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Commutator expansions and smoothing properties of generalized Benjamin-Ono equations

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ABSTRACT. — We derive series expansions and remainder estimates for commutators of the type $[D|D|^{2\mu}, h]$ where $D=d/dx$, $\mu \in \mathbb{R}^+$, and h is the operator of multiplication by a smooth function. Those results are instrumental in deriving smoothing properties and existence of solutions for generalized Benjamin-Ono equations (see the equation (1) below).

RÉSUMÉ. — On démontre des développements en série et des estimations de restes pour des commutateurs du type $[D|D|^{2\mu}, h]$ où $D=d/dx$, $\mu \in \mathbb{R}^+$ et h est l'opérateur de multiplication par une fonction régulière. Ces résultats sont utiles pour la démonstration de propriétés de lissage et d'existence de solutions pour des équations de Benjamin-Ono généralisées (*voir* l'équation (1) ci-dessous).

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In a previous paper [4] we studied the Cauchy problem for the generalized Benjamin-Ono equation

$$\partial_t u - \mathrm{L}\, u = \mathrm{D}\mathrm{V}'(u) \tag{1}$$

where u is a real function defined in $1+1$ dimensional space-time, $\mathrm{L} = \mathrm{D}\omega^{2\mu}$, $\mathrm{D} = d/dx$, $\omega = (-\mathrm{D}^2)^{1/2}$, $0 < \mu \leqq 1$ and $\mathrm{V} \in \mathscr{C}1(\mathbb{R}, \mathbb{R})$ with $\mathrm{V}(0) = \mathrm{V}'(0) = 0$. In the special case $\mathrm{V}'(u) = u^2$, the equation (1) reduces to the ordinary Benjamin-Ono equation for $\mu = 1/2$ and to the ordinary Korteweg-de Vries equation for $\mu = 1$. A large amount of work has been devoted to the Cauchy problem for (1) in the case $\mu = 1$ (*see* [6] for a review and [2]-[3] for recent results and for a bibliography), and to a lesser extent to the case $\mu = 1/2$ ([1], [5], [8]). An important role is played in the case $\mu = 1$ by the smoothing property of the equation (1), whereby under suitable circumstances, solutions with initial data $u(0) = u_0$ in the Sobolev space $\mathrm{H}^s \equiv \mathrm{H}^s(\mathbb{R})$ tend to lie not only in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathrm{H}^s)$ but also in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathrm{H}^{s+1}_{\mathrm{loc}})$ ([2], [6], [7]). The basis for that property is the fact that the operator $\mathrm{L} = -\mathrm{D}^3$ in the linear part of the equation (1) tends to produce commutators of a definite sign. In fact, if h is a smooth function,

$$[\mathrm{L}, h] = [-\mathrm{D}^3, h] = -3\,\mathrm{D}h'\mathrm{D} - h''' \tag{2}$$

and if h is non decreasing, the first and (more singular) term in the right hand side is a positive operator. The simplest instance of that property arises for $s = 0$. Proceeding formally and using (2) and integration by parts, one obtains from the equation (1).

$$\begin{aligned}\partial_t \langle u, hu \rangle + 3 \langle \mathrm{D}u, h'\mathrm{D}u \rangle - \langle u, h''' u \rangle &= 2 \langle u, h\,\mathrm{D}\mathrm{V}'(u) \rangle \\ &= -2 \int dx\, h' (u\,\mathrm{V}'(u) - \mathrm{V}(u)) \end{aligned} \tag{3}$$

where $\langle .,. \rangle$ denotes the scalar product in L^2, and by integration

$$\begin{aligned}\langle u, hu \rangle (t) + \int_0^t d\tau \{ 3 \langle \mathrm{D}u, h'\mathrm{D}u \rangle - \langle u, h''' u \rangle \} (\tau) \\ = \langle u_0, hu_0 \rangle - 2 \int_0^t d\tau \int dx\, h' (u\,\mathrm{V}'(u) - \mathrm{V}(u)) (\tau, x). \end{aligned} \tag{4}$$

In particular for $h \geqq 0$, $h' \geqq 0$, h' with compact support, $h'(x) = 1$ for x in some interval J, (4) provides an *a priori* estimate of u in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathrm{H}^1(\mathrm{J}))$ in terms of $\| u_0 \|_2$, where $\| \,.\, \|_2$ denote the L^2-norm, in so far as the integral in the right hand side can be suitably controlled.

