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Renormalized Bethe-Salpeter kernel and 2-particle structure in field theories

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ABSTRACT. — A proof of the existence, analyticity properties and bounds on renormalized Feynman-type integrals $(G\, o\, G\, o \ldots o\, G)_{\text{ren}}$, with renormalized Bethe-Salpeter kernels G at cach vertex, is given for non super-renormalizable theories with 2- and 4-point relevant or marginal operators, such as the Gross-Neveu model, and in a related axiomatic framework in which G is the basic quantity. Results on 2-particle structure (second sheet analyticity, 2-particle asymptotic completeness) are then made rigorous (at weak coupling) in an approach that starts from the expansion $F = \sum (G\, o \ldots o\, G)_{\text{ren}}$ of the 4-point Green function. The alternative derivation based on subtracted Bethe-Salpeter equations and the analysis of the links between the two approaches are also completed.

RÉSUMÉ. — Une preuve de l'existence, des propriétés d'analyticité et de bornes sur les intégrales renormalisées $(G\, o\, G \ldots o\, G)_{\text{ren}}$, avec des noyaux de Bethe-Salpeter renormalisés à chaque vertex, est donnée pour des théories non super-renormalisables dont les parties de renormalisation sont à 2 et à 4 points, telles que le modèle de Gross-Neveu, et dans un cadre axiomatique correspondant où G est la quantité de base. Les résultats sur la structure à deux particules sont alors démontrés rigoureusement à faible couplage dans une approche basée sur le développement $F = \sum (G\, o \ldots o\, G)_{\text{ren}}$ de la fonction de Green à 4 points. La démonstration alternative basée sur des équations de Bethe-Salpeter soustraites et l'analyse des liens entre les deux approches sont aussi complétées.

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INTRODUCTION

Throughout this paper, we consider even field theories with >0 lowest physical mass $\mu=\mu(\lambda)$ (where λ is the coupling or a related small parameter) corresponding to a pole of the momentum-space 2-point function $H_2(k;\lambda)$ at $k^2(\equiv k_0^2-\mathbf{k}^2)=\mu^2$. Our aim is to complete recent works ([1], [2], [3]) on the derivation of 2-particle structure (second-sheet analyticity, 2-particle asymptotic completeness) in *non super-renormalizable* models of constructive field theory, such as the massive Gross-Neveu (GN) model in dimension 2, with 2 and 4-point relevant or maginal operators, and in related axiomatic frameworks in which the Bethe-Salpeter (BS) kernel B, or the renormalized BS kernel G is taken as the basic quantity. In the latter, the analysis includes the reconstruction of the (connected, amputated) 4-point Green function F in terms of B, or of G.

Part I is an introductory review on the subjet. It starts with the simpler super-renormalizable case (Sect. 1), in which the analysis is based on the Bethe-Salpeter kernel B and makes use of its analyticity properties in complex momentum space. The 4-point function F is linked to B in the models, initially in euclidean space, via the BS equation

$$F=B+F\,o\,B=B+B\,o\,F \qquad (F\,o\,B\equiv {=}\!\!(F)\!\!{=}\!\!(B)\!\!{=}) \tag{1}$$

where o denotes Feynman-type convolution with 2-point functions on internal lines. Eq. (1) is then used to obtain desired results on F away from euclidean space. It is also used to define F in the axiomatic framework that starts from B, e. g. at weak coupling through the series

$$F=B+B\,o\,B+B\,o\,B\,o\,B+\dots \tag{2}$$

which is convergent in particular in euclidean space. Sect. 1 also includes an outline, useful for our later purposes, of the U-kernel method ([4], [5]) recently developed [5] for the treatment of the neighborhood of the 2-particle threshold where this series becomes divergent (due to kinematical factors) in space-time dimension 2 (or 3).

As explained in Sect. 2, the methods used in Sect. 1 cannot be directly applied to non super-renormalizable theories in view of ultraviolet divergences of the 2-particle convolution. An efficient way of avoiding these problems in models of constructive theory is based on the use of an alternative kernel $B_{(M)}$ which, however, is not intrinsic and therefore is not a satisfactory starting point in the axiomatic approaches we wish to consider. Approaches of interest in this work, which rely on the actual BS kernel B (if it exists) or on the renormalized kernel G, require a non trivial adaptation of the methods of Sect. 1. They start either ([1], [2]) from the expansion (derived perturbatively in [2])

$$F=G+(G\,o\,G)_{\text{ren}}+(G\,o\,G\,o\,G)_{\text{ren}}+\dots \tag{3}$$

of F in terms of renormalized Feynman-type integrals with kernels G at each vertex, or ([2], [3]) from a la Symanzik [6] subtracted Bethe-Salpeter equations relating F to B, or alternatively to G. Besides analyticity properties, large momentum properties of B or of G are required in either case to treat ultraviolet problems. These approaches, previous results and remaining problems to be treated in this work are presented in Sect. 3.

