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RAGE theorem for power bounded operators and decay of local energy for moving obstacles

by

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ABSTRACT. — We prove a RAGE type theorem for power bounded operators. This theorem enables us to obtain a local energy decay of the solutions of the wave equation in the exterior of a periodically moving non-trapping obstacle provided that the global energy is bounded. We study also the spectral properties of the monodromy operator in the case that the global energy is not bounded. For Dirichlet and Robin boundary problems for moving obstacles we establish the existence of the scattering operator assuming a local energy decay and a boundedness of the global energy. We treat simultaneously both cases of odd and even space dimension.

RÉSUMÉ. — On prouve un théorème de type RAGE pour des opérateurs à puissances bornées. Cet théorème permet d'obtenir une décroissance de l'énergie locale pour des solutions de l'équation des ondes dans l'extérieur d'un obstacle non captif se déplaçant périodiquement sous la condition que l'énergie globale reste bornée. On examine aussi les propriétés spectrales de l'opérateur de monodromie dans le cas où l'énergie globale n'est pas bornée. Pour Dirichlet et Robin problèmes pour des obstacles mouvants on montre l'existence de l'opérateur de diffusion sous l'hypothèse que l'énergie locale décroît et que l'énergie globale est bornée. On traite à la fois les deux cas de dimension d'espace impaire et paire.

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1. INTRODUCTION

The decay of the local energy of the solutions of the wave equation plays a crucial role in scattering theory. For Dirichlet and Neumann problems in the exteriour of a stationary obstacle the decay of the local energy has been treated by many authors (*see* [11], [12], [16], [14], [24] and the references given in these works). In particular, for stationary non-trapping obstacles we obtain a uniform rate of the decay provided that the initial data have compact support ([16], [14], [24]). For the examination of the decay of the local energy different approaches have been proposed. The progress in the analysis of the propagation of singularities for smooth domains with arbitrary geometry made by Melrose and Sjöstrand [15] (*see* also [10]) enables us to deduce the compactness of the Lax-Phillips operator $Z^a(t)=P^a_+ U(t) P^a_-$ for large $t>0$ provided the space dimension is odd and the obstacle non-trapping. Here $P^a_\pm$ are the orthogonal projections on the orthogonal complement of the Lax-Phillips spaces $D^a_\pm$ [11], while $U(t)$ is the unitary group related to the boundary problem. The non-trapping obstacles are defined by using the notion of generalized bicharacteristics (rays) of the wave operator. Rellich's theorem concerning the eigenvalues of the Laplace operator $-\Delta$ in the exteriour of a bounded domain implies that $Z^a(t)$ has no eigenvalues on the unit circle S^1. This leads immediatly to a decay of local energy.

In a series of papers [2], [3], [4], [5], [6] Cooper and Strauss examined the decay of the local energy for moving obstacle which can change its form and place with a speed less than the speed of the propagations of the solutions of the wave equation. Especially, in [4] they treated the case of odd space dimension and non-trapping obstacles. The latter condition implies the compactness of the local evolution operator $Z^a(t, 0)=P^a_+ U(t, 0) P^a$ for large $t>0$. Here $U(t, s)$ is the propagator related to the boundary problem. Cooper and Strauss obtained in [4] a decay of the local energy for initial data $f\in(D^a_-)^\perp$ provided that f satisfies a finite number of conditions. Since the space D^a_- is infinite dimensional, it is important to study the case $f\in D^a_-$. Moreover, for the composition of the wave operators W_- and W, needed for the existence of the scattering operator S (*see* section 5), we wish to determine the maximal space of initial data f for which we have a decay of local energy.

In this paper we study the above problem simultaneously for odd and even space dimensions. Our idea is to apply a RAGE type theorem for power bounded operators. A RAGE theorem for contraction semigroups has been proved by Simon [22]. For unitary groups this theorem is connected with the names of Ruelle, Amrein, Georgescu and Enss, while the abbreviation RAGE has been introduced in [21].

A linear operator V in a Hilbert space H is called power bounded if

$$\sup_{m \in \mathbb{N}} \| V^m \| < \infty, \tag{1.1}$$

$\| V \|$ being the operator norm in H. In section 2 we obtain a RAGE theorem for power bounded operators. In this theorem the linear space H_b spanned by the eigenvectors of the adjoint operator V^* with eigenvalues on S^1 plays an essential role.

Theorem 2.4 has been applied in [1] for the examination of the wave equation with time periodic potentials. Here we study obstacles $K(t)$ periodically moving with period $T>0$ and we take $V=U(T, 0)$. The operator V is power bounded if the global energy remains bounded as $t \to \infty$. This is expressed in our assumption (H_4) introduced in section 4. In sections 3 and 4 we study non-trapping moving obstacles determined by using the generalized geodesics of the wave operator. For such obstacles we show that the operator $\varphi U(t, 0) P^a_-$ is compact for large t, φ being a fixed smooth function with compact support. This result holds for odd and even space dimensions n and it is a natural generalization of the result of Melrose [14] for stationary non-trapping obstacles.

The crucial point in the proof of the decay of the local energy is the relation

$$\lim_{N \to \infty} \sum_{k=0}^{N-1} \| \varphi U(kT, 0) P^a_- f \|^2 = 0 \tag{1.2}$$

established for all $f \in H_b^\perp$. This relation implies the assumption $(LD)_+$ which guarantees the existence of the wave operator Wg for all $g \in H_b^\perp$ (*see* section 5). The proof of (1.2) is based on Theorem 2.4. For n odd a second application of Theorem 2.4 shows that the operator $Z^a(T, 0)$ has no eigenvalues $\lambda \in S^1$. This step replaces the argument related to Rellich's theorem for the Laplacian mentioned above. Thus for periodically moving non-trapping obstacles we obtain an exponential decay of the local energy for data $f \in H_b^\perp$ with compact support, provided (H_4) fulfilled. Moreover, $D^a_- \subset H_b^\perp$ and for n odd H_b is finite dimensional. Next, assuming (H_4), we prove that $H_b^\perp$ is the maximal space of data for which we have a decay of local energy. Recently, one of the authors investigated the same problem without assuming (H_4) (*see* [9]). He characterized the maximal space H_{sc} such that for all $f \in H_{sc}$ the energy of $U(t, 0) f$ is bounded as $t \to \infty$ and the local energy decreases for $t \to \infty$. The results in [9] are proved for the wave equation with a time periodic potential but with trivial modifications they hold for periodically moving non-trapping obstacles.

In section 4 we study the spectrum of V and the behaviour of the global energy for non-trapping periodically moving obstacles in the case that (H_4) is not fulfilled. Our results are similar to those for time periodic potentials given in [1]. The above two problems are open for trapping obstacles. In particular, it is interesting to examine the spectrum of V in the domain $\{ z \in \mathbb{C} : |z| > 1 \}$. On the other hand, for some class of trapping obstacles Popov and Rangelov established an exponential growth of the local energy [20].

In section 5 we discuss the existence of the wave operators W_- and W. We prove the existence of Wg provided assumptions (H_1), (H_2), (H_3) and $(LD)_+$ are fulfilled. For n even we exploit the local energy decay of the solutions of the Cauchy problem. This idea has been used previously in [1] and [19]. In particular, for periodically moving obstacles we obtain the inclusion

$$\overline{\operatorname{Ran} W_-} \subset H_b^{\perp}.$$

This enables us to compose W_- and W since Wg exists for all $g \in H_b^{\perp}$. On the other hand, the assumption (H_4) is necessary for the existence of W.

Finally, in section 6 we show that our techniques can be applied for Neumann and Robin boundary value problems. Thus we obtain a unified approach to different boundary problems which works for odd and even space dimensions. In particular, we can treat the wave equation with dissipative boundary condition in the exteriour of a stationary obstacle (*see* [13]).

A part of our results has been announced in [8], [17].

2. RAGE THEOREM FOR POWER BOUNDED OPERATORS

Let H be a Hilbert space with inner product (,) and norm $\| . \|$. We use the notation $\| . \|$ also for the norm of bounded operators in H. By V^* we denote the adjoint operator to V. First we shall prove that every power bounded operator is related to a partial isometry.

THEOREM 2.1. — *Let* V *be a power bounded operator in* H *and let*

$$K = \{ f \in H : \lim_{m \to \infty} V^m f = 0 \}.$$

Then there exists a self-adjoint bounded non-negative linear operator $A: H \to H$ *and a partial isometry* $U: H \to H$ *so that*

$$(a)\, A = V^* A V,$$
$$(b)\, \operatorname{Ker} A = K,$$
$$(c)\, UB = BV \quad \textit{with} \quad B = A^{1/2}.$$

Proof. — Consider the linear space $\mathscr{L}$ of all bilinear Hermitian forms $f : H \times H \to \mathbf{C}$ such that

$$f(x, y) = \overline{f(y, x)}.$$

We endow $\mathscr{L}$ with the weak topology having as a subbase the sets

$$\mathcal{O}_{x,y,g,\varepsilon}=\{f\in\mathscr{L}: |f(x,y)-g(x,y)|<\varepsilon\},$$

where $\varepsilon>0$, x, $y\in\mathrm{H}$ and $g\in\mathscr{L}$. Consider the set

$$\mathscr{R}=\{f\in\mathscr{L}: f(x,y)=(\mathrm{V}^k x, \mathrm{V}^k y)\ \text{for some}\ k\in\mathbb{N}\}$$

and denote by $\mathscr{R}_c$ the closure of the convex hull of $\mathscr{R}$. Clearly, for each $f\in\mathscr{R}_c$ we have $f(x,x)\geqq 0$.

Given an element $f\in\mathscr{L}$ in the form

$$\left.\begin{aligned} f=\sum_{k=0}^{m}\alpha_k f_k,\qquad f_k(x,y)=(\mathrm{V}^k x,\mathrm{V}^k y),\\ \alpha_k\geqq 0,\qquad \sum_{k=0}^{m}\alpha_k=1,\end{aligned}\right\}\tag{2.1}$$

we obtain the estimate

$$|f(x,y)|\leqq \mathrm{C}_0^2\|x\|\,\|y\|,\qquad \mathrm{C}_0=\text{const.}\tag{2.2}$$

This estimate remains valid for all $f\in\mathscr{R}_c$. Notice that the set K is invariant with respect to V. Moreover, it is easy to see that

$$\mathrm{K}=\{f\in\mathrm{H}: \underline{\lim}_{m\to\infty}\|\mathrm{V}^m f\|=0\}.\tag{2.3}$$

Denote by $\mathrm{K}^\perp$ the orthogonal complement of K in H. The representation (2.3) enables us to deduce the following property

$$\left.\begin{aligned}&\text{For every } x\in\mathrm{K}^\perp,\ x\neq 0,\ \text{there exists } \varepsilon_x>0\\ &\text{such that } f(x,x)\geqq\varepsilon_x \text{ for all } f\in\mathscr{R}_c.\end{aligned}\right\}\tag{2.4}$$

Combining the estimate (2.2) with Tychonoff's theorem for the compactness of product spaces, we conclude that $\mathscr{R}_c$ is a compact convex subset of $\mathscr{L}$. Introduce the linear operator

$$\mathscr{R}_c\ni f\to l(f)(x,y)=f(\mathrm{V}x,\mathrm{V}y).$$

It follows easily that $\mathscr{R}_c$ is invariant with respect to l. Thus we can apply Leray-Schauder-Tychonoff fixed point theorem. Let $f_0\in\mathscr{R}_c$ be a fixed point of l, that is

$$l(f_0)=f_0.\tag{2.5}$$

Since $f_0(x,y)$ is a bilinear bounded form, we can find a self-adjoint bounded operator A such that $f_0(x,y)=(\mathrm{A}x,y)$. Then with the aid of (2.5) we get (a).

