ANNALES DE L'I. H. P., SECTION A

TAKASHI ICHINOSE

# Essential selfadjointness of the Weyl quantized relativistic hamiltonian

*Annales de l'I. H. P., section A*, tome 51, nº 3 (1989), p. 265-297
<http://www.numdam.org/item?id=AIHPA_1989__51_3_265_0>

*Ann. Inst. Henri Poincaré,*

Vol. 51, n° 3, 1989, p. 265-298. *Physique théorique*

# Essential selfadjointness of the Weyl quantized relativistic hamiltonian

by

**Takashi ICHINOSE (*)**

Department of Mathematics, Kanazawa University,
920 Kanazawa, Japan

*Dedicated to Professor Takeyuki Hida on the occasion of his sixtieth birthday*

ABSTRACT. — It is shown that the relativistic quantum Hamiltonian $H_A^m$ associated, via the Weyl correspondence, with the relativistic classical Hamiltonian $\sqrt{(p-A(x))^2+m^2}$ with a general vector potential $A(x)$, is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and bounded from below by $m$. The core of proof lies in establishing a distributional inequality for $H_A^m$, an analogue to Kato's inequality for the nonrelativistic quantum Hamiltonian.

RÉSUMÉ. — On démontre que l'hamiltonien relativiste quantique $H_A^m$ associé, via la correspondance de Weyl, à l'hamiltonien relativiste classique $\sqrt{(p-A(x))^2+m^2}$ avec un potentiel vectoriel général $A(x)$, est essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^d)$ et semi-borné inférieurement par $m$. Le nœud de la preuve consiste à établir une inégalité pour $H_A^m$ au sens de

(*) Research supported in part by Grant-in-Aid for Scientific Research No. 62540096 and No. 63540106, Ministry of Education.

distributions, un analogue à l'inégalité de Kato pour l'hamiltonien non relativiste quantique.

---

## 1. INTRODUCTION

In the present paper we study the problem of essential selfadjointness of the Weyl quantized relativistic Hamiltonian

$$\mathrm{H}^m = \mathrm{H}_{\mathrm{A}}^m + \Phi \tag{1.1}$$

corresponding to the classical relativistic Hamiltonian

$$h^m(p, x) = h_{\mathrm{A}}^m(p, x) + \Phi(x) \equiv \sqrt{(p - \mathrm{A}(x))^2 + m^2} + \Phi(x), \qquad p \in \mathbb{R}^d, \quad x \in \mathbb{R}^d, \tag{1.2}$$

of a spinless particle of mass $m$ interacting with vector and scalar potentials $\mathrm{A}(x)$ and $\Phi(x)$. $\mathrm{A}(x)$ and $\Phi(x)$ are respectively $\mathbb{R}^d$-valued and $\mathbb{R}$-valued measurable functions defined in $d$-dimensional space $\mathbb{R}^d$.

In the previous paper [12], $\mathrm{H}_{\mathrm{A}}^m$ was defined as a Weyl pseudo-differential operator (see Berezin-Subin [1], Hörmander [8])

$$(\mathrm{H}_{\mathrm{A}}^m u)(x) = (2\pi)^{-d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)p} h_{\mathrm{A}}^m\left(p, \frac{x+y}{2}\right) u(y)\, dy dp, \qquad u \in \mathscr{S}(\mathbb{R}^d), \tag{1.3}$$

the integral on the right being an oscillatory integral. There $\mathrm{A}(x)$ was assumed to be bounded and continuous together with its derivatives up to sufficiently higher order, since the usual theory of pseudo-differential operators needs assumption of sufficient regularity of the symbol $h_{\mathrm{A}}^m(p, x)$. Then $\mathrm{H}_{\mathrm{A}}^m$ defines a linear operator in $\mathrm{L}^2(\mathbb{R}^d)$ with domain $\mathscr{S}(\mathbb{R}^d)$. It was shown ([12], cf. Shubin [22]) that $\mathrm{H}_{\mathrm{A}}^m$ is essentially selfadjoint on $\mathrm{C}_0^\infty(\mathbb{R}^d)$ and [12] that its unique selfadjoint extension is bounded from below by $m: \mathrm{H}_{\mathrm{A}}^m \geqq m$. Here the proof of the latter result is based on a path integral representation established there for the semigroup $\exp[-t(\mathrm{H}^m - m)]$, while in [9], [10] this path integral formula has been further used to discuss the nonrelativistic limit problem. Recently, Nagase-Umeda [20] have proved the essential selfadjointness of $\mathrm{H}_{\mathrm{A}}^m$, assuming $\mathrm{A}(x)$ to be continuous with bounded derivatives of the higher order than the first, so as to include the case of constant magnetic fields.

The aim of the present paper is to extend these results to the case of a less regular and unbounded vector potential $\mathrm{A}(x)$. Namely we only assume that

$$\mathrm{A}(x) \text{ and } \int_{0<|y|<1} |\mathrm{A}(x-y/2)-\mathrm{A}(x)|\,|y|^{-d}\,dy \quad \textit{are locally bounded.} \tag{1.4}$$

In particular, a *locally Hölder-continuous* function $\mathrm{A}(x)$ satisfies (1.4). Then redefining $\mathrm{H}_{\mathrm{A}}^m$ with an integral operator which is equivalent to the pseudo-differential operator (1.3) if the latter makes sense, we show that $\mathrm{H}_{\mathrm{A}}^m$ is essentially selfadjoint on $\mathrm{C}_0^\infty(\mathbb{R}^d)$. Here the mass $m$ is nonnegative; it may be zero. The assumption (1.4) is suggested by the path integral formula for $\exp[-t(\mathrm{H}^m-m)]$ obtained in the previous work [12], which is still valid in this case (see the discussion in [12], § 5). The problem of essential selfadjointness for (1.1), $\mathrm{H}^m$ with both vector and scalar potentials $\mathrm{A}(x)$ and $\Phi(x)$, is also discussed. Further it is shown to remain still valid that $\mathrm{H}_{\mathrm{A}}^m$ is bounded from below by $m$.

As the definition of $\mathrm{H}_{\mathrm{A}}^m$ with a vector potential $\mathrm{A}(x)$ satisfying (1.4), we propose the following:

$$\begin{aligned}(\mathrm{H}_{\mathrm{A}}^m u)(x) = m u(x) - \int_{|y|>0} [&e^{-iy\mathrm{A}(x+y/2)}u(x+y)-u(x) \\ &-\mathrm{I}_{\{|y|<1\}}\,y(\partial_x - i\mathrm{A}(x))u(x)]\,n^m(dy), \quad u\in\mathscr{S}(\mathbb{R}^d).\end{aligned} \tag{1.5}$$

Here $\mathrm{I}_{\{|y|<1\}}$ is the indicator function of the set $\{|y|<1\}$, and $n^m(dy)$ is a $\sigma$-finite measure on $\mathbb{R}^d\setminus\{0\}$ satisfying $\int_{|y|>0} [y^2/(1+y^2)]\,n^m(dy)<\infty$, called the Lévy measure. Note that when $\mathrm{A}(x)$ identically vanishes, then (1.5) is, via the Fourier transform, equivalent to the Lévy-Khinchin formula for $\sqrt{p^2+m^2}$ (e. g. [14], [21]). The right-hand side of (1.5) can be shown to coincide with that of (1.3) if $\mathrm{A}(x)$ is sufficiently smooth and has bounded derivatives $\partial^\alpha \mathrm{A}(x)$. The core of proof of the essential selfadjointness of $\mathrm{H}_{\mathrm{A}}^m$ and $\mathrm{H}^m$ consists in establishing a distributional inequality for $\mathrm{H}_{\mathrm{A}}^m$: If $v\in \mathrm{L}^2(\mathbb{R}^d)$ with $\mathrm{H}_{\mathrm{A}}^m v\in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R}^d)$, then

$$\mathrm{Re}\,[(\operatorname{sgn} v)\,\mathrm{H}_{\mathrm{A}}^m v] \geqq \sqrt{-\Delta+m^2}\,|v|, \tag{1.6}$$

in the sense of distributions, where $(\operatorname{sgn} v)(x)=\overline{v(x)}/|v(x)|$, for $v(x)\neq 0$, and $=0$, for $v(x)=0$. This may be regarded as an analogue to Kato's inequality for the nonrelativistic Schrödinger operator [16].

It should be noted that our $\mathrm{H}_{\mathrm{A}}^m$ differs from the square root

$$\sqrt{(-i\partial-\mathrm{A}(x))^2+m^2} \tag{1.7}$$

of the nonnegative selfadjoint operator $(-i\partial-\mathrm{A}(x))^2+m^2$. We are more interested in $\mathrm{H}_{\mathrm{A}}^m$ from the path integral point of view, because $\mathrm{H}_{\mathrm{A}}^m$ is suited to path integral but not (1.7) (see [12], cf. [18], IV A $b$). However, we shall not discuss which is physically more appropriate for a relativistic quantum Hamiltonian of a spinless particle in an magnetic field. We also mention that such a Hamiltonian was also treated, though with no vector potential, by several authors, Weder [25], Herbst [6], Daubechies-Lieb [3], Daubechies [2], Fefferman-de-la Llave [5].

In Section 2 the way of defining $H_A^m$ with $A(x)$ which satisfies (1.4) is presented. Section 3 is concerned with the regularity of solutions $u$ of $H_A^m u=f$, which is needed in Section 4 to prove an analogue to Kato's inequality in the distribution sense. Section 5 is devoted to the main result on the essential selfadjointness and semiboundedness of $H^m$ as well as $H_A^m$, to the effect that $H_A^m$ and $H^m$ are essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and their unique selfadjoint extensions are bounded from below by $m$, if $A(x)$ satisfies (1.4) and $\Phi(x)$ is locally square-integrable with $\Phi(x)\geqq 0$ a. e.

The result of the present paper has been announced in [11].

## 2. DEFINITION OF THE WEYL QUANTIZED RELATIVISTIC HAMILTONIAN

We are going to define the Weyl quantized Hamiltonian $H_A^m$ corresponding to the classical relativistic Hamiltonian $h_A^m(p,x)$ in (1.2). $m$ is a nonnegative constant.

Our starting point is the Lévy-Khinchin formula for the conditionally negative definite function $\sqrt{p^2+m^2}-m$ (e. g. [14], p. 65, or [21] Appendix 2 to XIII.12, p. 212-222):

$$\sqrt{p^2+m^2}=m-\int_{|y|>0}[e^{ipy}-1-ipy\,I_{\{|y|<1\}}]\,n^m(dy). \tag{2.1}$$

Here $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y|<1\}$, *i.e.* $I_{\{|y|<1\}}(z)=1$, if $|z|<1$, and $=0$, if $|z|\geqq 1$. $n^m(dy)$ is the *Lévy measure* which is a $\sigma$-finite measure on $\mathbb{R}^d\setminus\{0\}$ such that

$$\int_{|y|>0} y^2/(1+y^2)\,n^m(dy)<\infty.$$

It is given by

$$n^m(dy)=2(2\pi)^{-(d+1)/2}m^{(d+1)/2}\times|y|^{-(d+1)/2}K_{(d+1)/2}(m|y|)\,dy,\qquad m>0. \tag{2.2a}$$

$$n^0(dy)=\pi^{-(d+1)}\Gamma\left(\frac{d+1}{2}\right)|y|^{-(d+1)}dy,\qquad m=0, \tag{2.2b}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind of order $\nu$ and $\Gamma(z)$ the gamma function. $K_\nu(z)$ satisfies, for $\nu>0$,

$$0<K_\nu(z)\leqq C[z^{-\nu}\vee z^{-1/2}]e^{-z}, \qquad z>0. \tag{2.3}$$

with a constant $C>0$ (see [4], chap. VII, 7.2.5, (37), 7.2.6, (41), p. 9-10, and 7.4.1, (1), (4), p. 23-24]).

To see (2.2) recall (see [12], Eq. (4.2), p. 244) that the operator $\exp[-t(\sqrt{-\Delta+m^2}-m)]$ has the kernel $k_0^m(t, x-y)$, where

$$\begin{aligned} k_0^m(t, z) &= 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} e^{mt} t \\ &\quad \times (z^2+t^2)^{-(d+1)/4} K_{(d+1)/2}(m(z^2+t^2)^{1/2}), \qquad m>0. \quad (2.4a) \\ k_0^0(t, z) &= \pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right) t(z^2+t^2)^{-(d+1)/2}, \qquad m=0, \quad (2.4b) \end{aligned}$$

and use the fact (e. g. [14], Example 1) that

$$t^{-1} k_0^m(t, y)\,dy \to n^m(dy), \quad \text{as } t\downarrow 0.$$

*In this section, unless otherwise specified,* $A(x)$ *is an* $\mathbb{R}^d$*-valued measurable function in* $\mathbb{R}^d$ *satisfying* (1.4) *or equivalently*

$$a(K) \equiv \sup_{x\in K} |A(x)| < \infty, \tag{2.5a}$$

$$b(K) \equiv \sup_{x\in K} \int_{0<|y|<1} |A(x-y/2)-A(x)|\,|y|\, n^m(dy) < \infty, \tag{2.5b}$$

*for every compact subset* K *of* $\mathbb{R}^d$. It is seen, using the asymptotic behavior (2.3) of $K_\nu(z)$ as $z\to 0$, that (2.5) is satisfied if $A(x)$ is *locally Hölder-continuous, i. e.* $|A(x-y/2)-A(x)|\leqq b'(K)|y/2|^\alpha$, $x\in K$, $0<|y|<1$, with constants $0<\alpha<1$ and $b'(K)$ depending on K; in fact, $b(K)\leqq 2^{-\alpha} b'(K)\, n_\alpha^m$. Here and throughout we set

$$n_\infty^m = n^m(|y|\geqq 1) \equiv \int_{|y|\geqq 1} n^m(dy), \tag{2.6a}$$

$$n_\alpha^m = \int_{0<|y|<1} |y|^{1+\alpha} n^m(dy), \qquad 0<\alpha\leqq 1. \tag{2.6b}$$

The explicit expression (2.2) of $n^m(dy)$ with (2.3) shows that $n_\infty^m$ and $n_\alpha^m$ are uniformly bounded for $m\geqq 0$.

We shall denote by $H_0^m \equiv \sqrt{-\Delta+m^2}$ not only the linear map of the Sobolev space $H^s(\mathbb{R}^d)$ into $H^{s+1}(\mathbb{R}^d)$ but also the linear map $\mathscr{F}^{-1}\sqrt{p^2+m^2}\,\mathscr{F}$ of $\mathscr{S}'(\mathbb{R}^d)$ into $\mathscr{S}'(\mathbb{R}^d)$, where $\mathscr{F}$ and $\mathscr{F}^{-1}$ stand for the Fourier and inverse Fourier transforms.