New results are given in Part. II. The analyticity and large momentum properties of G, or B, to be used, which have been recently established [3] for the GN model and are assumed in the axiomatic framework, are stated in Sect. 1. An important part of the paper is devoted, in Sect. 2, to a subsequent proof of the existence, analyticity properties and bounds on the integrals $(G \circ G \circ \ldots \circ G)_{\text{ren}}$. (The latter are "$\mathscr{G}$-convolution renormalized integrals" in the sense introduced by J. Bros. Renormalization does not apply "inside" the kernels G. Original results of Bros, or more recent results by various authors, on these integrals are, however, insufficient in the present situation: *see* Sect. 3.1 in Part I). The analysis of 2-particle structure presented in [1], [2] is then made rigorous and completed in the approach that starts from the expansion (3). In view of results on individual terms $(G \circ \ldots \circ G)_{\text{ren}}$, this expansion is shown to make sense non perturbatively for the GN model, at weak coupling, for bounded values of the energy in euclidean momentum space and in a further 2-sheeted domain. It is used to define F in terms of G from the axiomatic viewpoint. The neighborhood of the 2-particle threshold (where this series is again divergent in dimension 2) is treated by an adaptation of the U–kernel method described in Sect. 1 of Part I.

The alternative approach based on subtracted Bethe-Salpeter equations has been proposed in [2] (in terms of G) and implemented in a precise way for the GN model in [3] (in terms of B or G). We explain in Sect. 3 how this approach can also be made rigorous in the axiomatic framework in which one starts either from B, or from G. As a byproduct of our methods, the algebraic links between the various approaches are exhibited in a way which completes the previous analysis of [1], [2]. In particular, a derivation of the subtracted BS equations in terms of G from Eq. (3), more direct and transparent than that given in [2], follows.

Notes. — 1. In contrast to the situation in the super-renormalizable case, where the series (2) is convergent at small coupling in the whole euclidean space (*see* Sect. 1 in Part I), the series (3) or the related series arising from one of the subtracted BS equations will not be shown to be convergent (and will not allow the definition of F, in the axiomatic framework) in the whole euclidean space, even at weak coupling: results are established only for bounded values of the channel energy $k_0 = (p_1 + p_2)_0$ (with $\mathbf{k} = \mathbf{p}_1 + \mathbf{p}_2$ e.g. fixed at zero). This is sufficient for the study of 2-particle structure, which involves local properties in complex momentum space. Values of k_0 can be chosen arbitrarily large but for smaller and smaller values of the coupling. A more complete study should be possible, but it would require the use of supplementary sign properties in the GN model, or similar assumptions in the axiomatic framework, in order to avoid the possible occurrence of "Landau ghosts" (poles) in euclidean space (*see* in this connection, the discussion given in [2]).

2. Results on the integrals $(G\,o\,G\ldots o\,G)_{ren}$ have also been obtained independently (J. Bros, private communication) by a different method to be presented in [20] (*see* in this connection Sect. 3.1. of Part I).

I. 2-PARTICLE STRUCTURE IN SUPER-RENORMALIZABLE AND NON SUPER-RENORMALIZABLE THEORIES: INTRODUCTORY REVIEW

1. The super-renormalizable case

1.1. *Models of constructive field theory*

The analysis is based on the Bethe-Salpeter (BS) kernel B in a $2 \to 2$ channel, e.g. $(1, 2) \to (3, 4)$, which is perturbatively the sum of 2-particle irreducible ($2-p.i.$) connected graphs in that channel. B, which can be defined non perturbatively in various ways in euclidean space ([7]-[10]), is shown ([8], [10]), for any $\varepsilon > 0$ and λ sufficiently small, to be analytic, in complex momentum space, in a strip around euclidean space going up to $s = (4\mu)^2 - \varepsilon$ in Minkowski space, where $s = (p_1 + p_2)^2$ is the squared center-of-mass energy of the channel. The 2-point function $H_2(k)$ is on the other hand shown to be analytic up to $k^2 = (3\mu)^2 - \varepsilon$ apart from its pole at μ^2. From these analyticity properties of B and H_2 and the decrease of H_2 in euclidean space (which ensures good convergence properties of the integrals), it is shown in [11] (with complements in [5]), via local distortions of integration contours (so as to avoid the poles of the 2-point functions) and Fredholm theory in complex space, that F can be analytically continued as a meromorphic function in a 2-sheeted (d even) or multisheeted (d odd) domain around the 2-particle threshold $s = 4\mu^2$, where d is the dimension of space-time, with possible poles in s to be interpreted as 2-particle bound states or resonances. Moreover, the unitary-type equation

$$F_+ - F_- = F_+ *_+ F_- \tag{4}$$

where $*_+$ denotes on-mass-shell convolution and which (essentially) characterizes (*see* [11]) 2-particle asymptotic completeness, is also derived in [11] in the region s real, $4\mu^2 < s < 16\mu^2 - \varepsilon$ provided there is no pole of F in that region. In (4), the signs + and - refer to values obtained at s real $>4\mu^2$ from the sides $\operatorname{Im} s>0$ and $\operatorname{Im} s<0$ respectively of the physical sheet. Eq. (4) follows from Eq. (1), which yields $F_+ = B + B o_+ F_+$, $F_- = B + F_- o_- B$ (using the analyticity of $B: B_+ = B_-$), and from the relation (*see* [11] and references therein) $o_+ - o_- = *_+$. For a new simple and unified algebraic derivation of Eq. (4) and of further formulae below, *see* App. 1 of [12]. See also a derivation of results similar to above for even $P(\varphi)_2$ models (by somewhat different methods) in [13].