To prove (b), take $x\in\operatorname{Ker}\mathrm{A}$ and set $x=x_0+r$ with $x_0\in\mathrm{K}$, $r\in\mathrm{K}^\perp$. We have

$$0=(\mathrm{A}x,x)=f_0(x_0,x_0)+2\operatorname{Re}f_0(x_0,r)+f_0(r,r).$$

On the other hand, by (a) we obtain

$$f_0(x_0, r) = (\mathrm{A}\, x_0, r) = (\mathrm{A V}^m x_0, \mathrm{V}^m r) \underset{m \to \infty}{\to} 0.$$

The same argument yields $f_0(x_0, x_0) = 0$. Thus $f_0(r, r) = 0$ and by (2.4) we deduce $r = 0$. Hence $\operatorname{Ker} \mathrm{A} \subset \mathrm{K}$. The converse inclusion is obvious. This completes the proof of (b).

Let $\mathrm{B} = \mathrm{A}^{1/2}$ be the operator determined by the spectral calculus. Clearly, $\mathrm{K} = \operatorname{Ker} \mathrm{A} = \operatorname{Ker} \mathrm{B}$ and $\overline{\operatorname{Ran} \mathrm{B}} \oplus \mathrm{K} = \mathrm{H}$. We define on $\operatorname{Ran} \mathrm{B} \oplus \mathrm{K}$ the operator U by

$$\mathrm{U}(\mathrm{B}x + k) = \mathrm{BV}x \qquad \text{for} \quad x \in \mathrm{H}, \quad k \in \mathrm{K}.$$

The definition is correct since K is invariant with respect to V and $x - y \in \operatorname{Ker} \mathrm{B}$ implies $\mathrm{V}(x - y) \in \operatorname{Ker} \mathrm{B}$. The equalities

$$\| \mathrm{U}(\mathrm{B}x) \|^2 = (\mathrm{BV}x, \mathrm{BV}x) = (\mathrm{A}\,x, x) = \| \mathrm{B}x \|^2$$

show that the operator U is an isometry on $\operatorname{Ran} \mathrm{B}$. Extending U on $\overline{\operatorname{Ran} \mathrm{B}}$ as an isometry, we obtain (c). The proof is complete.

Remark 2.2. — The proof of (a) and (c) is similar to the proof of Lemma XV.6.1 in [7], where the case that

$$\sup_{k \in \mathbb{Z}} \| \mathrm{V}^k \| < \infty$$

has been treated. In this case V is similar to a unitary operator.

Denote by $\sigma_p(\mathrm{L})$ the point spectrum of the operator L. Let F_b (resp. H_b) be the space spanned by the eigenvectors of V (resp. V*) with eigenvalues on the unit circle S^1.

COROLLARY 2.3. — *Let* V *be a power bounded operator in* H *and let* $\lambda \in \mathrm{S}^1$. *Then the following assertions are equivalent:*

(i) $\lambda \notin \sigma_p(\mathrm{V})$,

(ii) $\bar{\lambda} \notin \sigma_p(\mathrm{V}^*)$.

Moreover, if at least one of the spaces F_b, H_b *is finite dimensional then* $\dim \mathrm{F}_b = \dim \mathrm{H}_b$.

Proof. — Since V* is power bounded also, it suffices to show that (ii) implies (i). Assume that (ii) is fulfilled and suppose that there exists a vector $f \neq 0$ such that $\mathrm{V}\, f = \lambda\, f$ with $\lambda \in \mathrm{S}^1$. Let A be the operator related to V by Theorem 2.1. Then $\mathrm{A}\, f = \mathrm{V}^* \mathrm{A V}\, f = \lambda \mathrm{V}^* \mathrm{A}\, f$ and (ii) yields $\mathrm{A}\, f = 0$. This leads to $f = 0$ since $f = \lambda^{-m} \mathrm{V}^m f \to 0$ as $m \to \infty$.

Passing to the second part of the corollary, assume, for example, that $\dim \mathrm{F}_b < \infty$. Let $\{ f_i \}_{i=1, \ldots, \mathrm{N}}$ with $\mathrm{V}\, f_i = \lambda_i\, f_i$, $\lambda_i \in \mathrm{S}^1$, form a basis in F_b. With the above argument we deduce $\mathrm{A}\, f_i \in \mathrm{H}_b$. On the other hand, if $\sum_{i=1}^{\mathrm{N}} \alpha_i \mathrm{A}\, f_i = 0$ with some constants α_i, then $\sum_{i=1}^{\mathrm{N}} \alpha_i\, f_i \in \operatorname{Ker} \mathrm{A}$. Consequently, property (b) of Theorem 2.1 implies

$$\sum_{i=1}^{N} \alpha_i \lambda_i^m f_i \to 0 \quad \text{as } m \to \infty.$$

This leads to $\alpha_i = 0$ for $i = 1, \ldots, N$, hence $\dim F_b \leqq \dim H_b$. If we interchange the spaces F_b and H_b and repeat the same argument, we obtain $\dim H_b \leqq \dim F_b$. The proof is complete.

Now we turn to the main result in this section. Denote by $H_b^\perp$ the orthogonal complement of the space H_b in H.

THEOREM 2.4. — *Let* V *be a power bounded operator in* H *and let* $C: H_b^\perp \to H_b^\perp$ *be a compact operator. Then for each* $f \in H_b^\perp$ *we have*

$$\frac{1}{N} \sum_{m=0}^{N-1} \| C V^m f \|^2 \leqq \varepsilon(N) \| f \|^2, \tag{2.6}$$

where $\varepsilon(N) \to 0$ *as* $N \to \infty$.

Proof. — Obviously, the space $H_b^\perp$ is invariant with respect to V. Assume that $Vx = \lambda x \neq 0$, $\lambda \in S^1$, $x \in H_b^\perp$. Then the argument of Corollary 2.3 yields $Ax \in H_b$, where A is the same as in the latter corollary. Then $0 = (x, Ax) = \| Bx \|^2$ shows that $x \in \operatorname{Ker} A$ and it is easy to see that $x = 0$. Thus the restriction $V' = V|_{H_b^\perp}$ has no eigenvalues $\lambda \in S^1$. By Corollary 2.3 the same is true for the adjoint of V'. Without loss of generality, replacing H and V respectively by $H_b^\perp$ and V', we can assume that $H_b = \{0\}$.

It is sufficient to establish (2.6) for finite rank operators. By approximation with finite rank operators we can assume that C has rank one. Let C^* be the adjoint to C. Then

$$\| C V^k f \|^2 = \| (C^* C)^{1/2} V^k f \|^2$$

and $C^* C$ is rank one operator. Moreover, if C has the form $Cx = (x, h) g$ with $\| g \| = \| h \| = 1$, then $C^* C = (C^* C)^{1/2}$. Thus we are reduced to study the case that C is a rank one projection having the form

$$C f = (f, g) g, \qquad \| g \| = 1.$$

We must prove

$$\frac{1}{N} \sum_{m=0}^{N-1} | (V^m f, g) |^2 \leqq \varepsilon(N) \| f \|^2 \quad \text{for all } f \in H.$$

We write

$$| (V^m f, g) |^2 = (V^m f, g)(g, V^m f) = (V^{*m} C V^m f, f)$$

and the proof is reduced to that of the relation

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} V^{*m} C V^m = 0. \tag{2.7}$$

Consider the space $\mathscr{F}_2$ of Hilbert-Schmidt operators in H with inner product

$$(F, G)_2 = \operatorname{tr}(G^* F)$$

and norm $\| \,.\, \|_2$. Consider the map

$$\Phi: \quad \mathscr{F}_2 \ni L \to L - V^* L V \in \mathscr{F}_2.$$

Clearly,

$$\frac{1}{N} \sum_{m=0}^{N-1} V^{*m} (L - V^* L V) V^m f = \frac{1}{N} (L f - V^{*N} L V^N f) \to 0 \quad \text{as } N \to \infty.$$

Hence by an approximation argument it suffices to show that

$$\overline{\operatorname{Ran} \Phi} = \mathscr{F}_2, \tag{2.8}$$

where the closure is taken in $\mathscr{F}_2$.

To do this, assume that $0 \neq F \in \mathscr{F}_2$ is orthogonal to $\operatorname{Ran}\Phi$. Then for all $L \in \mathscr{F}_2$ we have

$$\begin{aligned} 0 = (F, \Phi(L))_2 &= \operatorname{tr}((L^* - V^* L^* V) F) \\ &= \operatorname{tr}(L^* F) - \operatorname{tr}(L^* VFV^*) = \operatorname{tr}(L^* (F - VFV^*)) = (F - VFV^*, L)_2 \end{aligned}$$

and we deduce

$$VFV^* = F. \tag{2.9}$$

Our aim is to prove that (2.9) implies $F = 0$. If (2.9) holds, then (2.9) remains valid for $F + F^*$ and for $F - F^*$. Without loss of generality in the sequel we may assume F self-adjoint.

Let A, B and U be the operators related to V by Theorem 2.1. We set $F_1 = BFB$ and we obtain

$$UF_1 U^* = F_1. \tag{2.10}$$

Assuming $F_1 \neq 0$ and replacing F_1 by $-F_1$, if it is necessary, we can suppose that $\| F_1 \|$ is an eigenvalue of F_1. Introduce the finite dimensional space $K_1 = \operatorname{Ker}(F_1 - \| F_1 \| \operatorname{Id})$. For $0 \neq x \in K_1$ we get

$$\begin{aligned} \| F_1 \| \, \| x \|^2 &= (\| F_1 \| x, x) = (F_1 x, x) = (UF_1 U^* x, x) \\ &= (F_1 U^* x, U^* x) \leqq \| F_1 \| \, \| U^* x \|^2 \leqq \| F_1 \| \, \| x \|^2. \end{aligned}$$

These inequalities yield $F_1 U^* x = \| F_1 \| U^* x$, hence K_1 is invariant with respect to U^*. Therefore, it is possible to find an element $y \in K_1$ such that $U^* y = \lambda y$, $y \neq 0$, $\lambda \in \mathbf{C}$. On the other hand, by equality (2.10) we obtain

$$\| F_1 \| \, \| y \|^2 = (F_1 y, y) = (F_1 U^* y, U^* y) = | \lambda |^2 \| F_1 \| \, \| y \|^2$$

and we must have $\lambda \in S^1$. The assertion (*c*) of Theorem 2.1 implies the equality $BU^* = V^* B$ and we conclude that

$$\lambda B y = BU^* y = V^* B y.$$

Since $H_b=\{0\}$, we deduce $By=0$ and $\|F_1\|y=F_1y=BFBy=0$. We obtain a contradiction with the choice of y, hence $F_1=0$. The latter yields

$$B^2FB^2F=AFAF=0. \tag{2.10}$$

On the other hand, the equality (2.9) combined with the assertion (a) of Theorem 2.1 leads to

$$VFAFV^*=VFV^*AVFV^*=FAF. \tag{2.11}$$

Next, we need the following.

LEMMA 2.5. — *Let* F *be a self-adjoint compact operator satisfying* (2.9). *Then the equation* $AF=0$ *implies* $F=0$.