In order to extend the available results for the case $\mu = 1$ of the KdV equation to the case of general $\mu > 0$, an important preliminary step consists in extending the almost positivity of commutators expressed by (2) to that case. In [4] we made a first step in that direction by providing a series expansion for the commutator $[\mathrm{L}, h]$ which yields a possible extension of (2). That extension however is sufficient only in the case $0<\mu\leqq 1$ (although the expansion itself is valid for any $\mu\geqq 0$), and does not lend itself to a generalization that would cover the case $\mu>1$. In the present paper, we provide a different expansion of the commutator $[L, h]$ and a set of estimates that make it possible to cover the case of arbitrary $\mu>0$. As a by product, we are able to improve and simplify some of the commutator estimates of [4]. The applications of those results to the Cauchy problem for the equation (1) are only briefly mentioned at the end of this paper and will be described in detail elsewhere.

We shall need somme additional notation. We denote the Fourier transform by

$$(\mathscr{F} u)(\xi)\equiv\hat{u}(\xi)=(2\pi)^{-1/2}\int dx\, u(x)\exp(-ix\xi).$$

We shall use the polar decomposition $D=H\omega$ of D, where H is the Hilbert transform. In Fourier transformed variables, the operators D, ω and H become the operators of multiplication by $i\xi$, by $|\xi|$ and by $i\varepsilon(\xi)=i\xi/|\xi|$ respectively. For any two operators P, Q, we denote the anticommutator by $[P, Q]_+=PQ+QP$, and for any operator P, we denote by $\operatorname{Ad}P$ the map $Q\to[P, Q]$.

The key of the expansion method consists in representing the commutator $[L, h]$ by its integral kernel in Fourier transformed variables. One obtains

$$\mathscr{F}([L, h]u)(\xi)=(2\pi)^{-1/2}\int d\xi'\,\Lambda(\xi, \xi')\,\hat{u}(\xi') \tag{5}$$

with

$$\Lambda(\xi, \xi')=i\hat{h}(\xi-\xi')(\varepsilon|\xi|^{2\mu+1}-\varepsilon'|\xi'|^{2\mu+1}) \tag{6}$$

where $\varepsilon=\varepsilon(\xi)$ and $\varepsilon'=\varepsilon(\xi')$. Let now $|\xi|=|\xi'|e^{2t}$ and $a=2\mu+1$. Then

$$\begin{aligned}\Lambda(\xi, \xi')&=i\hat{h}(\xi-\xi')|\xi\xi'|^{a/2}(\varepsilon e^{at}-\varepsilon' e^{-at})\\&=i\hat{h}(\xi-\xi')|\xi\xi'|^{a/2}((\varepsilon+\varepsilon')\sinh at+(\varepsilon-\varepsilon')\cosh at).\end{aligned} \tag{7}$$

It will be convenient to look for an expansion of Λ in terms of the successive derivatives of h, generated by powers of $\xi-\xi'$. Now

$$\xi-\xi'=|\xi\xi'|^{1/2}(\varepsilon e^{t}-\varepsilon' e^{-t})=|\xi\xi'|^{1/2}((\varepsilon+\varepsilon')\sinh t+(\varepsilon-\varepsilon')\cosh t). \tag{8}$$

It will turn out that the more important region in the subsequent estimates is the region $\varepsilon\varepsilon'=+1$. Comparing (7) and (8) then shows that what is needed is an expansion of $\sinh at$ as a series in $\sinh t$. This however is a standard problem in classical analysis. We only quote the relevant results, and refer to textbooks for the methods and for additional information (*see* for instance [9], Chap. VI and VII). One looks for a function f_1 (and also, for the sake of completeness and although that is not needed here, for a function f_0) such that

$$\left.\begin{aligned}\sinh at &= af_1(\sinh t)\\ \cosh at &= f_0(\sinh t)\end{aligned}\right\} \tag{9}$$

