infty$ of suitable finite-time states formed with the interacting fields and then drawing the t-limit in front of the sum. But whether such a method works is completely unknown at present. We hope to return to this problem in a later publication. In the present paper we follow a different course, that of studying the problem in perturbation theory. We shall find formal structures very similar to those underlying the KO-argument, which therefore can be transcribed into the new framework. This allows to examine the argument in a context in which sufficient information is available to discuss the

difficulties encountered in attempting to turn the formal proof into a rigorous one.

3. PERTURBATIVE DISCUSSION OF THE KO-ARGUMENT

Accessible to a perturbative treatment are states of the form

$$V = \sum_{n=0}^{\infty} \Sigma_{\{\alpha_i\beta_i\}} \int \Pi \, dp_i f_{\{\alpha_i\beta_i\}}(p_1, \ldots, p_n) \prod_i \Phi_{\alpha_i\beta_i}(p_i) \Omega, \qquad (3.1)$$

where $\Phi_{\alpha\beta}$ are the basic fields of sect. 2. The n-sum should extend only over finitely many terms in every finite order of perturbation theory. In physical states we cannot expect the weight functions f to be very smooth, e. g. they are certainly not test functions in the spaces $\mathscr{D}$ or $\mathscr{S}$ known from the theory of distributions ([2]). But we assume them to be sufficiently regular to make (V, V) exist to every order in the coupling constant g.

The terms in the perturbative expansion of a positive quantity need not be positive themselves, except the lowest one. We need therefore another characterization of positivity. We say that the formal power series (V, V) is perturbatively positive, if it can be written in the form

$$(V, V) = \lim_{\Lambda \to \infty} \Sigma_{i,j} \overline{A_i(\Lambda)} K_{ij}(\Lambda) A_j(\Lambda), \qquad (3.2)$$

where the A_i are themselves formal power series and (K_{ij}) is a positive matrix. The multi-index i contains also continuous variables, $i.\,e.$ momenta, for which the sum must be read as an integral. The introduction of a regularization parameter Λ is advisable for handling the IR problems: Λ will be an IR regularization, and the Λ-limit and the $i-j$-summation will not interchange.

If the KO-conjecture is correct, then we expect that (V, V) is perturbatively positive for V of the form (3.1), if $Q_B V = 0$ to all orders in g.

For calculating (V, V) we need perturbative expressions for the Wightman functions $(\Omega, \Phi_{\alpha_1\beta_1}(p_1) \ldots \Phi_{\alpha_n\beta_n}(p_n)\Omega)$. Such expressions have been derived by Ostendorf [11]. A slight generalization of these rules, adapted to our proposes, will be given in an appendix, without proofs.

In order not to burden our arguments with inessential formal complications, we assume, however, that the field products in (3.1) are time-ordered. The generalization to ordinary products is notationally involved

but otherwise straightforward. We are, then, confronted with vacuum expectation values of the type $(\Omega, T^*(p_1, \ldots, p_n) T(q_1, \ldots, q_m) \Omega)$, where T is a time-ordered product of n or m fields respectively.

The σ-th order of this expression is a sum over graphs, called G-graphs (for "generalized graphs"), which at first look like ordinary Feynman graphs with $n+m$ external lines and σ internal vertices. This line-vertex-structure will be called the skeleton of the graph. The skeleton is then cut into two non-overlapping subgraphs, called sectors, one containing the p_i-lines, the other one the q_j-lines. The cut runs only through lines, never through vertices. Different ways of cutting the same skeleton lead to different G-graphs. In the q-sector the ordinary Feynman rules hold, in the p-sector the adjoint of the ordinary rules. Cut A-lines carry the propagator $-(2\pi)^{-3} \delta^{ab} g_{\mu\nu} \delta_+(k)$. Cut ghost lines carry $\pm i(2\pi)^{-3} \delta^{ab} \delta_+(k)$ if the C-end is in the $\begin{Bmatrix} p \\ q \end{Bmatrix}$-sector. Cut Ψ-lines carry $(2\pi)^{-3} \gamma_0 (\slashed{k}+m) \delta_+(k, m)$ or $(2\pi)^{-3} (\slashed{k}-m) \gamma_0 \delta_+(k)$ if the Ψ-end is in the p-sector or the q-sector respectively. In all cases the line-momentum k is directed from the p-sector towards the q-sector. The γ-factors along a spinor loop (open or closed) are ordered in the $\Psi \to \bar{\Psi}$ or $\bar{\Psi}^* \to \Psi^*$ direction. The factors δ^{ab} are absent in QED.

Notice that the fermionic cut propagators $\gamma^0 (\slashed{k}+m) \delta_+(k)$ and $(\slashed{k}-m) \gamma^0 \delta_+(k)$ are positive 4×4-matrices.

Usually this graphical representation is derived by assuming the asymptotic conditions to hold, inserting a complete set of *out* states between T* and T, and applying the LSZ reduction formula. It is important to note that Ostendorf's derivation does not make use of these questionable assumptions. It uses only properties of the Wightman functions like the spectral condition and locality, which hold independently of the validity of asymptotic conditions. Nevertheless, also this method does not avoid the usual IR problems: the individual graphs do not exist, because of IR divergences. It is, however, expected that these divergences cancel in the sum over all G-graphs with the same skeleton. This cancellation has not yet been proved in general. But if it is not true, then perturbation theory cannot be applied to the Wightman functions, hence a perturbative treatment is hardly meaningful for the theory in question.