Now let $u \in \mathcal{S}(\mathbb{R}^d)$ and put $\hat{u} = \mathcal{F} u$. Then the inverse Fourier transform of $\hat{u}(p)$ multiplied by (2.1) is

$$
\begin{aligned}
(\mathrm{H}_0^m u)(x) &\equiv (\sqrt{-\Delta + m^2}\, u)(x) \\
&= mu(x) - \int_{|y|>0} [u(x+y) - u(x) - \mathrm{I}_{\{|y|<1\}} y\, \partial_x u(x)]\, n^m(dy). \quad (2.7)
\end{aligned}
$$

Definition 2.1. — The *Weyl quantized relativistic Hamiltonian* $\mathrm{H}_\mathrm{A}^m$ *corresponding to the symbol* $h_\mathrm{A}^m(p, x)$ in (1.2) is defined to be the integral operator:

$$
\begin{aligned}
(\mathrm{H}_\mathrm{A}^m u)(x) = mu(x) - \int_{|y|>0} & [e^{-iy\mathrm{A}(x+y/2)} u(x+y) - u(x) \\
& - \mathrm{I}_{\{|y|<1\}} y(\partial_x - i\mathrm{A}(x))u(x)]\, n^m(dy), \qquad u \in \mathcal{S}(\mathbb{R}^d). \quad (2.8)
\end{aligned}
$$

Obviously, if $\mathrm{A}(x) \equiv 0$, (2.8) reduces itself to (2.7). On the other hand, $\mathrm{H}_\mathrm{A}^m$ may be defined, as in [12], to be the pseudo-differential operator (1.3), *i. e.*

$$
\begin{aligned}
(\mathrm{H}_\mathrm{A}^m u)(x) \\
= (2\pi)^{-d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)p}\, h_\mathrm{A}^m\left(p, \frac{x+y}{2}\right) u(y)\, dy\, dp, \\
u \in \mathcal{S}(\mathbb{R}^d), \quad (2.9)
\end{aligned}
$$

if the right-hand side exists as an oscillatory integral (e. g. [8]).

Both the definitions yield the same $\mathrm{H}_\mathrm{A}^m$, which in the following lemma we content ourselves to check in a case including that of constant magnetic fields.

Lemma 2.2. — *Let $m$ be nonnegative. Assume* $\mathrm{A}(x)$ *is a* $\mathrm{C}^\infty$ *function satisfying*

$$
\left|
\begin{aligned}
&|\mathrm{A}(x)| \leqq \mathrm{C}_0(1+|x|^2)^{1/2}, \\
&|\partial^\alpha \mathrm{A}(x)| \leqq \mathrm{C}_\alpha, \qquad |\alpha| \geqq 1, \qquad x \in \mathbb{R}^d,
\end{aligned}
\right.
$$

*with constants* $\mathrm{C}_0$ *and* $\mathrm{C}_\alpha$. *Then:* (i) *The right-hand side of* (2.9) *exists as an oscillatory integral.* (ii) *The pseudo-differential operator* $\mathrm{H}_\mathrm{A}^m$ *defined through* (2.9) *coincides on* $\mathcal{S}(\mathbb{R}^d)$ *with the integral operator* $\mathrm{H}_\mathrm{A}^m$ *defined through* (2.8).

*Remark.* — It is for simplicity that in Lemma 2.2 we have assumed $\mathrm{A}(x)$ is $\mathrm{C}^\infty$. It can be seen that both the statements (i) and (ii) are valid for sufficiently smooth $\mathrm{A}(x)$ satisfying $|\partial^\alpha \mathrm{A}(x)| \leqq \mathrm{C}_\alpha$, $1 \leqq |\alpha| \leqq \mathrm{N}$, $x \in \mathbb{R}^d$, with N sufficiently large.

*Proof of Lemma* 2.2. — (i) The assertion is obvious if $m > 0$. We give here a proof which is valid for $m \geqq 0$. Let $\chi$ be a rotation-invariant $\mathrm{C}_0^\infty$ function with $0 \leqq \chi(p) \leqq 1$ in $\mathbb{R}^d$, $\chi(p) = 1$ on $\{|p| \leqq 1/2\}$ and $\chi(p) = 0$ on $\{|p| \geqq 1\}$. Put $\chi_\mathrm{R}(p) = \chi(p/\mathrm{R})$ for $\mathrm{R} > 0$ and $\psi_\mathrm{R}(p) = 1 - \chi_\mathrm{R}(p)$. Write (2.9) as a sum of two terms:

$$H_A^m u = H_1 u + H_2 u, \tag{2.10a}$$

where

$$(H_1 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} h_1(x, p, y)\, u(y)\, dy dp, \tag{2.10b}$$

$$(H_2 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} h_2(x, p, y)\, u(y)\, dy dp, \tag{2.10c}$$

with

$$h_1(x, p, y) = \chi_R\left(p - A\left(\frac{x+y}{2}\right)\right) h_A^m\left(p, \frac{x+y}{2}\right),$$

$$h_2(x, p, y) = \psi_R\left(p - A\left(\frac{x+y}{2}\right)\right) h_A^m\left(p, \frac{x+y}{2}\right).$$

It is easy to see by change of variables that the integral $H_1 u$ is absolutely convergent. $H_2 u$ is an oscillatory integral, whose existence follows from the basic theory of oscillatory integrals (e. g. [17], Chap. 1), because by assumption on $A(x)$, $h_2(x, p, y)$ is a $C^\infty$ function satisfying: for all multi-indices $\alpha$, $\beta$ and $\beta'$ there exists a constant $C_{\alpha\beta\beta'}$ independent of $m \geqq 0$ such that

$$|\partial_p^\alpha \partial_x^\beta \partial_y^{\beta'} h_2(x, p, y)| \leqq C_{\alpha\beta\beta'}(1+p^2)^{1/2}(1+x^2+y^2)^{1/2},$$
$$p \in \mathbb{R}^d, \qquad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

(ii) In view of (2.7) we can see, for $H_A^m u$ in (2.8), that

$$(H_A^m u)(x) = \left(H_0^m\left(\exp\left[i(x-\cdot)A\left(\frac{x+\cdot}{2}\right)\right] u(\cdot)\right)\right)(x), \; u \in \mathscr{S}(\mathbb{R}^d), \tag{2.11}$$

which may be said that apply $H_A^m$ to $u$ is nothing but apply $H_0^m$ to the appropriately "gauge transformed" $u$. Here note by assumption on $A(x)$ that $u \in \mathscr{S}(\mathbb{R}^d)$ implies that, for $x$ fixed, the function

$$y \to \exp\left[i(x-y)A\left(\frac{x+y}{2}\right)\right] u(y)$$

belongs to $\mathscr{S}(\mathbb{R}^d)$. Now, to show the statement (ii) of the lemma, we have only to show the right-hand side of (2.9) is equal, as oscillatory integrals, to the right-hand side of (2.11), that is,

$$(2\pi)^{-d}\iint e^{i(x-y)p}\sqrt{p^2+m^2}\exp\left[i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right]u(y)\,dydp.$$

Changing in (2.10 *b*, *c*) the integration variables $p'=p-\mathrm{A}\left(\frac{x+y}{2}\right)$ (writing $p$ again instead of $p'$), we get

$$(\mathrm{H}_1 u)(x)=(2\pi)^{-d}\iint e^{i(x-y)p}\chi_{\mathrm{R}}(p)\sqrt{p^2+m^2}\times\exp\left[i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right]u(y)\,dydp. \quad (2.12\,a)$$

and

$$\begin{aligned}(2\pi)^d(\mathrm{H}_2 u)(x) &\equiv \lim_{\varepsilon\to 0}\iint e^{i(x-y)p}\chi(\varepsilon p)\,h_2(x,p,y)\,u(y)\,dydp,\\ &= \lim_{\varepsilon\to 0}\iint \exp\left[i(x-y)\left(p+\mathrm{A}\left(\frac{x+y}{2}\right)\right)\right]\\ &\quad\times\chi\left(\varepsilon\left(p+\mathrm{A}\left(\frac{x+y}{2}\right)\right)\right)\psi_{\mathrm{R}}(p)\sqrt{p^2+m^2}\,u(y)\,dydp,\\ &= \lim_{\varepsilon\to 0}\iint e^{i(x-y)p}(1+p^2)^{-l}(1-\Delta_y)^l\left\{\chi\left(\varepsilon\left(p+\mathrm{A}\left(\frac{x+y}{2}\right)\right)\right)\right.\\ &\quad\left.\times\exp\left[i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right]\psi_{\mathrm{R}}(p)\sqrt{p^2+m^2}\,u(y)\right\}dydp.\end{aligned}$$

Here $l$ is an integer $>(d+1)/2$, and the last equality is due to integration by parts based on

$$e^{-iyp}=(1+p^2)^{-l}(1-\Delta_y)^l e^{-iyp}.$$

Since $\mathrm{A}(x)$ is continuous and the derivatives of $\mathrm{A}(x)$ are all bounded, it is seen for $x$ fixed that as $\varepsilon\to 0$, $\chi\left(\varepsilon\left(p+\mathrm{A}\left(\frac{x+y}{2}\right)\right)\right)$ converges to 1 uniformly on compact sets of both $p$ and $y$, and $\partial_y^\alpha\chi\left(\varepsilon\left(p+\mathrm{A}\left(\frac{x+y}{2}\right)\right)\right)$ with $|\alpha|\geqq 1$ converges to zero uniformly in both $p$ and $y$. Then we have by the Lebesgue dominated convergence theorem

$$(2\pi)^d (\mathrm{H}_2 u, (x) = \iint e^{i(x-y)p}(1+p^2)^{-l}(1-\Delta_y)^l \times \left\{ \exp\left[ i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right] \psi_{\mathrm{R}}(p)\sqrt{p^2+m^2}\, u(y) \right\} dy dp$$

$$= \lim_{\varepsilon \to 0} \iint e^{i(x-y)p}\chi(\varepsilon p)(1+p^2)^{-l}(1-\Delta_y)^l \{ \dots \} dy dp$$

$$= \lim_{\varepsilon \to 0} \iint e^{i(x-y)p}\chi(\varepsilon p)\psi_{\mathrm{R}}(p)\sqrt{p^2+m^2} \times \exp\left[ i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right] u(y)\, dy dp,$$

where the last equality is due to integration by parts, so that, as an oscillatory integral,

$$(\mathrm{H}_2 u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p}\psi_{\mathrm{R}}(p)\sqrt{p^2+m^2} \times \exp\left[ i(x-y)\mathrm{A}\left(\frac{x+y}{2}\right)\right] u(y)\, dy dp. \quad (2.12\,b)$$

Thus with (2.12 *a*, *b*) we have shown the assertion (ii), completing the proof of Lemma 2.2.

Next we shall see that $\mathrm{H}_{\mathrm{A}}^m$ defined by (2.8) maps $\mathrm{C}_0^\infty(\mathbb{R}^d)$ into $\mathrm{L}^2(\mathbb{R}^d)$, so that it can define a linear operator in $\mathrm{L}^2(\mathbb{R}^d)$ with domain $\mathrm{C}_0^\infty(\mathbb{R}^d)$. For $u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathbb{R}^d)$, write (2.8) as

$$\mathrm{H}_{\mathrm{A}}^m u = mu + \mathrm{I}_1 u + \mathrm{I}_2 u, \quad (2.13\,a)$$

where

$$(\mathrm{I}_1 u)(x) = -\int_{|y| \geq 1} [e^{-iy\mathrm{A}(x+y/2)} u(x+y) - u(x)]\, n^m(dy), \quad (2.13\,b)$$

$$(\mathrm{I}_2 u)(x) = -\int_{0<|y|<1} [e^{-iy\mathrm{A}(x+y/2)} u(x+y) - u(x) - y(\partial_x - i\mathrm{A}(x))u(x)]\, n^m(dy). \quad (2.13\,c)$$

LEMMA 2.3. – *Assume* $\mathrm{A}(x)$ *satisfies* (2.5) *or* (1.4). (i) *If* $u$ *is in* $\mathrm{C}^\infty \cap \mathrm{L}^2(\mathbb{R}^d)$, *then* $\mathrm{H}_{\mathrm{A}}^m u$ *is in* $\mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d)$. *More Precisely,*

$$\|\mathrm{I}_1 u\|_p \leq 2 n_\infty^m \|u\|_p, \qquad u \in \mathrm{L}^p(\mathbb{R}^d), \quad (2.14\,a)$$

*for each* $1 \leq p \leq \infty$, *where* $\|.\|_p$ *stands for the* $\mathrm{L}^p$ *norm, and for every compact subset* K *of* $\mathbb{R}^d$ *there exists a constant* $\mathrm{C}_{\mathrm{K}}$ *such that*

$$\|\mathrm{I}_2 u\|_{2,\mathrm{K}} \leq \mathrm{C}_{\mathrm{K}}[\|u\|_{2,\mathrm{K}_2} + \|\partial u\|_{2,\mathrm{K}_2}], \qquad u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathbb{R}^d). (2.14\,b)$$

(ii) *Let* $1 \leqq p \leqq \infty$. *If* $u$ *is in* $C_0^\infty(\mathbb{R}^d)$, *then* $H_A^m u$ *is in* $L^p(\mathbb{R}^d)$. *For every compact subset* K *of* $\mathbb{R}^d$ *there exists a constant* $C_K$ *such that*

$$\| H_A^m u \|_p \leqq C_K \left[ \| u \|_\infty + \sum_{j=1}^{d} \| \partial_j u \|_\infty + \sum_{j,k=1}^{d} \| \partial_j \partial_k u \|_\infty \right], \qquad (2.15)$$

*for all* $u \in C_0^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp} u \subseteq K$. *Here* $n_\infty^m$ *in* (2.14 *a*) *is the constant* (2.6 *a*), *the* $C_K$ *in* (2.14 *b*) *and* (2.15) *are constants depending on* K *and the behavior of* A(*x*) *in a neighbourhood of* K, *and for* $r > 0$,

$$K_r = \{ x \in \mathbb{R}^d ; \operatorname{dist}(x, K) \leqq r \}, \qquad (2.16)$$

$$\| f \|_{p, K} = \begin{cases} \left( \int_K | f(x) |^p \, dx \right)^{1/p}, & 1 \leqq p < \infty, \\ \operatorname{ess\,sup}_{p \in K} | f(x) |, & p = \infty, \end{cases} \qquad (2.17)$$

*with K a compact set in* $\mathbb{R}^d$.