The functions f_0 and f_1 should be even and odd respectively, and normalized to $f_0(0)=1$, $f'_0(0)=0$, $f_1(0)=0$, $f'_1(0)=1$. Since $(\partial_t^2-a^2)\sinh at=(\partial_t^2-a^2)\cosh at=0$ and since $\partial_t^2=(1+z^2)\partial_z^2+z\,\partial_z$ with $z=\sinh t$, the functions f_0 and f_1 should be solutions of the differential equation

$$(1+z^2)f''+zf'-a^2 f=0. \tag{10}$$

Now (10) is a standard equation of hypergeometric type. It has two singular points of regular Fuchs type at $z=\pm i$ (with an analytic solution and a square root type solution) and a singular point at infinity. The functions f_0 and f_1 as specified above exist, are analytic in obvious regions, for instance in the cut plane $\mathbb{C}\setminus\{\pm(i+i\,\mathbb{R}^+)\}$, and have power series expansions at the origin that converge in the unit disk. The coefficients of the expansions are easily determined. With the notation

$$f(z)=\sum_j c_j z^j \tag{11}$$

one finds $c_0=1$, $c_{2j+1}=0$ for all $j\geqq 0$ for f_0 and $c_1=1$, $c_{2j}=0$ for all $j\geqq 0$ for f_1, while in both cases the equation (10) yields the recursion relation

$$(j+1)(j+2)\,c_{j+2}=(a^2-j^2)\,c_j \tag{12}$$

which is solved by

$$c_{2j}=((2j)!)^{-1}\prod_{0\leqq k<j}(a^2-4k^2) \qquad \text{for} \quad f_0, \tag{13}$$

$$c_{2j+1}=((2j+1)!)^{-1}\prod_{0\leqq k<j}(a^2-(2k+1)^2) \qquad \text{for} \quad f_1. \tag{14}$$

In particular f_0 is a polynomial for a an even integer and f_1 is a polynomial for a an odd integer. Those polynomials are closely related to the Tchebyshev polynomials. From their definition, it follows that f_0 and f_1 are related by

$$f_0(z)=(1+z^2)^{1/2}f'_1(z), \qquad f_1(z)=a^{-2}(1+z^2)^{1/2}f'_0(z) \tag{15}$$

but those relations will not be used here. More important for our purposes is the fact that the remainders of arbitrary order in the series (11) can be given a simple integral representation. In fact from (10) (14) we obtain

$$((1+z^2)\partial_z^2+z\,\partial_z-a^2)\Big(f_1(z)-\sum_{0\leqq j\leqq n}c_{2j+1}z^{2j+1}\Big) = (a^2-(2n+1)^2)\,c_{2n+1}\,z^{2n+1} \tag{16}$$

so that $q_n(t)$ defined by

$$q_n(t)=f_1(\sinh t)-\sum_{0\leqq j\leqq n}c_{2j+1}(\sinh t)^{2j+1} \tag{17}$$

satisfies $q_n(0)=q_n'(0)=0$ and

$$(\partial_t^2-a^2)\,q_n(t)=(a^2-(2n+1)^2)\,c_{2n+1}(\sinh t)^{2n+1} \tag{18}$$

which is readily integrated to yield

$$aq_n(t)-(a^2-(2n+1)^2)\,c_{2n+1}\int_0^t d\tau\,(\sinh\tau)^{2n+1}\sinh(a(t-\tau)). \tag{19}$$

A similar result holds for f_0, but will not be needed here.

The relations (17) (19) are the basis for the expansion of the commutator $[L, h]$ and for all subsequent estimates. The commutator expansion is obtained as follows. One substitutes (17) (19) into (9) and then into (7), one uses the identity

$$(\varepsilon+\varepsilon')\,f(\sinh t)=(1+\varepsilon\varepsilon')\,f((\varepsilon\, e^t-\varepsilon'\, e^{-t})/2), \tag{20}$$

which holds for any odd function f, to transform the sum in (17), one changes the integration variable from τ to $\lambda=\tau/t$ in (19) and one reverses the path leading from (5) to (7). One obtains

$$2\,[\mathrm{H}\,\omega^{2\mu+1},\, h]=\mathrm{P}-\mathrm{HPH}+[\mathrm{H},\, \mathrm{Q}]_+ +[\omega^{2\mu+1},\, [\mathrm{H},\, h]]_+ \tag{21}$$

with

$$\mathrm{P}=(2\mu+1)\sum_{0\leqq j\leqq n} c_{2j+1}(-)^j\, 4^{-j}\,\omega^{\mu-j}\, h^{(2j+1)}\,\omega^{\mu-j}, \tag{22}$$

$$\mathrm{Q}=((2\mu+1)^2-(2n+1)^2)\,4^{-(n+1)}\,c_{2n+1}$$
$$\times\int_0^1 d\lambda\,\omega^{(\mu-n)\lambda}\,[\mathrm{Log}\,\omega,\, [\omega^{(2\mu+1)(1-\lambda)},\, (\mathrm{Ad}\,\omega^{\lambda})^{2n+1}\, h]]\,\omega^{(\mu-n)\lambda}. \tag{23}$$