In order to give a meaning to the individual graphs, we introduce the following IR regularization (^{3}). The δ-factors $\delta^4(\Sigma k_i)$ implementing

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(^{3}) Self-energy parts next to a cut line lead to divergent products of the type $(p^2 - m^2)^{-1} \delta(p^2 - m^2)$ even in purely massive theories. These divergences also cancel between graphs (*see* [11], p. 285), and they are also taken care of by our regularization. For convenience we subsume them under the term "IR-divergences".

momentum conservation in each vertex are replaced by $\delta_\Lambda(K) = \Lambda^4 \Delta(\Lambda K)$, where $\Delta(K)$ is a C^∞-function with support in $\{|K| \leqq 1\}$, such that $\int d^4 K \, \Delta(K) = 1$. δ_Λ converges to δ^4 in the limit $\Lambda \to \infty$.

There are, of course, also UV divergences present. These are dealt with as usual by renormalization. We shall not consider this problem explicitly, since it is known to be solvable and is not germane to our purposes.

In translating the KO argument into our graph language we proceed at first formally, without worrying about existence problems, in particular without introducing the Λ-regularization. As mentioned above we can, at this formal level, think of our graphs as having arisen from using the LSZ asymptotic conditions and reduction formulae. *I. e.* we can write the propagators of the cut lines in the form $(2\pi)^{-3/2} K^\alpha_{\beta\gamma} \delta_+(k)(2\pi)^{-3/2}$ for scalar lines, $(2\pi)^{-3/2} \gamma_0 (\not{k}+m) K^\alpha_{\beta\gamma} \delta_+(k)(\not{k}+m)(2\pi)^{-3/2}$ for $\Psi \to \bar\Psi$ lines and similarly for $\bar\Psi \to \Psi$ lines, where the kernels $K^\alpha_{\beta\gamma}$ are given by (2.10). The factors to the left of $K\delta_+$ are then included in the p-sector of the graph, those to the right of $K\delta_+$ in the q-sector. After this the q-sector is a contribution to $\prod\limits_{i=1}^{l} (k_i^2 - m_i^2) \tau(k_1, \ldots, k_l, q_1, \ldots, q_m)$, which expression is equal to $(\Omega, T(\Phi^{out}(k_1)\ldots\Phi^{out}(k_l)\Phi(q_1)\ldots\Phi(q_m))\Omega)$ on the mass shell $k_i^2 = m_i^2$, in obvious notation. An analogous result holds for the p-sector. The algebraic manipulations of KO can be reduced to the repeated use of the identity

$$\sum_{i=1}^{l} \pm (\Omega, T(\Phi^{out}(k_1)\ldots[Q_B, \Phi^{out}(k_i)]_\pm\ldots\Phi(q_m))\Omega)$$

$$+ \sum_{j=1}^{m} \pm (\Omega, T(\Phi^{out}(k_1)\ldots[Q_B, \Phi(q_j)]_\pm\ldots\Omega) = 0. \quad (3.3)$$

After replacing the commutators $[Q_B, \Phi^{out}]_\pm$ by their values (2.19), $[Q_B, \Phi]_\pm$ by the corresponding expressions for interacting fields (*see* below), these identities become the Ward-Takahashi-identities (WTI), which are identities between Green functions [*see* KO II, eq. (2.41)]. In this way the KO argument is translated into our graph language. The j-sum in (3.3) drops out in the sum over all graphs of a given order contributing to (V, V) because of the condition $Q_B V = 0$. Its exact form is therefore at present of no importance.

The decomposition (2.11) of the A-field into physical and unphysical parts is rendered by the following decomposition of the cut A propagators:

$$
\begin{array}{cccc}
& \text{B}-\text{B} & \text{S}-\text{B} & \text{B}-\text{S} \quad \text{phys}
\end{array}
$$

$$
-g_{00}\,\delta_+(k)=\delta_+(k)\left\{\frac{(-ik_0)(ik_0)}{\omega_k^2}+\frac{ik_0}{\omega_k^2}(ik_0)+(-ik_0)\frac{-ik_0}{\omega_k^2}+\ 0\right\}
$$

$$
-g_{0j}\,\delta_+(k)=\delta_+(k)\left\{\frac{(-ik_0)(ik_j)}{\omega_k^2}+\frac{ik_j}{\omega_k^2}(ik_0)+\ \ \ 0\ \ \ +\ 0\right\} \qquad (3.4)
$$

$$
-g_{ij}\,\delta_+(k)=\delta_+(k)\left\{\frac{(-ik_i)(ik_j)}{\omega_k^2}+\ \ \ 0\ \ \ +\ \ \ 0\ \ \ +G_{ij}\right\}
$$

with

$$
G_{ij}(k)=\delta_{ij}-\frac{k_i k_j}{\omega_k^2}. \qquad (3.5)
$$

The association of the various terms with the pairs occuring in (2.17) is indicated at the top.

The translated KO argument states that only terms containing exclusively physical cut lines contribute to (V, V). But the K-factors of these physical lines are positive, hence an expression of the form (3.2), without Λ-limit, results.

We shall now analyze whether these considerations can be rigourized by extending them to the regularized graphs and studying the limit $\Lambda \to \infty$. We still use the cut-line propagators $K^\alpha_{\beta\gamma}\delta_+$, including the other factors of the original prescription in the q- and p-sectors. Also we use the decomposition (3.4) for cut A-lines. The q-sector gives a regularized contribution to the Green function

$$
\hat{\tau}(k_1,\ldots,k_l;q_1,\ldots,q_m)=\prod_{i=1}^{l}(k_i^2-m_i^2)\,\tau(k_1,\ldots,k_l,q_1,\ldots,q_m), \qquad (3.6)
$$

and analogously for the p-sector. Because of the δ_+-factors of the cut lines only the values of $\hat{\tau}$ on the mass shell $k_i^2=m_i^2$ contribute. This mass shell restriction exists for finite Λ, but not for $\Lambda=\infty$. Also, the mass shell value $\hat{\tau}$ can no longer be expressed with the help of asymptotic fields Φ^{out}, since they do not exist.