At weak coupling, bounds in cst λ are established on B, related bounds (*see* e. g. [5] and references therein) follow on terms $B o B$, $B o B o B$, ... (defined in a 2-sheeted or multisheeted domain via local distortions of integration contours) and control on poles of F follows from convergence properties of the Neumann series (2) of F in terms of B. In particular, this series is convergent, and thus F has no pole, in a neighborhood of the real $s-$axis in the physical or second sheet, apart, in dimension $d=2$, 3, from a neighborhood of the 2-particle threshold $s=4\mu^2$. The latter is treated for (even) $P(\varphi)_2$ in [13], [14] and with a more general method, in the line of [11] and relying on further ideas and methods of [4], in [5]. This is the method adapted later in the non super-renormalizable case. We outline it below. The kernel U is defined in terms of B through the equation:

$$U = B + U \nabla B (= B + B \nabla U) \tag{5}$$

where, at d even (see the case d odd in [5]), $\nabla = o - \frac{1}{2} *$, $*$ denoting on mass-shell convolution at s real $>4\mu^2$ or its analytic continuation in s. Whereas o or $*$ generate at e. g. $d=2$ a kinematical factor $\sigma^{-1/2}$, $\sigma = s - 4\mu^2$, responsible for the divergence of the series (2), ∇ is regular and preserves analyticity. U is then defined, as a locally analytic function in a neighborhood of the real $s-$axis including the 2-particle threshold $s=4\mu^2$, via the convergent series

$$U = B + B \nabla B + B \nabla B \nabla B + \dots \tag{6}$$

Eqs. (1) (5) yield on the other hand ([4], [12]) the relation

$$F = U + \frac{1}{2} U * F \left(= U + \frac{1}{2} F * U \right) \tag{7}$$

from which an explicit expression of F in terms of U follows at $d=2$. As a consequence, one sees that there is no pole at s real $>4\mu^2$, and one pole [the zero of $a(s) - u(s; \lambda)$ where $a(s)$ is a kinematical factor of the form cst $\sqrt{s}\,\sigma^{1/2}$ and u is the on shell restriction of U] at s real $<4\mu^2$, either in the physical sheet (2-particle bound state) or in the second sheet, depending on a sign in the theory.

Remark. — An alternative derivation of Eq. (4), useful later, follows equally from Eq. (7), by taking into account the analyticity of U and the straightforward relation $\frac{1}{2}*_{+}-\frac{1}{2}*_{-}=*_{+}$ (at d even).

1.2. *Axiomatic approach based on the Bethe-Salpeter kernel*

All results above apply equally in an axiomatic framework (*see* e. g. [7]) in which the BS kernel B is considered as the basic quantity (depending on the parameter λ), which then plays the role of the potential in non relativistic theory (*see* in this connection [15]) and on which properties similar to above are assumed. F is then defined at small λ from B via Eq. (1) or the series (2) and results on 2-particle structure follow as above.

2. Ultraviolet problems in non super-renormalizable theories. The non intrinsic kernel $B_{(M)}$

In non super-renormalizable theories, ultraviolet problems prevent a similar use of the BS kernel and BS equation: integrals such as $B\,o\,F$, $B\,o\,B$, ... might be divergent and as noticed in [11], the BS kernel B might as a matter of fact not exist. It turns out [3] that a kernel B satisfying Eq. (2), initially in euclidean space, does exist for the GN model (in the actual theory without cut-off) and satisfies the same analyticity properties as in Sect. 1. (For the GN model, B has a perturbative content, but only in the theory with cut-off. It is then the sum of non renormalized graphs with bare couplings λ_ρ; $\lambda_\rho \to 0$ in the limit when the cut-off is removed). However, the convergence of the integrals $B\,o\,F$, $B\,o\,B$, ... is no longer due to inverse powers of the relative energy-momenta over which there is integration (arising in the super-renormalizable case from the decrease of the 2-point function), but only to inverse logarithms (linked to the property of asymptotic freedom of the model) in bounds on F and B. As a consequence, previous methods do not apply.

Whether B exists or not, one might then wish to first introduce an ultraviolet cut-off in the theory, to be ultimately removed. Fredholm theory in complex space [11] then does apply, but there is no control on poles of F, which might e. g. accumulate on the real $s-$axis in the limit when the cut-off is removed, for any given value of λ (Martin's type pathology [16]).

A way to avoid ultraviolet problems and to obtain desired results on 2-particle structure has been implemented in [17] for the GN model, in accordance with original ideas of [11]. It relies on an alternative kernel $B_{(M)}$, defined in [17] from phase space analysis, which is linked to the Green function F (of the actual theory without cut-off) via a modified, "regularized" BS equation of the form [11]

$$F = B_{(M)} + F\, o_{(M)}\, B_{(M)} \tag{9}$$

with a suitable *fixed* ultraviolet cut-off in the operation $o_{(M)}$. The same analyticity properties as in Sect. 1 are established on $B_{(M)}$ in [17] and results on 2-particle structure are then obtained in the same way as in Sect. 1.

This method can probably be applied to more general (renormalizable or non renormalizable) theories that might be defined in constructive theory. However, the kernel $B_{(M)}$, which does not have a simple perturbative content, is not intrinsic: it depends on the cut-off M, linked in the constructive framework to technical choices in phase-space analysis. Therefore such a kernel is not a good candidate to replace the BS kernel B in axiomatic approaches analogous to that of Sect. 1.2.