*Proof.* — (i) It is easy to verify (2.14 *a*) by use of the Hölder inequality. To show (2.14 *b*) let K be a compact set in $\mathbb{R}^d$. Let $\varphi(x)$ be a $C_0^\infty$ function with $0 \leqq \varphi(x) \leqq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on $K_1$ and $\operatorname{supp} \varphi \subseteq K_2$. Then (2.13 *c*) is rewritten for $x \in K$ as

$$\begin{aligned}
(I_2 u)(x) &= (I_2 (\varphi u))(x) \\
&= \lim_{\varepsilon \downarrow 0} \left\{ - \int_{\varepsilon \leqq |y| < 1} [e^{-iy A(x+y/2)} - 1 + iy A(x+y)] \varphi(x+y) u(x+y) n^m(dy) \right. \\
&\quad \left. + \int_{\varepsilon \leqq |y| < 1} i [y A(x+y) \varphi(x+y) u(x+y) - y A(x) \varphi(x) u(x)] n^m(dy) \right\} \\
&\quad - \int_{0 < |y| < 1} [\varphi(x+y) u(x+y) - \varphi(x) u(x) - y \partial_x (\varphi(x) u(x))] n^m(dy) \\
&\equiv \lim_{\varepsilon \downarrow 0} \{ i_1(\varepsilon)(x) + i_2(\varepsilon)(x) \} + i_3(x). \qquad (2.18)
\end{aligned}$$

Then by Fatou's lemma

$$\| I_2 u \|_{2, K} \leqq \sum_{j=1}^{2} \liminf_{\varepsilon \downarrow 0} \| i_j(\varepsilon) \|_{2, K} + \| i_3 \|_{2, K}. \qquad (2.19)$$

Now we estimate each term on the right of (2.19). First, for $i_1(\varepsilon)$ we have with $x \in K$,

$$| i_1(\varepsilon)(x) | \leqq \int_{\varepsilon \leqq |y| < 1} [2^{-1} |y|^2 | A(x+y/2) |^2 + |y| | A(x+y/2) - A(x+y) | ] | \varphi(x+y) u(x+y) | n^m(dy),$$

using

$$| e^{-is} - 1 + it | \leqq 2^{-1} s^2 + | t - s |, \qquad s, t \in \mathbb{R}.$$

We use the Schwarz inequality and then change the integration variables $x+y=x'$ (writing $x$ again instead of $x'$) to get

$$\|i_1(\varepsilon)\|_{2,\mathrm{K}} \leqq \left\{ \int_{\mathrm{K}_1} dx \left( \int_{\varepsilon \leqq |y| < 1} [2^{-1}|y|^2 |\mathrm{A}(x-y/2)|^2 + |y| |\mathrm{A}(x-y/2)-\mathrm{A}(x)|] n^m(dy) \right) \right.$$
$$\times \int_{\varepsilon \leqq |y| < 1} [2^{-1}|y|^2 |\mathrm{A}(x-y/2)|^2 + |y| |\mathrm{A}(x-y/2)-\mathrm{A}(x)|]$$
$$\left. \times |\varphi(x) u(x)|^2 n^m(dy) \right\}^{1/2}$$
$$\leqq (2^{-1} a(\mathrm{K}_2) n_1^m + b(\mathrm{K}_1)) \|\varphi u\|_2 \leqq \mathrm{C}(\mathrm{K}_1) \|u\|_{2,\mathrm{K}_2}. \quad (2.20)$$

with a constant $\mathrm{C}(\mathrm{K}_1)$ independent of $\varepsilon>0$, where $a(\mathrm{K}_2)$ and $b(\mathrm{K}_1)$ are the constants in (2.5) with $\mathrm{K}_2$ and $\mathrm{K}_1$ in place of K, respectively, and $n_1^m$ the constant (2.6 $b$) with $\alpha=1$.

Next, for $i_2(\varepsilon)$, we show

$$\|i_2(\varepsilon)\|_{2,\mathrm{K}} \leqq \|i_2(\varepsilon)\|_2 \leqq \mathrm{C}_0 \sum_{k=1}^{d} \|\varphi \mathrm{A}_k u\| \leqq \mathrm{C}_0 a(\mathrm{K}_2) \|u\|_{2,\mathrm{K}_2}, \quad (2.21)$$

with a constant $\mathrm{C}_0$ independent of $\varepsilon>0$. To do so, first note that the Lévy measure has a rotation-invariant density: $n^m(dy')=n^m(y')\,dy'$, as seen from its explicit expression (2.2). Let for $0<\varepsilon<1$ and $1\leqq k\leqq d$,

$$(\mathrm{N}_{k,\varepsilon} f)(x) = \int_{\varepsilon \leqq |y| < 1} f(x-y) y_k n^m(y)\,dy, f \in \mathrm{L}^2(\mathbb{R}^d). \quad (2.22)$$

With $\mathrm{N}_{k,\varepsilon}$ rewrite $i_2(\varepsilon)$ as

$$i_2(\varepsilon) = -i \sum_{k=1}^{d} \mathrm{N}_{k,\varepsilon}(\varphi \mathrm{A}_k u).$$

To establish (2.21) we show that

$$\|\mathrm{N}_{k,\varepsilon} f\| \leqq \mathrm{C}_0 \|f\|, \qquad f \in \mathrm{L}^2(\mathbb{R}^d), \quad (2.23)$$

with a constant $\mathrm{C}_0$ independent of $0<\varepsilon<1$ and $m\geqq 0$, and that the $\mathrm{L}^2$ limit of $\mathrm{N}_{k,\varepsilon} f$ as $\varepsilon \downarrow 0$ exists. To this end we use the theory of singular integrals; we have only to confirm that each $y_k n^m(y)$ is the Calderón-Zygmund kernel [24], Chap. II, § 3, Theorem 2, p. 35, that is,

$$|y_k n^m(y)| \leqq \mathrm{B} |y|^{-d}, \qquad |y|>0, \quad (2.24a)$$

$$\int_{|y| \geqq 2|y'|} |(y_k - y'_k) n^m(y-y') - y_k n^m(y)|\,dy \leqq \mathrm{B}, |y'|>0, \quad (2.24b)$$

$$\int_{\mathrm{R}_1 < |y| < \mathrm{R}_2} y_k n^m(y)\,dy = 0, \qquad 0<\mathrm{R}_1<\mathrm{R}_2<\infty, \quad (2.24c)$$

with a constant B independent of $m \geqq 0$. (2.24 $a$) is easy to see from (2.2) with the asymptotics (2.3) of the Bessel function $K_\nu(z)$ and (2.24 $c$) is clear from the rotational invariance of the density $n^m(y)$. To see (2.24 $b$), first estimate, when $|y| \geqq 2|y'|$, the integrand in the integral to get

$$
\begin{aligned}
&|(y_k - y'_k)\, n^m(y-y') - y_k\, n^m(y)| \\
&\quad = \left| \int_0^1 (d/d\theta)\,[(y_k - \theta y'_k)\, n^m(y-\theta y')]\, d\theta \right| \\
&\quad = \left| \int_0^1 [y'_k\, n^m(y-\theta y') + (y_k - \theta y'_k) \sum_{j=1}^{d} y'_j (\partial/\partial y_j)\, n^m(y-\theta y')]\, d\theta \right| \\
&\quad \leqq B_0 \int_0^1 \left( |y'_k|\,|y-\theta y'|^{-(d+1)} + \sum_{j=1}^{d} |y'_j|\,|y-\theta y'|^{-(d+1)} \right) d\theta \\
&\quad \leqq B_1 |y'|\,|y|^{-(d+1)},
\end{aligned}
$$

with constants $B_0$ and $B_1$ independent of $m \geqq 0$, because

$$|y-\theta y'| \geqq |y| - |y'| \geqq 2^{-1}|y|, \qquad \text{if} \quad |y| \geqq 2|y'|.$$

Then we can bound the left-hand side of (2.24 $b$) by

$$B_1 |y'| \int_{|y| \geqq 2|y'|} |y|^{-(d+1)}\, dy = B_2 |y'| \int_{2|y'|}^{\infty} |y|^{-2}\, d|y| = 2^{-1} B_2,$$

using the spherical coordinates, where $B_2$ is the constant for which $B_2/B_1$ is the area of the $(d-1)$-dimensional unit sphere.

Finally, for $i_3$, we have by (2.7), for $x \in K$,

$$
\begin{aligned}
i_3(x) = (H_0^m(\varphi u))(x) - m\,\varphi(x)\,u(x) \\
+ \int_{|y| \geqq 1} [\varphi(x+y)\,u(x+y) - \varphi(x)\,u(x)]\, n^m(dy). \quad (2.25)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|i_3\|_{2,K} \leqq \|i_3\|_2 \leqq \|\sqrt{-\Delta + m^2}\,(\varphi u)\|_2 + (m + 2n_\infty^m)\,\|\varphi u\|_2 \\
\leqq \|\partial(\varphi u)\|_2 + 2(m + n_\infty^m)\,\|\varphi u\|_2 \\
\leqq c_K [\|u\|_{2,K_2} + \|\partial u\|_{2,K_2}], \quad (2.26)
\end{aligned}
$$

with a constant $c_K$ dependent on K. Then (2.14 $b$) follows from (2.20), (2.21) and (2.26) with (2.19).

(ii) Let K be a compact set in $\mathbb{R}^d$, and let $u$ be a $C_0^\infty$ function with supp $u \subseteq K$. Then we see by (2.14 $a$) that $I_1 u$ is $L^2$ with

$$\|I_1 u\|_p \leqq 2 n_\infty^m \|u\|_p \leqq 2|K|^{1/p} n_\infty^m \|u\|_\infty,$$

and, since $I_2 u$ has compact support in $K_1$, we have with (2.18)

$$
\begin{aligned}
\|I_2 u\|_p = \|I_2 u\|_{p,K_1} \leqq |K_1|^{1/p} \|I_2 u\|_{\infty,K_1} \\
\leqq |K_1|^{1/p} \left[ \sum_{j=1}^{2} \liminf_{\varepsilon \downarrow 0} \|i_j(\varepsilon)\|_{\infty,K_1} + \|i_3\|_{\infty,K_1} \right].
\end{aligned}
$$

Here $|K|$ and $|K_1|$ are the volumes of K and $K_1$. We can estimate, similarly to the proof of (i),

$$\| i_1(\varepsilon)\|_{\infty, K_1} \leqq (2^{-1} a(K_2) n_1^m + b(K_1)) \|\varphi u\|_\infty \leqq C(K_1) \|u\|_\infty,$$

$$\| i_2(\varepsilon)\|_{\infty, K_1} \leqq (b(K) + b(K_1)) \|\varphi u\|_\infty + a(K) n_1^m \sum_{j=1}^{d} \|\partial_j(\varphi u)\|_\infty,$$

$$\leqq c_K \left[ \|u\|_\infty + \sum_{j=1}^{d} \|\partial_j u\|_\infty \right],$$

$$\| i_3(\varepsilon)\|_{\infty, K_1} \leqq n_1^m \sum_{j, k=1}^{d} \|\partial_j \partial_k(\varphi u)\|_\infty \leqq c_K \sum_{j, k=1}^{d} \|\partial_j \partial_k u\|_\infty.$$

Thus, recalling (2.13 *a*), we have shown (2.15). This ends the proof of Lemma 2.3.

## 3. REGULARITY

*Throughout this section,* $A(x)$ *is assumed to satisfy* (2.5) *or* (1.4). We have seen in Lemma 2.3 among other things that $H_A^m$ defined by (2.8) is a linear operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$. It is easy to see with the rotational invariance of the Lévy measure $n^m(dy)$ that $H_A^m$ is symmetric, *i. e.*

$$(H_A^m \varphi, \psi) = (\varphi, H_A^m \psi), \qquad \varphi, \psi \in C_0^\infty(\mathbb{R}^d). \tag{3.1}$$

For $u \in L^2(\mathbb{R}^d)$ we can define, in view of Lemma 2.3, a distribution $H_A^m u$ in $\mathscr{D}'(\mathbb{R}^d)$ through

$$(H_A^m u, \varphi) = (u, H_A^m \varphi), \qquad \varphi \in C_0^\infty(\mathbb{R}^d). \tag{3.2}$$

In this section we shall show regularity of the function $v \in L^2(\mathbb{R}^d)$ with $H_A^m v \in L^1_{loc}(\mathbb{R}^d)$. To this end we give a kind of integral representation of such $v$, the proof of which needs some task. The main result of this section is Theorem 3.6, which is needed in the next section.

Let $G_0^m(x-y)$ be the fundamental solution for the operator $H_0^m + 1 \equiv \sqrt{-\Delta + m^2} + 1$. A direct calculation with (2.4) yields

$$\begin{aligned} G_0^m(z) &= (2\pi)^{-d} \int_0^\infty dt \int e^{izp} e^{-t[(p^2+m^2)^{1/2}+1]} dp \\ &= \int_0^\infty e^{-(m+1)t} k_0^m(t, z)\, dt \\ &= 2(2\pi)^{-(d+1)/2} m^{(d+1)/2} \int_0^\infty t e^{-t} (z^2+t^2)^{-(d+1)/4} \\ &\qquad \times K_{(d+1)/2}(m(z^2+t^2)^{1/2})\, dt. \end{aligned} \tag{3.3}$$

Hence $G_0^m(z)$ is a positive $C^\infty$ function where $|z|>0$, and satisfies

$$G_0^m(z) \leqq C|z|^{1-d}, \qquad d>1, \tag{3.4a}$$
$$G_0^m(z) \leqq C(1+|\log|z||), \qquad d=1, \tag{3.4b}$$
$$|\partial_z G_0^m(z)| \leqq C|z|^{-d}, \qquad d>0, \tag{3.4c}$$

near $z=0$, with a constant C independent of $m \geqq 0$, and $\int G_0^m(z)\,dz = 1/(m+1)$, which can be seen with the aid of (2.3) (cf. [24], Chap. V).

Now put for $\varepsilon \geqq 0$,

$$G_{R,\varepsilon}^m(x) = (2\pi)^{-d/2} \chi_R(x) \left( \mathscr{F}^{-1} \left( \frac{\exp[-\varepsilon((p^2+m^2)^{1/2}+1)]}{(p^2+m^2)^{1/2}+1} \right) \right)(x)$$
$$= e^{-(m+1)\varepsilon} \chi_R(x) \int G_0^m(x-y)\, k_0^m(\varepsilon, y)\, dy, \qquad \varepsilon>0, \tag{3.5a}$$
$$G_{R,0}^m(x) = \chi_R(x)\, G_0^m(x), \qquad \varepsilon=0, \tag{3.5b}$$

where $\chi_R$ is the same cutoff function as in the proof of Lemma 2.2, $k_0^m(t, z)$ the same as (2.4) and $\mathscr{F}^{-1}$ means the inverse Fourier transform. From (3.5) it is evident that, for $\varepsilon \geqq 0$, $G_{R,\varepsilon}^m(x)$ is a nonnegative $C^\infty$ function where $|x|>0$ which identically vanishes outside $|x|<R$, while, for $\varepsilon>0$, it is a $C_0^\infty$ function in $\mathbb{R}^d$. In view of (2.7), put for $\varepsilon>0$.

$$\begin{aligned} S_{R,\varepsilon}^m(x-y) &= ([\sqrt{-\Delta+m^2}+1]\, G_{R,\varepsilon}^m(x-\cdot))(y) \\ &= ([\sqrt{-\Delta+m^2}+1]\, G_{R,\varepsilon}^m)(x-y) \\ &= (m+1)\, G_{R,\varepsilon}^m(x-y) - \int_{|y'|>0} [G_{R,\varepsilon}^m(x-y-y') - G_{R,\varepsilon}^m(x-y) \\ &\qquad - I_{\{|y'|<1\}}\, y'\, \partial_y G_{R,\varepsilon}^m(x-y)]\, n^m(dy'), \end{aligned} \tag{3.6}$$

which is a real-valued, bounded $C^\infty$ function.