The relations (21) (22) (23) form the appropriate extension of (2) to the case of arbitrary $\mu>0$. For $n\leqq\mu$, all the powers of ω are non negative, and those relations hold in the sense of quadratic forms on $\mathrm{H}^{2\mu+1}\times\mathrm{H}^{2\mu+1}$. The smoothing property of the equation (1) for general μ follows in the same way as in (3) from the contribution of the term $j=0$ in (22) to the right hand side of (21). In fact that term is simply

$$(2\mu+1)\,(\omega^\mu\, h'\,\omega^\mu-\mathrm{H}\,\omega^\mu\, h'\,\mathrm{H}\,\omega^\mu)$$

and its contribution to the diagonal matrix elements of (21) for non negative h' is

$$(2\mu+1)\,(\,\|h'^{1/2}\,\omega^\mu\, u\|_2^2+\|h'^{1/2}\,\mathrm{H}\,\omega^\mu\, u\|_2^2).$$

By the same arguments as in the case $\mu=1$, this yields (among others) the fact that solutions with initial data in L^2 actually have u and $\mathrm{H}u$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R},\, \mathrm{H}^\mu_{\mathrm{loc}})$, provided the other terms in (21) are suitably controlled. The terms with $j\neq 0$ in (22) have the same structure as the term $j=0$, with however lower powers of ω. We shall show elsewhere that they are controlled in a suitable sense by the term $j=0$ and thus do not spoil the argument. There remains the task of controlling the last two terms in (21). We shall now show that the last term and, for a suitable choice of n, also the last but one term are bounded operators in L^2, and we shall give an estimate for the operator norm of their sum. The crux of the argument is an estimate for $q_n(t)$ defined by (19).

LEMMA 1. — *Let n be a non negative integer and let $2n+1 \leqq a \leqq 2n+3$. Then*

$$2a\,|q_n(t)| \leqq (2\,|\sinh t\,|)^a \tag{24}$$

for all $t \in \mathbb{R}$, with equality if $a = 2n+3$. (If $a = 2n+1$ then $q_n(t)$ vasnishes identically).

Proof. — Since $q_n(t)$ is an odd function of t, it suffices to consider the case $t > 0$. We shall show that the function

$$y(t) = (\sinh t)^{-a}\, q_n(t) \tag{25}$$

is non decreasing in t. In fact

$$y'(t) = (a^2 - (2n+1)^2)\, c_{2n+1} (\sinh t)^{-(a+1)}\, \mathrm{Y}(t) \tag{26}$$

where

$$\mathrm{Y}(t) = \int_0^t d\tau\, (\sinh \tau)^{2n+1} \{ \sinh t \cosh (a(t-\tau)) - \cosh t \sinh (a(t-\tau)) \}$$
$$= \int_0^t d\tau\, (\sinh (t-\tau))^{2n+1} \sinh (t - a\tau) \tag{27}$$

by changing τ to $t-\tau$. We now use the following result.

LEMMA 2. — *Let $t > 0$ and $a \in \mathbb{R}$, $b+1 > 0$. Define*

$$\mathrm{Y}_{ab}(t) = \int_0^t d\tau\, (\sinh (t-\tau))^b \sinh (t - a\tau). \tag{28}$$

Then

$$\left.\begin{array}{lll} \mathrm{Y}_{ab}(t) < 0 & \text{if} & a-1 > b+1 > 0 \\ \mathrm{Y}_{ab}(t) = 0 & \text{if} & a-1 = b+1 > 0 \\ \mathrm{Y}_{ab}(t) > 0 & \text{if} & b+1 > a-1 \end{array}\right\} \tag{29}$$

Proof of Lemma 2. — It suffices to consider the case $a > 1$.

$$\mathrm{Y}_{ab}(t) = \int_0^t d\tau\, \{ (\sinh (t-\tau))^{b+1} \cosh (a-1)\tau$$
$$- (\sinh (t-\tau))^b \cosh (t-\tau) \sinh (a-1)\tau \} = \mathrm{A} - \mathrm{B} \tag{30}$$

in obvious notation, with $\mathrm{A} > 0$, $\mathrm{B} > 0$.

On the other hand for $a-1 > 0$, $b+1 > 0$,

$$0 = \int_0^t d\tau \frac{d}{d\tau} \{ (\sinh (t-\tau))^{b+1} \sinh (a-1)\tau \} = (a-1)\mathrm{A} - (b+1)\mathrm{B} \tag{31}$$

and the lemma follows immediately by comparing (30) and (31).