Carrying through nevertheless the KO-algebra, translated into WTI-manipulations as explained above, we run into two problems.

1st problem: In the WTI (3.3) the k_i-fields must now also be taken to be interacting fields, and for them the BRS-relations take the more complicated form

$$
[Q_B,\, A_\mu]=-k_\mu C-ig\,A_\mu\times C, \qquad \{Q_B,\, C\}=\frac{i}{2}g\,C\times C,
$$

$$
\{Q_B,\, \bar{C}\}=B, \qquad \{Q_B,\, \Psi\}=g\,C\Psi, \qquad \{Q_B,\, \bar{\Psi}\}=-g\,C\bar{\Psi}, \qquad (3.7)
$$

instead of (2.19). The field products on the right-hand side are suitably renormalized local products in x-space, hence convolution products in p-space. For the sake of consistency these convolutions are also IR-regularized, *i.e.* $\Phi^\alpha \Phi^\beta$ must be understood as

$$\int dk_1 \, dk_2 \, \Phi^\alpha(k_1) \Phi^\beta(k_2) \, \delta_\Lambda(k - k_1 - k_2).$$

The regularized q-sectors are then continuous functions of the variable k, hence they vanish on the mass shell after amputation with $(k^2 - m^2)$, m the appropriate mass, and thus the additional terms in (3.7) do not contribute to our expressions, because of the δ_+ factors of the cut lines.

2nd problem: The Λ-regularization destroys the validity of the WTI.

In perturbative QED the proof of the WTI can be reduced to the following consideration: let k, p, q, be three momenta incident at a given vertex, k being the photon momentum. Then $k^\mu \gamma_\mu = (\not{q} - m) - (\not{p} - m)$ if momentum is conserved at the vertex. The terms $(\not{q} - m)$, $(\not{p} - m)$ cancel one of the adjacent propagators, and the resulting expressions are cancelled by similar expressions from other graphs. In our regularization momentum is not conserved. Hence we get an additional term containing the factor $(q^\mu - p^\mu - k^\mu) \, \delta_\Lambda(q - p - k)$. Because of the support of δ_Λ we find that this product is of order Λ^{-1}, so that the terms violating the WTI are of this order.

In YM theories the situation is similar. In a graphical proof of the WTI one contracts, for example, the vertex factor $g_{\beta\gamma}(q_\alpha - r_\alpha)$ with p_α. Assuming $p + q + r = 0$ he finds $(r^2 - q^2) g_{\beta\gamma} + q_\beta q_\gamma - r_\beta r_\gamma$. The factors r^2, q^2, in the first terms remove the singularities of the adjacent q- or r-propagators respectively, and the resulting expressions cancel with expressions obtained similarly from other graphs. The factors q_β or r_γ of the last two terms are contracted into the next vertex, and the process is iterated. But here we have again used momentum conservation, hence in a regularized graph we obtain an additional contribution with a factor

$$(p + q + r) \, \delta_\Lambda(p + q + r) = O(\Lambda^{-1}).$$

In order to see whether these Λ^{-1}-factors are sufficient to get rid of the violation terms for $\Lambda \to \infty$, we have to study the strength of the IR-divergences of our graphs. They are determined by the strength of the mass shell singularities in the cut variables of the p- and q-sectors. The sectors are ordinary Feynman graphs. To fix the ideas, we shall consider the q-sectors.

Let us first study the case of QED. We call a function $F(\tau)$ weakly singular at $\tau = 0$, if $\tau^\varepsilon F(\tau)$ is continuous at $\tau = 0$ for all $\varepsilon > 0$. The following facts about Feynman graphs in QED are well known (for a rough proof see the analogous considerations for YM theories given below).

1. A sector integral $S(k_1, \ldots, k_l; q_1, \ldots, q_m)$, amputated with respect to the k_i with the full propagators, has at the mass shell $k_i^2 = m_i^2$ at most weak singularities. Stronger singularities may occur in partial sums of photon momenta if they lie on the mass shell and are all parallel. But this does not lead to more than logarithmic divergences in the G-graph, because of the low dimensionality of the exceptional manifold.

2. If $\Sigma(k)$ is the electron propagator after mass renormalization then $(k^2 - m^2) \Sigma(k)$ has a weak singularity at $k^2 = m^2$.

3. Photon self-energy parts ($=$ vacuum polarization terms) $\Pi_{\mu\nu}(k)$ are of the form

$$\Pi_{\mu\nu}(k) = k^2 g_{\mu\nu} A(k^2) + k_\mu k_\nu B(k^2), \qquad (3.8)$$

with A and B continuous at $k^2 = 0$. This is seen by noticing that the lowest intermediate state contributing to the absorbtive part is the 3γ-state (momenta k_1, k_2, k_3), and that the corresponding phase space vanishes sufficiently strongly at $k_1^2 = k_2^2 = k_3^2 = (k_1 + \ldots)^2 = 0$ for overcoming any IR divergences of the rest of the graph. An analyticity argument then gives the desired result. An iterated chain of self-energy parts:

$$\Pi_{\mu\nu_1}(k) \frac{g^{\nu_1\mu_2}}{k^2 + i\varepsilon} \Pi_{\mu_2\nu_2} \frac{g^{\nu_2\mu_3}}{k^2 + i\varepsilon} \ldots \Pi_{\nu_r\nu} \qquad (3.9)$$

is then again of the form

$$g_{\mu\nu} k^2 A'(k^2) + k_\mu k_\nu B'(k^2),$$

with A′, B′, continuous at $k^2 = 0$. The dangerous term is obviously the B′ one, since it leads, if the propagator to the right of the last Π-factor is included, to a singularity $k_\mu k_\nu (k^2 + i\varepsilon)^{-1}$ of the amputated q-sector. If the adjacent cut-line propagator is split according to (3.4), we see that this dangerous term does not contribute to the physical part because of $G_{ij}k^j \equiv 0$. Nor does it contribute dangerously to the S-B and B-B parts, because the factor $k^\mu k_\mu = k^2$ cancels the singularity. But in the B-S part we must expect problems.