Proof. — For fixed $x\in H$ consider the sequence $\{V^{*m}x\}_{m=0}^{\infty}$. Since F is compact, there exists a sequence $m_k\to\infty$ such that

$$\lim_{k\to\infty} FV^{*m_k}x=y.$$

The assumption $AF=0$ yields $Ay=0$, hence $\lim_{k\to\infty} V^{m_k}y=0$. According to (2.9), we obtain

$$\lim_{k\to\infty}(Fx-V^{m_k}y)=\lim_{k\to\infty}(V^{m_k}(FV^{*m_k}x-y))=0.$$

Consequently, $Fx=0$ and $F=0$. The proof is complete.

Now we may finish the proof of Theorem 2.4. We take together (2.10) and (2.11) and apply Lemma 2.5 for FAF. We obtain $FAF=0$ and we get $0=(FAFx, x)=\|BFx\|^2$. Thus $AF=0$ and we can apply Lemma 2.5 once more. Finally, $F=0$ and the proof of Theorem 2.4 is complete.

Remark 2.6. — The assertion of Theorem 2.4 holds if we assume that $C: H\to H$ is a compact operator. In fact, passing to rank one operators, we must establish

$$\frac{1}{N}\sum_{m=0}^{N-1}|(V^m x, y)|^2\leqq\varepsilon(N)\|x\|^2 \quad \text{for} \quad x\in H_b^{\perp}.$$

We write y in the form $y=y_1+y_2$ with $y_1\in H_b^{\perp}$, $y_2\in H_b$ and we have

$$\frac{1}{N}\sum_{m=0}^{N-1}|(V^m x, y_1)|^2\leqq\varepsilon(N)\|x\|^2.$$

Next we can apply the arguments of the proof of Theorem 2.4.

COROLLARY 2.7. — *Assume* H *separable. Then there exists a sequence* $n_k\to\infty$, *independent of f and g, such that for all* $f\in H_b^{\perp}$, $g\in H$

$$\lim_{n_k\to\infty}(V^{n_k}f, g)=0. \tag{2.12}$$

Proof. – Let $C f = \sum_{k=1}^{\infty} 2^{-k/2} (f, \Phi_k) \Phi_k$, $\{\Phi_k\}_{k=1}^{\infty}$ being an orthonormal basis in $H_b^{\perp}$. We set

$$g_k = \sum_{m=1}^{\infty} 2^{-m} \| C V^k \Phi_m \|^2$$

and apply Theorem 2.4. Thus we deduce

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N-1} g_k = 0.$$

It is easy to see that there exists a sequence $n_k \to \infty$ such that $g_{n_k} \to 0$. On the other hand, we obtain the identity

$$g_k = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-(n+m)} |(\Phi_n, V^k \Phi_m)|^2.$$

Consequently, we conclude that $(\Phi_n, V^{n_k} \Phi_m) \to 0$ as $n_k \to \infty$. By a density argument we complete the proof of (2.12).

3. NON-TRAPPING MOVING OBSTACLES

Let Q be an open domain in $\mathbb{R}^{n+1}$, $n \geqq 3$, with C^∞ smooth boundary ∂Q. Introduce the sets

$$\Omega(t) = \{x \in \mathbb{R}^n : (t, x) \in Q\}, \qquad K(t) = \mathbb{R}^n \setminus \Omega(t).$$

We denote by $\nu = (\nu_t, \nu_x)$ the exterior unit normal to ∂Q at $(t, x) \in \partial Q$ pointing into Q. We make the following assumptions:

(H_1) There exists $\rho > 0$ such that for each $t \in \mathbb{R}$ we have

$$\varnothing \neq K(t) \subset \{x : |x| < \rho\},$$

(H_2) For each $(t, x) \in \partial Q$ we have $|\nu_t| < |\nu_x|$.

The condition (H_1) means that the obstacle $K(t)$ stays in a fixed ball, while (H_2) says that the boundary ∂Q can move with a speed less than 1.

To define the class of non-trapping obstacles we need to consider the generalized bicharacteristics (rays) of the wave operator $\square = \partial_t^2 - \Delta_x$ which carry out C^∞ singularities of the solutions of Dirichlet and Neumann problems. These bicharacteristics have been introduced by Melrose and Sjöstrand [15]. For reader's convenience we shall use the notations introduced by Hörmander in Chapter XXIV, Section 24.3 [10]. In particular, we denote by Σ and G the characteristic and glancing sets of $\square$, respectively. Let I be an open interval in $\mathbb{R}$ and let B be a subset of I formed by isolated points. A generalized bicharacteristic (ray) γ of $\square$ is a map

$$I \setminus B \ni s \to \gamma(s) \in \Sigma$$

satisfying the properties (i)-(iii) of Definition 24.3.7 in [10]. The projections of generalized bicharacteristics on $\bar{Q} \cap \mathbb{R}^n$ are called generalized geodesics. Each such geodesic is a union of linear segments in $\bar{Q} \cap \mathbb{R}^n$ and gliding segments on $\partial Q \cap \mathbb{R}^n$. Let $-\infty < a < b < \infty$ and let

$$(a, b) \setminus B \ni s \to \gamma(s) = (t(s), x(s), \tau(s), \xi(s)) \tag{3.1}$$

be a generalized ray of $\square$. Here (τ, ξ) denote the variables dual to (t, x). Let

$$(a, b) \setminus B \ni s \to x(s) \in \bar{Q} \cap \mathbb{R}^n$$

be the corresponding geodesic. Since $\tau^2(s) - |\xi(s)|^2 = 0$ on $\gamma(s)$, we have $\tau(s) \neq 0$ on $\gamma(s)$. Therefore, $\dot{t}(s) \neq 0$ too, where by dot we denote the derivative with respect to s.

If the obstacle $K(t)$ does not depend on t, then $\tau(s) = \text{Const.}$ on each generalized ray of $\square$. Indeed, representing ∂Q locally by $\varphi(x) = 0$, it is easy to see that the function $\tau(s)$ is continuous at each $s \in B$. Normalizing $\tau(s)$ by $\tau(s) = 1/2$, the generalized rays in this case are parametrized by the time $t = s$. Then the length of the geodesic $\{x(s): s \in (a, b)\}$ is just $b - a$.

For moving obstacles we have $\dot{\tau}(s) = 0$ on the linear segments of $\gamma(s)$ lying in Q but, in general, $\tau(s)$ has a jump at $s \in B$. To explain this phenomenon, consider a linear segment γ_i with incident direction $-\xi_i/\tau_i$ such that $|\xi_i|^2 = \tau_i^2$. Let γ_i hit transversally ∂Q at $z \in \partial Q$ and let γ_r be the reflected linear segment with direction $-\xi_r/\tau_r$, where $|\xi_r|^2 = \tau_r^2$. From the definition of generalized rays we deduce

$$(\tau_r, \xi_r)|_{T_z(\partial Q)} = (\tau_i, \xi_i)|_{T_z(\partial Q)}.$$

Identifying $T_z^*(\partial Q)$ and $T_z(\partial Q)$ via the Euclidean metric in $\mathbb{R}^{n+1}$, we obtain

$$(\tau_r - \tau_i, \xi_r - \xi_i) = a(z)\,\nu(z).$$

A simple calculus yields

$$a(z) = -2[\alpha(z)\tau_i + \langle \xi_i, \hat{n}(z) \rangle] / |\nu_x(z)| (1 - \alpha^2(z)).$$

Here

$$\alpha(z) = -\frac{\nu_t(z)}{|\nu_x(z)|}, \qquad \hat{n}(z) = \frac{\nu_x(z)}{|\nu_x(z)|}.$$

Consequently,

$$\tau_r = \mu(z)\,\tau_i \tag{3.2}$$

with

$$\mu(z)=\left(1+\alpha^2+2\alpha\left\langle\frac{\xi_i}{\tau_i}, \hat{n}\right\rangle\right)\Big/(1-\alpha^2)(z).$$

Thus, assuming $\tau_i>0$, we get

$$\begin{array}{ll}\mu(z)>1 & \text{if} \quad \alpha(z)>0,\\ \mu(z)<1 & \text{if} \quad \alpha(z)<0.\end{array}$$

From this observation it follows that the speed $\dot{t}(s)=2\tau(s)$ can change. Nevertheless, it is important to note that the length of the projection on $\mathbb{R}_x^n$ of each generalized ray in the form (3.1) is equal to $|t(b)-t(a)|$. In fact, for $\gamma(s)\in T^*(Q)$ we obtain

$$|\dot{x}(s)|=2|\xi(s)|=2|\tau(s)|=|\dot{t}(s)|. \tag{3.3}$$

From the form of the glancing vector field H_p^G (*see* Definition 24.3.6 in [10]) we conclude that (3.3) remains valid for $\gamma(s)\in G$.

This implies easily the desired assertion.

After this discussion we introduce the class of non-trapping obstacles by the following.

DEFINITION 3.1. — The domain Q is called non-trapping if for each $R>\rho$ there exists $T_R>0$ such that there are no generalized geodesics of $\square$ with length T_R lying entirely within $\bar{Q}\cap\{x:|x|\leqq R\}$.

For stationary obstacles this definition has been introduced by Melrose [14].

Below we shall study the Dirichlet problem for $\square$ in Q. Consider the space $H(t)$ defined as the closure of functions

$$f=(f_1, f_2)\in C_0^\infty(\Omega(t))\times C_0^\infty(\Omega(t))$$

with respect to the energy norm

$$\|f\|_{H(t)}^2=\int_{\Omega(t)}(|\nabla_x f_1|^2+|f_2|^2)\,dx.$$

Consider the problem

$$\left.\begin{array}{r}\square u=0 \quad \text{in } Q,\\ u=0 \quad \text{on } \partial Q,\\ u(s,x)=f_1(x), \qquad \partial_t u(s,x)=f_2(x).\end{array}\right\} \tag{3.4}$$

DEFINITION 3.2. — A distribution $u(t, x)\in\mathscr{D}'(Q)$ is called a solution of (3.4) if the following conditions hold:

(*a*) for each $t\in\mathbb{R}$ we have $(u(t, x), \partial_t u(t, x))\in H(t)$ and extending u as 0 outside $\Omega(t)$ the functions

$$t\to\nabla_x u(t, .), \qquad t\to\partial_t u(t, .)$$

are continuous with values in $L^2(\mathbb{R}^n)$,

(*b*) $(u(s,.), \partial_t u(s,.)) = (f_1, f_2)$,

(*c*) $\Box u = 0$ in the sense of distributions.

In [4] it was shown that under the assumptions (H_1) and (H_2) for each $f \in H(s)$ there exists a unique solution

$$(u(t,x), \partial_t u(t,x)) = U(t,s) f$$

of the problem (3.4). The operator $U(t, s)$ is called a propagator of (3.4). This operator has the following properties:

$$U(t,s)U(s,r) = U(t,r) \quad \text{for all} \quad t, s, r \in \mathbb{R},$$
$$\| U(t,s) \|_{H(t)} \leqq C_A \| f \|_{H(s)} \quad \text{if} \quad |t| \leqq A, \quad |s| \leqq A.$$

Let H_0 be the space given as the closure of functions $f = (f_1, f_2) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\| f \|_0^2 = \int_{\mathbb{R}^n} (|\nabla_x f_1|^2 + |f_2|^2)\, dx.$$

Let $U_0(t)$ be the unitary group in H_0 related to the Cauchy problem for the wave equation in $\mathbb{R}^n$ (*see* [11], [12]). Introduce Lax-Phillips spaces

$$D_\pm^a = \{ f \in H_0 : U_0(t) f = 0 \text{ for } |x| \leqq a + t, \ \pm t \geqq 0 \}$$

and denote by $P_\pm^a$ the orthogonal projections on $(D_\pm^a)^\perp$. By $(.,.)$ we denote the inner product in H_0 and observe that each element f in $H(t)$ can be considered as an element of H_0 extending f as 0 for $x \in K(t)$. For the sake of completeness we state the following lemma established in [4].

