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Resonance theory of two-body Schrödinger operators

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ABSTRACT. — We discuss resonances of Schrödinger operators $H=-\Delta+V+W$, where V is dilation-analytic, short-range and W exponentially decaying. Resonances are defined as poles of the analytically continued resolvent of H and identified with poles of the analytically extended S-matrix. Resonance functions associated with a resonance k_0 are defined as certain exponentially growing solutions u of the Schrödinger equation $(H-k_0^2)u=0$, and an isomorphism is established between the space of resonance functions and the null space of the analytically continued inverse S-matrix.

RÉSUMÉ. — Nous discutons les résonances des opérateurs de Schrödinger $H=-\Delta+V+W$, où V est analytique par dilatation et à courte portée, et où W est à décroissance exponentielle. Les résonances sont définies comme les poles de la résolvante de H continuée analytiquement et identifiés avec les poles de la matrice S continuée analytiquement. Les fonctions de résonance associées à la résonance k_0 sont définies comme des solutions exponentiellement croissantes de l'équation de Schrödinger $(H-k_0^2)u=0$, et nous établissons un isomorphisme entre l'espace des fonctions de résonance et le noyau de la continuation analytique de l'inverse de la matrice S.

INTRODUCTION

Resonances of Schrödinger operators in $L^2(\mathbb{R}^3)$ of the form $H=-\Delta+Q$, where $Q=V+W$ and $V=\mathcal{O}(r^{-2-\varepsilon})$ is a multiplicative, dilation-analytic potential and W an exponentially decaying potential were studied in [5]. In the present paper the theory is extended and generalized to cover the case where V is short range (not necessarily local or symmetric) and the dimension $n \geqq 3$.

Basic to our approach is an extension of the limiting absorption principle for the resolvent $R(k)=(H-k^2)^{-1}$ from the real axis to the upper half-plane $\mathbb{C}^+$. This is studied in Section 1. It was established by Saitō [14] for a certain class of potentials. Using Saitō's result for $R_0(k)=(H_0-k^2)^{-1}$, where $H_0=-\Delta$, we extend this to a large class of short-range potentials Q (Theorem 1.4). From this we obtein an analytic extension to $\mathbb{C}^+$ of the trace operators $T(k)$ defining a spectral representation of the absolutely continuous part of H (Lemma 1.8) and, as a consequence, an analytic extension to $\mathbb{C}^+$ of the generalized eigenfunctions of H (Theorem 1.10). Recently Agmon [2] gave a simple proof of an extended limiting absorption principle for $R_0(k)$, using the Phragmèn-Lindelöf principle.

Section 2 deals with the analytic continuation of resolvent and scattering matrix to the lower half-plane. We start by considering short-range potentials V which satisfy an implicit analyticity condition, assuming that the S-matrix $S_1(k)$ of (H_0, H_1), where $H_1=H_0+V$, has an analytic extension $\tilde{S}_1(k)$ to a region $\mathcal{O}$ in the lower half-plane. Under this assumption it is proved that the resolvent $(H_1-k^2)^{-1}$ has an analytic continuation $\tilde{R}_1(k)$ to $\mathcal{O}$ as an operator from a space of exponentially decaying functions to its dual (Theorem 2.2). In this context we also prove the existence of a meromorphic continuation of the generalized eigenfunctions to the lower half-plane. The result on $\tilde{R}_1(k)$ makes it possible to study the perturbation of H_1 by an exponentially decaying term W and prove the existence of a meromorphic extension $\tilde{R}_2(k)$ to $\mathcal{O}$ of the resolvent $(H_2-k^2)^{-1}$, where $H_2=H_1+W$, and of the S-matrices $S_{12}(k)$ of (H_1, H_2) and $S_2(k)$ of (H_0, H_2), identifiying their poles (Theorem 2.4). The analytically continued resolvent $\tilde{R}_2(k)$ provides a natural definition of resonance functions associated with a resonance k_0 as solutions u of the Schrödinger equation $(-\Delta+V+W-k_0^2)u=0$. lying in the range of the residue of $\tilde{R}_2(k)$ at k_0. We furthermore study the pole expansion of $\tilde{R}_2(k)$ around k_0 and establish an isomorphism of this space of resonance functions with the null space of $\tilde{S}_2^{-1}(k_0)$ (Theorem 2.4). The function τ on S^{n-1} corresponding by this isomorphism to the resonance function f also enters into the asymptotic formula

$$f(r, \omega) \simeq c^{-1}(k_0, n)\, e^{ik_0 r} r^{(1-n)/2} \tau(\omega) \quad \text{for } r \to \infty \quad \text{(Theorem 2.5).}$$

For dilation-analytic, short-range potentials V the S-matrix $S_1(k)$ is known to have a meromorphic extension to the dilation angle S_α [4]. Thus the theory applies to an operator $H_2=H_0+V+W$ with $W=o(e^{-2br})$ and $\mathcal{O}=\{k\in S_\alpha \mid |\operatorname{Im} k|<b\}$ (neither V nor W need to be local). Any S_α-dilation-analytic potential V can be decomposed as a sum $V_{1\varepsilon}+V_{2\varepsilon}$ of S_α-dilation-analytic potentials such that $-\Delta+V_{1\varepsilon}$ is resonance-free outside

an ε-distance from the limiting halfline $e^{-2i\alpha}\mathrm{R}^{+}$ and $V_{2\varepsilon}$ decays faster than any exponential (Lemma 2.8). As a consequence we obtain as a main result the identification of the poles of the analytically continued resolvent $\tilde{\mathrm{R}}(k)$ and S-matrix $\tilde{\mathrm{S}}(k)$ of $(\mathrm{H}_0, \mathrm{H}_0+\mathrm{Q})$ for a potential $\mathrm{Q}=\mathrm{V}+\mathrm{W}$ (Theorem 2.9). Potentials Q which admit a decomposition of the type $\mathrm{Q}=\mathrm{V}+\mathrm{W}$ seem to constitute a rather large class. Thus, if Q is an "exterior analytic", short-range potential, let

$$\mathrm{V}(r, \cdot)=\mathrm{Q}(r+\mathrm{K}\,e^{-r^2}, \cdot).$$

Then one can show, using Cauchy's integral formula, that for K large enough V is dilation-analytic and short-range, while $\mathrm{W}=\mathrm{Q}-\mathrm{V}$ decays like e^{-r^2}.

The results of Section 1 and Theorem 2.2 are formulated and proved (in fact without any complication of the proof) for not necessarily symmetric potentials. This will be useful in [8], where we obtain precise results on the analyticity and asymptotic properties of resonance functions for dilation-analytic potentials (*cf.* Theorem 2.5 and Remark 2.6).

1. ANALYTIC EXTENSION OF THE FREE RESOLVENT BOUNDARY VALUES TO THE UPPER HALF-PLANE

We shall make use of some straightforward extensions of results of Saitō [14], which we formulate in the following Lemma. We refer to [14] for the proofs and to [7] Appendix 4 for some comments on the extension.

We start by introducing the basic Hilbert spaces and operators.

DEFINITION 1.1. — The weighted spaces $\mathrm{L}^2_{\delta, b}$ of complex-valued, measurable functions f on n-dimensional Euclidean space $\mathbb{R}^n$ are defined for $\delta, b \in \mathbb{R}$, setting $r=|x|$, by

$$\mathrm{L}^2_{\delta, b}=\mathrm{L}^2_{\delta, b}(\mathbb{R}^n)=\left\{ f \,\Big|\, \| f \|^2_{\delta, b}=\int_{\mathbb{R}^n} | f(x)|^2 (1+r^2)^{\delta} e^{2br}\, dx<\infty \right\}.$$

The weighted Sobolev spaces $\mathrm{H}^2_{\delta, b}$ are defined by

$$\mathrm{H}^2_{\delta, b}=\mathrm{H}^2_{\delta, b}(\mathbb{R}^n)=\{ f \,|\, \| f \|^2_{2, \delta, b}=\sum_{|\alpha|\leqq 2} \| \mathrm{D}^{\alpha} f \|^2_{\delta, b}<\infty \},$$

where

$$\mathrm{D}^{\alpha}=\prod_j (\partial_j)^{\alpha_j}, \qquad \alpha=(\alpha_1, \ldots, \alpha_n), \qquad |\alpha|=\sum_j \alpha_j.$$

We set

$$\mathrm{L}^2_{\delta, 0}=\mathrm{L}^2_{\delta}, \qquad \mathrm{H}^2_{\delta, 0}=\mathrm{H}^2_{\delta}$$

$\mathbb{C}$ = the complex plane

$$\mathbb{C}^{(\pm)} = \{ k = a + ib \mid a \in \mathbb{R},\ b_{\genfrac{}{}{0pt}{}{>}{(<)}} 0 \}$$

$$\tilde{\mathbb{C}}^{(\pm)} = \{ k = a + ib \mid k \neq 0,\ b_{\genfrac{}{}{0pt}{}{\geq}{(\leq)}} 0 \}.$$

$$\hat{\mathrm{D}}_j = \frac{\partial}{\partial x_j} + \frac{n-1}{2r^2} x_j - ik \frac{x_j}{r}$$

$$\hat{\mathrm{D}} u = \{ \hat{\mathrm{D}}_1 u, \ldots, \hat{\mathrm{D}}_n u \}.$$

S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $\mathfrak{h} = L^2(S^{n-1})$.

We assume throughout the paper that $n \geq 3$.

Let M be a subset of the complex plane $\mathbb{C}$. A statement is said to hold locally uniformly for $k \in M$ if it is true uniformly for k in any compact subset of M.

Given an operator T, we denote by $\mathscr{D}(T)$, $\mathscr{R}(T)$ and $\mathscr{N}(N)$ the domain, range and null space of T, respectively.

For $b \geq 0$ the operators $H_0^{\pm b}$ in $L^2_{0,\ \pm b}$ are defined [*cf.* Lemma A 1.1)] by

$$\mathscr{D}(H_0^{\pm b}) = H^2_{0,\ \pm b}$$

and

$$H_0^{\pm b} u = -\Delta u \qquad \text{for} \quad u \in \mathscr{D}(H_0^{\pm b}).$$

For $b = 0$, $H_0^0 = H_0$ is selfadjoint with $\sigma(H_0) = \overline{\mathbb{R}^+} = \{ k^2 \mid k \in \mathbb{R} \}$.