Λ-regularization removes the weak singularities of points 1 and 2. This is also true for graphs containing one of the unusual vertices introduced by WTI violation. If we multiply a regularized q-sector with the propagator $\delta_+(k)$ the IR-divergences coming from cut spinor lines are weak. ($F(\Lambda)$ is said to be weakly divergent if $\lim_{\Lambda \to \infty} \Lambda^{-\varepsilon} F(\Lambda) = 0$ for all $\varepsilon > 0$). In the KO algebra such divergences in the WTI violating terms are unimportant for $\Lambda \to \infty$ because of the Λ^{-1} factor from the violation vertex. Since the FP ghost fields are free, they do not cause any divergences.

The regularized form of the chain (3.9) is

$$\Pi_{\mu\nu_1}^\Lambda(k, k_1) \frac{g^{\nu_1\mu_2}}{k_1^2 + i\varepsilon} \Pi_{\mu_2\nu_2}^\Lambda(k_1, k_2) \ldots \Pi_{\nu_r\nu}^\Lambda(k_r, k_{r+1}). \qquad (3.10)$$

Since the Π^Λ are not covariant they are not of the simple form (3.8). However, if Π^Λ denotes the sum over all one-particle irreducible graphs of a given order, we find by applying the WTI:

$$\Pi_{\mu\nu}^\Lambda (k, l) = \delta^4 (k-l)(k^2 g_{\mu\nu} - k_\mu k_\nu) A(k^2) + O'(\Lambda^{-1}). \qquad (3.11)$$

$O'(\Lambda^n)$ denotes a term which is of order $O(\Lambda^{n-\varepsilon})$ for all $\varepsilon > 0$. Multiplying the chain (3.10), including an additional propagator at the right-hand end, with $\delta_+ (k)$, and possibly with a similar chain from the p-sector, we find a divergence of order Λ. As has been mentioned before, these divergences cancel in the sum over all G-graphs with the same skeleton.

It remains to be seen what effects the Λ-divergences of individual terms, in particular of the WTI violating terms, have on our KO application of the WTI. For QED the formal KO argument translates very simply into our language. We note first that the FP ghosts are free and that the condition $Q_B V = 0$ allows any number of $C's$ but no $\bar{C}'s$ in V. This means that no cut ghost lines are present in our graphs. As unphysical cut lines there remain the pairs B-S, S-B, B-B, each of whom contains at least one factor $B(k) = ik^\mu A_\mu (k)$. We can sum all the q-sectors (or p-sectors) of a given order with the same cut lines, without disturbing the KO algebra.

In such a sum the WTI expression for

$$(\Omega, T(B(k)\dots)\Omega) = ik^\mu (\Omega, T(A_\mu (k)\dots)\Omega)$$

contains only terms which can be discarded, either by the argument given for the j-sum in (3.3) (for external variables), or by the solution of the "1st problem" [for internal variables: see the remarks after (3.7)]. Hence the unphysical pairs do not contribute to (V, V).

To make this formal argument rigorous, we must look carefully at the S-B pairs, because we have seen that in this case the p-sector diverges if multiplied with the propagator $\delta_+ (k)$, so that our argument gives the undefined result $\infty \cdot 0$. Let us look at the regularized graphs, especially at the effects of chains (3.10) in $(\Omega, T(B(k)\dots)\Omega)$.

As contributions to $\Pi_{\mu\nu}^\Lambda (k, l)$ we have integrals of the form

$$J_{\mu\nu} = \int \prod_{i=1}^N du_i \, \delta_\Lambda (k-v_1) \, \delta_\Lambda (v_2 - l) \prod_{j=3}^M \delta_\Lambda (v_j) \, I_{\mu\nu} (u_1, \dots, u_N).$$

Here N is the number of internal lines, M is the number of vertices, v_j is the sum (with appropriate signs) of the momenta u_i impinging at the j-th vertex, and I is the product of the usual propagators and vertex factors. We introduce loop variables $l_1, \dots, l_L$, as we would in an unregularized graph, and use l_α and v_j as new integration variables. We obtain

$$J_{\mu\nu} = \int \prod dv_j \prod \delta_\Lambda (\,.\,.\,) \, \delta^4 (\Sigma v_j) \int \prod dl_\alpha I_{\mu\nu}' (v_j, l_\alpha).$$