3. Non super-renormalizable theories: approaches based on the (renormalized) BS kernel

Two approaches, based on the renormalized BS kernel G (which is perturbatively, in the GN model, the sum of renormalized $2-p.i.$ graphs with renormalized couplings) have been proposed for the theories mentioned in the introduction, as already indicated there. The BS kernel B (if it exists) can alternatively be used in the second one. We note that in the theories under consideration, B and G, if both exist as in the GN model [3], differ only by a constant (depending on λ): *see* [1], [3]. In axiomatic approaches, the kernel G has been considered in previous works such as [2] or [1] as the basic one, more likely to exist in general than B. As a matter of fact, the existence of B in the GN model appears to be linked to asymptotic freedom and one might think that G might exist, whereas B would not, in other hypothetical "a la φ_4^4" theories. However, the existence of such theories remains an open problem and is usually considered as rather doubtful.

As will appear, a crucial role will be played (in both approaches) not only (as in Sect. 1) by analyticity properties of G, or B, but also by large momentum properties of these kernels in euclidean space. More precisely, the latter apply to differences $G-\mathring{G}$, $\mathring{G}-{}_0\mathring{G}$, $\mathring{G}-\mathring{G}_0$ of values of G (or related differences of values of B), where an index *o* on the top, the left and the right means respectively that the channel energy-momentum, the relative energy-momentum of the two initial, or of the two final, particles is fixed at zero. The idea that these differences should have better asymptotic properties than G, or B, that should allow the treatment of ultraviolet problems, already implicit in [6], has been emphasized in [2] and made precise for the GN model in [3]. It is also the basis of our complementary analysis.

Throughout this section, we restrict our attention, as is natural for the theories considered, to even dimension $d: d=2$ as in the GN model or $d=4$ in other hypothetical theories.

3.1. *Expansion of* F *in terms of renormalized integrals* $(G\,o \ldots o\,G)_{\text{ren}}$

The first approach ([1], [2]) starts from the (*a priori* formal) expansion (3) of F. The hope is that it will play a role similar to the expansion (2) in Sect. 1. (Even if B exists, the series (2) itself is expected to be always divergent, even at small coupling, in the non super-renormalizable theories now considered). That the integrals $(G\,o \ldots o\,G)_{\text{ren}}$ might be well defined non perturbatively (in contrast to integrals $G\,o \ldots o\,G$ which are expected, and shown [3] in the GN model, to be divergent) was supported by the work [18], where the euclidean existence of such integrals is proved by a generalization of Zimmerman's forest formalism, modulo specified assumptions on large momentum properties of derivatives of the vertex functions: differentiation is assumed to improve the asymptotic behaviour. (For an extension of these results to the primitive domain in some cases, *see* [19]). However, these results are not sufficient. First, as in Sect. 1, one wishes to establish the existence of such integrals not only in euclidean space but also in a further 2-sheeted analyticity domain. Secondly, although the general idea of [18] will remain valid, some aspects of the large momentum properties assumed on the vertex functions (isotropic properties) cannot be expected for G, as noticed in [2] and as confirmed by the precise analysis of these properties made more recently in [3] for the GN model. Finally, precise bounds (analogous to those obtained on the terms $B\,o \ldots o\,B$ in Sect. 1) are needed for the subsequent derivation of convergence properties of the series.

These problems are left aside in [1] where the purpose on this topic is to state (and justify) suitable analyticity properties of the terms $(G\,o \ldots o\,G)_{\text{ren}}$, including discontinuity formulae around the 2-particle threshold intended to derive in turn the 2-particle asymptotic completeness equation (4). As a matter of fact, such discontinuity formulae are established there for terms $(G\,o_{\rho} \ldots o_{\rho}\,G)_{\text{ren}}$ in which an ultraviolet cut-off is included in each operation o, it is conjectured that they still hold in the limit when the cut-off is removed, and Eq. (4) is derived formally.

A proof of the existence and analyticity properties (including discontinuity formulae) of the integrals $(G\,o \ldots o\,G)_{\text{ren}}$ has then been outlined in [2] in the line of [18], under more general assumptions on the vertex functions G. However, the latter are still unsatisfactory, in view of the more recent analysis of [3] for the GN model. A further non trivial work [20] appears at that stage to be needed in this approach to adapt the proofs and to obtain desired bounds on the integrals. Our method, presented in Sect. 2 of Part II is different and simpler. In contrast to that of [18], [2], [20], which might potentially be applied to more general renormalized integrals that might be encountered in other contexts, it is more specifically adapted to the integrals $(G\,o\ldots o\,G)_{\mathrm{ren}}$ under consideration. The latter will be expressed in terms of the differences $G-\mathring{G}$, $\mathring{G}-\mathring{G}_0$, $\mathring{G}-{}_0\mathring{G}$ of values of G, to which the large momentum properties established for the GN model, or assumed, apply, and are then treated by a simple adaptation of a lemma of [3].

The neighborhood of $s=4\mu^2$, where the series (1) becomes divergent, will be treated by an adaptation of the U−kernel method outlined in Sect. 1 of Part I. A suitable definition of U in terms of G will be given to that purpose.

3.2. *Subtracted Bethe-Salpeter equations*

The second approach starts from "subtracted BS equations". Such equations have been formally derived from the BS equation in [6] where they involve the BS kernel B, essentially through differences $B-\mathring{B}$, $\mathring{B}-\mathring{B}_0$, ... As already mentioned, B and G if both exist, differ by a constant, so that differences of values of B, or of G, coincide, and hence Symanzik's equations can be equally expressed in terms of G, as noticed in [1] (where a slightly more refined argument introducing an ultraviolet cut-off, to be ultimately removed, is also given to cover the case when G would exist in the limit, but not B). An alternative formal derivation of the latter equations from Eq. (3), *i.e.* directly in terms of G, has been given in [2] where G is considered as the basic kernel and where one does not wish to introduce B (either with or without cut-off). The derivation of one of the relevant equations, given in [2] through an intermediate "renormalized BS equation", is, however, indirect. A more direct derivation follows from our expressions of the integrals $(G\,o\ldots o\,G)_{\mathrm{ren}}$ in terms of $G-\mathring{G}$, $\mathring{G}-{}_0\mathring{G}$, $\mathring{G}-\mathring{G}_0$.