For $b > 0$ the spectrum of $H_0^{\pm b}$ is the parabolic region $\mathscr{P}_b = \{ k^2 \mid k = a + ib_1,\ a \in \mathbb{R},\ |b_1| \leq b \}$, *see* Appendix 2. For $k = a + ib$, $b \geq 0$ and $k' = a + ib'$, $b' > b$ we set

$$R_0^{\pm b}(k') = (H_0^{\pm b} - k'^2)^{-1} \in \mathscr{B}(L^2_{0,\ \pm b}, H^2_{0,\ \pm b})$$

and introduce the operators

$$H^{+}_{0,k} = e^{\mp ikr} H_0^{\pm b} e^{\pm ikr} \quad \text{on} \quad \mathscr{D}(H^{+}_{0,k}) = H^2$$

$$R^{+}_{0,k}(k') = (H^{+}_{0,k} - k'^2)^{-1} = e^{\mp ikr} R_0^{\pm b}(k') e^{\pm ikr} \in \mathscr{B}(L^2, H^2).$$

With respect to the duality of $L^2_{0,b}$ and $L^2_{0,\ -b}$ we have

$$H_0^{-b} = (H_0^b)^*, \qquad H_0^b = (H_0^{-b})^* \tag{1.1}$$

$$R_0^{-b}(k') = R_0^b(-\bar{k}')^*, \qquad R_0^b(k') = R_0^{-b}(-\bar{k}')^* \tag{1.2}$$

$$H^{-}_{0,\ k} = H^{+*}_{0,\ -\bar{k}}, \qquad H^{+}_{0,\ k} = H^{-*}_{0,\ -\bar{k}} \tag{1.3}$$

$$R^{-}_{0,k}(k') = R^{+}_{0,\ -\bar{k}}(-\bar{k}')^*, \qquad R^{+}_{0,\ k}(k') = R^{-}_{0,\ -\bar{k}}(-\bar{k}')^*. \tag{1.4}$$

LEMMA 1.2. — *Let* $1 > \delta > \frac{1}{2}$, $k' = a + ib' \in \tilde{\mathbb{C}}^+$, $b' \geq b \geq 0$, $0 < \varepsilon < \frac{\pi}{2}$. *Let* $f \in L^2_{\delta,\ -b}$. *Then there exists a unique solution u of*

$$\begin{gathered} (-\Delta - k'^2) u = f \\ u \in L^2_{-\delta,\ -b} \\ \| \hat{\mathrm{D}} u \|_{\delta - 1,\ -b} < \infty \end{gathered} \tag{1.5}$$

The exists $C=C(\delta, \varepsilon)$ *such that the following estimates hold for* $b' \geqq b \geqq |a| \geqq 0$ *and for* $0 \leqq b < |a|$, $b \leqq b'$, $\left|\operatorname{Arg} k' - \frac{\pi}{2}\right| \geqq \varepsilon$

$$\|u\|_{-\delta,\,-b} \leqq \frac{C}{|k'|} \|f\|_{\delta,\,-b} \tag{1.6}$$

$$\|\hat{D}u\|_{\delta-1,\,-b} \leqq C \|f\|_{\delta,\,-b}. \tag{1.7}$$

Moreover, $u \in H^2_{-\delta,\,-b}$ *and*

$$\|u\|_{2,\,-\delta,\,-b} \leqq C(|k'| + |k'|^{-1}) \|f\|_{\delta,\,-b}. \tag{1.8}$$

Proof. — We refer to [14] for the proof of existence and uniqueness as well as (1.6) and (1.7) in the case $b'=b$ and to [7] Appendix 4 for remarks on the extension to $b' \geqq b$; the inequalities (1.6) and (1.7) are given in [7] Lemma A 4.5. By Lemma A 1.1, the norm $\|u\|_{2,\delta,b}$ is equivalent to the norm $\{\|u\|^2_{\delta,b} + \|\Delta u\|^2_{\delta,b}\}^{1/2}$. Then (1.8) follows from (1.6), since $\Delta u = -f - k'^2 u$. □

Clearly, for $b'>b$, $R_0^{-b}(k') \mid L^2_{\delta,-b} \in \mathscr{B}(L^2_{\delta,-b}, L^2_{\delta-1,\,-b})$ and hence, by Lemma A 1.1, $R_0^{-b}(k') \mid L^2_{\delta,-b} \in \mathscr{B}(L^2_{\delta,-b}, H^2_{\delta-1,\,-b})$. Thus, for $f \in L^2_{\delta,-b}$, $u = R_0^{-b}(k') f$ is the unique solution of (1.5).

By Lemma 1.2, there exists $C=C(\delta, \varepsilon)$ such that

$$\|R^-_{0,\,k}(k')\|_{\mathscr{B}(L^2_\delta,\, H^2_{-\delta})} < C(|k'| + |k'|^{-1}) \tag{1.9}$$

for $|a| \leqq b < b'$ and for $0 \leqq b < |a|$, $b' > b$, $\left|\operatorname{Arg} k' - \frac{\pi}{2}\right| < \varepsilon$.

By duality [*cf.* (1.4)],

$$\|R^+_{0,k}(k')\|_{\mathscr{B}(L^2_\delta,\, L^2_{-\delta})} = \|R^-_{0,-\bar{k}}(-\bar{k}')^*\|_{\mathscr{B}(L^2_\delta,\, L^2_{-\delta})}$$
$$= \|R^-_{0,-\bar{k}}(-\bar{k}')\|_{\mathscr{B}(L^2_\delta,\, L^2_{-\delta})} < C(|k'| + |k'|^{-1})$$

and hence, by Lemma A 2.1.

$$\|R^+_{0,k}(k')\|_{\mathscr{B}(L^2_\delta,\, H^2_{-\delta})} < C(|k'|^3 + |k'|^{-1}) \tag{1.10}$$

Letting $b' \downarrow b$, we obtain the boundary values of $R^+_{0,k}(k')$ on the spectrum $\mathscr{P}_b$ of $H^{\pm b}_{0,\,k}$.

THEOREM 1.3. — *The following limits exist in the operator norm topology, locally uniformly in* $\tilde{\mathbb{C}}^+$,

$$R_0^{\pm}(k) := \lim_{\varepsilon \downarrow 0} R^+_{0,k}(k+i\varepsilon) \quad \text{in } \mathscr{B}(L^2_\delta, H^2_{-\delta})$$

$$R_0^{\pm}(k+i0) := \lim_{\varepsilon \downarrow 0} R_0^{\pm b}(k+i\varepsilon) \quad \text{in } \mathscr{B}(L^2_{\delta,\,\pm b}, H^2_{-\delta,\,\pm b}).$$

We have

$$R_0^{\pm}(k) = e^{\mp ikr} R_0^{\pm}(k+i0)\, e^{\pm ikr} \tag{1.11}$$

The $\mathscr{B}(L^2_\delta, H^2_{-\delta})$*-valued functions* $R_0^\pm(k)$ *are analytic in* $\mathbb{C}^+$ *and continuous in* $\tilde{\mathbb{C}}^+$.

For $f \in L^2_{\delta, -b}$, $u = R_0^-(k+i0)f$ *is the unique solution of* (1.5) *with* $k' = k$.

Remark. — For $b=0$, $R_0^\pm(k+i0) = R_0 k + i0)$ is the usual limit of $R_0(k+i\varepsilon)$ in $\mathscr{B}(L^2_\delta, H^2_{-\delta})$.

Proof. — The statements about the existence of the limits $R_0^\pm(k)$ and $R_0^\pm(k+i0)$ are equivalent, and the identity (1.11) is obvious.

We consider the case of $R_0^-(k)$. For $f, g \in C_0^\infty(\mathbb{R}^n)$ the limit

$$\lim_{\varepsilon \downarrow 0} (f, R_{0,k}^-(k+i\varepsilon)g) = \lim_{\varepsilon \downarrow 0} (e^{-i\bar{k}r} f, (-\Delta-(k+i\varepsilon)^2)^{-1} e^{-ikr} g)$$

exists, locally uniformly in $\tilde{\mathbb{C}}^+$. This is obvious if $k \in K$, where K is a compact subset of $\tilde{\mathbb{C}}^+$ with $K \cap \mathbb{R} = \varnothing$, and if $K \cap \mathbb{R} \neq \varnothing$ it is easy to prove by local deformation of the integration contour in momentum space (*cf.* [5] Lemma A 1).

By the proof of Agmon [1] Theorem 4.1, this in view of (1.9) implies that the weak limit $R_0^-(k)$ is actually the limit of $R_{0,k}^-(k+i\varepsilon)$ in the operator norm topology of $\mathscr{B}(L^2_\delta, L^2_{-\delta})$, locally uniformly in k. By Lemma A 1.1 it follows that this limit is also attained in the operator norm of $\mathscr{B}(L^2_\delta, H^2_{-\delta})$, locally uniformly in k. The proof for $R_0^+(k)$ is similar, using (1.10).

Since $R_{0,k}^+(k+i\varepsilon)$ is analytic in $\mathbb{C}^+$ and continuous in $\tilde{\mathbb{C}}^+$ for fixed $\varepsilon > 0$, the analyticity and continuity properties of $R_0^\pm(k)$ follows from the fact that $R_{0,k}^+(k+i\varepsilon) \underset{\varepsilon \downarrow 0}{\to} R_0^\pm(k)$ in the norm of $\mathscr{B}(L^2_\delta, H^2_{-\delta})$, locally uniformly on $\tilde{\mathbb{C}}^+$.

The last statement follows from a Theorem of Ikebe and Saito [10] in the case $b=0$. For $b>0$ and $f \in L^2$, $u = R_0^-(k+i0)f = (-\Delta-k^2)^{-1} f \in H^2$, so u satisfies (1.5). This means that $R_0^-(k+i0)$ coincides on L^2 with the operator $(L-k^2)^{-1}$, defined in accordance with Lemma 1.2, which maps $f \in L^2_{\delta,-b}$ into the unique solution of (1.5). It then follows from (1.6) that $R_0^-(k+i0)$ is identical with $(L-k^2)^{-1}$ on $L^2_{\delta,-b}$, and the statement follows. □

We now introduce the interaction V and the Hamiltonian $H = -\Delta + V$. The basic assumptions on V are as follows. Let V be a linear operator in $L^2(\mathbb{R}^n)$ satisfying for a fixed $\delta_1 > \frac{1}{2}$ and every $b \in \mathbb{R}$ the condition

$$V \in \mathscr{C}(H^2_{-\delta_1, b}, L^2_{\delta_1, b}) \tag{A.1}$$

with the norms of V in $\mathscr{B}(H^2_{-\delta_1, b}, L^2_{\delta_1, b})$ locally bounded on $\mathbb{R}$.

The singular set Σ_r defined by

$$\Sigma_r = \{ k \in \mathbb{R} \setminus \{0\} \mid \mathscr{N}(1 + VR_0(k+i0) \neq \{0\}\} \tag{A.2}$$

is bounded and accumulates at most at 0.

We shall also refer to the class of dilation-analytic potentials defined as follows.

Let $\{U(\rho)\}_{\rho\in\mathbf{R}^+}$ be the group of dilations on $L^2(\mathbb{R}^n)$ defined by

$$(U(\rho)f)(x)=\rho^{(n-1)/2}f(\rho x).$$

Let $S_\alpha=\{\rho e^{i\varphi}\,|\,\rho>0,\ |\varphi|<\alpha\}$.

An H_0-compact, symmetric operator V is said to be S_α-dilation-analytic if the family $V(\rho)=U(\rho)VU(\rho^{-1})$ of H_0-compact operators has an analytic extension $V(z)$ to the angle S_α.

Remark. — Notice that we neither assume V to be symmetric nor local. If V is multiplicative, (A.1) is implied by the short-range condition $V\in\mathscr{C}(H^2_{-\delta_1}, L^2_{\delta_1})$. This also holds for example if V is a first order differential operator. The local boundedness ensures that the functions $V^\pm(k)$ defined below are analytic.

The condition (A.2) is natural in the context of the present work. Thus it is known to hold for V symmetric and satisfying $V\in\mathscr{C}(H^2_{-\delta+s}, L^2_{\delta+s})$ for $0\leqq s\leqq\delta$ (*cf.* [11]). It holds for $V(z)$ with V dilation-analytic and satisfying (A.1) (*cf.* [4] and Lemma 2.7), and it will be clear from what follows that (A.2) also holds for $V(z)+W$ with W exponentially decaying.

The results of this section can be generalized if condition (A.1) is replaced by

$$V\in\mathscr{C}H^2_{-\delta_1,b}, L^2_{\delta_1,b}) \qquad (A.1_{b_0})$$

with the norms of V in $\mathscr{B}(H^2_{-\delta_1,b}, L^2_{\delta_1,b})$ locally bounded on $(-b_0, b_0)$, where b_0 is fixed, $0<b_0\leqq\infty$.