The l-integral is UV-divergent. We renormalize it by the BPHZ procedure [12], subtracting the integrand at $v_1 = v_2 = 0$, but keeping the other v_i variable. After this the l-integral exists as a covariant function, $i.e.$ it is a sum of terms of the form $P_{\mu\nu}(v_j) I'(v_j)$, where $P_{\mu\nu}$ is a polynomial transforming as a tensor while I' is Lorentz invariant. We have either $P_{\mu\nu} = g_{\mu\nu}$, or $P_{\mu\nu} = v_{r,\mu} v_{s,\nu}$. In the first case we can draw $g_{\mu\nu}$ in front of the remaining v-integral and get a contribution to J of the form $g_{\mu\nu} J_1(k, l)$. In the second case, assume that $r > 2$. Because of the support of δ_Λ, the euclidean length of v_j is of order Λ^{-1}, so that we obtain a factor of order $O'(\Lambda^{-1})$ in the contribution to J. The same is true if $s > 2$. If $r > 2$ and $s > 2$ we get a contribution of order $O'(\Lambda^{-2})$. If $r = 1$ or 2, we find $v_r = k + O'(\Lambda^{-1})$. If $s = 1$ or 2 we have $v_s = l + O'(\Lambda^{-1})$. Collecting all contributions and using (3.11) and $k - l = O'(\Lambda^{-1})$, we find

$$\Pi^\Lambda_{\mu\nu}(k,\ l) = (k^2 g_{\mu\nu} - k_\mu k_\nu) A(k^2) \delta^4(k-l)$$
$$+ g_{\mu\nu} O'(\Lambda^{-1}) + k_\mu O'(\Lambda^{-1}) + l_\nu O'(\Lambda^{-1}) + O'(\Lambda^{-2}).$$

Let now $\Pi^\Lambda_{\mu\nu}(k, l)$ be the first factor in the chain (3.10). Contracting with k^μ annihilates the first term in the above expression. In the second term we get $k_\nu O'(\Lambda^{-1}) = l_\nu O'(\Lambda^{-1}) + O'(\Lambda^{-2})$. The l_ν-factor in here, and also in the fourth term of Π^Λ, is then contracted into the next Π^Λ of the chain, with similar results. If we are at the end of the chain, l_ν is contracted into the adjacent amputated many-line subgraph, which it turns into an expression of order $O'(\Lambda^{-1})$ because of the validity of the WTI at $\Lambda = \infty$. In the third term we get a factor k^2, which reduces the Λ-divergence by one order. Hence all the terms vanish for $\Lambda \to \infty$ as $O'(\Lambda^{-2})$, and this suffices to kill the $O(\Lambda)$ divergence possibly coming from the p-sector in a S-B pair.

As a result we find that the KO-argument works for QED. This is due to the following special features of QED.

1. The WTI have a very simple form, because of the absence of the cross products in (3.7) and because the ghost fields are free.

2. The IR singularities are mild. Serious problems are only created by the $k_\mu k_\nu (k^2)^{-2}$ term in the photon propagator, and these problems can be handled with the help of the WTI.

In order to see whether these considerations can be adapted to a pure YM theory, we need again information on the strengths of the mass shell singularities of Feynman graphs. We shall show that these singularities build up indefinitely in increasing orders of perturbation.

The gluon propagator in second order is of the same general form as in QED:

$$\frac{g_{\mu\nu}}{k^2} A(k^2) + \frac{k_\mu k_\nu}{(k^2)^2} B(k^2),$$

with A and B being weakly singular at the origin. The ghost propagator
in second order is $(k^2)^{-1} C(k^2)$ with C weakly singular at $k^2 = 0$.

Let J be an integral of the form

$$J = \int d^4 u\, F(u,\,k)(u^2 + i\varepsilon)^{-n}[(k-u)^2 + i\varepsilon]^{-m},$$

where F may be weakly singular at $u^2 = 0$ or $(k-u)^2 = 0$, but is otherwise
continuous. Then J is singular like $(k^2)^{-n-m+2}$ at $k^2 = 0$, up to weak
singularities [4].

Consider now a Feynman graph of higher order. Let k be an external
variable, whose line joins a 3-line vertex (the 4-line vertices also present
in YM theories can be treated analogously). Let u, $v = k - u$, be the other
momenta of that vertex. Assume that the u-line is a A-line carrying a self-
energy insertion of the second-order form just discussed. The k-dependence
of the graph is then given by

$$\int du\, F(\ldots,\,u,\,v)\left[\frac{g_{\rho\alpha}}{u^2}A(u^2) + \frac{u_\rho u_\alpha}{(u^2)^2}B(u^2)\right]\frac{1}{v^2}V(u,\,v),$$

where V is the vertex factor, and F summarizes the rest of the graph.
Assuming that F has only weak singularities at $u^2 = 0$ or $v^2 = 0$, we are in
the situation considered above. The A-term will lead to a k^{-2} singularity,
the B-term at first count to a $(k^2)^{-2}$ singularity. If the k- and v-lines are
ghost lines we find, however, that $u_\alpha V^\alpha$ contains the factor
$(u,\,k) = \frac{1}{2}(k^2 + u^2 - v^2)$, or $(u,\,v) = \frac{1}{2}(k^2 - u^2 - v^2)$. In both cases we get a
reduction of the singularity to k^{-2}. But if all three lines are A-lines such
a reduction does not occur: we remain with a $(k^2)^{-2}$ singularity, and this
singularity has no reason to be multiplied with a factor k_μ (if k is a A_μ-
momentum), at least if the graph in question belongs to a n-point function
with $n > 2$ [5]. Such a singularity emerges also if both the u- and v-
propagators contain B-type singularities. Similarly to the ghost case we
find that $u_\alpha v_\beta V^{\alpha\beta\gamma}$ contains a sum over squares u^2, v^2, k^2, which reduce
the expected $(k^2)^{-3}$ singularity by one order.

Next consider the same u-v-k-vertex, but assume now that the rest of
the graph contains second-order singularities in u^2 and/or v^2 of the kind
just deduced, not multiplied with factors u_α, v_β, which could be usefully

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[4] The qualifying statement "up to weak singularities" will henceforth be omitted if weak
singularities are as good as continuity.