A program for deriving desired results on F has then been proposed in [2]. A first equation, relating F to $\mathring{F}$ and to $G-\mathring{G}$, is intended to allow the analytic continuation of F away from euclidean space if $\mathring{F}$ is under control. A second equation, relating $\mathring{F}$ to differences $\mathring{G}-\mathring{G}_0$, ... is intended to first reconstruct $\mathring{F}$ in terms of G in the axiomatic framework. (These two equations will be found in Sect. 3 of Part II). This program has been carried out in [2] modulo assumptions on the existence of inverses of relevant operators, and has then been implemented in a precise way in [3] for the weakly coupled GN model with the following differences. On the one hand, one starts from the BS kernel B (which does exist). On the other hand, only the first à la Symanzik equation is used because F (hence $\mathring{F}$) is under control in euclidean space from the outset in the model.

The neighborhood of $s=4\mu^2$ is treated in [3] by a suitable definition of U in terms of B (different from that of Sect. 1, in order to avoid ultraviolet problems). A somewhat simpler definition will be given in Part II in that context.

We explain in Sect. 3 of Part II how to implement equally this method in an axiomatic framework that starts from B, or alternatively from G. The main remaining point to be treated is the prior reconstruction of $\mathring{F}$ via the second a la Symanzik equation. This follows from a lemma established in [3] for other purposes.

II. NON SUPER-RENORMALIZABLE THEORIES: COMPLEMENTARY RESULTS

1. Assumptions on the (renormalized) Bethe-Salpeter kernel

The following notations, analogous to those of [6], [2], [3] will be used. The 4-point function F, as also B, G are expressed in terms of the variables $k=p_1+p_2$, $z=\frac{p_1-p_2}{2}$, $z'=\frac{p_3-p_4}{2}$. As usually, we take k of the form $(k_0, \mathbf{o})$. Given a kernel $A(k, z, z')$, an index zero on the top, on the left and on the right means that k, z or z' respectively is fixed at zero. E. g.:

$$\mathring{A}_0(k, z, z') \equiv A(o, z, o) \tag{10}$$

On the other hand, $\omega(k, z)$ will denote the product $H_2(k_1)H_2(k_2)$ of 2-point functions $\left(k_1=\frac{k}{2}+z,\ k_2=\frac{k}{2}-z\right)$. According to previous notations $\mathring{\omega}(k, z)\equiv\omega(o, z)$.

Given kernels A_1, A_2, $A_1\,\omega\,A_2$ is the integral

$$A_1\,\omega\,A_2 \equiv A_1\,o\,A_2 = \int A_1(k, z, \zeta)\,A_2(k, \zeta, z')\,\omega(k, \zeta)\,d\zeta \tag{11}$$

As usually the integration contour, initially euclidean space, is locally distorted, so as to avoid the poles of 2-point functions in $\omega(k, \zeta)$, when k will vary away from euclidean space.

$A_1\,\mathring{\omega}\,A_2$ or $A_1(\omega-\mathring{\omega})A_2$ are defined similarly with $\omega(k, \zeta)$ replaced by $\omega(o, \zeta)$ or $\omega(k, \zeta)-\omega(o, \zeta)$ in the integration measure. $A_1\,\omega\,A_2\,\omega\,A_3\ldots$, with some ω' s replaced possibly by $\mathring{\omega}$ or $\omega-\mathring{\omega}$, is a corresponding multiple convolution integral over successive variables ζ_1, ζ_2, ... We consider a theory depending on a parameter λ, corresponding in the Gross-Neveu model to the renormalized coupling and in which G and H_2, or alternatively B and H_2, are known to exist and to satisfy the properties indicated below (GN model) or in which these functions are considered as the basic quantities, assumed to exist and to satisfy these properties. For definiteness, we state below relevant properties of G. If one starts in Sect. 3 from B, the same properties will be assumed with G replaced everywhere by B: in the GN model, where both B and G exist, the two sets of properties are established, as is natural in view of the form of these properties and of the fact that B and G differ only by a constant, which is of the order of λ at lowest order in λ.

Concerning H_2, decrease properties to be assumed depend on the dimension d of space-time. We consider the case $d=2$ (such as the GN model) and the hypothetical scalar case $d=4$.

Although better bounds (in which η is removed and which include inverse powers of ln) are established for the GN model in [3], we state below properties in a form that is sufficient for our purposes and might cover more general (hypothetical) theories.