The same conclusions are obtained under the assumption $(A.1_{b_0})$ on replacing $\mathbb{C}^+$ by $\mathbb{C}^+_{b_0}=\{k=a+ib\,|\,0<b<b_0\}$ and $\tilde{\mathbb{C}}^+$ by

$$\mathbb{C}^+_{b_0}=\{k\neq 0\,|\,0\leqq b\leqq b_0\}$$

and similarly with $\mathbb{C}^-$ and $\tilde{\mathbb{C}}^-$.

This allows for example potentials of the type $\sum_{i=1}^{N}(\varphi_i,\,.\,)\psi_i$ with φ_i, $\psi_i\in L^2_{0,b_0}$.

To simplify the presentation we formulate most results for $b=\infty$. We denote by V the above defined operator acting in any of the spaces $L^2_{0,\pm b}$.

The operator H is defined on $\mathscr{D}(H)=H^2(\mathbb{R}^n)$ by $Hu=-\Delta u+Vu$. H is closed on $\mathscr{D}(H)=\mathscr{D}(\Delta)$, since V is Δ-compact. The essential spectrum of H is $\overline{\mathbf{R}^+}$. Let $\Sigma_d=\{k\in\mathbb{C}^+\,|\,k^2$ is a (discrete) eigen value of H$\}$ and set $\Sigma=\Sigma_d\cup\Sigma_r$.

The operators $H^{\pm b}$ in $L^2_{0,\pm b}$ are defined by

$$H^{\pm b}=H_0^{\pm b}+V \quad \text{on } \mathscr{D}(H^{\pm b})=H^2_{0,\pm b}.$$

By (A.1) and Lemma 1.1, V is $H_0^{\pm b}$-compact, hence $H^{\pm b}$ is closed. By Lemma A 2.1,

$$\sigma_d(H^{\pm b})\setminus\mathscr{P}_b=\sigma_d(H)\setminus\mathscr{P}_b=(\Sigma_d^b)^2,$$

where

$$\Sigma_d^b=\{\,k'=a+ib'\,|\,k'\in\Sigma_d,\ b'>b\,\}.$$

For $k=a+ib$, $b\geqq 0$ and $k'=a+ib'$, $b'>b$, $k'\notin\Sigma_d^b$, we set

$$R^{\pm b}(k')=(H^{\pm b}-k'^2)^{-1}\in\mathscr{B}(L^2_{0,\ \pm b},\ H^2_{0,\ \pm b}).$$

For $k\in\tilde{\mathbb{C}}^+$ we introduce the following families of operators:

$$V^{\pm}(k)=e^{\mp ikr}\,V\,e^{\pm ikr}\in\mathscr{C}(H^2_{-\delta_1},\ L^2_{\delta_1})$$
$$H_k^{\pm}=e^{\mp ikr}\,H^{\pm b}\,e^{\pm ikr}\quad\text{on}\quad\mathscr{D}(H_k^{\pm})=H^2$$
$$R_k^{\pm}(k')=(H_k^{\pm}-k'^2)^{-1}=e^{\mp ikr}\,R^{\pm b}(k')\,e^{\pm ikr}\in\mathscr{B}(L^2,\ H^2).$$

Based on Theorem 1.3 and (A.1) we have the following result. Here and in what follows δ is assumed to satisfy $1/2<\delta\leqq\min\{\delta_1,\ 1\}$.

THEOREM 1.4. — *The following limits exists in the operator norm topology, locally uniformly in* $\tilde{\mathbb{C}}^+\setminus\Sigma$,

$$R^{\pm}(k):=\lim_{\varepsilon\downarrow 0}R_k^{\pm}(k+i\varepsilon)\quad\text{in}\ \mathscr{B}(L^2_\delta,\ H^2_{-\delta})$$
$$R^{\pm}(k+i0):=\lim_{\varepsilon\downarrow 0}R^{\pm b}(k+i\varepsilon)\quad\text{in}\ \mathscr{B}(L^2_{\delta,\ \pm b},\ H^2_{-\delta,\ \pm b}).$$

We have

$$R^{\pm}(k)=e^{\mp ikr}\,R^{\pm}(k+i0)\,e^{\pm ikr}\tag{1.12}$$

The operators $V^{\pm}(k)\,R_0^{\pm}(k)$ *are analytic,* $\mathscr{C}(L^2_\delta)$*-valued functions on* $\mathbb{C}^+$, *continuous on* $\tilde{\mathbb{C}}^+$. *The* $\mathscr{B}(L^2_\delta)$*-valued functions* $(1+V^{\pm}(k)\,R_0^{\pm}(k))^{-1}$ *are meromorphic in* $\mathbb{C}^+$ *with poles at* Σ_d *and continuous on* $\tilde{\mathbb{C}}^+\setminus\Sigma$, *Moreover,*

$$R^{\pm}(k)=R_0^{\pm}(k)\,(1+V^{\pm}(k)\,R_0^{\pm}(k))^{-1},\tag{1.13}$$

and $R^{\pm}(k)$ *are meromorphic,* $\mathscr{B}(L^2_\delta,\ H^2_{-\delta})$*-valued functions on* $\mathbb{C}^+$ *with poles at* Σ_d *and continuous on* $\tilde{\mathbb{C}}^+\setminus\Sigma$.

Let $k\in\tilde{\mathbb{C}}^+\setminus\Sigma$ *and* $f\in L^2_{\delta,\ -b}$. *The function* $u=R^-(k+i0)\,f$ *in* $H^2_{-\delta,\ -b}$ *is a solution of the equation*

$$(-\Delta+V-k^2)\,u=f\tag{1.14}$$

satisfying $\|\hat{D}u\|_{\delta-1,\ -b}<\infty$, *and* u *is uniquely determined as a solution of* (1.14) *such that* $u\in H^2_{-\delta,\ -b}$ *and* $\|\hat{D}u\|_{\delta-1,\ -b}<\infty$.

Proof. — It is proved in Lemma A 2.3 that

$$\{\,k\in\mathbb{C}^+\,|\,\mathscr{N}(1+V^{\pm}(k)\,R_0^{\pm}(k))\neq\{0\}\}=\Sigma_d.$$

Then the existence of $\lim_{\varepsilon \downarrow 0} R_k^{\pm}(k+i\varepsilon)$ for $k \in \tilde{\mathbb{C}}^+ \setminus \Sigma$ follows from Theorem 1.3 and the second resolvent equation

$$R_k^{\pm}(k+i\varepsilon) = R_{0,k}^{+}(k+i\varepsilon)(1+V^{\pm}(k)R_{0,k}^{+}(k+i\varepsilon))^{-1}.$$

The existence of $\lim_{\varepsilon \downarrow 0} R^{\pm}(k+i\varepsilon)$ and the identity (1.12) follow immediately from this.

By Theorem 1.3 and (A.1), $V^{\pm}(k)R_0^{\pm}(k)$ are analytic, $\mathscr{C}(L_\delta^2)$ – valued functions on $\mathbb{C}^+$, continuous on $\tilde{\mathbb{C}}^+$. By the analytic Fredholm Theorem and Lemma A 2.3, $(1+V^{\pm}(k)R_0^{\pm}(k))^{-1}$ are meromorphic, $\mathscr{B}(L_\delta^2)$-valued functions in $\mathbb{C}^+$ with poles at Σ_c, and $(1+V^{\pm}(k)R_0(k))^{-1}$ are continuous on $\tilde{\mathbb{C}}^+ \setminus \Sigma$. This together with Theorem 1.3 implies that $R^{\pm}(k)$ are given by (1.13) and are meromorphic, $\mathscr{B}(L_\delta^2, H_{-\delta}^2)$-valued functions in $\mathbb{C}^+$ with poles at Σ_c, continuous on $\tilde{\mathbb{C}}^+ \setminus \Sigma$.

To prove the last statement, assume that $u \in H_{-\delta,-b}^2$ is a solution of (1.14) with $f=0$ and $\| \hat{D}u \|_{\delta-1,-b} < \infty$. Then

$$(-\Delta - k^2)u = -Vu$$

where $-Vu \in L_{\delta,-b}^2$ by (A.1). Hence, by Theorem 1.3, u agrees with the unique solution $v = R_0^-(k+i0)(-Vu) \in H_{-\delta,-b}^2$ with $\| \hat{D}v \|_{\delta-1,-b} < \infty$ of

$$(-\Delta - k^2)v = -Vu,$$

so $u = R_0^-(k+i0)(-Vu)$. Then $w = Vu$ is a solution in $L_{\delta,-b}^2$ of $(1+VR_0^-(k+i0))w=0$. Since $k \notin \Sigma$, this implies $w=0$. Thus u is a solution of $(-\Delta-k^2)u=0$, so by Lemma 1.2 $u=0$, proving the uniqueness.

On the other hand, let $f \in L_{\delta,-b}^2$ and set

$$u = R^-(k+i0)f = \lim_{\varepsilon \downarrow 0} u_\varepsilon \quad \text{in } H_{-\delta,-b}^2$$

where $u_\varepsilon = R^{-b}(k+i\varepsilon)f$ is a solution of

$$(-\Delta - (k+i\varepsilon)^2 + V)u_\varepsilon = f.$$

$$R_k^{\pm}(k+i\varepsilon) = R_{0,k}^{+}(k+i\varepsilon)(1+V^{\pm}(k)R_{0,k}^{+}(k+i\varepsilon))^{-1}.$$

The existence of $\lim_{\varepsilon \downarrow 0} R^{\pm}(k+i\varepsilon)$ and the identity (1.12) follow immediately from this.

By Theorem 1.3 and (A.1), $V^{\pm}(k)R_0^{\pm}(k)$ are analytic, $\mathscr{C}(L_\delta^2)$-valued functions on $\mathbb{C}^+$, continuous on $\tilde{\mathbb{C}}^+$. By the analytic Fredholm Theorem and Lemma A 2.3, $(1+V^{\pm}(k)R_0^{\pm}(k))^{-1}$ are meromorphic, $\mathscr{B}(L_\delta^2)$-valued functions in $\mathbb{C}^+$ with poles at Σ_c, and $(1+V^{\pm}(k)R_0(k))^{-}$ are continuous on $\tilde{\mathbb{C}}^+ \setminus \Sigma$. This together with Theorem 1.3 implies that $R^{\pm}(k)$ are given by (1.13) and are meromorphic, $\mathscr{B}(L_\delta^2, H_{-\delta}^2)$-valued functions in $\mathbb{C}^+$ with poles at Σ_c, continuous on $\tilde{\mathbb{C}}^+ \setminus \Sigma$.

To prove the last statement, assume that $u \in H^2_{-\delta, -b}$ is a solution of (1.14) with $f=0$ and $\| \hat{D} u \|_{\delta-1, -b} < \infty$. Then

$$(-\Delta - k^2) u = -V u$$

where $-V u \in L^2_{\delta, -b}$ by (A.1). Hence, by Theorem 1.3, u agrees with the unique solution $v = R_0^- (k+i0)(-V u) \in H^2_{-\delta, -b}$ with $\| \hat{D} v \|_{\delta-1, -b} < \infty$ of

$$(-\Delta - k^2) v = -V u,$$

so $u = R_0^- (k+i0)(-V u)$. Then $w = V u$ is a solution in $L^2_{\delta, -b}$ of $(1 + VR_0^- (k+i0)) w = 0$. Since $k \notin \Sigma$, this implies $w=0$. Thus u is a solution of $(-\Delta - k^2) u = 0$, so by Lemma 1.2 $u=0$, proving the uniqueness.