[5] The numerator does, however, vanish sufficiently strongly at $k = 0$ to make its product
with $(k^2 + i\varepsilon)^{-2}$ well defined as a distribution. This remains true for the higher singularities
obtained by the following buildup.

contracted with the vertex factor V. Then no singularity reduction occurs, and we get $(k^2)^{-2}$ singularities for ghost variables, $(k^2)^{-3}$ singularities for A-variables. It is clear that by this mechanism higher and higher singularities build up in increasing order of perturbation.

These strong singularities of the unregularized sectors manifest themselves in correspondingly strong Λ-divergences of the regularized G-graphs. As mentioned before, we think that these divergences cancel in the sum over all G-graphs of a given order. But in applying the KO-algebra we find that the Λ^{-1} factor of the WTI violation vertex is almost never sufficient to overcome the divergence of any graph. And here there are no indications of a possible cancellation between graphs.

To see the problem more clearly, let us look at the sum over all those G-graphs of a given order, which are cut in only one line belong to a S-B or B-B pair. According to KO these graphs vanish formally by the following argument: The summed q-sector can be written as $(\Omega, B^{out} T(\Pi\Phi_{\alpha\beta})\Omega)$, with the $\Phi_{\alpha\beta}$ belonging to the state V. This expression can be written as $(\Omega, \{Q_B, \bar{C}^{out}\} T(\Pi \Phi_{\alpha\beta})\Omega) = (\Omega, \bar{C}^{out} Q_B T(\Pi\Phi_{\alpha\beta})\Omega) = 0$, because of $Q_B V = 0$. But this argument is wrong for the regularized graphs, due to the WTI violating terms. The violation graphs differ from ordinary graphs by the special form of the violation vertex, and by the fact that some vertices between this special one and the cut line, where application of the WTI started, are modified as explained in connection with the "2nd problem" stated earlier in this section. The cancellation of divergences that leads to the existence of the original expression for (V, V) occurs between graphs with the same skeleton, differing only by the position of the cut. But in the violation graphs the position of the cut is fixed: it must lie immediately to the left of the modified part. Hence the ordinary mechanism of cancellation cannot work. And another mechanism is not in sight.

It must be admitted that the non-existence of such an alternative mechanism has not yet been demonstrated, e. g. with the help of a concrete example. Such examples are difficult to come by because the divergences become virulent only in high orders of perturbation.

That serious difficulties exist in YM theories is also shown by the following consideration. If the KO conjecture is right, then (V, V) can be written as a sum over the G-graphs containing only cut lines belonging to physical pairs. *I. e.* no ghost lines are cut, and all cut A-lines carry G_{ij}-propagators. The graphs of that type are individually divergent. An analogous cancellation to that in the sum over the original G-graphs would have to involve graphs with the same skeleton. But a given line in a skeleton carries a $g_{\mu\nu}$-propagator if it occurs inside a sector, a G_{ij}-propagator if it is cut. Hence the divergences from the $k_i k_j$-contributions cannot be expected to cancel, and thus the very existence of the physical

expression for (V, V) becomes doubtful. An analogous problem does not exist in QED, because there the $k_i k_j$-terms do not actually contribute to the full expression, again because of the simple form of the WTI. This is shown like the vanishing of the B-terms in our discussion of QED. Note that this argument does not rely on a particular regularization scheme. Indeed, the problems discussed before should not be blamed on the use of an unsuitable regularization. Other schemes have been considered, with similar results.

Finally, let us remark that we cannot hope the bad mass shell singularities to disappear on summing over all orders or perturbation theory. This might be hoped for in analogy to the S-matrix in QED, which, after summing over the leading singularities in all orders, is zero rather than infinite. If in a YM model perturbation theory makes sense at all, *i. e.* if the perturbative expansions of the Green functions are asymptotic expansions of their exact values, then the exact Green function has mass shell singularities of arbitrarily high order ([6]). This is seen as follows. Let us concentrate on the variable $x = p_i^2$, and ignore all the other variables. The Green function τ is then written as $\tau = F(x, g)$. If its perturbative expansion is asymptotic in g, we find in order σ:

$$F(x, g) = F_{\sigma-1}(x, g) + g^\sigma \alpha(x)(x + i\varepsilon)^{-N} + g^{\sigma+1} R_\sigma(x, g), \quad (3.12)$$

where $F_{\sigma-1}$ collects the terms up to order $\sigma-1$, and α is at most weakly singular at $x = 0$. N is chosen such that $\alpha(0) \neq 0$. Assume that a singularity of order N occurs for the first time in order σ, so that $F_{\sigma-1}$ has only a $O'(x^{-N+1})$ singularity. Assume that F has at most a singularity $O'(x^{-N+1})$ (this is the possibility we want to disprove), and that F, F_σ, α, have no singularities outside $x = 0$ in some bounded neighbourhood I of $x = 0$. The latter is then also true for R_σ, and R_σ has at $x = 0$ a singularity of order $O'(x^{-N})$. Finally, assume that $R_\sigma(x, g)$ be continuous in g in the interval $G = [0, g_0]$ for some $g_0 > 0$. In view of the singularity at $x = 0$ this cannot mean pointwise continuity. Instead we demand that $|x|^{N+\varepsilon} R_\sigma(x, g)$ be jointly continuous in x and g in $I \times G$ for all $\varepsilon > 0$ ([7]). If such a condition is not satisfied, then perturbation theory hardly makes sense, since we could not control, even in principle, the error made in working with finite orders.

([6]) Non-perturbative discussions of the mass shell behaviour of the gauge field propagator in various gauges can be found in [13] and earlier works quoted there. Arguments are given that even the propagator has worse singularities than the canonical ones.

([7]) We could also start from the apparently weaker requirement that $\int dx\, R(x, g)\, \varphi(x)$ be continuous in G for all sufficiently regular test functions φ with support in I. A contradiction can then be derived by a similar but somewhat lengthier argument to the one given here.