ASSUMPTION 1. — Given $\varepsilon>0$, there exists $\lambda_0>0$ such that, $\forall \lambda<\lambda_0$, G is an analytic function of k_0, z, z' for $|\mathrm{Re}\, k_0| \leqq 4\mu-\varepsilon$, $|\mathrm{Im}\, z|$, $|\mathrm{Im}\, z'| \leqq \mu-\varepsilon$ and satisfies in this domain the following bounds for some $\eta>0$ $(\eta<1/4)$ independent of λ:

(a) $$|(G-\mathring{G})(k_0, z, z'; \lambda)| < f(k_0)\lambda^2 \times \mathrm{Inf}\left\{\frac{1}{(1+|z|)^{1-\eta}}, \frac{1}{(1+|z'|)^{1-\eta}}\right\} \tag{12}$$

(b) $$|(\mathring{G}-\mathring{G}_0)(k_0, z, z'; \lambda)| < f(k_0)\lambda^2\left(\frac{1+|z'|}{1+|z|}\right)^{1-\eta}, \quad |z'| \leqq |z| \tag{13}$$

$$|(\mathring{G}-{}_0\mathring{G})(k_0, z, z'; \lambda)| < f(k_0)\lambda^2\left(\frac{1+|z|}{1+|z'|}\right)^{1-\eta}, \quad |z| \leqq |z'| \qquad (13')\ (14)$$

(c) $$|G(k_0, z, z'; \lambda)| < f(k_0)\lambda$$

where f is bounded by a constant when k_0 varies in a given bounded region.

ASSUMPTION 2. — $\forall \lambda<\lambda_0$, $H_2(p, \lambda)$ is analytic in a strip around euclidean space going up to $p^2=(3\mu)^2-\varepsilon$ in Minkowski space, apart from a pole at $p^2=\mu^2$, $\mu=\mu(\lambda)=\mu_0+0(\lambda)$, $\mu_0>0$. It decreases like $1/|p|^2$ $(d=4)$ or $1/|p|$ $(d=2)$ as p tends to infinity in euclidean directions. Moreover:

$$|H_2(k+z)-H_2(z)| \left\{\begin{array}{ll} < f(k_0)/(1+|z|)^3 & (d=4) \\ < f(k_0)/(1+|z|)^2 & (d=2) \end{array}\right\}. \tag{15}$$

Remarks. — 1. The bounds (15) can be equivalently stated on derivatives of H_2, in the spirit of [2].

2. Assumption 2 yields corresponding bounds on $(\omega-\mathring{\omega})(k, z)$ as $z \to \infty$ in euclidean directions. E. g. at $d=4$

$$|\omega(k, z)-\omega(o, z)| < g(k_0)/(1+|z|)^5 \tag{16}$$

where g is bounded when k_0 varies in a bounded region.

2. Renormalized integrals $(G\,o\,G\ldots o\,G)_{\text{ren}}$ and related results

2.1. *Definitions and preliminary relations*

Given the kernel G, $(G\,o\,G\ldots o\,G)^{(n)}_{\text{ren}}$ (n factors G, $n \geqq 2$) is formally the sum of all terms associated with Zimmerman's forests, with the convention that renormalization does not apply "inside" each factor G. I. e. it is the sum, over all possible sets of brackets with no overlap, of corresponding contributions. By convention $[A] = -A(0, 0, 0)$. E. g.:

$$(G\,o\,G)_{\text{ren}} = G\,o\,G + [G\,o\,G] \tag{17}$$

$$(G\,o\,G\,o\,G)_{\text{ren}} = G\,o\,G\,o\,G + [G\,o\,G]\,o\,G + G\,o\,[G\,o\,G] + [G\,o\,G\,o\,G] + [[G\,o\,G]\,o\,G] + [G\,o\,[G\,o\,G]] \tag{18}$$

where e. g.:

$$[G\,o\,G]\,o\,G \equiv [G\,o\,G](1\,o\,G) \tag{19}$$

This definition is so far only formal: each individual contribution is expected to be infinite. The algebraic manipulations described below will transform it into the expression (23) at $k=0$ and (25) at $k \neq 0$ that will be finite and will define $(G\,o\ldots o\,G)_{\text{ren}}$ (initially in euclidean space). Alternatively, an ultraviolet cut-off is first introduced in each operation o, *i. e.* o is replaced by a suitable operation o_ρ. Initial contributions to $(G\,o_\rho\ldots o_\rho\,G)_{\text{ren}}$ are then well defined, the above algebraic procedure is legitimate and yields alternative expressions of $(G\,o_\rho\ldots o_\rho\,G)_{\text{ren}}$ from which the existence of the $\rho \to \infty$ limit follows.

Let $\overline{[\ \ ]}^{(p)}$, $p \geqq 2$, denote the sum of all contributions to $(G\,o\,G\ldots o\,G)^{(p)}_{\text{ren}}$ (p factors G) in which an overall bracket [ ] includes all G's. E. g.:

$$\overline{[G\,o\,G]} = [G\,o\,G] \tag{20}$$

$$\overline{[G\,o\,G\,o\,G]} = [G\,o\,G\,o\,G] + [[G\,o\,G]\,o\,G] + [G\,o\,[G\,o\,G]] \tag{21}$$

Then:

$$(G\,o\ldots o\,G)^{(n)}_{\text{ren}} = \sum \ldots G\,\omega\,G\ldots\omega\,G\,\omega\,\overline{[\ \ ]}\,\omega\,G\ldots\omega\,G\,\omega\,\overline{[\ \ ]}\,\omega\,\overline{[\ \ ]}\,\omega\ldots \tag{22}$$

where the sum runs over all possible multiple convolutions of factors G and $\overline{[\ \ ]}$ in arbitrary order and number (and arbitrary numbers of G's inside each factor $\overline{[\;\;]}$), except that the total number of factors G, either explicit or inside the brackets $\overline{[\;\;]}$ must be equal to n.