LEMMA 3.3. — *Let* $t \geqq s$, $a \geqq \rho$. *Then*

(i) $U(t,s) f = U_0(t-s) f$ *for* $f \in D_+^a$, $U(s,t) f = U_0(s-t) f$ *for* $f \in D_-^a$,

(ii) $(U(t,s) f, g) = (f, U(t,s) g)$ *if either* $f \in D_+^a$ *or* $g \in D_-^a$,

(iii) *if* $f \in H(s) \cap (D_-^a)^\perp$, *then* $U(t,s) f \in (D_-^a)^\perp$,

(iv) *if* $f \in H(s)$ *and* $a \leqq b \leqq a + t - s$, *then*

$$P_+^b U(t,s) P_+^a f = P_+^b U(t,s) f.$$

Next, notice that if $t > a$ and $|x| \leqq t - a$, then

$$U_0(t) P_-^a f = 0 \quad \text{for } n \text{ odd}, \tag{3.5}$$
$$U_0(t) P_-^a f \in C^\infty \quad \text{for } n \text{ even}. \tag{3.6}$$

These properties follow from the translation representation of the group $U_0(t)$. For the proofs the reader should consult [11] for n odd and Lemma III.1, [1] for n even.

To describe the propagation of singularities it is convenient to consider the compressed generalized bicharacteristics of $\Box$ and the wave front $WF_b(u)$ (*see* [10] for the corresponding definitions). The next result due to Melrose and Sjöstrand [15] plays an essential role in the proof of the decay of local energy.

THEOREM 3.4 (Melrose and Sjöstrand). — *Let $u \in \mathscr{D}'(Q)$ be a solution of the problem*

$$\left.\begin{aligned} \square\, u = 0 \, in \, Q, \\ u\,\big|_{\partial Q \cap \{t > t_0\}} \in C^\infty \end{aligned}\right\} \tag{3.7}$$

in the sense of distributions. Let $\hat{z} = (\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi}) \in WF_b(u)$, $\hat{t} > t_0$. Then there exists a maximal compressed generalized bicharacteristic $\tilde{\gamma}(s) = (t(s), x(s), \tau(s), \xi(s))$ passing through $\hat{z}$ and staying in $WF_b(u)$ untill $t(s) > t_0$.

Now we turn to the main result in this section.

THEOREM 3.5. — *Assume (H_1) and (H_2) fulfilled, $n \geqq 3$ and $a \geqq \rho$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} \varphi \subset \{x : |x| \leqq a\}$. Therefore, if Q is non-trapping, then for $t > 10a + T_{4a}$ the operator $\varphi U(t, 0) P_-^a$ acting from $H(0)$ into $H(t)$ is compact.*

Proof. — Below we treat the case n even since for n odd the proof is simpler. Fix ε, $0 < \varepsilon < a - \rho/2$, sufficiently small, and choose a smooth function $\psi(x)$ such that $\psi(x) = 0$ for $|x| \leqq \rho + \varepsilon$, $\psi(x) = 1$ for $|x| \geqq \rho + 2\varepsilon$. Set $M(t, s) = U(t, s) - U_0(t - s)$. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi = 1$ for $|x| \leqq 3a$ and $\chi = 0$ for $|x| \geqq 4a$. By a finite speed of propagation argument we deduce

$$\begin{gathered} M(t, t-2a) g = M(t, t-2a) \chi g, \\ \operatorname{supp} M(2a, 0) P_-^a f \subset \{x : |x| \leqq 3a\}. \end{gathered}$$

Write

$$\begin{aligned} \varphi U(t, 0) P_-^a = {} & \varphi M(t, t-2a) \chi U(t-2a, 2a) (U(2a, 0) - \psi U_0(2a)) P_-^a \\ & + \varphi M(t, t-2a) \chi U(t-2a, 2a) \psi U_0(2a) P_-^a \\ & + \varphi U_0(2a) U(t-2a, 0) P_-^a = I_1 + I_2 + I_3. \end{aligned} \tag{3.8}$$

Since $U(2a, 0) - \psi U_0(2a) = \chi(U(2a, 0) - \psi U_0(2a))$, to prove the compactness of I_1 it suffices to show that the operator $\chi U(t-2a, 2a) \chi$ is compact for $t > 4a + T_{4a}$. Let $u(t, x)$ be the first component of $U(t-2a, 2a) \chi f$. We claim that for $t > 4a + T_{4a}$ and $|x| \leqq 4a$ we have $u \in C^\infty$. To proof this, assume that there exists a point

$$\hat{z} = (\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi}) \in WF_b(u)$$

such that $\hat{t} > 4a + T_{4a}$, $|\hat{x}| \leqq 4a$. By Theorem 3.4 there exists a maximal compressed generalized bicharacteristic $\tilde{\gamma}(s) = (t(s), x(s), \tau(s), \xi(s))$ of $\square$ passing through $\hat{z}$ for $s = 0$ and lying in $WF_b(u)$. The non-trapping assumption implies that the projection of $\tilde{\gamma}(s)$ on $\mathbb{R}^n$ leaves the set $B_{4a} = \bar{Q} \cap \{x : |x| \leqq 4a\}$ for $t(s) < \hat{t} - T_{4a}$. Thus for $t(s) = 4a$ we obtain

$$(4a, \tilde{x}, \tilde{\tau}, \tilde{\xi}) \in \tilde{\gamma}(s) \in WF_b(u) \tag{3.9}$$

for some $|\tilde{x}| > 4a$. On the other hand, $u(4a, x) = \chi f_1$ and in view of the finite speed of propagations we conclude that $u(t, x) \in C^\infty$ for (t, x) sufficiently close to $(4a, \tilde{x})$. This leads to a contradiction with (3.9). This proves the claim. Now, let $\{f_m\}$ run over a bounded set in $H(2a)$. Rellich compactness theorem implies the existence of a subsequence of $\chi U(t-2a, 2a)\chi f_m$ which is convergent in $H(t-2a)$ provided $t>4a+T_{4a}$ is fixed. Thus the operator I_1 is compact for such t.

To treat I_3, notice that the property (iii) of Lemma 3.3 implies

$$U(t-2a,0)P_-^a f \in (D_-^a)^\perp \quad \text{for} \quad t>2a.$$

Therefore for $t>2a$ an application of (3.6) yields

$$\varphi U_0(2a) U(t-2a,0) P_-^a f \in C^\infty$$

and we obtain as above the compactness of I_3.

To study I_2, we shall show that for $t>10a+T_{4a}$ the operator

$$\begin{aligned}&\chi U(t-2a,2a)\psi U_0(2a)P_-^a\\ &\quad =\chi[U(t-2a,2a)-U_0(t-4a)]\psi U_0(2a)P_-^a+\chi U_0(t-4a)\psi U_0(2a)P_-^a\end{aligned}$$

is compact. We write

$$\begin{aligned}&\chi U_0(t-4a)\psi U_0(2a)P_-^a f\\ &\qquad =\chi U_0(t-2a)P_-^a f-\chi U_0(t-4a)(1-\psi)U_0(2a)P_-^a f. \quad (3.10)\end{aligned}$$

We apply (3.6) once more and conclude that the left-hand side of (3.10) is C^∞ for $t>10a$. Let $v(t, x)$ be the first component of $(U(t-2a,2a)-U_0(t-4a))\psi U_0(2a)P_-^a f$. Then

$$\left\{\begin{array}{l}\square v=0 \quad \text{in } Q,\\ v\big|_{\partial Q\cap\{t>10a\}}\in C^\infty\end{array}\right.$$

and we are in position to apply Theorem 3.4. Assume that there exists $\hat z=(\hat t,\hat x,\hat\tau,\hat\xi)\in WF_b(u)$ such that $\hat t>10a+T_{4a}$, $|\hat x|\leqq 4a$. Then we can find a compressed generalized bicharacteristic $\tilde\gamma(s)=(t(s),x(s),\tau(s),\xi(s))$ of $\square$ which lies in $WF_b(v)$ if $t(s)\geqq 10a$. In particular, if $t(s_1)=10a$, then $\tilde\gamma(s_1)\in WF_b(v)$ and the non-trapping assumption guarantees that the projection of $\tilde\gamma(s_1)$ does not belong to B_{4a}. According to the propagation of singularities in the open domain Q, we can extend $\tilde\gamma(s)$ for $t<10a$ preserving the property $\tilde\gamma(s)\in WF_b(v)$. Thus we may find a point $(4a,\tilde x,\tilde\tau,\tilde\xi)\in WF_b(v)$ and this leads to a contradiction. Consequently, $\chi v(t,x)\in C^\infty$ for $t>10a+T_{4a}$ and I_2 is compact for such t. This finishes the proof for n even.

For n odd we get $U_0(2a)P_-^a f=0$ for $|x|\leqq a$. Then we obtain

$$\begin{aligned}\varphi U(t,0)P_-^a=\varphi M(t,t-2a)\chi U(t-2a,2a)\chi M(2a,0)P_-^a&\\ +\varphi M(t,t-2a)\chi U(t-2a,2a)U_0(2a)P_-^a&\\ +\varphi U_0(2a)U(t-2a,0)P_-^a.&\quad (3.11)\end{aligned}$$

By (3.5) for $t>4a$ we get

$$\left.\begin{array}{l}U(t-2a,2a)U_0(2a)P_-^a f\in D_+^a,\\ U_0(2a)U(t-2a,0)P_-^a f\in D_+^a.\end{array}\right\}\quad (3.12)$$

Hence the last two terms in the right-hand side of (3.11) vanish and $\varphi \mathrm{U}(t, 0) \mathrm{P}_{-}^{a}$ is compact for $t>4a+\mathrm{T}_{4a}$. The proof is complete.

For n odd we introduce the local evolution operator

$$\mathrm{Z}^{a}(t, s)=\mathrm{P}_{+}^{a} \mathrm{U}(t, s) \mathrm{P}_{-}^{a}, \qquad a \geqq \rho, \quad t \geqq s.$$

Let $\mathrm{H}(s)=\mathrm{K}^{a}(s) \oplus \mathrm{D}_{+}^{a} \oplus \mathrm{D}_{-}^{a}$. By using Lemma 3.3, it is easy to see that

$$\left.\begin{array}{r}\mathrm{Z}^{a}(t, s) \mathrm{H}(s) \subset \mathrm{K}^{a}(t), \qquad \qquad \\ \mathrm{Z}^{a}(t, r) \mathrm{Z}^{a}(r, s)=\mathrm{Z}^{a}(t, s), \qquad t \geqq r \geqq s.\end{array}\right\} \tag{3.13}$$

Replacing in (3.11) φ by P_{+}^{a} and repeating the above argument, we obtain the following result established in [4].