On the other hand, let $f \in L^2_{\delta, -b}$ and set

$$u = R^- (k+i0) f = \lim_{\varepsilon \downarrow 0} u_\varepsilon \quad \text{in } H^2_{-\delta, -b}$$

where $u_\varepsilon = R^{-b} (k+i\varepsilon) f$ is a solution of

$$(-\Delta - (k+i\varepsilon)^2 + V) u_\varepsilon = f.$$

This implies $\Delta u_\varepsilon \underset{\varepsilon \downarrow 0}{\to} \Delta u$ in $L^2_{-\delta, -b}$, hence $V u_\varepsilon \underset{\varepsilon \downarrow 0}{\to} V u$ by (A.1), and it follows that u is a solution of (1.14). Since

$$u = R_0^- (k+i0)(1 + VR_0^- (k+i0))^{-1} f,$$

the fact that $\| \hat{D} u \|_{\delta-1, b} < \infty$ follows from the last statement of Theorem 1.3. □

DEFINITION 1.5. — The trace operator $T_0(k) \in \mathscr{B}(L^2_\delta, \mathfrak{h})$ is defined for $k \in \mathbb{R} \setminus \{0\}$ and $f \in L^2_\delta$, by

$$(T_0(k) f)(\omega) = F f(k, \cdot), \tag{1.15}$$

where F is the Fourier-Plancherel transform; we set

$$T_0^+(k) = T_0(k) e^{ikr}. \tag{1.16}$$

$T_0^*(\bar{k})$ may be considered as an operator in $\mathscr{B}(\mathfrak{h}, H^2_{-\delta})$, given for $k \in \mathbb{R} \setminus \{0\}$ and $\sigma \in \mathfrak{h}$ by

$$(T_0^*(\bar{k}) \sigma)(x) = (2\pi)^{-n/2} \int_{S^{n-1}} \sigma(\omega) e^{ik\omega \cdot x} d\omega \tag{1.17}$$

and similarly we have

$$T_0^{+*}(\bar{k}) = e^{-ikr} T_0^*(\bar{k}) \in \mathscr{B}(\mathfrak{h}, H^2_{-\delta}). \tag{1.18}$$

For $k > 0$, in $\mathscr{B}(L^2_\delta, H^2_{-\delta})$

$$R_0(k+i0) - R_0(-k+i0) = \pi i k^{n-2} T_0^*(\bar{k}) T_0(k). \tag{1.19}$$

LEMMA 1.6. — *The* $\mathscr{B}(L^2_\delta, \mathfrak{h})$*-valued function* $T_0^+(\bar{k})$ *has a continuous extension to* $\tilde{\mathbb{C}}^+$*, analytic in* $\mathbb{C}^+$ *and given by* (1.15) *and* (1.16).

The $\mathscr{B}(\mathfrak{h}, H^2_{-\delta})$*-valued function* $T_0^{+*}(\bar{k})$ has a continuous extension to $\tilde{\mathbb{C}}^-$, *analytic in* $\mathbb{C}^-$ *and given by* (1.17) *and* (1.18).

Proof. – The second statement is proved in Appendix 3, and the first follows by taking adjoints.

DEFINITION 1.7. – The perturbed trace operators $T_V(k) \in \mathscr{B}(L^2_\delta, \mathfrak{h})$ are defined for $k \in \mathbb{R} \setminus (\{0\} \cup \Sigma_r\}$ by

$$T_V(k) = T_0(k)(1 - VR(k + i0)) \tag{1.20}$$

and we set

$$T_V^+(k) = T_V(k) e^{ikr} = T_0^+(k)(1 - V^+(k) R^+(k)) \tag{1.21}$$

The perturbed adjoint trace operators $T^*_{V^*}(\bar{k}) \in \mathscr{B}(\mathfrak{h}, H^2_{-\delta})$ are defined for $k \in \mathbb{R} \setminus (\{0\} \cup \{-\Sigma_r\})$ by

$$T^*_{V^*}(\bar{k}) = (1 - R(-k + i0) V) T^*_0(\bar{k}) \tag{1.22}$$

and we set

$$T^{+*}_{V^*}(\bar{k}) = e^{-ikr} T^*_{V^*}(\bar{k}) = (1 - R^-(-k) V^-(-k)) T_0^{+*}(\bar{k}) \tag{1.23}$$

Remark. – If V^* satisfies (A.1) (for example if V is multiplicative) then $T^*_{V^*}(\bar{k})$ is the adjoint of the trace operator for $H_0 + V^*$ defined by V^* in (1.20).

LEMMA 1.8. – *The* $\mathscr{B}(L^2_\delta, \mathfrak{h})$*-valued function* $T_V^+(k)$ *has a continuous extension to* $\tilde{\mathbb{C}}^+ \setminus \Sigma$, *meromorphic in* $\mathbb{C}^+$ *with poles at* Σ_d *and given by* (1.21).

The $\mathscr{B}(\mathfrak{h}, H^2_{-\delta})$*-valued function* $T^{+*}_{V^*}(\bar{k})$ *has a continuous extension to* $\tilde{\mathbb{C}}^- \setminus \{-\Sigma\}$, *meromorphic in* $\mathbb{C}^-$ *with poles at* $-\Sigma_d$ *and given by* (1.23).

Proof. – This follows from Theorem 1.4 and Lemma 1.6. □

DEFINITION 1.9. – For $k \in \mathbb{R} \setminus (\{0\} \cup \Sigma_r)$ and $0 \neq \sigma \in \mathfrak{h}$, the family of generalized eigenfunctions $\psi(k, \sigma, \cdot) \in H^2_{-\delta}$ is defined by

$$\psi(k, \sigma, \cdot) = T^*_{V^*}(-\bar{k}) R \sigma = (1 - R^-(k + i0) V) T^*_0(\bar{k}) \sigma \tag{1.24}$$

and we set

$$\psi^+(k, \sigma, \cdot) = T^{+*}_{V^*}(-\bar{k}) R \sigma = (1 - R^-(k) V^-(k)) T_0^{+*}(-\bar{k}) R \sigma \tag{1.25}$$

where $(R\sigma)(\omega) = \sigma(-\omega)$.

THEOREM 1.10. – (*a*) $\psi^+(k, \sigma, \cdot)$ *has a* $H^2_{-\delta}$*-valued continuous extension to* $\tilde{\mathbb{C}}^+ \setminus \Sigma$, *meromorphic in* $\mathbb{C}^+$ *with poles contained in* Σ_d, *given by* (1.25).

(*b*) *For any* $\delta > \frac{1}{2}$ *and* $k \in \tilde{\mathbb{C}}^+ \setminus \Sigma$, $\psi^+(k, \sigma, \cdot) \notin L^2_{\delta - 1}$.

Proof. – (*a*) This follows from Theorem 1.4 and Lemma 1.6. (*b*) If

$$\psi = \psi(k, \sigma, \cdot) := e^{-ikr} \psi^+(k, \sigma, \cdot) \in L^2_{\delta - 1, -b}$$

then $\Delta\psi = -k^2\psi + V\psi \in L^2_{\delta-1,\,-b}$, so by Lemma A 1.1, $\psi \in H^2_{\delta-1,\,-b}$ and hence

$$\frac{\partial}{\partial x_j}\psi \in L^2_{\delta-1,\,-b} \quad \text{for } j=1, \ldots, n.$$

Since $\hat{D}_j = \frac{\partial}{\partial x_j} + B_j$, where $B_j \in \mathscr{B}(H^2_{\delta-1,\,-b}, L^2_{\delta-1,\,-b})$, we conclude that $\hat{D}_j\psi \in L^2_{\delta-1,\,-b}$ or $\|\hat{D}\psi\|_{\delta-1,\,-b} < \infty$.

Since ψ satisfies (1.14), this implies by the uniqueness statement of Theorem 1.4 that $\psi = 0$, a contradiction, so $\psi \notin L^2_{\delta-1,\,-b}$ and $\psi^+(k, \sigma, \cdot) \notin L^2_{\delta-1}$. Here we have used Lemma A 2.3.

2. ANALYTIC CONTINUATION OF RESOLVENT AND EIGENFUNCTIONS

Assume that V satisfies (A.1) and (A.2). For any bounded open interval $I \subset \mathbb{R}^+$ such that $\bar{I} \cup (-\bar{I}) \subset \mathbb{R} \setminus \Sigma_r$ the results of Kako and Yajima [11] show that the local wave operators $W_\pm(I)$ exist and are complete. The local scattering operator $S(I) = W_+^{-1}(I)\, W_-(I)$ is a bicontinuous isomorphism of $E_0(I)\, L_2$ onto itself, where $\{E_0(\cdot)\}$ is the spectral measure of $-\Delta$. Since $S(I)\, E_0(B) = E_0(B)\, S(I)$ for any Borel set $B \subset I$, $S(I)$ has a spectral representation via the Fourier transform, and it is given explicitly for $f \in L^2$ by

$$(FS(I) f)(k, \cdot) = S(k)\, T_0(k) f \tag{2.1}$$

where $S(k)$ is the bounded operator on $\mathfrak{h}$ defined for $k \in \mathbb{R}^+ \setminus \Sigma_r$ by

$$S(k) = I - \pi i k^{n-2}\, T_0(k)(V - VR(k+i0)\, V)\, T_0^*(k) \tag{2.2}$$

The connection between the scattering matrix $S(k)$ and the trace operators $T_V(\pm k)$ is given by the following formula.

LEMMA 2.1. — *For $k \in \mathbb{R}^+ \setminus \Sigma_r$ such that $-k \notin \Sigma_r$*

$$T_V(k) = S(k)\, RT_V(-k). \tag{2.3}$$

Proof. — Let I be as above. The operators $F_\pm(I) = W_\pm^{-1}(I)$ have the following representation for $f \in L_{2,\,\delta}$,

$$(FF_+(I) f)(k, \cdot) = T_V(k) f, \qquad k \in I \tag{2.4}$$

$$(FF_-(I) f)(k, \cdot) = RT_V(-k) f, \qquad k \in I \tag{2.5}$$

Since $S(I)\, F_-(I) = F_+(I)$, we obtain (2.3) for $k \in I$. Letting I vary over all open bounded intervals in $\mathbb{R}^+$ such that $\bar{I} \cup (-\bar{I}) \subset \mathbb{R} \setminus \Sigma$, we obtain (2.3) for all $k \in \mathbb{R}^+ \setminus \Sigma_r$ such that $-k \notin \Sigma_r$.

We now introduce the basic additional analyticity condition on H.

(A.3) The S-matrix $S(k)$ has a $\mathscr{B}(\mathfrak{h})$-valued extension $\tilde{S}(k)$ from an interval $I \subset \mathbb{R}^+$ with $I \cup (-I) \subset \mathbb{R} \setminus \Sigma_r$ to a domain $\mathcal{O} \subset \mathbb{C}^- \setminus (-\Sigma_d)$ with $\partial\mathcal{O} \cap \mathbb{R} = I$, such that $\tilde{S}(k)$ is analytic in $\mathcal{O}$ and continuous in $I \cup \mathcal{O}$.