The following lemmas will be useful.

LEMMA 1. – $(k=0)$

$$\begin{aligned}(\mathrm{G}\,o\ldots o\,\mathrm{G})^{(n)}_{\mathrm{ren}}(0, z, z') = {} & (\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\,\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\ldots\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\\ & +(\mathring{\mathrm{G}}_0)\,\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\,\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\ldots\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\\ & +\sum(\mathring{\mathrm{G}}-{}_0\mathring{\mathrm{G}})\,\mathring{\omega}\ldots\mathring{\omega}\,(\mathring{\mathrm{G}}-{}_0\mathring{\mathrm{G}})\,\mathring{\omega}\,(\mathring{\mathrm{G}}_0)\,\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\,\mathring{\omega}\ldots\mathring{\omega}\,(\mathring{\mathrm{G}}-\mathring{\mathrm{G}}_0)\\ & +(\mathring{\mathrm{G}}-{}_0\mathring{\mathrm{G}})\,\mathring{\omega}\,(\mathring{\mathrm{G}}-{}_0\mathring{\mathrm{G}})\ldots\mathring{\omega}\,(\mathring{\mathrm{G}}-{}_0\mathring{\mathrm{G}})\,\mathring{\omega}\,\mathring{\mathrm{G}}_0 \qquad (23)\end{aligned}$$

with a total number of n factors in each term.

Remark. – An anologous expression in which the roles of $\mathring{\mathrm{G}}_0$ and ${}_0\mathring{\mathrm{G}}$ are exchanged also holds.

Proof of Lemma 1. – Let $\widehat{[\;\;]}$ denote the sum of contributions to $\overline{[\;\;]}$ including, inside their overall bracket, no internal sequence of the form $\ldots[\;\;]\,o\ldots o\,[\;\;]\ldots$ with two or more sub-brackets $[\;\;]$. We first show that

$$\begin{aligned}(\mathrm{G}\,o\ldots o\,\mathrm{G})^{(n)}_{\mathrm{ren}}(0, z, z') = {} & \mathring{\mathrm{G}}\,\mathring{\omega}\ldots\mathring{\omega}\,\mathring{\mathrm{G}}\\ & +\sum\mathring{\mathrm{G}}\,\mathring{\omega}\ldots\mathring{\mathrm{G}}\,\mathring{\omega}\,\widehat{[\;\;]}^{(p)}\,\mathring{\omega}\,\mathring{\mathrm{G}}\ldots\mathring{\omega}\,\mathring{\mathrm{G}}\\ & +\Sigma\,\mathring{\mathrm{G}}\,\mathring{\omega}\ldots\mathring{\omega}\,\mathring{\mathrm{G}}\,\mathring{\omega}\,\widehat{[\;\;]}^{(p)}+\Sigma\,\widehat{[\;\;]}^{(p)}\,\mathring{\omega}\,\mathring{\mathrm{G}}\ldots\mathring{\omega}\,\mathring{\mathrm{G}}+\widehat{[\;\;]}^{(p)} \qquad (24)\end{aligned}$$

with a total number of n factors G in each term, including $p \geqq 2$ factors G inside the brackets.

In fact, at $k=0$, a term in (22) with an extremal factor $\overline{[\;\;]}$ on the left and a (different) extremal factor $\overline{[\;\;]}$ on the right is equal to its value at $k=z=z'=0$ (since $k=0$ and since it does not depend on z, z', extremal factors being constant). Hence, this term and the same term with a further overall bracket cancel each other. Formula (24) follows from a simple application of this remark.

To identify (23) and (24), one may e. g. start from (23), and expand all terms. One reobtains (i) the term $\mathring{\mathrm{G}}\,\mathring{\omega}\ldots\mathring{\omega}\,\mathring{\mathrm{G}}$ (ii) terms including one or more factors ${}_0\mathring{\mathrm{G}}$, one or more factors $\mathring{\mathrm{G}}_0$ on the right of the former ones and zero, one or more factors $\mathring{\mathrm{G}}$ (iii) terms with factors $\mathring{\mathrm{G}}_0$ and possibly factors $\mathring{\mathrm{G}}$, but without factors ${}_0\mathring{\mathrm{G}}$. The sum of the terms (i) and (ii) is just equal to the right-hand such of (24) in view of the definition of $\widehat{[\;\;]}$. The terms obtained in (iii) from different sources cancel each other.

Q.E.D.

LEMMA 2 (1). — $(k \neq 0)$

$$(25)\quad (G o \ldots o G)^{(n)}_{\text{ren}} = \sum \ldots \mathring{F}^{(n_i)} (\omega - \mathring{\omega}) \mathring{F}^{(n_{i+1})} \ldots \mathring{F}^{(\cdot)} \omega (G - \mathring{G}) \omega (G - \mathring{G}) \ldots \omega \mathring{F}^{(\cdot)} \ldots$$

$$\mathring{F}^{(p)} \equiv G o G \ldots o G)^{(p)}_{\text{ren}} \Big|_{k=0} \qquad (26)$$

where, in (25), the sum runs over all possible multiple convolutions of factors $\mathring{F}^{(\cdot)}$ and $G - \mathring{G}$ in arbitrary order and number, (and arbitrary numbers n_i in the factors $\mathring{F}^{(\cdot)}$), except that the total number of factors $G - \mathring{G}$ and of factors G inside each $\mathring{F}^{(\cdot)}$ must be equal to n. Two successive factors are separated by ω except that two successive factors $\mathring{F}^{(n_i)}$, $\mathring{F}^{(n_{i+1})}$ are separated by $\omega - \mathring{\omega}$.