Q.E.D.

End of the proof of Lemma 1. — It follows from Lemma 2 that $y(t)$ is increasing in t for $t>0$, $2n+1<a<2n+3$ and constant for $a=2n+3$. The inequality (24) for general $t>0$ then follows from the fact that it is saturated for $t \to \infty$, since both members are equivalent to exp(at) in that limit.

Q.E.D.

We are now in a position to estimate the last two terms of (21) in operator norm. We denote by $|||\,.\,|||$ the norm of a bounded operator in L^2 and by $\|\,.\,\|_1$ the norm in $L^1(\mathbb{R})$.

PROPOSITION 1. — *Let* $\mu>0$, *let* $n=[\mu]$ (*the integral part of* μ), *and let* Q *be defined by* (23) *with h a smooth function* (*for instance with* h' *in the Schwartz space* $\mathcal{S}$). *Then* $[H, Q]_+$ *and* $[\omega^{2\mu+1}, [H, h]]_+$ *are bounded operators in* L^2 *and satisfy*

$$|||[H, Q]_+ + [\omega^{2\mu+1}, [H, h]]_+ ||| \leqq 2(2\pi)^{-1/2} \|\widehat{\omega^{2\mu} h'}\|_1 \qquad (32)$$

Proof. — We estimate the integral kernels of the operators under consideration in Fourier transformed variables. Those kernels have disjoint supports, in the regions $\varepsilon\varepsilon'=1$ and $\varepsilon\varepsilon'=-1$ respectively. By the same argument as that leading from (5) to (7), the integral kernel of the sum is

$$(2\pi)^{-1/2}\, i\hat{h}(\xi-\xi')\,|\xi\xi'|^{a/2}\,((\varepsilon+\varepsilon')\,2\,a\,q_n(t)+(\varepsilon-\varepsilon')\,2\cosh at) \qquad (33)$$

Now for all $a \geqq 1$ and all $t \in \mathbb{R}$

$$2\cosh at \leqq (2\cosh t)^a \qquad (34)$$

since

$$(d/dt)(\cosh at)(\cosh t)^{-a} = a(\cosh t)^{-(a+1)}\sinh(a-1)\,t \qquad (35)$$

has the sign of t and (34) is satisfied for $t=0$ and saturated for $t \to \pm\infty$. From (34) and from Lemma 1 it follows that (33) is estimated by

$$\begin{aligned}
|\,.\,| &\leqq (2\pi)^{-1/2}\,|\hat{h}(\xi-\xi')|\,|\xi\xi'|^{a/2} \\
&\qquad \times (|\varepsilon+\varepsilon'|\,|2\sinh t|^a + |\varepsilon-\varepsilon'|\,(2\cosh t)^a) \\
&= 2(2\pi)^{-1/2}\,|\hat{h}(\xi-\xi')|\,|\xi\xi'|^{a/2}\,|\varepsilon e^t - \varepsilon' e^{-t}|^a \\
&= 2(2\pi)^{-1/2}\,|\hat{h}(\xi-\xi')|\,|\xi-\xi'|^a
\end{aligned} \qquad (36)$$

from which (32) follows through the Young inequality.

Q.E.D.

We now compare the results of this paper with those of Section 2 of [4]. The representation (10) of [4] should be compared with the special case $n=0$ of (21) (22) (23), which can be slightly rearranged to yield

$$\begin{aligned}
R_\mu(h) &= [H\,\omega^{2\mu+1}, h] - (2\mu+1)\,\omega^\mu\,[H\,\omega, h]\,\omega^\mu \\
&= (1/2)\,[H, Q]_+ + (1/2)\,\{[\omega^{2\mu+1}, [H, h]]_+ \\
&\qquad - (2\mu+1)\,\omega^\mu\,[\omega, [H, h]]_+\,\omega^\mu\}
\end{aligned} \qquad (37)$$

with

$$Q = \mu(\mu+1)\int_0^1 d\lambda\,\omega^{\mu\lambda}\,[\mathrm{Log}\,\omega, [\omega^{(2\mu+1)(1-\lambda)}, [\omega^\lambda, h]]]\,\omega^{\mu\lambda}. \qquad (38)$$