The continuity assumption for the remainder $R_{\sigma-1}$ means that

$$g^{-\sigma}|x|^{N-1+\varepsilon}\{F(x, g)-F_{\sigma-1}(x, g)\}$$

is continuous in $I \times G$. Hence $(x+io)^{-1}\alpha(x)+g|x|^{N-1+\varepsilon}R(x, g)$ is continuous. But this is impossible, since the first term diverges for $x=g \to 0$, while the second term remains bounded in this limit. Hence the singularity of F at $x=0$ must be stronger than x^{-N+1}, and since N is arbitrarily large, the singularity of F must be arbitrarily strong, *i. e.* of infinite order.

4. THE PHYSICAL MEANING OF FIELD EQUATIONS

The classical equations of motion of a pure YM theory are

$$D_\mu F^{\mu\nu}=0, \tag{4.1}$$

where $F^{\mu\nu}$ is given by (1.3) and D_μ is the covariant derivative. In the quantized version the non-linear terms must be properly renormalized.

We can ask whether it is reasonable to expect validity of (4.1), if sandwiched between physical states:

$$(V_1, D_\mu F^{\mu\nu} V_2)=0, \tag{4.2}$$

for $V_i \in \mathscr{V}_{ph}$. This is true for the Maxwell equations in QED. The Maxwell equations, however, involve only observables: the field strengths $F^{\mu\nu}$ and the electric currents j^μ. But the fields entering the YM equation (4.1) are not gauge invariant and hence no observables. In this situation the field equation has apparently no direct (observable) physical meaning. This is fortunate if the KO characterization of $\mathscr{V}_{ph}$ is accepted, because (4.1) does not hold between physical states in the sense of KO: the field equation acquires a ghost term $ig(\partial^\nu \bar{c} \times c)$. But despite the non-observability of $F^{\mu\nu}$, the occurence of this ghost term is somewhat disturbing, since it destroys the gauge invariance of the dynamics even on $\mathscr{V}_{ph}$, so that we no longer know what gauge invariance of the theory means.

It is therefore legitimate to look for ways of ascribing a direct physical meaning to the field equation (4.1). This can be done as follows. In a complete field theory the observables are represented by hermitian, gauge invariant, operators which are functions of the basic fields. The field equations establish relations between the basic fields, and these relations induce relations between observables. It is reasonable to demand that these relations between observables are satisfied in nature, and to consider this as the observational content of the field equations. According to this argument we can, e. g., demand that the operator

$$\mathcal{O}^{(v)}(x)=\sum_a J_a^v(x) J_a^v(x), \qquad J_a^v=(D_\mu F^{\mu\nu})_a \tag{4.3}$$

(no summation over v) represents the trivial observable zero. This means that $(V_1, \mathcal{O}^{(v)} V_2) = 0$ for physical states $V_{1,2}$. Furthermore, the observable $\mathcal{O}^{(v)}$ must map $\mathscr{V}_{ph}$ into itself, so that also $(V_1, \mathcal{O}^{(v)}(x) \mathcal{O}^{(\mu)}(y) V_2) = 0$.

We will now show that this is not the case in perturbation theory, already in the lowest non-trivial order. We work again in the Feynman gauge, and we renormalize field products at a point by Wick ordering. This is sufficient for the low orders in which we work. We assume $(\Omega, A_\mu, \Omega) = 0$, which is true in perturbation theory.

In the Feynman gauge the fields A_μ satisfy the equations of motion

$$D^\mu F_{\mu\nu} + \partial_\nu \partial_\mu A^\mu + ig\, \partial_\nu \bar{c} \times c = 0, \tag{4.4}$$

hence

$$J_\nu(x) = -\partial_\nu \partial_\mu A^\mu - ig\, \partial_\nu \bar{c} \times c. \tag{4.5}$$

Consider the two-point function $W_{\alpha\beta}^{ab}(x, y) = (\Omega, J_\alpha^a(x) J_\beta^b(y)\Omega)$. It vanishes in orders g^0 and g^1, but not in order g^2, hence its exact value is $\neq 0$ if its perturbative expansion is asymptotic for $g \to 0$. In order to prove this non-vanishing we study the g^2-contribution to W in p-space. Using covariance, and omitting the δ^4-factor expressing momentum conservation, we find something of the form

$$\delta^{ab} w_{\alpha\beta}(p) = \delta^{ab} [p_\alpha p_\beta\, a(p^2) + g_{\alpha\beta}\, b(p^2)]\, \theta(p_0). \tag{4.6}$$

(p is the momentum conjugate to x.) The A^μ-term in (4.5) contributes only to the a-term in (4.6), because the index v occurs in a factor p_ν. Hence the b-term is wholly due to the ghost part of J. The ghost contribution to w is obtained from the g^0-term in

$$-(\Omega, (\partial_\alpha \bar{c}(x) \times c(x))(\partial_\beta \bar{c}(y) \times c(y))\, \Omega)$$

by Fourier transform. This term contains only the free ghost fields and can be easily calculated from the ghost propagators (2.2). The result is, omitting irrelevant numerical factors, and assuming $p_0 > 0$:

$$w_{\alpha\beta}^{gh}(p) = \int dk\, \delta_+(k)\, \delta_+(p-k)\, k_\alpha(p-k)_\beta.$$

For covariance reasons, this is of the form

$$w_{\alpha\beta}^{gh}(p) = g_{\alpha\beta}\, a'(p^2) + p_\alpha p_\beta\, b'(p^2).$$

Using $k^2 \delta_+(k) = (p-k)^2 \delta_+(p-k) = 0$ we find

$$g^{\alpha\beta} w_{\alpha\beta}^{gh} = p^2 a' + 4 b' = \frac{1}{2} p^2\, I,$$

$$p^\alpha p^\beta w_{\alpha\beta}^{gh} = p^2(p^2 a' + b') = \frac{1}{4}(p^2)^2\, I,$$

with $I(p^2) = \int dk\, \delta_+(k)\, \delta_+(p-k) > 0$ in $p^2 \geq 0$, $p^0 \geq 0$. From these equations

we obtain $b'(p^2) = \dfrac{1}{12} I(p^2) \neq 0$, hence $b(p^2) \neq 0$, which implies $W^{ab}_{\alpha\beta} \neq 0$ in

second order if $a = b$ and $\alpha = \beta$.