THEOREM 2.2. — *Assume that* (A.1)-(A.3) *are satisfied.*

(*a*) *The* $\mathscr{B}(L^2_\delta, H^2_{-\delta})$*-valued function* $e^{-ikr} R(k+i0) e^{-ikr}$ *has a continuous extension* $e^{-ikr} \tilde{R}(k) e^{-ikr}$ *from* I *to* $I \cup \mathcal{O}$*, analytic in* $\mathcal{O}$ *and given by*

$$e^{-ikr} \tilde{R}(k) e^{-ikr} = e^{-ikr} R(-k) e^{-ikr} + \pi i k^{n-2} T^{+*}_{V^*}(\bar{k}) \tilde{S}(k) R T^+_V(-k). \quad (2.6)$$

(*b*) *For each* $\sigma \in \mathfrak{h}$ the $H^2_{-\delta}$*-valued function* $e^{-ikr} \psi(k, \sigma, \cdot)$ *has a continuous extension* $e^{-ikr} \tilde{\psi}(k, \sigma, \cdot)$ *from* I *to* $I \cup \mathcal{O}$ *analytic in* $\mathcal{O}$ *and given by*

$$e^{-ikr} \tilde{\psi}(k, \sigma, \cdot) = e^{-ikr} T^*_{V^*}(\bar{k}) \tilde{S}(k) \sigma \quad (2.7)$$

Moreover,

$$e^{-ikr} \tilde{\psi}(k, \sigma, \cdot) \notin L^2_{\delta-1} \quad \textit{for any} \quad \delta > \frac{1}{2}.$$

(*c*) *The* $\mathscr{B}(L^2_\delta, \mathfrak{h})$*-valued function* $T_V(k) e^{-ikr}$ *has a continuous extension* $\tilde{T}_V(k) e^{-ikr}$ *from* I *to* $I \cup \mathcal{O}$*, analytic in* $\mathcal{O}$ *and given by*

$$\tilde{T}_V(k) e^{-ikr} = \tilde{S}(k) R T^+_V(-k).$$

Proof. — (*a*) This follows from (2.3), Lemma 1.8 and the following identity, valid in $\mathscr{B}(L^2_\delta, H^2_{-\delta})$ for $k \in I$,

$$R(k+i0) - R(-k+i0) = \pi i k^{n-2} T^*_{V^*}(k) T_V(k). \quad (2.8)$$

(*b*) It follows from the identity

$$\psi(k, \sigma, \cdot) = T^*_{V^*}(k) S(k) \sigma, \qquad k \in \mathbb{R}^+ \setminus (\Sigma_r \cup (-\Sigma_r))$$

to be established below and Lemma 1.8 that $e^{-ikr} \psi(k, \sigma, \cdot)$ has an extension with the stated properties, given by (2.7). The last statement follows from the uniqueness of solutions of (1.14) proved in Theorem 1.4, *see* the proof of Theorem 1.10.

To prove (2.9), we note that by (1.24) this amounts to showing that for all $\sigma \in \mathfrak{h}$

$$T^*_{V^*}(-k) R \sigma = T^*_{V^*}(k) S(k) \sigma \quad (2.10)$$

Since $\mathscr{R}(RT_V(-k))$ is dense in $\mathfrak{h}$, it suffices to prove (2.10) for all σ of the form $\sigma = RT_V(-k) u$, $u \in L^2_\delta$, or, in view of (2.3), to prove

$$T^*_{V^*}(-k) T_V(-k) u = T^*_{V^*}(k) T_V(k) u, \qquad u \in L^2_\delta. \quad (2.11)$$

The identity (2.11) in turn is an easy consequence of (1.22), (1.19) and the resolvent equations.

(*c*) The statement is obvious from (2.3) and Lemma 1.8. □

In the rest of this section we develop the perturbation theory of (H_1, H_1+W) where $H_1=H_0+V$, V satisfies (A.1)-(A.3) and W is exponentially decaying.

THEOREM 2.3. — *Let V be a symmetric operator in L^2 satisfying* $(A.1_{b_0})$, (A.2) *and* (A.3) *for a fixed b_0 with $\mathcal{O}\subset\{k=a+ib\,|\,a>0, -b_0<b<0\}$. Let W be a symmetric operator in L^2 satisfying the condition*

$$\text{(A.4)}\qquad W\in\mathscr{C}(H^2_{0,\,-b_0}, L^2_{0,\,b_0}).$$

Let

$$H_2=H_1+W \quad \text{on } \mathscr{D}(H_2)=\mathscr{D}(H_1)=\mathscr{D}(H_0)=H^2$$

and

$$R_2(k)=(H_2-k^2)^{-1} \qquad \text{for} \quad k^2\in\rho(H_2).$$

Let $S_{12}(k)$ be the scattering matrix of the pair (H_1, H_2) associated with the spectral representation of $H_{1,\,ac}$ defined by $T_1(k)$ (cf. [12]), *where $T_1(k)$ is the perturbed trace operator $T_V(k)$ of Definition 1.7. Let $S_1(k)$ and $S_2(k)$ be the scattering matrices of (H_0, H_1) and (H_0, H_2) respectively.*

$$\text{Let } \Sigma_{di}=\{i\beta\,|\,-\beta^2\in\sigma_d(H_i)\},\qquad i=1,2.$$

(*a*) *$R_2(k)$ considered as a $\mathscr{B}(L^2_{0,\,b_0}, H^2_{0,\,-b_0})$-valued function on $\mathbb{C}^+$ (with poles at Σ_{d2}) has a meromorphic continuation $\tilde{R}_2(k)$ across I to $\mathcal{O}$ given by*

$$\tilde{R}_2(k)=\tilde{R}_1(k)(I+W\tilde{R}_1(k))^{-1}. \qquad (2.12)$$

(*b*) *$S_{12}(k)$ has a continuous extension $\tilde{S}_{12}(k)$ to $(I\cup\mathcal{O})\diagdown\mathscr{R}$, where $\mathscr{R}$ is the set of poles of $\tilde{R}_2(k)$, meromorphic in $\mathcal{O}$ and given by*

$$\tilde{S}_{12}(k)=I-\pi i k^{n-2}\tilde{T}_1(k)(W-W\tilde{R}_2(k)W)T_1^*(\bar{k}) \qquad (2.13)$$

(*c*) *For $k_0\in\mathscr{R}\diagdown(-\Sigma_{d1})$, $\mathcal{N}(\tilde{S}_{12}^{-1}(k_0))$ and $\mathcal{N}(1+W\tilde{R}_1(k_0))$ are isomorphic via the map*

$$\mathcal{N}(I+W\tilde{R}_1(k_0))\ni\Omega\to\sigma=\tilde{T}_1(k_0)\Omega\in\mathcal{N}(\tilde{S}_{12}^{-1}(k_0)) \qquad (2.14)$$

with the inverse

$$\Omega=-\pi i k_0^{n-2}(I-WR_2(-k_0))WT_1^*(\bar{k}_0)\sigma. \qquad (2.15)$$

(*Here $\tilde{S}_{12}^{-1}(k_0)$ denotes the analytic extension of $S_{12}^{-1}(k)$, evaluated at $k=k_0$*).

(*d*) *$S_2(k)$ has a continuous extension $\tilde{S}_2(k)$ to $(I\cup\mathcal{O})\diagdown\mathscr{R}$, meromorphic in $\mathcal{O}$ and given by*

$$\tilde{S}_2(k)=\tilde{S}_{12}(k)\tilde{S}_1(k). \qquad (2.16)$$

(*e*) *The functions $\tilde{R}_2(k)$, $\tilde{S}_{12}(k)$ and $\tilde{S}_2(k)$ have the same poles and of the same orders in $\mathcal{O}$.*

Proof. — We refer to [5] Theorem 2.5 and Lemma 2.6 for the proof of (*a*)-(*d*).

To prove (*e*), we notice that $V+W$ satisfies $(A.1_{b_0})$, (A.2) and (A.3), so we can apply the above theory to $H_2=H_0+V+W$. Replacing V by $V+W$ in Theorem 2.2, we get

$$\tilde{R}_2(k)=R_2(-k)+\pi i k^{n-2} T_2^*(\bar{k})\tilde{S}_2(k) R T_2(-k). \qquad (2.17)$$

Since $\tilde{T}_1(k)$, $T_1^*(\bar{k})$, $T_2(-k)$, $T_2^*(\bar{k})$ and $\tilde{S}(k)$ are regular in $\mathcal{O}$, we obtain (*e*) from (2.13), (2.16) and (2.17).

We shall now discuss in more detail the pole expansion of $\tilde{R}_2(k)$ around a resonance k_0 (*i. e.* $k_0\in\mathcal{R}$). With a slight abuse of notation we write b instead of the number b_0 appearing in Theorem 2.3. We choose $\delta>0$ such that $S(k_0, \delta)=\{k\,|\,|k-k_0|<\delta\}\subset\mathcal{O}$ and such that k_0 is the only pole in $S(k_0, \delta)$ of the $\mathcal{B}(L^2_{0,b}, H^2_{0,-b})$-valued function $\tilde{R}_2(k)$.

We have for $k\in S(k_0, \delta)\setminus\{k_0\}$

$$H^{-b}\tilde{R}_2(k)=\tilde{R}_2(k)H^b \qquad (2.18)$$

and

$$(H^{-b}-k^2)\tilde{R}_2(k)=I \quad \text{on } L^2_{0,b}. \qquad (2.19)$$

Let $C=\left\{k\,|\,|k-k_0|=\frac{\delta}{2}\right\}$ and define $P\in\mathcal{B}(L^2_{0,b}, H^2_{0}, -b)$ by

$$P=-\frac{1}{2\pi i}\int_C \tilde{R}_2(k)\,dk^2 \qquad (2.20)$$

By (2.19) and (2.20), $\mathcal{R}(P)\subseteq\mathcal{D}((H^{-b}-k_0^2)^j)$ for $j=1, 2, \ldots$ and

$$(H^{-b}-k_0^2)^j P=-\frac{1}{2\pi i}\int_C (k^2-k_0^2)^j\tilde{R}_2(k)\,dk^2 \qquad (2.21)$$

Note that by (2.16) $H^{-b}P=PH^b$. Since $\tilde{W}R_1(k)$ is a $\mathcal{C}(L^2_{0,b})$-valued, analytic function on $S(k_0, \delta)$, $(I+W\tilde{R}_1(k))^{-1}$ has a pole at k_0 and hence $\tilde{R}_2(k)=\tilde{R}_1(k)(I+W\tilde{R}_1(k))^{-1}$ has a pole expansion around k_0

$$\tilde{R}_2(k)=L(k)-\frac{P}{k^2-k_0^2}-\sum_{j=2}^{N}\frac{D_j}{(k^2-k_0^2)^j} \qquad (2.22)$$

where $L(k)$ is regular at k_0. By (2.21),

$$D_j=-\frac{1}{2\pi i}\int_C (k^2-k_0^2)^{j-1}\tilde{R}_2(k)\,dk^2=(H^{-b}-k_0^2)^{j-1}P \qquad (2.23)$$

and hence

$$\tilde{R}_2(k)=L(k)-\sum_{j=1}^{N}\frac{(H^{-b}-k_0^2)^{j-1}P}{(k^2-k_0^2)^j}. \qquad (2.24)$$

Note also that the range of $(H^{-b}-k_0^2)^j P$ is a proper subspace of the range of $(H^{-b}-k_0^2)^{j-1}P$ for $j=1, \ldots, N$.