Equivalently, the integral kernel of $R_\mu(h)$ in Fourier transformed variables is

$$(2\pi)^{-1/2}\, i\hat{h}(\xi-\xi')\{\varepsilon|\xi|^{2\mu+1}-\varepsilon'|\xi'|^{2\mu+1}-(2\mu+1)|\xi\xi'|^{\mu}(\xi-\xi')\}$$
$$=(2\pi)^{-1/2}\, i\hat{h}(\xi-\xi')|\xi\xi'|^{\mu+1/2}$$
$$\times\{(\varepsilon+\varepsilon')4\mu(\mu+1)\int_0^t d\tau \sinh\tau \sinh((2\mu+1)(t-\tau))$$
$$+(\varepsilon-\varepsilon')(\cosh(2\mu+1)t-(2\mu+1)\cosh t)\} \quad (39)$$

If μ is rational and especially if μ is a simple fraction, the expansion (10) of [4] is simpler that (37) (38), as is clear for instance in the special case $\mu=1/2$ (*see* (15) of [4]). On the other hand the expansion (37) (38) has the advantage of being insensitive to the arithmetic properties of μ and to admit a generalisation to higher orders (namely to arbitrary $n\geqq 0$). The proof of Proposition 1 can be easily extended to the case of $R_\mu(h)$, which contains the additional term with $(\varepsilon-\varepsilon')(2\mu+1)\cosh t$ in (39) (see the proof of Lemma 3 below for the treatment of that term), and for $0\leqq\mu\leqq 1$ yields

$$|||R_\mu(h)|||\leqq(2\pi)^{-1/2}\|\widehat{\omega^{2\mu}h'}\|_1 \quad (40)$$

which is similar to (12) of [4], but with a better (actually optimal) constant. Interestingly enough however, for $0\leqq\mu\leqq 1$, the estimate (40) can be derived directly from the definition of $R_\mu(h)$ without using the expansions (10) of [4] or (37) (38) above, as we now show. In fact, the integral kernel of $R_\mu(h)$ is almost by definition

$$(2\pi)^{-1/2}\, i\hat{h}(\xi-\xi')|\xi\xi'|^{\mu+1/2}\{(\varepsilon+\varepsilon')(\sinh(2\mu+1)t-(2\mu+1)\sinh t)$$
$$+(\varepsilon-\varepsilon')(\cosh(2\mu+1)t-(2\mu+1)\cosh t)\} \quad (41)$$

and the estimate (40) follows directly from the next Lemma.

LEMMA 3. — *The following inequalities hold:*

$$2|\sinh(2\mu+1)t-(2\mu+1)\sinh t|\leqq(2|\sinh t|)^{2\mu+1} \quad (42)$$

for $0\leqq\mu\leqq 1$ *and all* $t\in\mathbb{R}$, *and*

$$-4\mu(\cosh t)^{2\mu+1}\leqq 2(\cosh(2\mu+1)t-(2\mu+1)\cosh t)\leqq(2\cosh t)^{2\mu+1} \quad (43)$$

for all $\mu\geqq 0$ *and all* $t\in\mathbb{R}$.

Proof. — The inequality (42) is the special case $n=0$ of (24). We give here a more direct proof for that case. It is sufficient to consider the case $t>0$. We define

$$y(t)=(\sinh t)^{-(2\mu+1)}(\sinh(2\mu+1)t-(2\mu+1)\sinh t). \quad (44)$$

Then $y(t)\geqq 0$ and

$$y'(t)=(2\mu+1)(\sinh t)^{-2(\mu+1)}(\mu\sinh 2t-\sinh 2\mu t)\geqq 0$$

for $t \geqq 0$ and $\mu \leqq 1$, so that $y(t)$ is increasing in t for $t \geqq 0$. The inequality (42) for general $t \geqq 0$ then follows from the positivity of $y(t)$ and from the fact that it is saturated for $t \to \infty$.

The inequality (43) follows similarly from the fact that

$$(d/dt)(\cosh t)^{-(2\mu+1)}(\cosh(2\mu+1)t-(2\mu+1)\cosh t) \\ =(2\mu+1)(\cosh t)^{-2(\mu+1)}(\mu \sinh 2t+\sinh 2\mu t) \quad (45)$$

has the sign of t and from the fact that (43) is saturated on the left for $t=0$ and on the right for $t \to \pm\infty$.

Q.E.D.

With the expansion (21) (22) (23) and the estimate (32) available, the existence results of solutions and the smoothing properties of the equation (1) stated in Propositions 3.1 and 4.1 of [4] extend in a straighforward fashion from the special case $0<\mu \leqq 1$ to the general case $\mu>0$. The complete statements and proofs will be presented elsewhere.

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