With $u \in \mathscr{S}(\mathbb{R}^d)$ let for $\varepsilon \geqq 0$

$$(G_{R,\varepsilon}^m u)(x) = \int G_{R,\varepsilon}^m(x-y)\, u(y)\, dy, \tag{3.7}$$

and for $\varepsilon>0$

$$(S_{R,\varepsilon}^m u)(x) = \int S_{R,\varepsilon}^m(x-y)\, u(y)\, dy. \tag{3.8}$$

*Remark.* – Here, to define $G_{R,\varepsilon}^m(x)$ we have made a large-momentum cutoff of $G_0^m(x)$. We might as well use another cutoff of $G_0^m(x)$ near the origin $x=0$ (e. g. [13], Appendix):

$$G_{R,\varepsilon}^{m'}(x) = \chi_R(x)(1-\chi_\varepsilon(x))\, G_0^m(x), \qquad 0<\varepsilon<R.$$

However, the merit of such a choice of $G^m_{R,\varepsilon}(x)$ as in (3.5) lies in allowing one to appeal more to Fourier analysis so as to simplify the proofs which follow.

In the following two lemmas we shall observe some properties of $G^m_{R,\varepsilon}$, (3.7), and $S^m_{R,\varepsilon}$, (3.8), as operators in the $L^2$ space. So $\|.\|$ stands for the $L^2$ norm there.

LEMMA 3.1. — *Let $\varepsilon \geqq 0$. The constants C below are, though all different in general, independent of $m \geqq 0$ and $\varepsilon$.*

(i)

$$0 \leqq G^m_{R,\varepsilon}(x) \leqq C|x|^{1-d}, \qquad d>1, \tag{3.9a}$$

$$0 \leqq G^m_{R,\varepsilon}(x) \leqq C(1+|\log|x||), \qquad d=1, \tag{3.9b}$$

*near $x=0$.*

(ii)

$$\|\sqrt{-\Delta+m^2}\, G^m_{R,\varepsilon} u\| \leqq C\|u\|, \qquad u \in \mathscr{S}(\mathbb{R}^d). \tag{3.10}$$

*In particular, $G^m_{R,\varepsilon}$ defines a bounded linear operator on $L^2(\mathbb{R}^d)$.*

(iii) *For $u \in L^2(\mathbb{R}^d)$,*

$$\|\sqrt{-\Delta+m^2}\,(G^m_{R,\varepsilon} u - G^m_{R,0} u)\| \to 0, \qquad \varepsilon \downarrow 0, \tag{3.11}$$

*uniformly on bounded subsets of m in $[0, \infty)$.*

*Proof.* — To simplify notation we suppress the superscript "$m$" to write $G_{R,\varepsilon}$, $G_0(x)$ and $k_0(t,x)$ for $G^m_{R,\varepsilon}$, $G^m_0(x)$ and $k^m_0(t,x)$, respectively.

(i) (3.9) follows from (3.4), since $G_{R,\varepsilon}(x) \leqq G_0(x)$.

(ii) Let $\hat{x}_R$ be the Fourier transform of $\chi_R$. By the Plancherel theorem we have with $\|u\|=1$,

$$\begin{aligned}\|(-\Delta+m^2)^{1/2} G_{R,\varepsilon} u\| &= \|(p^2+m^2)^{1/2}\mathscr{F}(G_{R,\varepsilon}u)\| \\ &\leqq (2\pi)^{d/2}\sup_p [(p^2+m^2)^{1/2}|\hat{G}_{R,\varepsilon}(p)|] \\ &= \sup_p \left|\int \frac{(p^2+m^2)^{1/2}\exp[-\varepsilon((p-\xi)^2+m^2)^{1/2}-\varepsilon]}{((p-\xi)^2+m^2)^{1/2}+1}\hat{\chi}_R(\xi)\,d\xi\right| \\ &\leqq \int (1+|\xi|)\,|\hat{\chi}_R(\xi)|\,d\xi,\end{aligned}$$

because $|((p-\xi)^2+m^2)^{1/2}-(p^2+m^2)^{1/2}| \leqq |\xi|$.

(iii) Similarly to the proof of (ii) we have

$$\begin{aligned}&\|(-\Delta+m^2)^{1/2}(G_{R,\varepsilon}u - G_{R,0}u)\|^2 \\ &\quad = \|(p^2+m^2)^{1/2}[\mathscr{F}(G_{R,\varepsilon}u)-\mathscr{F}(G_{R,0}u)]\|^2 \\ &\quad \leqq (2\pi)^{-d}\int\left(\int (1+|\xi|)\,|\hat{\chi}_R(\xi)|\right. \\ &\qquad\left.\times\left(1-\exp[-\varepsilon((p-\xi)^2+m^2)^{1/2}-\varepsilon]\right)d\xi\right)^2 |\hat{u}(p)|^2\,dp,\end{aligned}$$

which tends to zero as $\varepsilon \downarrow 0$, uniformly for bounded $m \geqq 0$, by the Lebesgue dominated convergence theorem. This proves Lemma 3.1.

LEMMA 3.2. — (i) *If* $\varepsilon > 0$, *then*

$$\| S^m_{R,\varepsilon} u \| \leqq C \| u \|, \qquad u \in \mathcal{S}(\mathbb{R}^d), \tag{3.12}$$

*with a constant* C *independent of* $m \geqq 0$ *and* $\varepsilon$. *In particular*, $S^m_{R,\varepsilon}$ *defines a bounded linear operator on* $L^2(\mathbb{R}^d)$.

(ii) *For* $u \in L^2(\mathbb{R}^d)$,

$$\| S^m_{R,\varepsilon} u - S^m_{R,\varepsilon} u \| \to 0, \qquad \varepsilon, \varepsilon' \downarrow 0, \tag{3.13 a}$$

*uniformly on bounded subsets of* $m$ *in* $[0, \infty)$, *so that*

$$S^m_R \equiv \text{S-}\lim_{\varepsilon \to 0} S^m_{R,\varepsilon} \tag{3.13 b}$$

*is a bounded linear operator on* $L^2(\mathbb{R}^d)$. *The bounded linear operator* $Q^m_R$ *on* $L^2(\mathbb{R}^d)$ *defined by*

$$Q^m_R u \equiv \int Q^m_R (x-y)\, u(y)\, dy = u - S^m_R u \tag{3.14}$$

*satisfies that*

$$\| \sqrt{-\Delta + m^2}\, Q^m_R u \| \leqq C \| u \|, \qquad u \in L^2(\mathbb{R}^d), \tag{3.15}$$

*with a constant C independent of* $m \geqq 0$.

*Proof.* — We suppress "$m$" of $S^m_{R,\varepsilon}$, $G^m_{R,\varepsilon}$ and $Q^m_R$. Since by (3.6)

$$S_{R,\varepsilon} = [\sqrt{-\Delta + m^2} + 1]\, G_{R,\varepsilon}, \tag{3.16}$$

(3.12) and (3.13 *a*) follow from (3.10) and (3.11) in Lemma 3.1. Obvious is that $S_R$ and so $Q_R$ is a bounded operator on $L^2(\mathbb{R}^d)$. We have (3.15), since $Q_R$ is a pseudo-differential operator with symbol

$$q_R(p) = (2\pi)^{-d} \int \left( 1 - \frac{(p^2 + m^2)^{1/2} + 1}{((p-\xi)^2 + m^2)^{1/2} + 1} \right) \hat{\chi}_R(\xi)\, d\xi$$

and so $(p^2 + m^2)^{1/2} q_R(p)$ is uniformly bounded. This proves Lemma 3.2.

The following lemma studies some further properties of $G^m_{R,\varepsilon}$ as an operator in the $L^p$ space. Recall $K_r$, (2.16), and $\| f \|_{p,K}$, (2.17).

LEMMA 3.3. — *Let* $1 \leqq p < \infty$. *Let* $\psi \in C^\infty_0(\mathbb{R}^d)$ *and* $K = \operatorname{supp} \psi$. (i) *If* $\varepsilon \geqq 0$, *then*

$$\| \psi G^m_{R,\varepsilon} f \|_p \leqq C \| f \|_{p, K_{R+1}}, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{3.17 a}$$

*and*

$$\| \sqrt{-\Delta + m^2}\, \psi G^m_{R,\varepsilon} f \|_p \leqq C \| f \|_{p, K_{R+1}}, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{3.17 b}$$

*with a constant* C *dependent on* $\psi$ *but independent of* $m \geqq 0$ *and* $\varepsilon$. *In particular*, $\psi G^m_{R,\varepsilon}$ *and* $\sqrt{-\Delta + m^2}\, \psi G^m_{R,\varepsilon}$ *define bounded linear operators of* $L^p_{\text{loc}}(\mathbb{R}^d)$ *into* $L^p(\mathbb{R}^d)$.

(ii) *For* $f \in L^p_{\text{loc}}(\mathbb{R}^d)$,

$$\| \psi G^m_{R,\varepsilon} f - \psi G^m_{R,0} f \|_p \to 0, \qquad \varepsilon \downarrow 0, \tag{3.18}$$

*uniformly on bounded subsets of* $m$ *in* $[0, \infty)$.

*Proof.* — We write $G_{R,\varepsilon}$, $G_0(x)$ and $k_0(t, x)$ for $G^m_{R,\varepsilon}$, $G^m_0(x)$ and $k^m_0(t, x)$. Let $\psi \in C^\infty_0(\mathbb{R}^d)$. Put $K = \operatorname{supp} \psi$ and let $\varphi$ be a $C^\infty_0$ function with $0 \leqq \varphi(x) \leqq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on $K_R$ and $\operatorname{supp} \varphi \subseteq K_{R+1}$.

(i) *Proof of* (3.17 *a*). — Let $f \in \mathcal{S}(\mathbb{R}^d)$. Since $G_{R,\varepsilon}(x-y)$ identically vanishes outside $|x-y| < R$, we have $G_{R,\varepsilon} f = G_{R,\varepsilon}(\varphi f)$ on K, so that

$$\| \psi G_{R,\varepsilon} f \|_p = \| \psi G_{R,\varepsilon}(\varphi f) \|_p \leqq \| \psi \|_\infty \| G_{R,\varepsilon}(\varphi f) \|_p.$$

Therefore, the proof of (3.17 *a*) is reduced to that of

$$\| G_{R,\varepsilon} u \|_p \leqq \| u \|_p, \qquad u \in L^p(\mathbb{R}^d). \tag{3.19}$$

since $\| \varphi f \|_p \leqq \| f \|_{p, K_{R+1}}$. But this follows by Young's inequality with

$$\int G_{R,\varepsilon}(x)\, dx \leqq (m+1)^{-1} \leqq 1.$$

*Proof of* (3.17 *b*). — By the same argument as above, the proof is reduced to that of

$$\| (-\Delta + m^2)^{1/2} \psi G_{R,\varepsilon} u \|_p \leqq C \| u \|_p, \qquad u \in L^p(\mathbb{R}^d), \tag{3.20}$$

with a constant C independent of $m \geqq 0$ and $\varepsilon \geqq 0$. In view of (3.5), let with $u \in L^p(\mathbb{R}^d)$,

$$(K_{R,\varepsilon} u)(x) = e^{-(m+1)\varepsilon} \int \chi_R(x-y)\, k_0(\varepsilon, x-y)\, u(y)\, dy \tag{3.21 a}$$

or

$$(\mathcal{F}(K_{R,\varepsilon} u))(p)$$
$$= (2\pi)^{-d/2} \int \hat{\chi}_R(\eta) \exp[-\varepsilon((p-\eta)^2 + m^2)^{1/2} - \varepsilon]\, d\eta\, \hat{u}(p). \tag{3.21 b}$$

It is easy to see by Young's inequality that $K_{R,\varepsilon}$ is a bounded linear operator on $L^p(\mathbb{R}^d)$:

$$\| K_{R,\varepsilon} u \|_p \leqq e^{-(m+1)\varepsilon} \| u \|_p \leqq \| u \|_p. \tag{3.22}$$

Therefore showing (3.20) is equivalent to showing

$$\| [(-\Delta + m^2)^{1/2} \psi G_{R,\varepsilon} - \psi K_{R,\varepsilon}] u \|_p \leqq C \| u \|_p, \qquad u \in L^p(\mathbb{R}^d), \tag{3.23}$$

with a constant C independent of $m \geqq 0$ and $\varepsilon \geqq 0$. The operator in the bracket [. . .] on the left of (3.23) is represented as the pseudo-differential operator

$$(b(X, D) u)(x) = (2\pi)^{-d/2} \int e^{ixp}\, b(x, p)\, \hat{u}(p)\, dp, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{3.24 a}$$

with symbol

$$b(x,p)=(2\pi)^{-d}\iint e^{ix\xi}\left(\frac{((p+\xi)^2+m^2)^{1/2}}{((p-\eta)^2+m^2)^{1/2}+1}-1\right)$$
$$\times\exp[-\varepsilon((p-\eta)^2+m^2)^{1/2}-\varepsilon]\,\hat{\psi}(\xi)\,\hat{\chi}_{\mathrm{R}}(\eta)\,d\xi\,d\eta$$
$$\equiv b_1(x,p)+b_2(x,p). \tag{3.24b}$$

Here

$$b_1(x,p)=(2\pi)^{-d}\iint e^{ix\xi}(\chi_{\mathrm{R}}(p+\xi)+\psi_{\mathrm{R}}(p+\xi)\chi_{\mathrm{R}}(p-\eta))(\ldots)$$
$$\times\exp[-\varepsilon((p-\eta)^2+m^2)^{1/2}-\varepsilon]\,\hat{\psi}(\xi)\,\hat{\chi}_{\mathrm{R}}(\eta)\,d\xi\,d\eta, \tag{3.25a}$$

$$b_2(x,p)=(2\pi)^{-d}\iint e^{ix\xi}\psi_{\mathrm{R}}(p+\xi)\,\psi_{\mathrm{R}}(p-\eta)(\ldots)$$
$$\times\exp[-\varepsilon((p-\eta)^2+m^2)^{1/2}-\varepsilon]\,\hat{\psi}(\xi)\,\hat{\chi}_{\mathrm{R}}(\eta)\,d\xi\,d\eta, \tag{3.25b}$$

with the same cutoff functions $\chi_{\mathrm{R}}(p)$ and $\psi_{\mathrm{R}}(p)=1-\chi_{\mathrm{R}}(p)$ as in the proof of Lemma 2.2. Note here that

$$(\chi_{\mathrm{R}}(p+\xi)+\psi_{\mathrm{R}}(p+\xi)\chi_{\mathrm{R}}(p-\eta))+\psi_{\mathrm{R}}(p+\xi)\,\psi_{\mathrm{R}}(p-\eta)=1.$$