From (2.24) we get

$$(1+W\tilde{R}_1(k))^{-1}=1-W\tilde{R}_2(k)=1-WL(k)+\sum_{j=1}^{N}\frac{W(H^{-b}-k_0^2)^{j-1}P}{(k^2-k_0^2)^j} \qquad (2.25)$$

Writing

$$\tilde{R}_1(k)=\sum_{m=0}^{\infty}(k^2-k_0^2)^m L_m \qquad (2.26)$$

we obtain from (2.25) and (2.26) the following formula for the residue P of $\tilde{R}_2(k)$ at k_0

$$P=-\sum_{j=0}^{N-1}L_j W(H^{-b}-k_0^2)^j P. \qquad (2.27)$$

DEFINITION. — The space $\mathscr{F}$ of resonance functions of H_2 at k_0 is defined by

$$\mathscr{F}=\{f\in\mathscr{R}(P)\,|\,(H^{-b}-k_0^2)f=0\}.$$

THEOREM 2.4. — *Under the conditions of Theorem* 2.3 *and the assumption that* $\tilde{S}_1(k_0)$ *is invertible the following holds.*

(*a*) $\mathscr{F}$ *is isomorphic to* $\mathscr{N}(I+W\tilde{R}_1(k_0))$ *via the map* W *with the inverse* $-\tilde{R}_1(k_0)$.

(*b*) $\pi i k_0^{n-2}T_2^*(\bar{k}_0)\tilde{T}_1(k_0)=\tilde{R}_1(k_0)$ *on* $\mathscr{N}(1+W\tilde{R}_1(k_0))$.

(*c*) $\mathscr{N}(\tilde{S}_2^{-1}(k_0))$ *is isomorphic to* $\mathscr{F}$ *via the map* $T_2^*(\bar{k}_0)$ *with the inverse* $-\pi i k_0^{n-2}\tilde{T}_1(k_0)W$.

Proof. — (*a*) Suppose $f=Pg\in\mathscr{F}$. Then by (2.27) $f=-\tilde{R}_1(k_0)Wf$, so $Wf\in\mathscr{N}(1+W\tilde{R}_1(k_0))$. Suppose on the other hand that $\Phi\in\mathscr{N}(1+W\tilde{R}_1(k_0))$. Then $f=\tilde{R}_1(k_0)\Phi\in\mathscr{N}(H^{-b}-k_0^2)$. Moreover, $\tilde{R}_1(k_0)\Phi=\lim_{k\to k_0}\tilde{R}_2(k)(1+W\tilde{R}_1(k))\Phi\in\mathscr{R}(P)$, since only the singular part of $\tilde{R}_2(k)$ contributes. Thus, $-\tilde{R}_1(k_0)\Phi\in\mathscr{F}$ and $W(-\tilde{R}_1(k_0)\Phi)=\Phi$. This proves (*a*).

(*b*) For $\Phi\in\mathscr{N}(I+W\tilde{R}_1(k_0))$,

$$\begin{aligned}\pi i k_0^{n-2}T_2^*(\bar{k}_0)\tilde{T}_1(k_0)\Phi &=\pi i k_0^{n-2}(I-R_2(-k_0)W)T_1^*(k_0)\tilde{T}_1(k_0)\Phi\\ &=(I-R_2(-k_0)W)(\tilde{R}_1(k_0)-R_1(-k_0))\Phi\\ &=\tilde{R}_1(k_0)\Phi-R_1(-k_0)\Phi+R_2(-k_0)(I+WR_1(-k_0))\Phi=\tilde{R}_1(k_0)\Phi.\end{aligned}$$ □

(*c*) By Theorem 2.3 (*c*), the map $\tilde{T}_1(k_0)$ is an isomorphism of $\mathscr{N}(1+W\tilde{R}_1(k_0))$ onto $\mathscr{N}(\tilde{S}_{12}^{-1}(k_0))$, which equals $\mathscr{N}(\tilde{S}_2^{-1}(k_0))$ by (2.16). By (*a*) and (*b*), $T_2^*(\bar{k})\tilde{T}_1(k_0)$ is an isomorphism of $\mathscr{N}(1+W\tilde{R}_1(k_0))$ onto $\mathscr{F}$ with inverse $-\pi i k_0^{n-2}W$, and (*c*) follows. □

Remark. — The assumption that $\tilde{S}_1(k_0)$ be invertible is always satisfied if V is S_α-dilation-analytic, *see* Theorem 2.9 and Remark 2.10.

The function $\tau \in \mathcal{N}(\tilde{S}_2^{-1}(k_0))$ connected with f through the isomorphism of Theorem 2.4 (*c*) is directly related as follows to the asymptotic behavior of $f = f(r, \omega)$ for $r \to \infty$.

THEOREM 2.5. — *Assume that the conditions of Theorem* 2.4 *hold and that* $f \in \mathcal{F}$ *satisfies*

$$f(r, \cdot) = r^{(1-n)/2} e^{ik_0 r}(\tilde{\tau} + \varepsilon_1(r)) \tag{2.28}$$

$$\frac{d}{dr}\{ r^{(n-1)/2} f(r, \cdot)\} = ik_0 e^{ik_0 r}(\tilde{\tau} + \varepsilon_2(r)) \tag{2.29}$$

where $\tilde{\tau} \in \mathfrak{h}$ *and* $\varepsilon_i(r) \xrightarrow[r \to \infty]{} 0$ *in* $\mathfrak{h}$, $i = 1, 2$.

Then

$$\tilde{\tau} \in \mathcal{N}(\tilde{S}_2^{-1}(k_0)) \quad \textit{and} \quad f = c(k_0, n) T_2^*(\bar{k}_0)\tilde{\tau},$$

where

$$c(k_0, n) = k_0^{(1-n)/2}(-i)^{(1-n)/2}(2\pi)^{1/2}.$$

Proof. — By Theorem 2.4(*c*) and (1.22) there exists $\tau \in \mathcal{N}(\tilde{S}_2^{-1}(k_0))$ such that

$$f = T_2^*(\bar{k}_0)\tau = (1 - R^-(-k_0 + i0) V) T_0^*(\bar{k}_0)\tau \tag{2.30}$$

Inserting (2.30) first in (2.29) and then in (2.28) we obtain

$$\begin{aligned}
\frac{d}{dr}\{ r^{(n-1)/2} T_0^*(\bar{k}_0)\tau \} - ik_0 e^{ik_0 r}(\tilde{\tau} + \varepsilon_2(r)) \\
&= \frac{d}{dr}\{ r^{(n-1)/2} R^-(-k_0 + i0) VT_0^*(\bar{k}_0)\tau \} \\
&= \left(\frac{d}{dr} + ik_0\right)\{ r^{(n-1)/2} R^-(-k_0 + i0) VT_0^*(\bar{k}_0)\tau \} \\
&\quad - ik_0 r^{(n-1)/2}\{ T_0^*(\bar{k}_0)\tau - r^{(1-n)/2} e^{ik_0 r}(\tilde{\tau} + \varepsilon_1(r))\}
\end{aligned} \tag{2.31}$$

Setting

$$F(r, \cdot) = e^{-ik_0 r}\left(\frac{d}{dr} + ik_0\right)\{ r^{(n-1)/2} R^-(-k_0 + i0) VT_0^*(\bar{k}_0)\tau \},$$

we can write (2.31) as

$$\begin{aligned}
e^{-ik_0 r}\frac{d}{dr}\{ r^{(n-1)/2} T_0^*(\bar{k}_0)\tau \} \\
+ ik_0 r^{(n-1)/2} e^{-ik_0 r} T_0^*(\bar{k}_0)\tau - 2ik_0\tilde{\tau} \\
= F(r, \cdot) + ik_0(\varepsilon_1(r) + \varepsilon_2(r))
\end{aligned} \tag{2.32}$$

By Theorem 1.4, $r^{(1-n)/2}\,F(r,\cdot)\in L^2_{\delta-1}$, hence there exists a sequence $r_p\to\infty$ such that $\|F(r_p,\,.)\|_{\mathfrak{h}}\to 0$, so in $\mathfrak{h}$

$$\left[e^{-ik_0 r}\frac{d}{dr}\{r^{(n-1)/2}\,T_0^*(\bar{k}_0)\,\tau\}+ik_0\,r^{(n-1)/2}\,e^{-ik_0 r}\,T_0^*(\bar{k}_0)\,\tau\right]_{r=r_p} \to 2\,ik_0\,\tau \quad \text{for} \quad p\to\infty. \tag{2.33}$$

Now we use the representation of $T_0^*(\bar{k}_0)$ given by (1) of Appendix 3 and take inner product of (2.33) with any spherical harmonic Y_{mj}. Taking into account the well known asymptotics of $J_m(z)$, we obtain $(\tau-c(k_0,\,n)\,\tilde{\tau},\,Y_{mj})=0$ for all m, j and hence $\tau=c(k_0,\,n)\,\tilde{\tau}$.

Remark 2.6. — The asymptotic formulas (2.28) and (2.29) as well as Theorem 2.5 were proved for exponentially decreasing potentials in [15] and [16]. For dilation-analytic, short-range potentials (2.28) and (2.29) are proved in [8], utilizing the joint analyticity in k and the dilation parameter z. For potentials satisfying the conditions of Theorem 2.3 these results are proved by a different method in [6] for $n=3$. The proof can be extended to $n\geqq 3$.

Based on these asymptotics, exponential decay in time of resonance states, defined as suitably cut-off resonance functions, was proved in [16] for exponentially decreasing potentials. Using the asymptotics established in [6] ([8]) the results and proofs of [16] section 4 carry over to the classes of potentials treated in [6] ([8]).

The assumption (A.2) is satisfied with $\mathcal{O}=S_\alpha\cap C^-$ for V satisfying (A.1) and S_α-dilation-analytic. This is proved in [3] under an additional assumption, which is made to ensure that Σ_r coincides with the set $\{k\in\mathbb{R}\,|\,k\neq 0,\ k^2\in\sigma_p(H)\}$. This, however, can be proved under the condition (A.1), using the dilation-analyticity, as indicated in the following lemma.

LEMMA 2.7. — *Assume that* V *is* S_α*-dilation-analytic and for some* $\delta>\frac{1}{2}$, $V(z)$ *is a* $\mathscr{C}(H^2_{-\delta},\,L^2_\delta)$*-valued, analytic function in* S_α. *Then*

$$\Sigma_r=\{k\in\mathbb{R}\setminus\{0\}\,|\,k^2\in\sigma_p(H)\}.$$

Proof. — We consider the case $k>0$, the proof is similar for $k<0$. We define projections $P(z)$ onto the algebraic null spaces of $1+z^2\,V(z)\,R_0(z,\,\lambda)$ for $\operatorname{Arg} z>0$ by

$$P(z)=-\frac{1}{2\pi i}\int_C(-\mu+V(z)\,R_0(z,\,k))^{-1}\,d\mu$$

where C is a small circle centered at -1, and prove

$$P(z) \underset{z \to 1}{\to} P = -\frac{1}{2\pi i}\int_C (-\mu + VR_0(k+i0)^{-1} d\mu.$$

(We refer for details to [4] Lemmas 2.3 and 2.4 and note, incidentally, that the proof is also valid under our assumption.) This implies the Lemma, since it is known that $k^2 \in \sigma_p(H)$ if and only if $P(z) \neq 0$ for $\operatorname{Arg} z > 0$. □

It is also known [4] that (A.3) holds under the assumption of Lemma 2.7. Thus, Theorem 2.2 holds under these assumptions with $\mathcal{O} = S_\alpha \cap \mathbb{C}^-$, and Theorem 2.3 holds under the assumptions (A.1_{b_0}) with $\mathcal{O} = \{ z \in S_\alpha \mid -b_0 < \operatorname{Im} z < 0 \}$ and (A.4). The following lemma enables us to obtain a precise result, formulated as Theorem 2.9, about the identity of the poles of the S-matrix and the poles of the resolvent for an operator $H = H_0 + V + W$ with V and W satisfying (A.1_{b_0}) and (A.4) respectively.