THEOREM 3.6. — *Assume* (H_1) *and* (H_2) *fulfilled,* $n \geqq 3$ *odd,* $a \geqq \rho$ *and* Q *non-trapping. Then for* $t>4a+\mathrm{T}_{4a}$ *the operator* $\mathrm{Z}^{a}(t, 0)$ *acting from* $\mathrm{H}(0)$ *into* $\mathrm{H}(\mathrm{t})$ *is compact.*

4. THE BEHAVIOUR OF THE LOCAL AND GLOBAL ENERGY FOR PERIODICALLY MOVING NON-TRAPPING OBSTACLES

In this section we use the notations of the previous one. Moreover, in the sequel we make the assumption

(H_3) There exists $\mathrm{T}>0$ such that $\mathrm{K}(t+\mathrm{T})=\mathrm{K}(t)$ for all $t \in \mathbb{R}$. This condition implies

$$\left.\begin{array}{r}\mathrm{U}(t+\mathrm{T}, s+\mathrm{T})=\mathrm{U}(t, s), \qquad \qquad \qquad \\ \mathrm{Z}^{a}(t+\mathrm{T}, s+\mathrm{T})=\mathrm{Z}^{a}(t, s) \quad \text{for all } t, s \in \mathbb{R}.\end{array}\right\} \tag{4.1}$$

Introduce the monodromy operator $\mathrm{V}=\mathrm{U}(\mathrm{T}, 0)$ and denote by $\mathrm{H}_b \subset \mathrm{H}(0)$ the space spanned by the eigenfunctions of the adjoint operator V^* with eigenvalues $\lambda \in \mathrm{S}^1$. For brevity of notations we denote $\mathrm{H}(0)$ by H and the norm in H by $\|.\|$.

Our next assumption means that the energy of $\mathrm{U}(t, s) f$ is globally bounded for $t \geqq s$.

(H_4) There exists a constant $\mathrm{C}_0>0$, independent of f, t and s, such that for each $f \in \mathrm{H}(s)$ and all $t \in \mathbb{R}$, $s \in \mathbb{R}$, $t \geqq s$ we have

$$\|\mathrm{U}(t, s) f\|_{\mathrm{H}(t)} \leqq \mathrm{C}_0\|f\|_{\mathrm{H}(s)}.$$

Clearly, (H_4) implies that V is power bounded.

In the following theorem we establish a decay of the local energy.

$$\|\mathrm{U}(t, 0) f\|_{\mathrm{R}, t}^{2}=\int_{\Omega(t) \cap\{|x| \leqq \mathrm{R}\}}\left(|\nabla_x u(t, x)|^2+|u_t(t, x)|^2\right) dx,$$

where $\mathrm{U}(t, 0) f=(u(t, x), u_t(t, x))$.

THEOREM 4.1. — *Assume* (H_1)-(H_4) *fulfilled*, $n \geqq 3$ *and* Q *non-trapping. Then for each fixed* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *and each fixed* $f \in H_b^\perp$ *we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \|\varphi U(kT, 0) f\|^2 = 0. \tag{4.2}$$

THEOREM 4.2. — *Under the assumptions of Theorem* 4.1 *for n odd we have*

$$\lim_{t \to \infty} \|\varphi U(t, 0) f\| = 0. \tag{4.3}$$

Moreover, for each $R \geqq \rho$ *there exist constants* $C > 0$, $\delta > 0$, *depending only on* Q *and* R, *so that if* $f \in H_b^\perp$ *and* $\operatorname{supp} f \subset \{x : |x| \leqq R\}$; *then*

$$\|U(t, 0) f\|_{R, t} \leqq C e^{-\delta t} \|f\|. \tag{4.4}$$

The proof of these results are based on Theorems 3.5, 3.6 combined with Theorem 2.4. Since these proofs are exactly the same as those of Theorems 2 and 3 in [1], we omit them.

The relations (4.2) and (4.3) show that for $f \in H_b^\perp$ and each fixed $R \geqq \rho$ we have

$$\liminf_{t \to \infty} \|U(t, 0) f\|_{R, t} = 0, \tag{4.5}$$

where for n odd in (4.5) we can take an arbitrary sequence $t_k \to \infty$. Applying the existence of the wave operator W established in section 5, the same observation is valid for n even.

Let F_b (resp. Z_b^a) be the space spanned by the eigenvectors of V (resp. $Z^a(T, 0)$) with eigenvalues $\lambda \in S^1$.

PROPOSITION 4.3. — *Assume n odd*, (H_1)-(H_4) *fulfilled and* Q *non-trapping. Then*

$$F_b = Z_b^a, \tag{4.6}$$

$$\dim F_b = \dim H_b = \dim Z_b^a. \tag{4.7}$$

Proof. — For m large the operator $Z^a(mT, 0)$ is compact and its spectrum is formed by eigenvalues with finite multiplicities. By a standard argument the same assertion is true for the spectrum of $Z^a(T, 0)$. This shows that $\dim Z_b^a < \infty$. Now let $Vf = \lambda f \neq 0$ with $|\lambda| = 1$. Applying Lemma III.4 in [1], we obtain $Z^a(T, 0) f = \lambda f$. Thus $F_b \subset Z_b^a$. Next, suppose that there exists $0 \neq f \in Z_b^a \cap H_b^\perp$. Then

$$f = \sum_{i=1}^{M} \alpha_i f_i \quad \text{with} \quad Z^a(T, 0) f_i = \lambda_i f_i,$$
$$|\lambda_i| = 1.$$

By (4.3) we obtain

$$\lim_{k \to \infty} Z^a(kT, 0)f = \lim_{k \to \infty} \sum_{i=1}^{M} \alpha_i \lambda_i^k f_i = 0.$$

The latter relation implies $\alpha_1 = \ldots = \alpha_M = 0$. So $Z_b^a \cap H_b^\perp = \{0\}$ and $\dim Z_b^a \leqq \dim H_b$. Since $\dim F_b < \infty$, we can apply Corollary 2.3 to get (4.7). The proof is complete.

Preserving the assumptions of Theorem 4.2, we write $K^\rho(0)$ as a direct sum $K^\rho(0) = G_b^\rho + Z_b^\rho$, where the spaces G_b^ρ and Z_b^ρ are invariant with respect to $Z = Z^\rho(T, 0)$ and

$$\| Z^k |_{G_b^\rho} \| \leqq C e^{-\delta k} \quad \text{with} \quad C > 0, \quad \delta > 0.$$

A simple argument yields

$$H_b^\perp = G_b^\rho + D_-^\rho + D_+^\rho. \tag{4.8}$$

Indeed, first we obtain the inclusion

$$G_b^\rho + D_-^\rho + D_+^\rho \subset H_b^\perp. \tag{4.9}$$

Since $D_\pm \subset H_b^\perp$, to see that $G_b^\rho \subset H_b^\perp$ take $g \in G_b^\rho$ and $f \in H_b$ such that $V^* f = \lambda f$ with $\lambda \in S^1$. Then

$$\bar{\lambda}^m (g, f) = (V^m g, f) = (Z^m g, f) \to 0 \quad \text{as } m \to \infty.$$

Combining (4.9) and $Z_b^\rho \cap H_b^\perp = \{0\}$, we deduce (4.8).

Now, it is easy to show that $H_b^\perp$ is the maximal space $\mathscr{H} \subset H$ such that

(i) $D_+^\rho \subset \mathscr{H}$,

(ii) for each $f \in \mathscr{H}$ and each $R \geqq \rho$ we have

$$\lim_{t \to \infty} \| U(t, 0) f \|_{R, t} = 0.$$

In fact, if H_b is not maximal, then there exists $0 \neq g \in Z_b$ such that

$$\lim_{k \to \infty} \| U(kT, 0) g \|_{R, 0} = \lim_{k \to \infty} \| Z(kT, 0) g \| = 0.$$

This leads easily to $g = 0$ and the assertion is proved.

Remark 4.4. — Cooper and Strauss established in [4] the decay of the local energy for periodically moving non-trapping obstacles. However, their results are valid for $f \in G_b^\rho + D_+^\rho$, only. The latter space has not a finite codimension. To deal with D_-^ρ we use essentially Theorem 2.4. On the other hand, the relation (4.5) is important for the proof of the existence of the wave operator Wg for $g \in H_b^\perp$ (*see* section 5).

In the sequel we assume conditions (H_1)-(H_3) fulfilled, while the global energy can be unbounded. First we study the spectrum $\sigma(V) = \sigma_p(V) \cup \sigma_c(V) \cup \sigma_r(V)$ of V. Here $\sigma_p(V)$ is the point spectrum formed by the eigenvalues of V, $\sigma_r(V)$ is the residual spectrum formed by $\lambda \notin \sigma_p(V)$ for which $\mathrm{Ran}(\lambda \, \mathrm{Id} - V)$ is not dense in H and

$\sigma_c(V)=\sigma(V)\diagdown(\sigma_p(V)\cup\sigma_r(V))$. For $\lambda\neq 0$ we denote by E_λ the space generated by the generalized eigenvectors of V related to $\lambda\in\sigma_p(V)$. Similarly, let $G^a_{\lambda,\pm}$ be the space generated by the generalized eigenvectors of the operators $Z^a_\pm$ related to λ. Here we set

$$Z^a_+=P^a_+\,VP^a_-,\qquad Z^a_-=P^a_-\,V^{-1}\,P^a_+,\quad a\geqq\rho.$$

THEOREM 4.5. — *Assume n odd,* (H_1)-(H_3) *fulfilled and* Q *non-trapping. Then* $\sigma_p(V)$ *is formed by a finite number eigenvalues* λ_j, $j=1,\ldots,M$ *with finite algebraic multiplicities. For each* $a\geqq\rho$ *we have the following properties:*

(*a*) *if* $|\lambda_j|>1$, *then* P^a_+ *is an isomorphism from* E_{λ_j} *onto* $G^a_{\lambda_j,+}$,

(*b*) *if* $|\lambda_j|=1$, *then* $E_{\lambda_j}\subset G^a_{\lambda_j,+}$,

(*c*) *if* $|\lambda_j|<1$, *then* P^a *is an isomorphism from* E_{λ_j} *onto* $G^a_{1/\lambda_j,-}$.

Moreover,

$$\sigma_c(V)\cup\sigma_r(V)\subset S^1. \tag{4.10}$$

A similar result holds for the adjoint operator V^*.

Proof. — We assume in the sequel $a\geqq\rho$ fixed. For simplicity we write Z, $P_\pm$ instead of $Z^a(T,0)$, $P^a_\pm$. The proof of the assertions (*a*)-(*c*) is exactly the same as that of the corresponding assertions in Theorem 6 [1]. For the analysis of $\sigma_c(V)$ and $\sigma_r(V)$ we fix k such that $kT>4\rho+T_{4\rho}$ and we write

$$V^k=Z^k+(\mathrm{Id}-P_+)V^kP_-+V^k(\mathrm{Id}-P_-)=Z^k+A_++A_-. \tag{4.11}$$

Clearly, $A_-Z=A_-A_+=ZA_+=0$. Moreover, the operators A_+ and A_- are power bounded since

$$\left.\begin{aligned} A^m_+&=[(\mathrm{Id}-P_+)U_0(kT)P_-]^{m-1}(\mathrm{Id}-P_+)V^kP_-,\\ A^m_-&=V^k(\mathrm{Id}-P_-)[U_0(kT)(\mathrm{Id}-P_-)]^{m-1}.\end{aligned}\right\} \tag{4.12}$$

On the other hand,

$$\|(A_++A_-)^m\|=\left\|\sum_{j=0}^m A^{m-j}_+A^j_-\right\|\leqq C(m+1)$$

with a constant C independent of m. We conclude that the spectral radius of $A=A_++A_-$ is less or equal to 1. Given $\lambda\in\mathbf{C}$, $|\lambda|>1$, we get

$$V^k-\lambda=Z^k+A-\lambda=(A-\lambda)(\mathrm{Id}+(A-\lambda)^{-1}Z^k).$$

Since Z^k is compact, an application of the analytic Fredholm theorem for $\mathrm{Id}+(A-\lambda)^{-1}Z^k$ shows that in the domain $\{z\in\mathbf{C}:|z|>1\}$ the spectrum of V^k contains isolated eigenvalues with finite multiplicities only. Similar argument works for V^* and V^{-1}. Thus we conclude that the essential spectrum of V is included in S^1. The proof is complete.