To show (3.23) we need to show for $1\leqq p<\infty$ that

$$\|b_1(\mathrm{X},\mathrm{D})u\|_p\leqq \mathrm{C}\|u\|_p, \tag{3.26a}$$
$$\|b_2(\mathrm{X},\mathrm{D})u\|_p\leqq \mathrm{C}\|u\|_p,\qquad u\in\mathscr{S}(\mathbb{R}^d), \tag{3.26b}$$

with a constant C independent of $m\geqq 0$ and $\varepsilon\geqq 0$. We can show (3.26 *b*), using a general result [19], Theorem 3, since the symbol $b_2(x,p)$ satisfies that for every multi-index $\alpha$ there exists a constant $\mathrm{C}_\alpha$ independent of $m\geqq 0$ and $\varepsilon\geqq 0$ such that for $|\alpha|\leqq d+1$,

$$|\partial_p^\alpha b_2(x,p)|\leqq \mathrm{C}_\alpha(1+p^2)^{-(|\alpha|+1)/2},\qquad x\in\mathbb{R}^d,\quad p\in\mathbb{R}^d.$$

To get (3.26 *a*) write $b_1(\mathrm{X},\mathrm{D})$ as

$$(b_1(\mathrm{X},\mathrm{D})u)(x)=\int \mathrm{K}(x,x-y)\,u(y)\,dy, \tag{3.27}$$

where

$$\mathrm{K}(x,z)=(2\pi)^{-d}\int e^{izp}\,b_1(x,p)\,dp$$
$$=(2\pi)^{-2d}\iiint e^{izp}\,a(x,p,\xi,\eta)\,d\xi\,d\eta\,dp. \tag{3.28}$$

with

$$a(x,p,\xi,\eta)=e^{ix\xi}(\chi_R(p+\xi)+\psi_R(p+\xi)\chi_R(\eta))\left(\frac{((p+\xi)^2+m^2)^{1/2}}{(\eta^2+m^2)^{1/2}+1}-1\right)$$
$$\times\exp[-\varepsilon((\eta^2+m^2)^{1/2}+1)]\,\hat{\psi}(\xi)\,\hat{\chi}_R(p-\eta).$$

Then it is easy to verify that $K(x,z)$ is a bounded function. Therefore to show (3.26 *a*) we have only to show that for $0<\delta<1$ there exists a constant $C_\delta$ depending on $\delta$ such that

$$|K(x,z)|\leqq C_\delta|z|^{-d-1+\delta},\qquad (x,z)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{3.29}$$

To do so let $\alpha$ be a multi-index with $|\alpha|=d$. Then we have by integration by parts

$$(2\pi)^{2d}z^\alpha K(x,z)=i^d\iiint e^{izp}\,\partial_p^\alpha a(x,p,\xi,\eta)\,d\xi\,d\eta\,dp$$
$$=-i^{d+1}\iiint\sum_{j=1}^{d}z_j|z|^{-2}\,\partial_{p_j}(e^{izp}-e^{-iz\xi})\,\partial_p^\alpha a(x,p,\xi,\eta)\,d\xi\,d\eta\,dp$$
$$=i^{d+1}\iiint\sum_{j=1}^{d}z_j|z|^{-2}(e^{izp}-e^{-iz\xi})\,\partial_{p_j}\partial_p^\alpha a(x,p,\xi,\eta)\,d\xi\,d\eta\,dp.$$

Hence, noting that $|e^{izp}-e^{-iz\xi}|\leqq 2|z|^\delta|p+\xi|^\delta$ for $0<\delta<1$, we obtain

$$|z^\alpha K(x,z)|\leqq(2\pi)^{-2d}|z|^{-1+\delta}$$
$$\times\iiint_{\{|p+\xi|\leqq R\text{ or }|\eta|\leqq R\}}|p+\xi|^\delta\,|\partial_{p_j}\partial_p^\alpha a(x,p,\xi,\eta)|\,d\xi\,d\eta\,dp.$$

The integral on the right is convergent, because

$$|\partial_{p_j}\partial_p^\alpha a(x,p,\xi,\eta)|\leqq C|\hat{\psi}(\xi)|\left[\sum_{l=0}^{d}|p+\xi|^{-l}\right]\left[\sum_{l=0}^{d+1}\sum_{|\beta|=l}|\partial_p^\beta\hat{\chi}_R(p-\eta)|\right],$$

with a constant C independent of $m\geqq 0$ and $\varepsilon\geqq 0$. This yields (3.29).

(ii) *Proof of* (3.18). — By the same argument as used with $\varphi$ in the proof of (i) it suffices to show that for $u\in L^p(\mathbb{R}^d)$,

$$\|G_{R,\varepsilon}u-G_{R,0}u\|_p\to 0,\qquad \varepsilon\downarrow 0. \tag{3.30}$$

To get (3.30) we see by applying Young's inequality

$$\|G_{R,\varepsilon}u-G_{R,0}u\|_p\leqq\int|G_{R,\varepsilon}(x)-G_{R,0}(x)|\,dx\,\|u\|_p.$$

Therefore we have only to show that

$$g(\varepsilon)\equiv\int|G_{R,\varepsilon}(x)-G_{R,0}(x)|\,dx\to 0,\qquad \varepsilon\downarrow 0. \tag{3.31}$$

uniformly on bounded $m \geqq 0$. We have from (3.5)

$$g(\varepsilon) \leqq \int k_0(\varepsilon, y)\, dy \int \chi_R(x) \,|\, [e^{-(m+1)\varepsilon} G_0(x-y) - G_0(x) \,|\, dx$$

$$\leqq (1 - e^{-(m+1)\varepsilon}) + \int k_0(\varepsilon, y)\, dy \int \chi_R(x) \,|\, G_0(x-y) - G_0(x)] \,|\, dx,$$

because $\int k_0(\varepsilon, y)\, dy = (m+1) \int G_0(x)\, dx = 1$. The first term in the last member above tends to zero as $\varepsilon \downarrow 0$, uniformly on bounded $m \geqq 0$. Therefore we have to prove that so does the second term, which is equal, by changing the integration variables $y = \varepsilon^{1/2} y'$ (writing $y$ instead of $y'$), to

$$\int dy\, k_0(\varepsilon, \varepsilon^{1/2} y)\, \varepsilon^{d/2} \int \chi_R(x) \,|\, [G_0(x - \varepsilon^{1/2} y) - G_0(x)] \,|\, dx$$

$$= \int_{|y|>r} dy \ldots + \int_{|y| \leqq r} dy \ldots \equiv g_1(\varepsilon) + g_2(\varepsilon).$$

Let $\delta > 0$ be arbitrary. Then there exists $r_0 > 0$ such that for $r \geqq r_0$, $\int_{|y|>r} k_0(\varepsilon, \varepsilon^{1/2} y)\, \varepsilon^{d/2}\, dy < \delta/2$, uniformly in $0 < \varepsilon < 1$ and $m \geqq 0$. For the proof see [9], Lemma 3.3, where the uniformity in $m$ is not mentioned, but the proof there is still valid. It follows that $g_1(\varepsilon) < \delta$. On the other hand, we can see $g_2(\varepsilon)$ satisfies that for $|y| \leqq r_0$.

$$g_2(\varepsilon) \leqq \int_{-R}^{R} |\, [G_0(x - \varepsilon^{1/2} y) - G_0(x)] \,|\, dx,$$

which tends to zero as $\varepsilon \downarrow 0$, since $G_0(x)$ is integrable. Thus we have shown (3.31), ending the proof of Lemma 3.3.

Now we are in a position to derive an integral representation for $v$ in $L^2$ with $H_A^m v$ in $L^1_{loc}$. Let $\varepsilon > 0$. By Lemma 3.1, $G_{R,\varepsilon}^m(x-y)$ is, for $x$ fixed, a real-vlaued $C_0^\infty$ function in $y$, so that by (3.2),

$$([H_A^m + 1] v, G_{R,\varepsilon}^m(x - \cdot)) = (v, [H_A^m + 1] G_{R,\varepsilon}^m(x - \cdot)) \qquad (3.32a)$$

or

$$\int G_{R,\varepsilon}^m(x-y)\, ([H_A^m + 1] v)(y)\, dy$$

$$= \int \overline{([H_A^m + 1] G_{R,\varepsilon}^m(x - \cdot))(y)}\, v(y)\, dy. \qquad (3.32b)$$

We use the expression (2.8) for $H_A^m$ to rewrite the kernel of the integral on the right-hand side of (3.32$b$):

$$([H_A^m + 1] G_{R,\varepsilon}^m(x - \cdot))(y) = S_{R,\varepsilon}^m(x-y) - \overline{E_{R,\varepsilon}^m(x,y)} - \overline{F_{R,\varepsilon}^m(x,y)}, \quad (3.33)$$

where $S^m_{R,\varepsilon}(x-y)$ is defined by (3.6), $\overline{E^m_{R,\varepsilon}(x,y)}$ and $\overline{F^m_{R,\varepsilon}(x,y)}$ are respectively the complex conjugates of

$$E^m_{R,\varepsilon}(x,y)=\int_{|y'|>0}[e^{iy'A(y+y'/2)}-1-I_{\{|y'|<1\}}iy'A(y)] \times G^m_{R,\varepsilon}(x-y-y')\,n^m(dy'), \quad (3.34)$$

and

$$F^m_{R,\varepsilon}(x,y) = \int_{0<|y'|<1} iy'A(y)[G^m_{R,\varepsilon}(x-y-y')-G^m_{R,\varepsilon}(x-y)]\,n^m(dy'). \quad (3.35)$$

The following two lemmas are concerned with the integral operators with kernels $E^m_{R,\varepsilon}(x,y)$ and $F^m_{R,\varepsilon}(x,y)$:

$$(E^m_{R,\varepsilon}u)(x)=\int E^m_{R,\varepsilon}(x,y)\,u(y)\,dy, \quad (3.36)$$

$$(F^m_{R,\varepsilon}u)(x)=\int F^m_{R,\varepsilon}(x,y)\,u(y)\,dy. \quad (3.37)$$

There $\|.\|$ stands for the $L^2$ norm.

LEMMA 3.4. – *Let* $\psi\in C_0^\infty(\mathbb{R}^d)$. (i) *If* $\varepsilon\geqq 0$, *then* $E^m_{R,\varepsilon}u$ *exists with* $u\in L^2(\mathbb{R}^d)$, *and*

$$\|\psi E^m_{R,\varepsilon}u\|\leqq C_1\|u\|, \qquad u\in L^2(\mathbb{R}^d), \quad (3.38\,a)$$

*while if* $\varepsilon>0$, *then*

$$\|\partial(\psi E^m_{R,\varepsilon}u)\|\leqq C_2\|u\|, \qquad u\in L^2(\mathbb{R}^d), \quad (3.38\,b)$$

*with constants* $C_1$ *and* $C_2$ *dependent on* $\psi$ *but independent of* $m\geqq 0$ *and* $\varepsilon$.

(ii) *For every* $u\in L^2(\mathbb{R}^d)$,

$$\|\psi E^m_{R,\varepsilon}u-\psi E^m_{R,0}u\|\to 0, \qquad \varepsilon\downarrow 0, \quad (3.39\,a)$$
$$\|\partial(\psi E^m_{R,\varepsilon}u)-\partial(\psi E^m_{R,\varepsilon'}u)\|\to 0, \qquad \varepsilon,\varepsilon'\downarrow 0, \quad (3.39\,b)$$

*uniformly on bounded subsets of* $m$ *in* $[0,\infty)$.

LEMMA 3.5. – *Let* $\psi\in C_0^\infty(\mathbb{R}^d)$. (i) *If* $\varepsilon\geqq 0$, *then* $F^m_{R,\varepsilon}u$ *exists with* $u\in L^2(\mathbb{R}^d)$, *and*

$$\|\psi F^m_{R,\varepsilon}u\|\leqq C_1\|u\|, \qquad u\in L^2(\mathbb{R}^d), \quad (3.40\,a)$$

*while if* $\varepsilon>0$, *then*

$$\|\partial(\psi F^m_{R,\varepsilon}u)\|\leqq C_2\|u\|, \qquad u\in L^2(\mathbb{R}^d), \quad (3.40\,b)$$

*with constants* $C_1$ *and* $C_2$ *dependent on* $\psi$ *but independent of* $m\geqq 0$ *and* $\varepsilon$.

(ii) *For every* $u\in L^2(\mathbb{R}^d)$,

$$\|\psi F^m_{R,\varepsilon}u-\psi F^m_{R,0}u\|\to 0, \qquad \varepsilon\downarrow 0, \quad (3.41\,a)$$

$$\| \partial(\psi F^m_{R,\varepsilon} u) - \partial(\psi F^m_{R,\varepsilon'} u) \| \to 0, \qquad \varepsilon, \varepsilon' \downarrow 0, \tag{3.41b}$$

*uniformly on bounded subsets of m in* $[0, \infty)$.

Based on Lemmas 3.1-5, we now give the main theorem of this section, Theorem 3.6. The proofs of Lemmas 3.4 and 3.5 will be postponed to the end of the proof of Theorem 3.6. We shall write for $x \neq y$,

$$G^m_R(x-y) = G^m_{R,0}(x-y), \qquad E^m_R(x,y) = E^m_{R,0}(x,y),$$
$$F^m_R(x,y) = F^m_{R,0}(x,y).$$

THEOREM 3.6. – *Let* $v \in L^2(\mathbb{R}^d)$ *and let* $H^m_A v \in L^1_{loc}(\mathbb{R}^d)$. *Then* $v$ *admits an integral representation*

$$\begin{aligned} v(x) = \int G^m_R(x-y)([H^m_A+1]v)(y)\,dy + \int Q^m_R(x-y)\,v(y)\,dy \\ + \int E^m_R(x,y)\,v(y)\,dy + \int F^m_R(x,y)\,v(y)\,dy \\ \equiv G^m_R[H^m_A+1]v + Q^m_R v + E^m_R v + F^m_R v. \end{aligned} \tag{3.42}$$

*Here* $G^m_R \equiv G^m_{R,0}$, $Q^m_R$, $E^m_R \equiv E^m_{R,0}$ *and* $F^m_R \equiv F^m_{R,0}$ *are defined by* (3.7), (3.14), (3.36) *and* (3.37), *respectively. Consequently,* $v$ *has a decomposition* $v = v_1 + v_2$ *such that, for every* $\psi \in C^\infty_0(\mathbb{R}^d)$, both $\psi v_1$ *and* $\sqrt{-\Delta + m^2}\,\psi v_1$ *are in* $L^1(\mathbb{R}^d)$, *and both* $\psi v_2$ *and* $\sqrt{-\Delta+m^2}\,\psi v_2$ *are in* $L^2(\mathbb{R}^d)$.