DEFINITION. — Let $0 < \alpha < \frac{\pi}{2}$ and assume that V is S_α-dilation-analytic. Let $\beta = \frac{\pi}{2\alpha}$ and define for $\varepsilon > 0$ the S_α-dilation-analytic operator V_ε by

$$V_\varepsilon = \exp(-\varepsilon r^\beta)\, V \exp(-\varepsilon r^\beta).$$

LEMMA 2.8. — *Let $0 < \kappa' < \alpha$ and $\gamma > 0$ be given. Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there are no resonances and no eigenvalues of $H_0 + V - V_\varepsilon$ in $T_{\kappa', \gamma} = \{ \eta \in \mathbb{C} \mid \operatorname{Im}(e^{2i\kappa'}\eta) > \gamma \}$.*

Proof. — It suffices to prove for some κ, $\kappa' < \kappa < \alpha$,

$$\| (V(e^{i\kappa}) - V_\varepsilon(e^{i\kappa}))(e^{-2i\kappa} H_0 - \eta)^{-1} \| \underset{\varepsilon \downarrow 0}{\to} 0 \qquad (2.34)$$

uniformly for $\eta \in T_{\kappa', \gamma}$. This follows from

$$(1 - \exp(-\varepsilon r^\beta e^{i\kappa\beta}))\, V(e^{i\kappa})(H_0 + 1)^{-1} \underset{\varepsilon \downarrow 0}{\to} 0 \qquad (2.35)$$

and

$$V(e^{i\kappa})(1 - \exp(-\varepsilon r^\beta e^{i\kappa\beta}))(H_0 + 1)^{-1} \underset{\varepsilon \downarrow 0}{\to} 0 \qquad (2.36)$$

Using the compactness of $V(e^{i\kappa})(H_0+1)^{-1}$, (2.35) is clear. As for (2.36), we have

$$\begin{aligned} V(e^{i\kappa})(1 - \exp(-\varepsilon r^\beta e^{i\kappa\beta}))(H_0+1)^{-1} \\ = V(e^{i\kappa})(H_0+1)^{-1} \{ 1 - \exp(-\varepsilon r^\beta e^{i\kappa\beta}) \\ + (\Delta \exp(-\varepsilon r^\beta e^{i\kappa\beta}))(H_0+1)^{-1} \\ + 2(\nabla \exp(-\varepsilon r^\beta e^{i\kappa\beta})) \cdot \nabla (H_0+1)^{-1} \} \underset{\varepsilon \downarrow 0}{\to} 0. \end{aligned}$$ □

THEOREM 2.9. — *Let* $0<\alpha<\frac{\pi}{2}$, $b_0>0$, $\delta>\frac{1}{2}$ *be given. Assume that* V *is a symmetric operator satisfying* $(A.1_{b_0})$ and $V_\delta:=(1+r^2)^{\delta/2}V(1+r^2)^{\delta/2}$ *is an* H_0*-compact,* S_α*-dilation-analytic operator,* i.e. $V_\delta(z)\in\mathscr{C}(H^2, L^2)$ *for* $|\operatorname{Arg} z|<\alpha$. *Let* W *be a symmetric operator satisfying* (A.4).

Then $R(k)=(H_0+V+W-k^2)^{-1}$ *has a meromorphic,* $\mathscr{B}(L^2_{0,b_0}, H^2_{0,-b_0})$*-valued, meromorphic continuation* $\tilde{R}(k)$ *from* $\mathbb{C}^+$ *across* $\mathbb{R}^+$ *to* $S^{b_0}_\alpha=\{k\in S_\alpha \mid -b_0<\operatorname{Im} k<0\}$. *The* S*-matrix* $S(k)$ *of* (H_0, H_0+V+W) *has a continuous extension* $\tilde{S}(k)$ *to* $\mathbb{R}^+\cup S^{b_0}_\alpha$, *meromorphic in* $S^{b_0}_\alpha$. *Moreover, the poles of* $\tilde{R}(k)$ *and* $\tilde{S}(k)$ *in* $S^{b_0}_\alpha$ *coincide and are of the same order.*

Proof. — Let $\alpha>\kappa>0$ and $\gamma>0$ be given, and choose $\varepsilon_0>0$ as in Lemma 2.8. Set

$$H_{1\,\varepsilon}=H_0+V-V_\varepsilon, \qquad H=H_{1\,\varepsilon}+(V_\varepsilon+W)=H_0+V+W.$$

Let $I_{\kappa,\gamma}=(\gamma/\sin 2\kappa, \infty)$. By Lemma 2.8, $H_{1\,\varepsilon}$ has no eigenvalues, hence by Lemma 2.7 no singular points, in $I^{1/2}_{\kappa,\gamma}$ and no resonances in $T^{1/2}_{\kappa,\gamma}$. Then by [4] Theorem 4.1, $\tilde{S}_{1\,\varepsilon}(k)$ has an analytic extension from $I^{1/2}_{\kappa,\gamma}$ to $T^{1/2}_{\kappa,\gamma}$. By Theorem 2.3 (*a*) and (*d*) with W replaced by $V_\varepsilon+W$ we obtain the meromorphic continuation $\tilde{R}(k)$ of $R(k)$ across $I^{1/2}_{\kappa,\gamma}$ to $T^{b_0}_{\kappa,\gamma}:=\{k\in T^{1/2}_{\kappa,\gamma} \mid -b_0<\operatorname{Im} k<0\}$ and the meromorphic extension $\tilde{S}(k)$ of $S(k)$ from $I^{1/2}_{\kappa,\gamma}$ to $T^{b_0}_{\kappa,\gamma}$. Moreover, by Theorem 2.4 (*e*) the poles of $\tilde{R}(k)$ and $\tilde{S}(k)$ and their orders agree in $T^{b_0}_{\kappa,\gamma}$. Letting $\kappa\uparrow\alpha$ and $\gamma\downarrow 0$, we have $T^{b_0}_{\kappa,\gamma}\uparrow S^{b_0}_\alpha$, and the Theorem is proved. □

Remark 2.10. — Under the assumptions of Theorem 2.9, $S_2(k)$ has a meromorphic extension $\tilde{S}_2(k)$ to $\{k\in S_\alpha \mid |\operatorname{Im} k|<b_0\}$, regular for $\operatorname{Im} k\geqq 0$. It was shown in [4] that $S_1(k)$ has a meromorphic extension $\tilde{S}_1(k)$ to S_α, regular in $S_\alpha\cap\bar{\mathbb{C}}^+$. Moreover, $S_{12}(k)$ has a meromorphic extension $\tilde{S}_{12}(k)$ to S_α, given for $\operatorname{Im} k\leqq 0$ by (2.13) and for $\operatorname{Im} k\geqq 0$ by

$$\tilde{S}_{12}(k)=1-\pi i k^{n-2}T_1(k)(W-WR_2(k)W)\tilde{T}_1^*(\bar{k}),$$

where $\tilde{S}_{12}(k)$ is regular for $\operatorname{Im} k\geqq 0$. The extension $\tilde{S}_2(k)$ is then defined by (2.16).

APPENDIX 1

LEMMA A 1.1. — *For* $\delta, b\in\mathbb{R}$ *the following four norms are equivalent; locally uniformly in* $k=a+ib$

$$\|u\|_{1,a}=\|e^{ikr}(1+r^2)^{\delta/2}u\|_{H^2}$$
$$\|u\|_{2,a}=\|e^{ikr}(1+r^2)^{\delta/2}u\|_{\Delta}$$

$$\|u\|_3=\Big(\sum_{|\alpha|\leqq 2}\|D^\alpha u\|^2_{\delta,\,b}\Big)^{1/2}$$

$$\|u\|_4=(\|u\|^2_{\delta,\,b}+\|\Delta u\|^2_{\delta,\,b})^{1/2}.$$

Proof. — We set $a=0$. The extension to the general case is obvious. We use, setting $f_{\delta,\,b}(r)=e^{br}(1+r^2)^{\delta/2}$

$$\frac{\partial}{\partial x_i}(f_{\delta,\,b}u)=\frac{\partial}{\partial x_i}f_{\delta,\,b}\cdot u+f_{\delta,\,b}\frac{\partial u}{\partial x_i} \tag{1}$$

$$\frac{\partial^2}{\partial x_i\partial x_j}(f_{\delta,\,b}u)=\frac{\partial^2}{\partial x_i\partial x_j}f_{\delta,\,b}\cdot u+\frac{\partial}{\partial x_i}f_{\delta,\,b}\frac{\partial u}{\partial x_j}+\frac{\partial}{\partial x_j}f_{\delta,\,b}\frac{\partial u}{\partial x_i}+f_{\delta,\,b}\frac{\partial^2 u}{\partial x_i\partial x_j} \tag{2}$$

From (1) follows

$$\left\|\frac{\partial}{\partial x_i}(f_{\delta,\,b}u)\right\|\leqq C\left(\|u\|_{\delta,\,b}+\left\|\frac{\partial u}{\partial x_i}\right\|_{\delta,\,b}\right) \tag{3}$$

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{\delta,\,b}\leqq C\left(\left\|\frac{\partial}{\partial x_i}(f_{\delta,\,b}u\right\|+\|u\|_{\delta,\,b}\right) \tag{4}$$

From (2) and (3) follows

$$\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}(f_{\delta,\,b}u)\right\|\leqq C\left(\|u\|_{\delta,\,b}+\left\|\frac{\partial u}{\partial x_j}\right\|_{\delta,\,b}+\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{\delta,\,b}\right) \tag{5}$$

From (2) and (4) follows

$$\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{\delta,\,b}\leqq C\left(\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}(f_{\delta,\,b}u)\right\|+\left\|\frac{\partial}{\partial x_i}(f_{\delta,\,b}u)\right\|+\|u\|_{\delta,\,b}\right) \tag{6}$$

From (3) and (5) follows

$$\|u\|_1\leqq C\|u\|_3.$$

From (4) and (6) follows

$$\|u\|_3\leqq C\|u\|_1.$$

It is well known that $\|u\|_1$ and $\|u\|_2$ are equivalent.
Clearly,

$$\|u\|_4\leqq C\|u\|_3.$$

Finally,

$$\|u\|_2\leqq C\;\|f_{\delta,\,b}u\|_{f_{\delta,\,b}\Delta f^{-1}_{\delta,\,b}}\leqq C(\|u\|_{\delta,\,b}+\|\Delta u\|_{\delta,\,b})\leqq C\|u\|_4.$$

Here we have used

$$f_{\delta,\,b}\Delta f^{-1}_{\delta,\,b}=\Delta+2f_{\delta,\,b}\nabla f^{-1}_{\delta,\,b}\cdot\nabla+f_{\delta,\,b}\Delta f^{-1}_{\delta,\,b},$$

and noting that the last two terms are $\Delta-\varepsilon$-bounded, the equivalence of the Δ-norm and the $f_{\delta,\,b}\Delta f_{\delta,\,b}^{-1}$-norm follows. □

APPENDIX 2

We refer to the definitions of $H_0^{\pm b}$, $H^{\pm b}$, $H_{0,\,k}^{\pm}$, $H_k^{\pm}$, $R_{0,\,k}^{+}(k+i\varepsilon)$, $R_k^{\pm}(k+i\varepsilon)$ and $R_0^{\pm}(k)$, $R^{\pm}(k)$ given in the beginning of Section 1.