From this result we find in order g^4, using $W^{ab}_{\alpha\beta} = \delta^{ab}\, \bar{W}_{\alpha\beta}$:

$$(\Omega,\, \mathcal{O}^{(\nu)}(x)\, \mathcal{O}^{(\nu)}(y)\, \Omega) = \sum_{a,\,b} (\Omega,\, J^\nu_a(x)\, J^\nu_b(y)\, \Omega)^2$$

$$= \sum_a \delta^{aa} \{\bar{W}^{\nu\nu}(x,\, y)\}^2 \neq 0. \quad (4.7)$$

As a result we find that $\mathscr{V}_{\mathrm{ph}}$ cannot be defined by requiring the validity of relations between observables, which are induced by the field equations in their gauge invariant form (4.1). At least this is not possible if perturbation theory is correct as an asymptotic expansion, and if $\Omega \in \mathscr{V}_{\mathrm{ph}}$.   •

APPENDIX

Generalized Ostendorf rules

In this appendix we state a generalization of the graph rules derived by Ostendorf [11] for Wightman functions and partially time-ordered functions. For simplicity, we consider only the theory of a single, scalar, field $\Phi(x)$. The extension to more complicated cases is straightforward.

Let $X = \{x_{\alpha_1},\, \ldots,\, x_{\alpha_r}\}$ be a set of r 4-vectors. Let $T^\pm(X)$ denote the $\left\{\begin{array}{c} \text{time-ordered} \\ \text{anti-time-ordered} \end{array}\right\}$ product of the fields $\Phi(x_{\alpha_1}),\, \ldots,\, \Phi(x_{\alpha_r})$. Consider the vacuum expectation value

$$W(X_1 \mid \ldots \mid X_n) = (\Omega,\, T^{\sigma_1}(X_1) \ldots T^{\sigma_n}(X_n)\, \Omega), \quad (A.1)$$

where the sets X_i are mutually non-overlapping and $\sigma_i = \pm$. Then the perturbative expression of W in order g^N can be written as a sum over graphs defined as follows.

Draw first an ordinary Feynman graph of the theory in question, with Σr_i external and N internal points. This graph is called a "skeleton". It is then partitioned into non-overlapping subgraphs, called "sectors", such that the external points of a set X_i belong all to the same sector, but variables of different X_i to different sectors. In general there exist also sectors not containing external points. These sectors are called "internal". To each sector S we affix a number $s(S)$, such that the following rules

hold:

(i) The sector containing the points of X_i carries the number i.

(ii) For the internal sector S the number $s(S)$ is non-integer and lies between the maximal and the minimal number of its neighbours, *i. e.* of those sectors that are directly linked to S by a line of the graph. If $\sigma_i \neq \sigma_{i+1}$ there are no internal sectors with $i < s(S) < i+1$.

If two partitions differ only by the numbering of their sectors, not by their topology, then they are only considered different if for at least one pair of neighbouring sectors S, S', we have $s(S) > s(S')$ in one partition, $s(S) < s(S')$ in the other one.

The sectors are either T^+-sectors or T^--sectors. The external sector with number i is a T^{σ_i}-sector. The internal sector with $i < s(S) < i+1$, $\sigma_i = \sigma_{i+1}$, is a T^{σ_i}-sector.

With a partitioned graph we associate an integral as follows.

(*a*) Inside a T^+-sector the normal Feynman rules hold: Feynman propagators are associated to the lines, the vertex factors of the model in question to the vertices.

(*b*) Inside a T^--sector the complex-conjugate of the normal Feynman rules hold.

(*c*) A line leading from sector S' to sector $S'' \neq S'$ carries, in p-space, the propagator $(2\pi)^{-3} \theta(\pm p_0) \delta(p^2 - m^2)$. The upper sign applies if $s(S') < s(S'')$, the lower sign in the other case.

(*d*) Each internal sector contributes a factor (-1).

W_N is the sum over all different partioned graphs of order N, integrated over the internal variables.

The proof, which will not be given here, is obtained by a simple extension of Ostendorf's methods. Its crucial points are:

1. The splitting requirement

$$(\Omega, \ldots T^+(X, Y) \ldots \Omega) = (\Omega, \ldots T^+(X) T^+(Y) \ldots \Omega)$$

if $x_i^0 > y_j^0$ for all $x_i \in X$, $y_j \in Y$, is satisfied by our representation, as is its T^--analogue.

2. The Wightman function $(\Omega, \Phi(x_1) \ldots \Phi(x_n) \Omega)$ can be considered as a special case of our rules in many different ways, since each $\Phi(x_i)$ can be taken as a time-ordered or an anti-time-ordered product with a single factor. It can be shown that our representation does not depend on this choice of signs. More exactly, the sum over all graphs with the same skeleton is not dependent on the signs, while the individual partitioned graphs obviously are. If we consider all $\Phi's$ as T^+-products, we recover Ostendorf's rules.

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