*Proof.* – We simplify notation to suppress the superscript "$m$" of $H^m_A$, $G^m_{R,\varepsilon}$, $Q^m_R$, $S^m_{R,\varepsilon}$, $E^m_{R,\varepsilon}$ and $F^m_{R,\varepsilon}$. Let $v \in L^2$ and $H_A v \in L^1_{loc}$. Then we get for $\varepsilon > 0$ from (3.32) and (3.33)

$$S_{R,\varepsilon} v = G_{R,\varepsilon}[H_A+1]v + E_{R,\varepsilon} v + F_{R,\varepsilon} v. \tag{3.43}$$

By Lemma 3.2 (ii), (3.13) and (3.14), the left-hand side of (3.43), $S_{R,\varepsilon} v$, converges to $v - Q_R v$ in $L^2$, as $\varepsilon \downarrow 0$. As to the right-hand side of (3.43), $G_{R,\varepsilon}[H_A+1]v$ converges to $G_R[H_A+1]v$ in $L^1_{loc}$, by Lemma 3.3 (ii), (3.18), $E_{R,\varepsilon} v$ to $E_R v$ in $L^2_{loc}$, by Lemma 3.4 (ii), (3.39*a*) and $F_{R,\varepsilon} v$ to $F_R v$ in $L^2_{loc}$, by Lemma 3.5 (ii), (3.41*a*). This proves (3.42). Once (3.42) is established, the remaining assertion is immediately seen by taking $v_1 = G^m_R[H_A+1]v$ and $v_2 = Q_R v + E_R v + F_R v$, since $(-\Delta+m^2)^{1/2}\psi G_R[H_A+1]v$ is in $L^1(\mathbb{R}^d)$ [by Lemma 3.3 (i), (3.17*b*)], and the three $Q_R v$, $\psi E_R v$ and $\psi F_R v$ are all in the Sobolev space $H^1(\mathbb{R}^d)$ [by Lemma 3.2(ii), (3.15), Lemma 3.4 (ii), (3.39*a*, *b*) and Lemma 3.5 (ii), (3.41 *a*, *b*)]. Thus Theorem 3.6 is proved.

Finally we prove Lemmas 3.4 and 3.5.

*Proof of Lemma* 3.4. – We write $G_{R,\varepsilon}$, $E_{R,\varepsilon}$ and $n(dy')$ for $G^m_{R,\varepsilon}$, $E^m_{R,\varepsilon}$ and $n^m(dy')$. Let $\psi \in C^\infty_0(\mathbb{R}^d)$. Put $K = \operatorname{supp}\psi$ and let $\varphi$ be a $C^\infty_0$ function with $0 \leqq \varphi(x) \leqq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on $K_R$ and $\operatorname{supp}\varphi \subseteq K_{R+1}$.

(i) *Proof of* (3.38*a*). Changing, in (3.34), the integration variables $y + y' = z$ (writing $y$ instead of $z$) and using the rotational invariance of the Lévy measure $n(dy')$, we have

$$(E_{R,\varepsilon}u)(x) = \int G_{R,\varepsilon}(x-y) f_1(y)\,dy + \int G_{R,\varepsilon}(x-y) f_2(y)\,dy$$
$$\equiv (E^1_{R,\varepsilon}u)(x) + (E^2_{R,\varepsilon}u)(x), \quad (3.44)$$

where

$$f_1(y) = \int_{|y'|\geqq 1} [e^{-iy'A(y+y'/2)} - 1]\, u(y+y')\, n(dy'), \quad (3.45\,a)$$

$$f_2(y) = \int_{0<|y'|<1} [e^{-iy'A(y+y'/2)} - 1 + iy'A(y+y')]\, u(y+y')\, n(dy'). \quad (3.45\,b)$$

Then we see by Lemma 3.1 (ii), (3.10), that $E^1_{R,\varepsilon}u$ on the right of (3.44) satisfies

$$\| E^1_{R,\varepsilon}u \| = \| G_{R,\varepsilon} f_1 \| \leqq C \| f_1 \| \leqq 2 C n_\infty \| u \|,$$

with the constants $n_\infty \equiv n^m_\infty$ in (2.6 *a*) and C in (3.10).

Similarly we have for $E^2_{R,\varepsilon}u$,

$$\| \psi E^2_{R,\varepsilon}u \| = \| \psi G_{R,\varepsilon}(\varphi f_2) \| \leqq C \| \psi \|_\infty \| \varphi f_2 \| \leqq C \| \psi \|_\infty \| f_2 \|_{2, K_{R+1}},$$

where $\|.\|_{2,K_{R+1}}$ stands for the local $L^2$ norm in (2.17). Therefore the proof of (3.38 *a*) will be accomplished if we can show

$$\| f_2 \|_{2,K_{R+1}} \leqq C(K_{R+1}) \| u \|_{2,K_{R+2}} \leqq C(K_{R+1}) \| u \|, \quad (3.46)$$

with a constant $C(K_{R+1})$. The proof of (3.46) will be done by a analogous argument used to prove (2.20).

*Proof of* (3.38 *b*). – Let $\varepsilon > 0$ and put $\partial_j = \partial_{x_j}$, $1 \leqq j \leqq d$. We can see similarly to the proof of (3.38 *a*) above

$$\| \partial_j E^1_{R,\varepsilon}u \| = \| \partial_j G_{R,\varepsilon} f_1 \| \leqq C \| f_1 \| \leqq 2 C n_\infty \| u \|,$$

and by Lemma 3.1 (ii) and by (3.46),

$$\begin{aligned}
\| \partial_j(\psi E^2_{R,\varepsilon}u) \| &= \| \partial_j(\psi G_{R,\varepsilon}(\varphi f_2)) \| \\
&\leqslant \| \partial_j \psi \|_\infty \| G_{R,\varepsilon}(\varphi f_2) \| + \| \psi \|_\infty \| \partial_j(G_{R,\varepsilon}(\varphi f_2)) \| \\
&\leqq C \| \partial_j \psi \|_\infty \| \varphi f_2 \| + C \| \psi \|_\infty \| \varphi f_2 \| \\
&\leqq C_\psi C \| f_2 \|_{2,K_{R+1}} \leqq C_\psi C C(K_{R+1}) \| u \|,
\end{aligned}$$

with $C_\psi = \| \psi \|_\infty + \| \partial_j \psi \|_\infty$.

(ii) *Proof of* (3.39 *a, b*). – By the same arguments as in the proof of (i), we can see that for $\varepsilon > 0$,

$$\| E^1_{R,\varepsilon}u - E^1_R u \| = \| G_{R,\varepsilon} f_1 - G_R f_1 \|,$$
$$\| \psi E^2_{R,\varepsilon}u - \psi E^2_R u \| = \| \psi E^2_{R,\varepsilon}(\varphi f_2) - \psi E^2_R(\varphi f_2) \|,$$

and for $\varepsilon, \varepsilon' > 0$,

$$\| \partial_j E^1_{R,\varepsilon}u - \partial_j E^1_{R,\varepsilon'}u \| = \| \partial_j G_{R,\varepsilon} f_1 - \partial_j G_{R,\varepsilon'} f_1 \|,$$
$$\| \partial_j(\psi E^2_{R,\varepsilon}u) - \partial_j(\psi E^2_{R,\varepsilon'}u) \| = \| \partial_j(\psi E^2_{R,\varepsilon}(\varphi f_2)) - \partial_j(\psi E^2_{R,\varepsilon'}(\varphi f_2)) \|,$$

all the four of which tend to zero as $\varepsilon \downarrow 0$ and $\varepsilon, \varepsilon' \downarrow 0$, uniformly on bounded $m \geqq 0$, by Lemma 3.1 (iii). This shows (3.39 *a*, *b*), ending the proof of Lemma 3.4.

*Proof of Lemma* 3.5. – We write again $G_{R,\varepsilon}$, $F_{R,\varepsilon}$ and $n(dy) = n(y)\,dy$ for $G^m_{R,\varepsilon}$, $F^m_{R,\varepsilon}$ and $n^m(dy) = n^m(y)\,dy$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Put $K = \operatorname{supp}\psi$ and let $\varphi$ be a $C_0^\infty$ function with $0 \leqq \varphi(x) \leqq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on $K_{R+1}$ and $\operatorname{supp}\varphi \subseteq K_{R+2}$.

(i) First we show that for $\varepsilon \geqq 0$, $F_{R,\varepsilon} u$ exists with $u \in L^2(\mathbb{R}^d)$. Put

$$F_{R,\varepsilon}(x, y) = i \sum_{k=1}^{d} A_k(y)\, L_{k,R,\varepsilon}(x-y), \tag{3.47 a}$$

with

$$L_{k,R,\varepsilon}(x) = \int_{0<|y|<1} y_k [G_{R,\varepsilon}(x-y) - G_{R,\varepsilon}(x)]\, n(dy), 1 \leqq k \leqq d. \tag{3.47 b}$$

Since $L_{k,R,\varepsilon}$ has compact support, we have only to show that for some constant $0 < \delta < 1$,

$$|L_{k,R,\varepsilon}(x)| \leqq c_0 |x|^{\delta-d}, \tag{3.48}$$

near $x = 0$, with a constant $c_0$ independent of $m \geqq 0$ and $\varepsilon \geqq 0$. Write (3.47 *b*) as

$$L_{k,R,\varepsilon}(x) = \int_{Y_1} + \int_{Y_2} + \int_{Y_3} \equiv L_1(x) + L_2(x) + L_3(x), \tag{3.49}$$

dividing the integration region $0 < |y| < 1$ into the three parts:

$$Y_1 = \{y; 0 < |y| < 1 \wedge (|x|/2)\}, Y_2 = \{y; 0 < |y| < 1, |x-y| < |x|/2\}$$

and the rest $Y_3$.

In the following argument, the constants such as $c_1$, $c_2$, $c_0$, are independent of $m \geqq 0$ and $\varepsilon \geqq 0$. Since

$$L_1(x) = -\int_{Y_1} n(dy) \int_0^1 y_k \sum_{j=1}^{d} y_j \partial_{x_j} G_{R,\varepsilon}(x - \theta y)\, d\theta,$$

we have by (3.4 *c*) and (2.2) together with (2.3),

$$\begin{aligned} |L_1(x)| &\leqq c_1 \int_{Y_1} |y|^{1-d}\, dy \int_0^1 |x - \theta y|^{-d}\, d\theta \\ &\leqq 2^d c_1 |x|^{-d} \int_{0<|y|<|x|/2} |y|^{1-d}\, dy \leqq c_0 |x|^{1-d}, \end{aligned} \tag{3.50}$$

because $|y|<|x|/2$ implies $|x-\theta y|\geqq|x|-|y|\geqq|x|/2$. For $L_2(x)$, we have by (3.4 *a*, *b*) and (2.2), in case $d>1$,

$$\begin{aligned}|L_2(x)|&\leqq\int_{Y_2}|y|(|G_{R,\varepsilon}(x-y)|+|G_{R,\varepsilon}(x)|)\,n(dy)\\&\leqq c_1\int_{Y_2}|y|^{-d}(|x-y|^{1-d}+|x|^{1-d})\,dy\\&\leqq 2^d c_1|x|^{-d}\int_{|x-y|<|x|/2}(|x-y|^{1-d}+|x|^{1-d})\,dy\leqq c_0|x|^{1-d},\end{aligned}\tag{3.51 a}$$

because $|x-y|<|x|/2$ implies $|y|>|x|/2$, and similarly, in case $d=1$,

$$|L_2(x)|\leqq c_1(1+|\log|x||)\leqq c_0|x|^{\delta-1},\tag{3.51 b}$$

with $0<\delta<1$. Since $|y|\geqq|x|/2$ and $|x-y|\geqq|x|/2$ in $Y_3$, we have by (3.4 *a*, *b*) and (2.2), in case $d>1$,

$$\begin{aligned}|L_3(x)|&\leqq c_1\int_{Y_3}|y|^{-d}(|x-y|^{1-d}+|x|^{1-d})\,dy\\&\leqq(2^d+1)c_1|x|^{1-d}\int_{1>|y|\geqq|x|/2}|y|^{-d}\,dy\\&\leqq c_2|x|^{1-d}(1+|\log|x||)\leqq c_0|x|^{\delta-d},\end{aligned}\tag{3.52 a}$$

with $0<\delta<1$, and similarly, in case $d=1$,

$$|L_3(x)|\leqq c_1(1+|\log|x||)^2\leqq c_0|x|^{\delta-1},\tag{3.52 b}$$

with $0<\delta<1$. Thus (3.50), (3.51 *a*, *b*) and (3.52 *a*, *b*) with (3.49) yield (3.48).

Next we show (3.40 *a*, *b*). We give a unified proof using the theory of singular integrals, though it is seen that (3.47) and (3.48) imply (3.40 *a*).

*Proof of* (3.40 *a*). — For $\varepsilon\geqq 0$, we have

$$\begin{aligned}\psi(x)(F_{R,\varepsilon}u)(x)&=\psi(x)(F_{R,\varepsilon}(\varphi u))(x)\\&=i\psi(x)\sum_{k=1}^{d}\int_{0<|y'|<1}(f_{k,\varepsilon}(x-y')-f_{k,\varepsilon}(x))\,y'_k\,n(dy')\\&=i\psi(x)\sum_{k=1}^{d}\lim_{\delta\downarrow 0}\int_{\delta\leqq|y'|<1}(f_{k,\varepsilon}(x-y')-f_{k,\varepsilon}(x))\,y'_k\,n(y')\,dy'\\&=i\psi(x)\sum_{k=1}^{d}\lim_{\delta\downarrow 0}N_{k,\delta}(f_{k,\varepsilon}),\end{aligned}\tag{3.53}$$

with $N_{k,\delta}$ in (2.22) with $\delta$ in place of $\varepsilon$, because $n(y')$ is rotation-invariant. Here

$$f_{k,\varepsilon}(x)=(G_{R,\varepsilon}(\varphi A_k u))(x)=\int G_{R,\varepsilon}(x-y)\,\varphi(y)\,A_k(y)\,u(y)\,dy,\tag{3.54}$$

which is a $C_0^\infty$ function. Then it follows from (3.53) with (2.23) that for $\varepsilon \geqq 0$,

$$\|\psi F_{R,\varepsilon} u\| = \|\psi F_{R,\varepsilon}(\varphi u)\| \leqq \|\psi\|_\infty \|F_{R,\varepsilon}(\varphi u)\| = C_\psi \lim_{\delta \downarrow 0} \|\sum_{k=1}^{d} N_{k,\delta}(f_{k,\varepsilon})\|$$
$$\leqq C_\psi C_0 \sum_{k=1}^{d} \|f_{k,\varepsilon}\| = C_\psi C_0 \sum_{k=1}^{d} \|G_{R,\varepsilon}(\varphi A_k u)\|$$
$$\leqq C_\psi C_0 C \sum_{k=1}^{d} \|\varphi A_k u\| \leqq C_\psi C_0 C a(K_{R+2}) \|u\|,$$

with $C_\psi = \|\psi\|_\infty$, $C_0$ a constant independent of $0 < \delta < 1$ and $m \geqq 0$, and $a(K_{R+2})$ in (2.5 $a$) with $K_{R+2}$ in place of K. Here we have used, in the second last inequality, Lemma 3.1 (ii). This shows (3.40 $a$).