It is natural to expect that in (4.10) we have an equality. The reader should consult [1] for the case of a periodic potential.

It is important to investigate the spectrum of V for periodically moving trapping obstacles. Popov and Rangelov proved in [20] that for a class of trapping obstacles the spectral radius of $Z^\rho(T, 0)$ is greater than 1. However, it is an open problem to prove that in this case $Z^\rho(T, 0)$ has eigenvalues λ with $|\lambda| > 1$.

In the next theorem we show that there exists a subspace $\mathscr{H} \subset H$ with finite codimension such that $\| V^m|_{\mathscr{H}} \|$ is bounded by a fixed polynomial of m.

THEOREM 4.6. — *Assume the assumptions of Theorem* 4.5 *fulfilled. Then there exists a decomposition* $H = \mathscr{H} + \mathscr{F}$ *as a direct sum so that*

(1) $\mathscr{H}$ *and* $\mathscr{F}$ *are invariant with respect to* V *and* V^{-1},

(2) $\dim \mathscr{F} < \infty$,

(3) *there exist constants* $C_0 > 0$ *and* $q_0 \in \mathbb{N}$, *independent of* m, *such that*

$$\| V^m|_{\mathscr{H}} \| \leqq C_0 |m|^{q_0} \quad \textit{for all } m \in Z \setminus \{0\}.$$

Proof. — We follow the proof of Theorem 11 in [1] making some simplifications. Let $\mathscr{F}_+$ (resp. $\mathscr{F}_-$) be the space generated by the generalized eigenvectors of V (resp. V^-) related to the eigenvalues λ, $|\lambda| > 1$. By using Theorem 4.5, we can write H as a direct sum $H = \mathscr{H} + \mathscr{F}_+ + \mathscr{F}_-$, where $\mathscr{H}$, $\mathscr{F}_\pm$ are invariant with respect to V and V^{-1} and $\dim \mathscr{F}_\pm < \infty$. Therefore

$$\sigma(V|_{\mathscr{H}}) \cup \sigma(V^{-1}|_{\mathscr{H}}) \subset S^1.$$

Consider the spaces $D'_{a,\pm} = \mathscr{H} \cap D^a_\pm$, $a \geqq \rho$, and denote by $P'_{a,\pm}$ the orthogonal projections in $\mathscr{H}$ on the orthogonal complement of $D'_{a,\pm}$ in $\mathscr{H}$. Set

$$Z'_a = P'_{a,+} V P'_{a,-}, \qquad \hat{Z}_a = P'_{a,-} V^{-1} P'_{a,+}.$$

For $mT > 4a + T_{4a}$ the operators Z'^m_a, Z^m_a are compact in $\mathscr{H}$ and for fixed $a \geqq \rho$ there exist $q_1, q_2 \in \mathbb{N}$ so that

$$\| Z'^m_a \| \leqq C_0 m^{q_1}, \qquad \| \hat{Z}^m_a \| \leqq C_0 m^{q_2} \qquad \text{for} \quad m \in \mathbb{N} \setminus \{0\}.$$

These assertions are established in [1].

We consider the operator $V' = V|_{\mathscr{H}}$ and for V', Z'_a, $P'_{a,\pm}$ we write the representations (4.11) and (4.12) with

$$A'_+ = (\mathrm{Id} - P'_{\rho,+}) V P'_{\rho,-}$$
$$A'_- = V(\mathrm{Id} - P'_{\rho,-}).$$

Therefore

$$V'^m = (Z'_\rho + A'_+ + A'_-)^m = \sum_{j=0}^{m} \sum_{k=0}^{m-j} (A'_+)^k (Z'_\rho)^{m-j-k} (A'_-)^j$$

which yields

$$\| V'^m \| \leqq C_1 m^2 m^{q_1} = C_1 m^{q_1+2}.$$

For $m<0$ we apply the estimate for $\hat{Z}_a^m$. The proof is complete.

Next we obtain a sufficient condition for (H_4).

THEOREM 4.7. — *Under the assumptions of Theorem* 4.5 *the following assertions are equivalent:*

(*a*) $\sigma(Z^\rho(T, 0)) \cap \{ z \in \mathbf{C} : |z| \geqq 1 \} = \varnothing$,

(*b*) *for each* $\psi \in C_0^\infty(\mathbb{R}^n)$ *and each* $f \in H(0)$ *we have*

$$\lim_{t \to \infty} \| \psi U(t, 0) f \| = 0.$$

Moreover, these conditions imply (H_4).

Proof. — Assume (*a*) fulfilled. According to Theorem 4 in [4], we conclude that for each $a \geqq \rho$ we have

$$\sigma(Z^a(T, 0)) \cap \{ z \in \mathbb{C} : |z| \geqq 1 \} = \varnothing.$$

By using a standard argument, this property implies the estimate

$$\| Z^a(t, s) \| \leqq C_a e^{-\delta_a (t-s)} \quad \text{for} \quad t \geqq s \tag{4.13}$$

with constants $C_a > 0$, $\delta_a > 0$ independent of t and s. Given $f \in C_0^\infty(\Omega(0)) \times C_0^\infty(\Omega(0))$ we choose $a_0 \geqq \rho$ so that for all $a \geqq a_0$ we have $(\mathrm{Id} - P_-^a) f = 0$. For fixed $\psi \in C_0^\infty(\mathbb{R}^n)$ we choose a sufficiently large so that $\operatorname{supp} \psi \subset \{ x : |x| \leqq a \}$. Then

$$\psi U(t, 0) f = \psi P_+^a U(t, 0) P_-^a f = \psi Z^a(t, 0) f$$

and we deduce (*b*) from (4.13). For $g \in H$ we obtain (*b*) by an approximation argument.

Now assume (*b*). Let $\theta(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(x) = 0$ for $|x| \leqq \rho + 1$, $\theta(x) = 1$ for $|x| \geqq \rho + 2$. Then we get

$$(\partial_t - i G_0) \theta U(t, 0) f = (0, -2 \langle \nabla \theta, \nabla u(t) \rangle - (\Delta \theta) u(t)) = L_\theta u(t).$$

Here G_0 is the generator of the group $U_0(t)$, $u(t)$ is the first component of $U(t, 0) f$ and L_θ is a first order differential operator with smooth coefficients having compact support. It is not hard to see that

$$P_+^\rho U(t, 0) f = P_+ (1 - \theta) U(t, 0) f + P_+^\rho U_0(t) \theta f + \int_0^t P_+^\rho U_0(t - \tau) L_\theta u(\tau) \, d\tau.$$

In fact the integration in the latter integral is over the interval $t - 2\rho - 2 \leqq \tau \leqq t$. Thus, exploiting (*b*), we conclude that

$$\lim_{t \to \infty} \| P_+^\rho U(t, 0) g \| = 0$$

for all $g \in H$. This leads immediately to (*a*).

In the sequel we shall show that the assumption (a) implies (H_4). Take again $f \in C_0^\infty(\Omega(0)) \times C_0^\infty(\Omega(0))$ and denote by $u_0(t)$ the first component of $U_0(t)f$. We have the equality

$$U(t, 0)f = U(t, 0)(1-\theta)f + \theta U_0(t)f - \int_0^t U(t, \tau) L_\theta u_0(\tau)\, d\tau.$$

Choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ for $|x| \leqq \rho + 2$ and $\operatorname{supp} \varphi \subset \{x : |x| \leqq a\}$. By (4.13) we derive

$$\left\| \int_0^t \varphi U(t, \tau) \varphi L_\theta u_0(\tau)\, d\tau \right\| \leqq C(Y(t) e^{-\delta_a t} * \| Y(t) L_\theta u_0(t) \|),$$

$Y(t)$ being the Heaviside function. On the other hand, Lemma VI.3 in [1] shows that for each $\psi \in C_0^\infty(\mathbb{R}^n)$ we have the estimate

$$\int_{-\infty}^{\infty} \| \psi U_0(t) f \|^2\, dt \leqq C(\psi) \| f \|^2, \tag{4.14}$$

where the constant $C(\psi)$ depends only on ψ.

We take together the estimates (4.13) and (4.14) and we apply Young's inequality to the above convolution. Thus we deduce

$$\int_0^{\infty} \| \varphi U(t, 0) f \|^2\, dt \leqq C(\varphi) \| f \|^2 \tag{4.15}$$

and this yields

$$\| \varphi U(t, 0) f \| \leqq C_1(\varphi) \| f \|, \tag{4.16}$$

where the constants $C(\varphi)$ and $C_1(\varphi)$ depend on φ, only.

Setting $(1-\varphi) U(t, 0) f = V(t, x) = (v(t), \partial_t v(t))$, we have

$$\Box v(t) = 2 \langle \nabla \varphi, \nabla u(t) \rangle + (\Delta \varphi) u(t) = h(t),$$

$u(t)$ being the first component of $U(t, 0)f$. We integrate the equality

$$\partial_t v(t) h(t) = (\partial_t v) \Box v = \frac{1}{2} \partial_t (|v_t|^2 + |\nabla v|^2) - \sum_{j=1}^{n} \partial_{x_j} (v_{x_j} v_t)$$

over the domain $[0, t] \times \mathbb{R}^n$ and by (4.15) we deduce

$$\| V(t, x) \|^2 = \| V(0, x) \|^2 + 2 \int_0^t \int_{\mathbb{R}^n} v_t(\tau, x) h(\tau)\, d\tau \leqq C_2(\varphi) \| f \|^2.$$

The last estimate and (4.16) lead to

$$\| U(t, 0) f \| \leqq C_3(\varphi) \| f \|.$$

By approximation we obtain (H_4) and the proof is complete.

5. EXISTENCE OF THE WAVE OPERATORS W_-, W

Throughout this section we assume $n \geqq 3$ and the conditions (H_1), (H_2) and (H_4), introduced in the previous sections, fulfilled. Our aim is to prove the existence of the operators

$$W_- f = \lim_{t \to \infty} U(0, -t) J(-t) U_0(-t) f, \qquad f \in H_0,$$

$$W g = \lim_{t \to \infty} U_0(-t) U(t, 0) g, \qquad g \in \mathscr{H} \subset H(0).$$

Here $J(t): H_0 \to H(t)$ is the orthogonal projection onto $H(t)$, considered as a subspace of H_0, while $\mathscr{H}$ is some subspace of $H(0)$ described below. We use freely the notations of the previous sections.

THEOREM 5.1. — *For each $f \in H_0$, $W_- f$ exists. Moreover, if (H_3) holds, then*

$$\overline{\operatorname{Ran} W_-} \subset H_b^\perp \ominus H_\infty^- \tag{5.1}$$

with $H_\infty^- = \{ f \in H(0) : \lim_{m \to \infty} V^{*m} f = 0 \}$.