Consider first $H_0^{\pm b}$, acting in $L^2_{0,\,\pm b}$. From the representation of $(-\Delta-k^2)^{-1}$ as a convolution operator by a function behaving asymptotically as $cr^{(1-n)/2}e^{ikr}$ for $r\to\infty$ [3] it follows that $\sigma(H_0^{\pm b})\subseteqq\mathscr{P}_b$.

On the other hand, for $|\mathrm{Im}\,k|<b$ the functions $e^{ik\omega x}$ are eigenfunctions of H_0^{-b}, so $\mathscr{P}_b=\sigma(H_0^{-b})$ and hence, by duality $\mathscr{P}_b\subseteqq\sigma(H_0^{b})$, so $\sigma(H_0^{\pm b})=\mathscr{P}_b$. By analytic Fredholm theory this implies that $\sigma_e(H^b)$, $\sigma_e(H^{-b})\subseteqq\mathscr{P}_b$. (If both V and V* satisfy (A.1) then it follows easily that in fact $\sigma_e(H^b)=\sigma_e(H^{-b})=\mathscr{P}_b$.) The operators $H(k)=e^{-ikr}H^b e^{ikr}$ from an entire analytic self-adjoint family of type A of operators in L^2 with common domain H^2 and $H(k)=H_k^+$ for $\mathrm{Im}\,k\geqq 0$, $H(k)=H_{-k}^-$ for $\mathrm{Im}\,k\leqq 0$, $H(0)=H$. For b fixed the operators $H(a+ib)$ are unitarily equivalent, hence the spectrum $\sigma(H(k))$ depends only on b. This together with the analyticity of $H(k)$ implies

LEMMA A 2.1. — *Let $b>0$ be fixed. For $k=a+ib'$, $|b'|\leqq b$, we have* $\sigma_d(H(k))\diagdown\mathscr{P}_b=\sigma_d(H)\diagdown\mathscr{P}_b$.

For $k_0^2\in\sigma_d(H)\diagdown\mathscr{P}_b$, $k_0=a_0+ib_0$, let $\{\Phi_1,\ldots,\Phi_m\}$ *be a basis of* $\mathcal{N}(H-k_0^2)$. Then $\{e^{-ikr}\Phi_j\}_{j=1}^m$ forms a basis of $\mathcal{N}(H(k)-k_0^2)$ for $|\mathrm{Im}\,k|<b_0$. *Any* Φ *in* $\mathcal{N}(H^{-b}-k_0^2)$ belongs to $L^2_{0,\,b_1}$ *for every* $b_1<b_0$.

Proof. — The first statement is proved above.

Let $k_0^2\in\sigma_d(H)\diagdown\mathscr{P}_b$. For $|\mathrm{Im}\,k|<b_0$ let C_k be a circle separating k_0^2 from the rest of the spectrum of $H(k)$, and define $P(k)=-\dfrac{1}{2\pi i}\displaystyle\int_{C_k}(H(k)-\mu)^{-1}\,d\mu$. Then $P(k)$ is an analytic, $\mathscr{B}(L^2)$-valued function of k defined for $|\mathrm{Im}\,k|<b_0$. By a standard method this is used to construct a basis $\{\Phi_i(k)\}_{i=1}^m$ of eigenfunctions of $H(k)$ corresponding to the eigenvalue k_0^2, such that $\Phi_i(0)=\Phi_i$, and the second statement of the Lemma follows. The last statement follows easily from the above discussion. □

LEMMA A 2.2. — *Let $k=a+ib$, $k_1=a+ib_1$, $0<b_1<b$. Then*

$$R_0^{\pm}(k)=\lim_{b_1\uparrow b} R_{0,\,k_1}^{\pm}(k)\quad \text{in } \mathscr{B}(L^2_\delta,\,H^2_{-\delta}). \tag{1}$$

Proof. — Fix $k=a+ib$, $b>0$, and let $\varepsilon>0$ be given.

By Theorem 1.3 there exists $\varepsilon'>0$ such that for $k_1=a+ib_1$, $b_1\in(b-\varepsilon', b)$,

$$\| R_0^{\pm}(k_1)-R_{0,\,k_1'}^{\pm}(k)\|_{\mathscr{B}(L_\delta^2,\, H_{-\delta}^2)}<\varepsilon/2 \tag{2}$$

and such that

$$\| R_0^{\pm}(k)-R_0^{\pm}(k_1)\|_{\mathscr{B}(L_\delta^2,\, H_{-\delta}^2)}<\varepsilon/2 \tag{3}$$

and (1) follows. □

LEMMA A 2.3. — *Assume that* (A 1) *holds. Then*

$$\Sigma_d=\{k\in\mathbb{C}^+ \mid \mathcal{N}(1+V^{\pm}(k)R_0^{\pm}(k))\neq\{0\}\}.$$

Proof. — This is proved using Lemma A 2.1 and A 2.2 in the same way as [5] Lemma A 4.

APPENDIX 3

We shall complete the proof of Lemma 1.6 using the following expansion, *cf.* [9].

For all $r>0$, $k\in\mathbb{C}\setminus\{0\}$ and ω, $\omega'\in S^{n-1}$

$$e^{ikr\,\omega\cdot\omega'}=(2\pi)^{n/2}\sum_{m=0}^{\infty}(i)^m(kr)^{1-(n/2)}J_{p_m}(kr)\times\sum_{j=1}^{N(n,\,m)}Y_{mj}(\omega)\overline{Y_{mj}(\omega')},\qquad p_m=\frac{n}{2}+m-1. \tag{1}$$

The set $\{Y_{mj}\mid 0\leqq m,\ 1\leqq j\leqq N(n,m)\}$ is a basis in $\mathfrak{h}$ of spherical harmonics, and $J_p(z)$ are Bessel functions which satisfy: $J_p(z)$ are entire analytic for $p=1, 2, \ldots$, $z^{-1/2}J_p(z)$ are entire analytic for $p=1/2, 3/2, \ldots$

$$|J_p(z)|\leqq\begin{cases} C_p|z|^p, & \text{for } |z|\leqq 1\\ C_p e^{|\operatorname{Im} z|}|z|^{-1/2}, & \text{for } |z|>1\end{cases}$$

and for $p\geqq 5/2$

$$J_{p-2}(z)-J_p(z)=2\frac{d}{dz}J_{p-1}(z) \tag{2}$$

$$J_{p-2}(z)+J_p(z)=2(p-1)z^{-1}J_{p-1}(z). \tag{3}$$

We define $T_0^*(\bar{k})$ for $k\in\mathbb{C}\setminus\{0\}$ to be the integral operator which has the kernel (1). It has to be proved that $T_0^{+\,*}(\bar{k}):=e^{-ikr}T_0^*(\bar{k})$ is continuous on $\tilde{\mathbb{C}}^-$ as a $\mathscr{B}(\mathfrak{h}, L_{-\delta}^2)$-valued function, $\delta>\dfrac{1}{2}$.

Boundedness. — Using (1) we obtain the following bound on the norm.

$$\| e^{-ikr} T_0^*(\bar{k}) \|_{\mathscr{B}(\mathfrak{h},\, L^2_{-\delta})} \leqq 2\pi |k|^{2-n}$$
$$\sup_{m \geqq 0} \int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_{p_m}(kr)|^2,$$
$$k = a + ib, \qquad \rho(r) = (1+r^2)^{1/2}.$$

To see that the r. h. s. is finite we procced as follows, *cf.* [9] :

We replace $\frac{d}{dz}$ by $\frac{1}{k}\frac{d}{dr}$ on the r. h. s. of (2), complex conjugate the equation and finally multiply by (3). The real part of the resulting equation reads

$$|J_{p-2}(kr)|^2 - |J_p(kr)|^2 = 2(p-1)\, |k|^{-2} r^{-1} \frac{d}{dr} |J_{p-1}(kr)|^2. \tag{4}$$

From (4) we obtain for any $R > 0$, $p \geqq \frac{5}{2}$

$$\begin{aligned}\int_0^R dr\, r\, \rho^{-2\delta}(r)\, e^{2br} \{ |J_{p-2}(kr)|^2 - |J_p(kr)|^2 \} \\ = 2(p-1) |k|^{-2} \rho^{-2\delta}(R)\, e^{2bR} |J_{p-1}(kR)|^2 \\ + 2(p-1) |k|^{-2} \int_0^R dr \{ 2r\delta\rho^{-2}(r) - 2b \} \\ \times \rho^{-2\delta}(r)\, e^{2br} |J_{p-1}(kr)|^2. \end{aligned} \tag{5}$$

In particular

$$\int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2 \leqq \int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_{p-2}(kr)|^2, \tag{6}$$

which proves boundedness.

Continuity. — We use (1). Continuity in each "sector" follows easily, so it suffices to show that

$$\int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2 \to 0 \qquad \text{for} \quad p \to \infty$$

uniformly in $k \in K$ where K is any given compact set in $\tilde{\mathbb{C}}^-$.

By Dini's theorem and (6) the problem is reduced to proving for fixed $k \in \tilde{\mathbb{C}}^-$ that

$$\int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2 \to 0 \qquad \text{for} \quad p \to \infty. \tag{7}$$

Let $k \in \tilde{\mathbb{C}}^-$ and $R > 0$ be given. Define

$$a_p = \int_0^R dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2, \qquad p = \frac{1}{2}, 1, \ldots$$

Then by (5), $\{a_p\}$ is convergent along any of the four sequences $p = \frac{1}{2}\left(1, \frac{3}{2}, 2\right) + 2q$, $q \in \mathbb{N} \cup \{0\}$, and

$$a_{p-2} - a_p \geqq 2(p-1)|k|^2 2\delta\rho^{-2}(R)\, a_{p-1}. \tag{8}$$

This implies that $a_p \to 0$ for $p \to \infty$.

Let $\varepsilon > 0$ be given and choose $\frac{1}{2} < \delta' < \delta$. Since (6) holds for δ replaced by δ' there exists a constant $C > 0$ such that

$$\int_0^\infty dr\, r\, \rho^{-2\delta'}(r)\, e^{2br} |J_p(kr)|^2\, dr < C, \qquad p \geqq \frac{5}{2}.$$

Now we choose $R > 0$ such that

$$C\rho^{2(\delta'-\delta)}(R) \leqq \varepsilon/2.$$

Then for all $p \geqq \frac{5}{2}$

$$\begin{aligned}\int_0^\infty dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2 &= \int_0^R + \int_R^\infty \\ &\leqq \int_0^R + \rho^{2(\delta'-\delta)}(R) \int_R^\infty dr\, r\, \rho^{-2\delta'}(r)\, e^{2br} |J_p(kr)|^2 \\ &\leqq \int_0^R dr\, r\, \rho^{-2\delta}(r)\, e^{2br} |J_p(kr)|^2 + \varepsilon/2.\end{aligned}$$

By (8) the r. h. s. is smaller than ε for all sufficiently large p. This proves (7).

To complete the proof of Lemma 1.6 we need to verify that $T_0^{+*}(\bar{k})$ is analytic in $\mathbb{C}^-$. This follows easily from the continuity and weak analyticity.

Since $(-\Delta - k^2) T_0^*(\bar{k}) = 0$, we can use Lemma A 1.1 and obtain that $L^2_{-\delta}$ can be replaced by $H^2_{-\delta}$, *i. e.* $T_0^{+*}(\bar{k}) \in \mathscr{B}(\mathfrak{h}, H^2_{-\delta})$, $\delta > \frac{1}{2}$. □

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