*Proof of* (3.40 $b$). – The proof proceeds in the same way as above. We have, this time for $\varepsilon > 0$,

$$\|\partial_j(\psi F_{R,\varepsilon} u)\| = \|\partial_j(\psi F_{R,\varepsilon}(\varphi u))\|$$
$$\leqq \|\partial_j \psi\|_\infty \|F_{R,\varepsilon}(\varphi u)\| + \|\psi\|_\infty \|\partial_j(F_{R,\varepsilon}(\varphi u))\|.$$

Here $F_{R,\varepsilon}(\varphi u)$ in the first term of the third member above has already been dealt with in the proof of (3.40 $a$). For the second term, since $\partial_j N_{k,\delta}(f_{k,\varepsilon}) = N_{k,\delta}(\partial_j f_{k,\varepsilon})$ and the $L^2$-limits of $N_{k,\delta}(f_{k,\varepsilon})$ and $N_{k,\delta}(\partial_j f_{k,\varepsilon})$ for $\delta \downarrow 0$ exist, we have

$$\|\partial_j(F_{R,\varepsilon}(\varphi u))\| = \lim_{\delta \downarrow 0} \|\sum_{k=1}^{d} N_{k,\delta}(\partial_j f_{k,\varepsilon})\|$$
$$\leqq C_0 \sum_{k=1}^{d} \|\partial_j f_{k,\varepsilon}\| = C_0 \sum_{k=1}^{d} \|\partial_j(G_{R,\varepsilon}(\varphi A_k u))\|$$
$$\leqq C_0 C \sum_{k=1}^{d} \|\varphi A_k u\| \leqq C_0 C a(K_{R+2}) \|u\|.$$

This shows (3.40 $b$).

(ii) *Proof of* (3.41 $a$). We have for $\varepsilon > 0$,

$$\|\psi F_{R,\varepsilon} u - \psi F_R u\| = \|\psi F_{R,\varepsilon}(\varphi u) - \psi F_R(\varphi u)\|$$
$$\leqq \|\psi\|_\infty \|F_{R,\varepsilon}(\varphi u) - F_R(\varphi u)\|$$
$$= C_\psi \lim_{\delta \downarrow 0} \|\sum_{k=1}^{d} N_{k,\delta}(f_{k,\varepsilon} - f_{k,0})\|$$
$$\leqq C_\psi C_0 \sum_{k=1}^{d} \|f_{k,\varepsilon} - f_{k,0}\|$$
$$= C_\psi C_0 \sum_{k=1}^{d} \|G_{R,\varepsilon}(\varphi A_k u) - G_R(\varphi A_k u)\|,$$

with $C_\psi = \|\psi\|_\infty$, which tends to zero as $\varepsilon \downarrow 0$, uniformly for bounded $m \geqq 0$, by Lemma 3.1 (iii). This shows (3.41 *a*).

*Proof of* (3.41 *b*). – The proof is similar to the above. In fact, we have for $\varepsilon, \varepsilon' > 0$,

$$\begin{aligned} \|\partial_j(\psi F_{R,\varepsilon} u) - \partial_j(\psi F_{R,\varepsilon'} u)\| &= \|\partial_j(\psi F_{R,\varepsilon}(\varphi u)) - \partial_j(\psi F_{R,\varepsilon'}(\varphi u))\| \\ &\leqq \|\partial_j \psi\|_\infty \|F_{R,\varepsilon}(\varphi u) - F_{R,\varepsilon'}(\varphi u)\| \\ &\quad + \|\psi\|_\infty \|\partial_j F_{R,\varepsilon}(\varphi u) - \partial_j F_{R,\varepsilon'}(\varphi u)\|. \end{aligned}$$

As $\varepsilon$ and $\varepsilon'$ go to zero, the first term in the last member tends to zero, uniformly for bounded $m \geqq 0$, because $F_{R,\varepsilon}(\varphi u) \to F_R(\varphi u)$, $\varepsilon \downarrow 0$, as seen above. The second term tends to zero, too. In fact, we have

$$\begin{aligned} \|\partial_j F_{R,\varepsilon} u - \partial_j F_{R,\varepsilon'} u\| &= \lim_{\delta \downarrow 0} \Big\| \sum_{k=1}^{d} N_{k,\delta}(\partial_j f_{k,\varepsilon} - \partial_j f_{k,\varepsilon'}) \Big\| \\ &\leqq C_0 \sum_{k=1}^{d} \|\partial_j f_{k,\varepsilon} - \partial_j f_{k,\varepsilon'}\| \\ &= C_0 \sum_{k=1}^{d} \|\partial_j (G_{R,\varepsilon}(\varphi A_k u)) - \partial_j (G_{R,\varepsilon'}(\varphi A_k u))\|, \end{aligned}$$

which tends to zero, as $\varepsilon, \varepsilon' \downarrow 0$, uniformly for bounded $m \geqq 0$, by Lemma 3.1 (iii), again. This prove (3.41 *b*), ending the proof of Lemma 3.5.

## 4. KATO'S INEQUALITY

For the nonrelativistic quantum Hamiltonian, *i.e.* the nonrelativistic Schrödinger operator, with magnetic fields, Kato [16] established a distributional inequality, which is now called Kato's inequality. In this section we are going to show an analogue for our relativistic quantum Hamiltonian $H_A^m$ with magnetic fields. *The vector potential* $A(x)$ *is assumed to satisfy* (2.5) *or* (1.4) as in the previous sections.

In [12], §4.4, we established, though assuming a sufficient regularity and boundedness of $A(x)$, a path integral representation for the semigroup $\exp[-t(H_A^m - m)]$ to derive a domination relation between two positive selfadjoint semigroups $\exp[-t(H_A^m - m)]$ and $\exp[-t(H_0^m - m)]$:

$$(f, \exp[-t(H_A^m - m)]f) \leqq (|f|, \exp[-t(H_0^m - m)]\,|f|),\ t \geqq 0, \qquad f \in L^2(\mathbb{R}^d),$$

which is equivalent to an abstract operator version of Kato's inequality (see [23], [7]). The distributional inequality is, however, a stronger statement in the sense that we are before knowing $H_A^m$ to be essentially selfadjoint or selfadjoint. The following theorem may be considered as Kato's inequality for the relativistic quantum Hamiltonian $H_A^m$ in (1.5).

THEOREM 4.1. — *Assume* $A(x)$ *satisfies* (2.5) *or* (1.4). *Let* $m \geqq 0$. *If* $v$ *is in* $L^2(\mathbb{R}^d)$ *with* $H_A^m v$ *in* $L_{loc}^1(\mathbb{R}^d)$, *then the following distributional inequality holds*

$$\operatorname{Re}[(\operatorname{sgn} v) H_A^m v] \geqq H_0^m |v| = \sqrt{-\Delta + m^2}\,|v|. \tag{4.1}$$

*Here* sgn $v$ *is a bounded function in* $\mathbb{R}^d$ *defined by*

$$(\operatorname{sgn} v)(x) = \begin{cases} \overline{v(x)}/|v(x)|, & \text{if } v(x) \neq 0, \\ 0, & \text{if } v(x) = 0. \end{cases}$$

Thanks to the expression (1.5) or (2.8) for $H_A^m$, the proof of Theorem 4.1 proceeds in a similar way to that of the original Kato's inequality [16]. We need the following lemma, whose proof needs Theorem 3.6. We write $f^\delta = \rho_\delta * f$ for a function $f \in L_{loc}^1$, where $\rho_\delta(x) = \delta^{-d} \rho(x/\delta)$, and $\rho(x)$ is a nonnegative $C_0^\infty$ function with supp $\rho \subseteq \{|x| \leqq 1\}$ and $\int \rho(x)\,dx = 1$.

LEMMA 4.2. — *Let* $v \in L^2(\mathbb{R}^d)$ *and let* $H_A^m v \in L_{loc}^1$. *Then* $v^\delta \to v$ *in* $L^2$ *and* $H_A^m v^\delta \to H_A^m v$ *in* $L_{loc}^1$ *as* $\delta \downarrow 0$.

*Proof.* — We simply write $H_A$, $H_0$ and $n(dy)$ for $H_A^m$, $H_0^m$ and $n^m(dy)$. Let $v \in L^2(\mathbb{R}^d)$ and let $H_A v \in L_{loc}^1(\mathbb{R}^d)$. Since $v^\delta$ is $C^\infty$ and $L^2$, we know by Lemma 2.3 that $H_A v^\delta$ is $L_{loc}^2$ and so $L_{loc}^1$. It is well-known that $v^\delta \to v$ in $L^2$ as $\delta \downarrow 0$. It suffices to show that $H_A v^\delta \to H_A v$ in $L_{loc}^1$. Then, with the decomposition (2.13 *a*) of $H_A$, $H_A = m + I_1 + I_2$, the assumption implies that $v \in L^2$ and $I_2 v \in L_{loc}^1$, because $I_1 v \in L^2$ by (2.14 *a*). Since $I_1$ is, by (2.14 *a*) again, a bounded linear operator on $L^2(\mathbb{R}^d)$, so that $I_1 v^\delta \to I_1 v$ in $L^2$, we have only to show that

$$I_2 v^\delta \to I_2 v \quad \text{in } L_{loc}^1, \qquad \delta \downarrow 0, \tag{4.2}$$

or equivalently,

$$I_2 v^\delta \to I_2 v \quad \text{in } \mathscr{D}', \qquad \delta \downarrow 0, \tag{4.3 a}$$

$$I_2 v^\delta - I_2 v^{\delta'} \to 0 \quad \text{in } L_{loc}^1, \qquad \delta, \delta' \downarrow 0. \tag{4.3 b}$$

It is easy to verify (4.3 *a*), since $v^\delta \to v$ in $L^2$ and $(I_2 v^\delta, \varphi) = (v^\delta, I_2 \varphi)$, $(I_2 v, \varphi) = (v, I_2 \varphi)$, for $\varphi \in C_0^\infty(\mathbb{R}^d)$, by the rotational invariance of the Lévy measure $n(dy)$. To see (4.3 *b*) let K be an arbitrary compact subset of $\mathbb{R}^d$. We show first that there exists a constant $C_K$ [dependent on K and the behavior of $A(x)$ in a neighbourhood of K] such that

$$\|I_2 u\|_{1,K} \leqq C_K [\|H_0 u\|_i + \|u\|_2], \qquad i = 1, 2, \tag{4.4}$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ with supp $u \subseteq K_4$. Recall that $u \in C_0^\infty(\mathbb{R}^d)$ implies $H_0 u \in L^i(\mathbb{R}^d)$, by Lemma 2.3 with $A(x) \equiv 0$. Here $\|\cdot\|_i$ stands for the $L^i$ norm, $i = 1, 2$. The following argument proceeds similarly to the proof of Lemma 2.3 (i). Let $\varphi$ be a $C_0^\infty$ function with $0 \leqq \varphi(x) \leqq 1$ in $\mathbb{R}^d$, $\varphi(x) = 1$ on $K_4$ and supp $\varphi \subseteq K_5$. Then from (2.18) we have by Fatou's lemma, this time, in the local $L^1$ norm,

$$\| I_2 u \|_{1,K} \leqq \sum_{j=1}^{2} \liminf_{\varepsilon \downarrow 0} \| i_1(\varepsilon) \|_{1,K} + \| i_3 \|_{1,K}. \tag{4.5}$$

For $i_1(\varepsilon)$ and $i_2(\varepsilon)$ we obtain from (2.20) and (2.21) with the Schwarz inequality

$$\| i_1(\varepsilon) \|_{1,K} \leqq | K |^{1/2} \| i_1(\varepsilon) \|_{2,K} \leqq c_1 \| u \|_2, \tag{4.6a}$$
$$\| i_2(\varepsilon) \|_{1,K} \leqq | K |^{1/2} \| i_2(\varepsilon) \|_{2,K} \leqq c_2 \| u \|_2, \tag{4.6b}$$

where $c_1$ and $c_2$ are constants independent of $\varepsilon > 0$. For $i_3$ we see from (2.25) that if $H_0 u$ is $L^1$, then

$$\begin{aligned} \| i_3 \|_{1,K} \leqq \| i_3 \|_1 \leqq \| H_0(\varphi u) \|_1 + (m+2n_\infty) \| \varphi u \|_1, \\ \leqq \| H_0 u \|_1 + | K_4 |^{1/2} (m+2n_\infty) \| u \|_2, \end{aligned} \tag{4.6c}$$

while if $H_0\, u$ is $L^2$, then by (2.26)

$$\begin{aligned} \| i_3 \|_{1,K} \leqq | K |^{1/2} \| i_3 \|_{2,K} \leqq | K |^{1/2} \| i_3 \|_2 \\ \leqq | K |^{1/2} [\| H_0 u \|_2 + (m+2n_\infty) \| u \|_2], \end{aligned} \tag{4.6d}$$

with the constant $n_\infty \equiv n_\infty^m$ in (2.6a) and $|K|$ the volume of K. Then (4.4) follows from (4.6a, b, c, d) with (4.5).

Now we shall finish the proof of (4.3b). Let $\psi$ be a $C_0^\infty$ function with $0 \leqq \psi(x) \leqq 1$ in $\mathbb{R}^d$, $\psi(x)=1$ on $K_2$ and $\operatorname{supp}\psi \subseteq K_3$. It is here that we need Theorem 3.6, by which $v = v_1 + v_2$ where $\psi v_1$, $H_0(\psi v_1) \in L^1$ and $\psi v_2$, $H_0(\psi v_2) \in L^2$. As $\psi v$ and $\psi v_2$ are $L^2$, so is $\psi v_1$. Let $0 < \delta$, $\delta' < 1$. Then $(\psi v_1)^\delta$ and $(\psi v_2)^\delta$ are $C_0^\infty$ with support in $K_4$, $H_0(\psi v_1)^\delta = (H_0(\psi v_1))^\delta$ is $L^1$, and $H_0(\psi v_2)^\delta = (H_0(\psi v_2))^\delta$ is $L^2$. Noting $v_i^\delta = (\psi v_i)^\delta$ on $K_1$ and $I_2 v_i^\delta = I_2(\psi v_i)^\delta$ on K, we use (4.4) with $u = (\psi v_i)^\delta - (\psi v_i)^{\delta'}$, $i=1, 2$, to get

$$\begin{aligned} \| I_2 v^\delta - I_2 v^{\delta'} \|_{1,K} &= \| \sum_{i=1}^{2} (I_2(\psi v_i)^\delta - I_2(\psi v_i)^{\delta'} \|_{1,K} \\ &\leqq \sum_{i=1}^{2} \| I_2(\psi v_i)^\delta - I_2(\psi v_i)^{\delta'} \|_{1,K} \\ &\leqq C_K \sum_{i=1}^{2} [\| H_0(\psi v_i)^\delta - H_0(\psi v_i)^{\delta'} \|_i + \| (\psi v_i)^\delta - (\psi v_i)^{\delta'} \|_2] \\ &= C_K \sum_{i=1}^{2} [\| (H_0(\psi v_i))^\delta - (H_0(\psi v_i))^{\delta'} \|_i + \| (\psi v_i)^\delta - (\psi v_i)^{\delta'} \|_2], \end{aligned}$$

which tends to zero as $\delta$, $\delta' \downarrow 0$. This proves (4.3b), completing the proof of Lemma 4.2.