Proof. — Let $\theta(x)$ be the function introduced in the proof of Theorem 4.7. Consider the operator

$$W_{-,\theta} f = \lim_{t \to \infty} U(0, -t) \theta U_0(-t) f.$$

We claim that for $f \in D(G_0)$ we have

$$W_\theta(t) f - W_\theta(s) f = \int_s^t U(0, -\sigma) \Phi U_0(-\sigma) f \, d\sigma \tag{5.2}$$

with $W_\theta(t) f = U(0, -t) \theta U_0(-t) f$ and $\Phi = i(G_0 \theta - \theta G_0)$. Notice that Φ is a (2×2) matrix whose elements are first order differential operators having smooth coefficients with compact support. To obtain the above representation observe that for $|s-t|$ sufficiently small we have

$$U(0, -\sigma) \theta g = U(0, -s) U_0(\sigma - s) \theta g.$$

This enables us to write

$$\frac{d}{d\sigma}(U(0, -\sigma) \theta U_0(-\sigma) f) = \frac{d}{d\sigma}(U(0, -s) U_0(\sigma - s) \theta U_0(-\sigma) f)$$
$$= i U(0, -\sigma)(G_0 \theta - \theta G_0) U_0(-\sigma) f, \qquad f \in D(G_0)$$

and the claim is proved.

Now by (H_4) we obtain

$$\| U(0, -t) \Phi U_0(-t) f \| \leqq C_0 \| \Phi U_0(-t) f \|.$$

Let $h \in \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ has Fourier transform $\hat{h}(\xi)$ with $0 \notin \operatorname{supp} \hat{h}(\xi)$. Then for each integer $N \geqq 0$ we get

$$\| \Phi U_0(-t) h \| \leqq C_{h, N} (1 + |t|)^{-N}. \tag{5.3}$$

To establish (5.3) it suffices to show that

$$\sup_{|x| \leqq \rho+2} \left| \int_{\mathbb{R}^n} e^{i(t|\xi| \pm \langle x, \xi \rangle)} \hat{h}(\xi) \, d\xi \right| \leqq C_{h, N} (1 + |t|)^{-N}.$$

This estimate follows easily after an integration by parts with respect to σ in the integral

$$\int_{|\omega|=1} \left[\int_0^\infty e^{i\sigma(t \pm \langle x, \omega \rangle)} \hat{h}(\sigma\omega) \sigma^{n-1} \, d\sigma \right] d\omega.$$

From (5.2) and (5.3) we deduce the convergence of the integral in the right-hand side of (5.2) and this implies the existence of $W_{-,\theta} h$. By a density argument we obtain the existence of $W_{-,\theta} f$ for all $f \in H_0$. Next we write

$$U(0, -t)(J(-t) - \theta) U_0(-t) h = U(0, t) J(-t)(1 - \theta) U_0(-t) h$$

and we apply estimate (5.3) with $1 - \theta$ instead of Φ. Thus $W_{-,\theta} h = W_- h$ and by a density argument we complete the proof of the existence of $W_- f$.

To establish (5.1), take $g \neq 0$ such that $V^* g = \lambda g$, $|\lambda| = 1$. We have

$$(W_- f, g) = \lim_{m \to \infty} (U(0, mT) U_0(-mT) f, g) = \lim_{m \to \infty} \bar{\lambda}^m (U_0(-mT) f, g).$$

Since the spectrum of G_0 is absolutely continuous, $U_0(t) f$ goes weakly to 0 as $t \to -\infty$. Thus $(W_- f, g) = 0$. For $g \in H_\infty^-$ the assertion is trivial. The proof is complete.

The operator W_- is connected with the evolution as $t \to -\infty$. Indeed, if $W_- f = h$, then

$$\begin{aligned} \| U_0(-t) f - U(-t, 0) h \|_0 &\leqq \| J(-t) U_0(-t) f - U(-t, 0) h \|_{H(-t)} \\ &\quad + \| (J(-t) - \mathrm{Id})(1 - \theta) U_0(-t) f \|_0 \\ &\leqq C_0 \| U(0, -t) J(-t) U_0(-t) f - h \| + C_1 \| (1 - \theta) U_0(-t) f \|_0 \underset{t \to \infty}{\to} 0. \end{aligned}$$

Here $\| . \|_0$ denotes the norm in H_0.

Remark 5.2. — The existence of $W_- f$ for n odd has been established by Strauss [23].

Below we turn to the existence of Wg. For this purpose we need an assumption concerning the decay of the local energy.

$$(\mathrm{LD})_+ \qquad \left\{ \begin{array}{l} \text{For each } R \geqq \rho \text{ and each } f \in \mathscr{H} \text{ we have} \\ \qquad \liminf_{t \to \infty} \| U(t, 0) f \|_{R, t} = 0. \end{array} \right.$$

First we shall prove the following.

LEMMA 5.3. — *The condition* $(\mathrm{LD})_+$ *is equivalent to*

$$(\mathrm{LD})'_+ \quad \begin{cases} \textit{For each } \varphi \in C_0^\infty(\mathbb{R}^n) \textit{ and each } f \in \mathcal{H} \textit{ we have} \\ \liminf_{t \to \infty} \| \varphi U(t, 0) f \|_{H(t)} = 0. \end{cases}$$

Proof. — It is obvious that $(\mathrm{LD})'_+$ implies $(\mathrm{LD})_+$. On the other hand, we have

$$\int_{\Omega(t)} |\nabla_x \varphi|^2 |u(t, x)|^2 dx \leqq C \| \varphi U(t, 0) f \|^2_{H(t)},$$

where $U(t, 0) f = (u(t, x), u_t(t, x))$. Thus $(\mathrm{LD})'_+$ implies a decay of the local L^2 norm of $u(t, x)$, while $(\mathrm{LD})_+$ guarantees a decay of the local L^2 norm of $\nabla_x u(t, x)$ *and* $u_t(t, x)$. Below we assume $(\mathrm{LD})_+$ fulfilled.

For simplicity of the notations we write $\|.\|_R$, $\|.\|$ instead of $\|.\|_{R,t}$, $\|.\|_{H(t)}$. Moreover, we extend $u(t, x)$ as 0 for $x \in K(t)$ preserving the continuity of $\nabla_x u(t, x)$ and $u_t(t, x)$ in $L^2(\mathbb{R}^n)$.

Fixing $f \in \mathcal{H}$, by (H_4) we deduce

$$\| U(t, 0) f \| \leqq C_1(f) \quad \text{for all } t \geqq 0.$$

Next we fix $\varepsilon > 0$, $R_0 \geqq \rho$. It suffices to show that there exists a sequence $t_j \nearrow \infty$, depending on g, ε and R_0 such that

$$\lim_{t_j \to \infty} \| U(t_j, 0) f \|_{R_0} = 0, \tag{5.4}$$

$$\lim_{t_j \to \infty} \| u(t_j, x) \|_{L^2(B_R)} < \varepsilon, \tag{5.5}$$

where $B_R = \{ x : |x| \leqq R \}$.

For each fixed $R \geqq \rho$ by $(\mathrm{LD})_+$ we can find a sequence $t_j \nearrow \infty$, depending on R, so that (5.4) holds with $R_0 = R$. For $n \geqq 3$ we recall the estimate

$$\| u(t, x) \|^2_{L^2(B_R)} \leqq \frac{R^2}{2(n-2)} \| \nabla_x u(t, x) \|^2_{L^2(B_R)}$$

(*see* Lemma 1.1 in Chapter IV, [11]),. Thus we obtain

$$\| u(t_j, x) \|_{H^1(B_R)} \leqq C(f, R) \quad \text{for all } t_j.$$

By Reliich's compacteness theorem we choose a subsequence $t_{j_k} \nearrow \infty$ so that

$$\lim_{t_{j_k} \to \infty} u(t_{j_k}, x) = g_R \quad \text{in } L^2(B_R). \tag{5.6}$$

Applying (5.4) with $R_0 = R$, we conclude that $\nabla g_R = 0$ in the sense of distributions. This implies $g_R = c_R = \text{Const}$. Clearly, for t_{j_k} large enough we have

$$\| u(t_{j_k}, x) \|^2_{L^2(B_R)} \geqq (c_R^2/2) \text{ mes } (B_R).$$

Since $u(t_{j_j}, x) \in H_0$ for each t_{j_k}, we can apply the above estimate for the $L^2(B_R)$ norm. Thus we get

$$\| u(t_{j_k}, x) \|^2_{L^2(B_R)} \leqq C_2(f) R^2$$

with a constant $C_2(f)$ independent of R and t_{j_k}. Letting $R \to \infty$, we deduce $\lim\limits_{R \to \infty} c_R = 0$. Since R_0 and ε are fixed, we can choose $R \geqq R_0$ sufficiently large and after this we may determine the sequence $t_{j_k} \nearrow \infty$ so that the relation (5.6) is true with $c_R \leqq \varepsilon/2$. This implies (5.5) and the proof is complete.

Now we turn to the main result in this section.

THEOREM 5.4. — *Assume* $n \geqq 3$ *and the conditions* (H_1), (H_2), (H_3) *and* $(LD)_+$ *fulfilled. Then for each* $f \in \mathscr{H}$, $W f$ *exists.*

Proof. — Consider the operator

$$v_s(t) f = U(t, 0) f - U_0(t-s) U(s, 0) f, \qquad f \in \mathscr{H}, \quad t \geqq 0, \quad s \geqq 0.$$

As in [1], it is easy to see that the proof is reduced to the following assertion:

$$\left.\begin{array}{l} \text{for each fixed } \varepsilon > 0 \text{ there exist } T_1 > 0 \text{ and } T_2 > 0 \\ \text{such that} \\ \qquad \sup\limits_{t \geqq T_2} \| v_{T_1}(t) f \| < \varepsilon. \end{array}\right\} \tag{5.7}$$

For n even we shall use the decay of the local energy of the solution of the Cauchy problem $U_0(t) f = (u(t, x), u_t(t, x))$. Let

$$\| U_0(t) f \|^2_{R_1} = \int_{|x| \leqq R_1} (| \nabla_x u(t, x) |^2 + | u_t(t, x) |^2) \, dx.$$

Then if $\operatorname{supp} f \subset \{ x : |x| \leqq R \}$, there exists a constant $C_{R, R_1} > 0$ depending only on R, R_1 and n such that

$$\| U_0(t) f \|_{R_1} \leqq C_{R, R_1} (1 + |t|)^{-n} \| f \|_0, \tag{5.8}$$

$$\| (U_0(t) f)_1 \|_{L^2(|x| \leqq R_1)} \leqq C_{R, R_1} (1 + |t|)^{-n+1} \| f \|_0, \tag{5.9}$$

$(U_0(t) f)_1$ being the first component of $U_0(t) f$. The proof of these estimates is trivial (*see* Lemma IV.1 in [1]).