*Proof of Theorem* 4.1. — We write $H_A$, $H_0$ and $n(dy)$ for $H_A^m$, $H_0^m$ and $n^m(dy)$.

I. First suppose $v$ is $C^\infty$ and $L^2$. Then by Lemma 2.2, $H_A v$ is $L^2_{loc}$, and so $L^1_{loc}$. For $\varepsilon>0$ let

$$v_\varepsilon(x)=\sqrt{|v(x)|^2+\varepsilon^2}. \tag{4.7}$$

Then $v_\varepsilon$ is $C^\infty$ and $v_\varepsilon(x)\geqq\varepsilon$. A direct calculation shows that

$$(v_\varepsilon(x)^2-\varepsilon^2)(v_\varepsilon(x+y)^2-\varepsilon^2)\leqq(v_\varepsilon(x)v_\varepsilon(x+y)-\varepsilon^2)^2$$

or

$$|v(x)||v(x+y)|\leqq v_\varepsilon(x)v_\varepsilon(x+y)-\varepsilon^2.$$

Subtracting $|v(x)|^2$ yields

$$-|v(x)||v(x+y)|+|v(x)|^2\geqq -v_\varepsilon(x)v_\varepsilon(x+y)+v_\varepsilon(x)^2. \tag{4.8}$$

Now we use (2.8) to get

$$\begin{aligned}
&2\operatorname{Re}[\overline{v(x)}([H_A-m]v)(x)]=\overline{v(x)}([H_A-m]v)(x)+v(x)\overline{([H_A-m]v)(x)}\\
&=-\int_{|y|>0}(\overline{v(x)}[e^{-iy A(x+y/2)}v(x+y)-v(x)-I_{\{|y|<1\}}y(\partial_x-iA(x))v(x)]\\
&\quad+v(x)[\overline{e^{iy A(x+y/2)}v(x+y)}-\overline{v(x)}-I_{\{|y|<1\}}y(\partial_x+iA(x))\overline{v(x)}])\,n(dy)\\
&\geqq 2\int_{|y|>0}[-|v(x)||v(x+y)|+|v(x)|^2+2^{-1}I_{\{|y|<1\}}y\,\partial|v(x)|^2]\,n(dy).
\end{aligned}$$

Since $\partial v_\varepsilon(x)^2=\partial(|v(x)|^2+\varepsilon^2)=\partial|v(x)|^2$, it is seen with (4.8) that

$$\begin{aligned}
&\operatorname{Re}[\overline{v(x)}([H_A-m]v)(x)]\\
&\geqq\int_{|y|>0}[-v_\varepsilon(x)v_\varepsilon(x+y)+v_\varepsilon(x)^2+2^{-1}I_{\{|y|<1\}}y\,\partial v_\varepsilon(x)^2]\,n(dy)\\
&=v_\varepsilon(x)([H_0-m]v_\varepsilon)(x),
\end{aligned} \tag{4.9}$$

pointwise and so in the distribution sense. It follows that

$$\operatorname{Re}[(\overline{v(x)}/v_\varepsilon(x))[H_A-m]v]\geqq[H_0-m]v_\varepsilon. \tag{4.10}$$

II. In the general case where $v\in L^2$ and $H_A v\in L^1_{loc}$, let $v^\delta=\rho_\delta * v$. Then $v^\delta$ is $C^\infty$ and $L^2$, so that by Lemma 2.3, $H_A v^\delta$ is $L^2_{loc}$ and hence $L^1_{loc}$. Then by (4.10) we have

$$\operatorname{Re}[(\overline{v^\delta}/(v^\delta)_\varepsilon)[H_A-m]v^\delta]\geqq[H_0-m](v^\delta)_\varepsilon, \tag{4.11}$$

for each $\varepsilon>0$ and $\delta>0$. For $\varepsilon>0$ fixed let $\delta\downarrow 0$. Then $v^\delta\to v$ in $L^2$. Taking a subsequence if necessary, we may suppose that $v^\delta\to v$ in $L^2$ as well as a. e. It is easy to see that $|(v^\delta)_\varepsilon-v_\varepsilon|\leqq||v^\delta|-|v||\leqq|v^\delta-v|$, so that $(v^\delta)_\varepsilon\to v_\varepsilon$ in $L^2$ as well as a. e., as $\delta\downarrow 0$. Hence $\overline{v^\delta}/(v^\delta)_\varepsilon\to\bar v/v_\varepsilon$ a. e. and $H_0(v^\delta)_\varepsilon\to H_0 v_\varepsilon$ in $\mathscr{D}'$. On the other hand, we know by Lemma 4.2 that $H_A v^\delta\to H_A v$ in $L^1_{loc}$. Since $|\overline{v^\delta}/(v^\delta)_\varepsilon|\leqq 1$, it follows by the Lebesgue dominated convergence theorem that $(\overline{v^\delta}/(v^\delta)_\varepsilon)[H_A-m]v^\delta\to(\bar v/v_\varepsilon)[H_A-m]v$ in $L^1_{loc}$, as $\delta\downarrow 0$, so that (4.10) holds for $v$. Now let $\varepsilon\downarrow 0$. Then $\bar v/v_\varepsilon\to\operatorname{sgn} v$ a. e. with $|\bar v/v_\varepsilon|\leqq 1$, so that the left-hand side of (4.10) converges to $\operatorname{Re}[(\operatorname{sgn} v)[H_A-m]v]$ a. e., while the right-hand side of (4.10) converges to $[H_0 - m]|v|$ in $\mathscr{D}'$. Hence we get

$$\operatorname{Re}[(\operatorname{sgn} v)[H_A - m] v] \geqq [H_0 - m] |v|$$

and hence (4.1), having proved Theorem 4.1.

*Remark.* — When both $A(x)$ and $v(x)$ are $C^\infty$, Theorem 4.1 follows from that in the case $A(x) \equiv 0$ together with the fact (2.11) with $v$ in place of $u$. The proof of Theorem 4.1 with $A(x) \equiv 0$ is comparatively easy, because then Theorem 3.6 and so Lemma 4.2 is evident.

## 5. ESSENTIAL SELFADJOINTNESS

We now show the essential selfadjointness of the Weyl quantized relativistic Hamiltonian $H^m = H_A^m + \Phi$ in (1.1), with $H_A^m$ in (1.5) or (2.8).

THEOREM 5.1. — *Let* A *be an* $\mathbb{R}^d$*-valued measurable function in* $\mathbb{R}^d$ *satisfying* (2.5) *or* (1.4) *and let* $\Phi$ *be in* $L^2_{\text{loc}}(\mathbb{R}^d)$ *with* $\Phi(x) \geqq 0$ *a. e. Then:* (i) $H^m = H_A^m + \Phi$, *and, in particular,* $H_A^m$, *is essentially selfadjoint on* $C_0^\infty(\mathbb{R}^d)$.

(ii) *The unique selfadjoint extension of* $H_A^m$, *denoted again by the same* $H_A^m$, *is bounded from below by* $m$: $H_A^m \geqq m$.

*Proof.* — We write $H_A$ and $H_0$ for $H_A^m$ and $H_0^m$. (i) We already know $H_A$ is symmetric on $C_0^\infty(\mathbb{R}^d)$ as in (3.1). From the proof of Theorem 4.1, $([H_A - m] u, u) \geqq 0$ for $u \in C_0^\infty(\mathbb{R}^d)$. In fact, integrating (4.9) with $u$ in place of $v$, we have with $u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}$,

$$\operatorname{Re}([H_A - m] u, u) \geqq ([H_0 - m] u_\varepsilon, u_\varepsilon) \geqq 0,$$

because $w\ \Phi\ u_\varepsilon - \varepsilon$ is $C_0^\infty$ and $(H_0 - m) u_\varepsilon = (H_0 - m) w$.

Now we show that $H_A + \Phi$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. Since $H_A + \Phi + 1$ is a strictly positive symmetric operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$, it suffices to show that $v \in L^2(\mathbb{R}^d)$ with

$$(H_A + \Phi + 1)^* v = 0 \tag{5.1}$$

implies that $v = 0$. (5.1) means

$$(H_A + \Phi + 1) v = 0 \tag{5.2}$$

in the sense of distributions. It follows that

$$H_A v = -\Phi v - v$$

is $L^1_{\text{loc}}$, because $\Phi + 1$ is $L^2_{\text{loc}}$ and $v$ is $L^2$. By Theorem 4.1,

$$H_0 |v| \leqq \operatorname{Re}[(\operatorname{sgn} v) H_A v] = -\operatorname{Re}((\operatorname{sgn} v)(\Phi v + v)) = -(\Phi + 1) |v|.$$

Hence $(H_0+1)|v| \leqq 0$ in the distribution sense, or $((H_0+1)|v|, \varphi) \leqq 0$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geqq 0$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi \geqq 0$. Then $(H_0+1)^{-1}\psi$ is in the Sobolev space $H^1(\mathbb{R}^d)$ and nonnegative. Choose a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^d)$ with $\varphi_n \geqq 0$ for all $n$ which converges to $(H_0+1)^{-1}\psi$ as $n \to \infty$. Since $(H_0+1)|v|$ is in the Sobolev space $H^{-1}(\mathbb{R}^d)$, we have

$$0 \leqq (|v|, \psi) = ((H_0+1)|v|, (H_0+1)^{-1}\psi) = \lim_{n \to \infty} ((H_0+1)|v|, \varphi_n) \leqq 0,$$

whence $(|v|, \psi) = 0$. It follows that $|v| = 0$ or $v = 0$.

(ii) As seen in the above proof of (i), $H_A$ is, on $C_0^\infty(\mathbb{R}^d)$, bounded from below by $m$ and essentially selfadjoint. So the assertion is obvious. This proves Theorem 5.1.

ACKNOWLEDGEMENT

The author would like to thank Professor M. Nagase for a useful discussion concerning the estimate (3.29) for the kernel (3.28).

## REFERENCES

[1] F. A. BEREZIN and M. A. ŠUBIN, *Symbols of Operators and Quantization*, Coll. Math. Soc. Janos Bolyai 5, Hilbert Space Operators, 21-52, Tihany, 1970.

[2] I. DAUBECHIES, One-Electron Molecules with Relativistic Kinetic Energy: Properties of the Discrete Spectrum, *Commun. Math. Phys.*, Vol. **94**, 1984, pp. 523-535.

[3] I. DAUBECHIES and E. H. LIEB, One Electron Relativistic Molecules with Coulomb Interaction, *Commun. Math. Phys.*, Vol. **90**, 1983, pp. 497-510.

[4] A. ERDÉLYI, *Higher Transcendental Functions*, Vol. **2**, McGraw-Hill, New York, 1953.

[5] C. FEFFERMAN and R. DE LA LLAVE, Relativistic Stability of Matter-I, *Rev. Mat. Iberoamericana*, Vol. **2**, 1986, pp. 119-213.

[6] I. W. HERBST, Spectral Theory of the Operator $(p^2+m^2)^{1/2}-Ze^2/r$. *Commun. Math. Phys.*, Vol. **53**, 1977, pp. 285-294; Errata, *Ibid.*, Vol. **55**, 1977, p. 316.

[7] H. HESS, R. SCHRADER and D. A. UHLENBROCK, Domination of Semigroups and Generalization of Kato's Inequality, *Duke Math. J.*, Vol. **44**, 1977, pp. 893-904.

[8] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators III*, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.

[9] T. ICHINOSE, The Nonrelativistic Limit Problem for a Relativistic Spinless Particle in an Electromagnetic Field, *J. Functional Analysis*, **73**, 1987, pp. 233-257.

[10] T. ICHINOSE, *Path Integral for a Weyl Quantized Relativistic Hamiltonian and the Nonrelativistic Limit Problem*, in *Differential Equations and Mathematical Physics; Lecture Notes in Mathematics*, Springer, No. **1285**, 1987, pp. 205-210.

[11] T. ICHINOSE, Kato's Inequality and Essential Selfadjointness for the Weyl Quantized Relativistic Hamiltonian, *Proc. Japan Acad.*, **64A**, 1988, pp. 367-369.

[12] T. ICHINOSE and H. TAMURA, Imaginary-Time Path Integral for a Relativistic Spinless Particle in an Electromagnetic Field, *Commun. Math. Phys.*, Vol. **105**, 1986, pp. 239-257; Path Integral for the Weyl Quantized Relativistic Hamiltonian, *Proc. Japan Acad.*, Vol. **62A**, 1986, pp. 91-93.

[13] T. IKEBE and T. KATO, Uniqueness of the Self-Adjoint Extension of Singular Elliptic Differential Operators, *Arch. Rat. Mech. Anal.*, Vol. **9**, 1962, pp. 77-92.

[14] N. IKEDA and S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North-Holland/Kodansha, Amsterdam, Tokyo, 1981.

[15] N. IKEDA and S. WATANABE, On Some Relations Between the Harmonic Measure and the Lévy Measure for a Certain Class of Markov Processes, *J. Math. Kyoto Univ.*, Vol. **2**, 1962, pp. 79-95.

[16] T. KATO, Schrödinger Operators with Singular Potentials, *Israel J. Math.*, Vol. **13**, 1972, pp. 135-148.

[17] H. KUMANO-GO, *Pseudo-Differential Operators*, The M.I.T. Press, Cambridge, Massachusetts, 1981.

[18] M. M. MIZRAHI, The Weyl Correspondence and Path Integrals, *J. Math. Phys.*, Vol. **16**, 1975, pp. 2201-2206.

[19] M. NAGASE, The $L^p$-Boundedness of Pseudo-Differential Operators with Non-Regular Symbols, *Comm. in P.D.E.*, Vol. **2**, 1977, pp. 1045-1061.

[20] M. NAGASE and T. UMEDA, On the Essential Self-Adjointness of Quantum Hamiltonians of Relativistic Particles in Magnetic Fields, *Sci. Rep. Col. Gen. Educ. Osaka Univ.*, Vol. **36**, 1987, pp. 1-6.

[21] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1975.

[22] M. A. SHUBIN, *Pseudodifferential Operators and Spectral Theory*. Springer, Berlin-Heidelberg, 1987.

[23] B. SIMON, An Abstract Kato's Inequality for Generators of Positivity Preserving Semigroups, *Indiana Univ. Math. J.*, Vol. **26**, 1977, pp. 1067-1073.

[24] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[25] R. A. WEDER, Spectral Analysis of Pseudodifferential Operators, *J. Functional Analysis*, **20**, 1975, pp. 319-337.

(*Manuscript received* 21 *septembre* 1988)
(*Version révisée reçue le* 24 *janvier* 1988.)