Now we turn to the analysis of $v_s(t) f$. Fix $\varepsilon > 0$ and choose $g \in C_0^\infty(\Omega(0)) \times C_0^\infty(\Omega(0))$ such that

$$\| f - g \|_0 \leqq \frac{\varepsilon}{6 C_0^2}, \qquad \operatorname{supp} g \subset \{ x : |x| \leqq R \}, \qquad R > \rho.$$

Here $C_0 \geqq 1$ is the constant introduced in (H_4). By using the function $\theta(x)$, introduced in the previous section, we write

$$v_s(t)g = p_s(t)g + q_s(t)g$$

with

$$p_s(t)g = U(t, 0)g - \theta U_0(t-s)U(s, 0)g,$$
$$q_s(t)g = (\theta - 1)U_0(t-s)U(s, 0)g.$$

To study $p_s(t)$, notice that in $\Omega(t)$ we have

$$\left[\frac{d}{dt} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\right] p_s(t)g = h_s(t)g = (0,\ 2\langle \nabla\theta, \nabla u_0(t)g\rangle + u_0(t)g\,\Delta\theta).$$

Here $u_0(t) = (U_0(t-s)U(s, 0)g)_1$ denotes the first component.
Therefore,

$$p_s(t)g = U(t, s+d)p_s(s+d)g + \int_{s+d}^{t} U(t, \tau)h_s(\tau)g\,d\tau, \qquad t \geqq s+d, \quad (5.10)$$

where $d > R$ will be chosen below. Consider the function

$$w_s(t, x) = \begin{cases} U_0(t-s)U(s, 0)g, & t \geqq s > 0, \\ U(t, 0)g, & t < s \end{cases}$$

and choose a function $\chi(t, x) \in C^\infty$ such that $\chi = 0$ for $|x| \leqq R+\delta$, $0 \leqq t \leqq s+\delta$ and $\chi = 1$ for $|x| \geqq R+2\delta$ or $t \geqq s+2\delta$. Here δ, $0 < \delta < d/2$, is chosen sufficiently small. Setting

$$r_s(t)g = \left[\frac{d}{dt} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\right](\chi w_s(t, x)),$$

we obtain

$$\operatorname{supp} r_s(t)g \cap \{t \geqq 0\} \subset \{R+\delta \leqq |x| \leqq R+2\delta,\ s+\delta \leqq t \leqq s+2\delta\},$$
$$\| r_s(t)g \|_0 \leqq C(\chi)\|g\|, \qquad t \geqq 0.$$

On the other hand,

$$(\chi w_s)(t, x) = \int_0^t U_0(t-\tau)r_s(\tau)g\,d\tau, \qquad t > s+R.$$

For $n \geqq 4$ we apply (5.8), (5.9) and we integrate with respect to τ. Thus estimating the local energy of χw_s we get

$$\| h_s(t)g \|_0 \leqq C_R(t-s-2\delta)^{-n+2} \qquad \text{for} \quad t > s+R.$$

Here and below we denote by C_R, C'_R, C''_R some constants depending on R, only. The condition (H_4) and (5.10) for $t \geqq s+d$ yield

$$\| p_s(t)g \|_0 \leqq C_0[\| v_s(s+d)g \|_0 + \| q_s(s+d)g)g \|_0] + C_1(d-2\delta)^{-n+3},$$

where C_1 does not depend on s and d. Since $\operatorname{supp}(1-\theta)\subset\{x : |x|<\rho+2\}$, the same argument yields

$$\| q_s(t) g \|_0 \leqq C_2 (t-s-2\delta)^{-n+2}, \qquad t \geqq s+d.$$

We choose $d_\varepsilon>0$, independent of $s>0$, so that for $t\geqq s+d_\varepsilon$

$$C_1 (d_\varepsilon-2\delta)^{-n+3} + \| q_s(t) g \|_0 + C_0 \| q_s(s+d_\varepsilon) g \|_0 < \varepsilon/6.$$

Consequently,

$$\| v_s(t) f \|_0 \leqq \varepsilon/3 + \| v_s(t) g \|_0 \leqq \frac{5\varepsilon}{6} + C_0 \| v_s(s+d_\varepsilon) f \|_0.$$

For $n=3$ we apply Huygens' principle and we conclude that the above estimate remains valid for suitably chosen $d_\varepsilon>0$.

Fixing $d_\varepsilon>0$, we have $v_s(s+d_\varepsilon) f=0$ for $|x|\geqq\rho+d_\varepsilon$. Now let $\varphi\in C_0^\infty(\mathbb{R}^n)$ be such that $\varphi=1$ for $|x|\leqq d_\varepsilon+\rho+1$, $\varphi=0$ for $|x|\geqq d_\varepsilon+\rho+2$. Then

$$\begin{aligned} U(s+d_\varepsilon, 0) f - U_0(d_\varepsilon) U(s, 0) f &= U(s+d_\varepsilon, s)\varphi U(s, 0) f \\ &\quad + U_0(d_\varepsilon)(1-\varphi) U(s, 0) f - U_0(d_\varepsilon) U(s, 0) f \\ &= [U(s+d_\varepsilon, s) - U_0(d_\varepsilon)]\varphi U(s, 0) f. \end{aligned}$$

Exploiting the condition $(LD)'_+$, we can find $T_1>0$ so that

$$\| \varphi U(T_1, 0) f \| < \frac{\varepsilon}{12 C_0^2}.$$

Therefore,

$$C_0 \| v_{T_1}(T_1+d_\varepsilon) f \|_0 \leqq 2 C_0^2 \| \varphi U(T_1, 0) f \| \leqq \frac{\varepsilon}{6}.$$

The latter estimate implies (5.7) with $T_2=T_1+d_\varepsilon$. The proof is complete.

Remark 5.5. — The existence of Wg for n odd has been established by Strauss [23] under an assumption slightly stronger than $(LD)_+$. By using Lemma 5.3 we can deduce from $(LD)_+$ the assumpation in [23]. For n even we use the idea exploited in [1], [19].

For periodically moving non-trapping obstacles we can apply the results of section 4. Thus assuming the global energy bounded and applying (5.1), we can compose the operators W_- and W. This leads to the existence of the scattering operator $S=W\circ W_-$ for such obstacles. Here the fact that Wg exists for all $g\in H_b^\perp$ plays an essential role for the composition of W and W_-. On the other hand, the assumption (H_4) is necessary for the existence of Wg.

6. NEUMANN AND ROBIN BOUNDARY VALUE PROBLEMS FOR MOVING OBSTACLES

In this section we discuss briefly the Neumann and Robin problems for the wave equation in the exteriour of a moving obstacle satisfying the assumptions (H_1) and (H_2). Let $\nu^*=(-\nu_t, \nu_x)$ be the conormal vector field and let $\zeta=(\zeta_t, \zeta_x)$ be a fixed vector field tangential to ∂Q and such that $|\zeta_x|<|\zeta_t|$. Consider the problem

$$\left.\begin{array}{l} \square u=0 \quad \text{in } Q, \\ \dfrac{\partial u}{\partial \nu^*}-\alpha\dfrac{\partial u}{\partial \zeta}+\beta u=0 \quad \text{on } \partial Q, \\ u(s, x)=f_1(x), \qquad u_t(s, x)=f_2(x). \end{array}\right\} \tag{6.1}$$

Here $\alpha \geqq 0$ and β are C^∞ smooth real-valued functions on ∂Q. Since the boundary ∂Q is not characteristic for the wave operator, we can interpret the boundary conditions in the sense of distributions. Let $L(t)$ be the closure of functions

$$f \in C_0^\infty(\overline{\Omega(t)}) \times C_0^\infty(\overline{\Omega(t)})$$

with respect to energy norm $\|.\|_{H(t)}$. We define the solution of (6.1) by the following.

DEFINITION 6.1. — A function $u(t, x) \in H^1_{loc}(Q)$ is called solution of (6.1) if the following conditions are satisfied:

(i) $(u(t, .), u_t(t, .)) \in L(t)$ for each $t \in \mathbb{R}$ and when extended as 0 in $K(t)$ the functions $t \to \nabla_x u(t, .)$, $t \to u_t(t, .)$ are continuous in $L^2(\mathbb{R}^n)$,

(ii) $(u(s, .), u_t(s, .))=(f_1, f_2)$ and $u(t, x)$ satisfies the boundary condition in (6.1) in the sense of distributions,

(iii) $\square u=0$ in the sense of distributions.

The existence and uniqueness of solution of (6.1) has been treated in [6]. We can introduce the propagator

$$L(s) \ni f \to U(t, s) f=(u(t, x), u_t(t, x)) \in L(t).$$

For each $\varphi \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ we define the restriction operator by $R(t)\varphi=\varphi|_{\Omega(t)}$. Obviously, $\|R(t)\varphi\|_{L(t)} \leqq \|\varphi\|_{H_0}$ and we can extend $R(t)$ as a bounded operator from H_0 into $L(t)$. We denote by $R^*(t)$ the adjoint operator and we introduce the wave operators

$$W_- f=\lim_{t\to\infty} U(0, -t) R(-t) U_0(-t) f, \qquad f \in H_0,$$
$$W f=\lim_{t\to\infty} U_0(-t) R(t) U(t, 0) f, \qquad f \in \mathscr{H} \subset L(0).$$

It is easy to extend the results of section 5 for Neumann and Robin problems provided the conditions (H_1), (H_2), (H_4) and $(LD)_+$ for the propagator $U(t, s)$ are fulfilled. Below we discuss briefly the modifications needed for the existence of W f. Consider the operator

$$v_s(t)\,f = \theta\,\mathrm{U}(t, 0)\,f - \mathrm{U}_0(t-s)\,\theta\,\mathrm{U}(s, 0)\,f, \qquad t \geqq s, \quad s \geqq 0,$$

$\theta(x)$ being the function of the previous section. Repeating the proof of Theorem 5.4 we can prove the assertion (5.7) replacing $v_s(t)\,f$ by

$$p_s(t)\,f = \mathrm{U}(t, 0)\,f - \theta\,\mathrm{U}_0(t-s)\,\theta\,\mathrm{U}(s, 0)\,f.$$

We write

$$v_s(t)\,f = \theta\,p_s(t)\,f - (1-\theta^2)\,\mathrm{U}_0(t-s)\,\theta\,\mathrm{U}(s, 0)\,f.$$

The second term in the right-hand side of the last equality can be handled by the estimates (5.8), (5.9). Thus we conclude that (5.7) holds for $v_s(t)\,f$ defined above. This implies easily the convergence of $\mathrm{U}_0(-t)\,\theta\,\mathrm{U}(t, 0)\,f$ as $t \to \infty$. It remains to deal with

$$\lim_{t \to \infty} \mathrm{U}_0(-t)\,\mathrm{R}^*(t)(1-\theta)\,\mathrm{U}(t, 0)\,f.$$

We write

$$(1-\theta)\,\mathrm{U}(t, 0)\,f = (1-\theta)\,p_s(t)\,f + (1-\theta)\,\theta\,\mathrm{U}_0(t-s)\,\theta\,\mathrm{U}(s, 0)\,f$$

and apply (5.7) for $p_s(t)\,f$. Finally,

$$\lim_{t \to \infty} \mathrm{U}_0(-t)\,\mathrm{R}^*(t)(1-\theta)\,\mathrm{U}(t, 0)\,\mathrm{f} = 0$$

and this completes the proof of the existence of W f. Thus we can develop a scattering theory for problem (6.1).

For periodically moving non-taping obstacles the operator $\varphi\,\mathrm{U}(t, 0)\,\mathrm{P}^a_-\,f$ is compact for sufficiently large t. The proof of this assertion is based on the arguments given in section 4 combined with the analogue of Theorem 3.4 for Neumann and Robin boundary conditions (*see* [15]). Consequently, if (H_3) and (H_4) hold, we may establish $(\mathrm{LD})_+$ for non-tapping obstacles.

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