Proof of Lemma 2. — Start e. g. from (22), replace explicit factors G by $(G - \mathring{G}) + \mathring{G}$, ω by $(\omega - \mathring{\omega}) + \mathring{\omega}$, $\overline{\text{expand}}$ and replace the sum of all possible sequences of factors $\mathring{G}$ and [ ] separated by $\mathring{\omega}$'s, with a given total number p of factors G inside or outside the brackets, by $\mathring{F}^{(p)}$.

2.2. *Applications.*

THEOREM 1 $(d = 2, 4, \ \lambda < \lambda_0)$. — The terms $(G o G \ldots o G)_{\text{ren}}$ (n factors G) are well defined analytic functions in a common 2-sheeted domain around $s = 4\mu^2$, including euclidean space and the physical sheet up to $s = (4\mu)^2 - \varepsilon$. They satisfy bounds of the form:

$$\left| (G o G \ldots o G)^{(n)}_{\text{ren}} (k, z, z'; \lambda) \right| < C_1(k) (C_2(k) \lambda)^n [(1 + |z|)(1 + |z'|)]^{2n} \qquad (27)$$

where $C_1(k)$, $C_2(k)$ are bounded when $s = k_0^2$ varies in given compact bounded regions $\mathcal{R}$ in the 2-sheeted domain of analyticity, outside at $d = 2$ values of s in a complex neighborhood V of $s = 4\mu^2$.

Proof. — At $k = 0$, it is sufficient to use formula (23) and the bounds on the terms $(\mathring{G} - \mathring{G}_0) \mathring{\omega} (\mathring{G} - \mathring{G}_0) \ldots \mathring{\omega} (\mathring{G} - \mathring{G}_0)$, $1 \mathring{\omega} (\mathring{G} - \mathring{G}_0) \ldots \mathring{\omega} (\mathring{G} - \mathring{G}_0)$, $(\mathring{G} -_0 \mathring{G}) \mathring{\omega} (\mathring{G} - \mathring{G}_0) \mathring{\omega} \ldots \mathring{\omega} \mathring{G}_0$ (q factors $\mathring{G} - \mathring{G}_0$, or $\mathring{G} -_0 \mathring{G}$, $q \leq n$) given in the proof of lemma 3 of [3] or obtained in the same way. Bounds of the form (27) at $k = 0$ follow on all individual terms of (23) and in turn on the sum (since the number of individual terms is n).

At $k \neq 0$, it is sufficient to use Lemma 2 together with the previous bounds on $\mathring{F}^{(p)}$ $(p \leq n)$ and the decrease of $\omega - \mathring{\omega}$ or $G - \mathring{G}$ as $\zeta \to \infty$. (The number of terms in (25) is bounded e. g. by 4^n). Analytic continuation in a 2-sheeted domain is made in the usual way by distorting locally integration

(1) After this work was completed, we have noticed from discussions with J. Bros that this lemma is closely related to a formula given in Appendix C of [2].

contours so as to avoid the poles of the 2-point functions in the ω's. As $s \to 4\mu^2$, $C_1(k)$ and $C_2(k)$ can be seen to become infinite at $d=2$, using e. g. the decomposition of o as $\nabla + \frac{1}{2} *$ where ∇ is regular (preserves analyticity) and gives no problem, but where $*$ yields (*see* Sect. 2) a factor $\sigma^{(d-3)/2}$, *i.e.* $\sigma^{-1/2}$ at $d=2$. C_1, C_2 remain bounded at $d=4$, where $\sigma^{(d-3)/2} \equiv \sigma^{1/2}$.

THEOREM 2. — (i) For s varying in a given bounded region, $s \notin V$, there exists $\lambda_0' > 0$ such that, $\forall \lambda < \lambda_0'$, the series $G + \sum_{n \geq 2} (G o \ldots o G)_{\text{ren}}^{(n)}$ is convergent and defines F as an analytic function in a corresponding 2-sheeted domain. (ii) $\exists \lambda_0'' > 0$ such that, $\forall \lambda < \lambda_0''$, F can be analytically continued to values of s in V with no pole at $d=4$, one pole at s real $< 4\mu^2$ at $d=2$, either in the physical or second sheet (depending on sign properties). (iii) The 2-particle asymptotic completeness relation (3) holds at s real, $4\mu^2 < s < 16\mu^2 - \varepsilon$.

Proof. — (i) direct corollary of Theorem 1.

(ii) ($d=2$) We define the kernel U by the relation

$$U = G + (G \nabla G)_{\text{ren}}^{(\text{mod})} + (G \nabla G \nabla G)_{\text{ren}}^{(\text{mod})} + \ldots \tag{28}$$

where ∇ has already been introduced in Sect. 1 and $(G \nabla G \ldots \nabla G)_{\text{ren}}^{(\text{mod})}$ is defined as $(G o G \ldots o G)_{\text{ren}}$ with o replaced by ∇, except that operations o *inside* brackets are unchanged. (∇ denotes, at d even, integration with the same integration measure $\omega(k, \zeta)\, d\zeta$ but over a non distorted contour which remains invariant as $s \to 4\mu^2$). Since ∇ is regular (= preserves analyticity), all individual terms in (28) are *analytic* (with no singularity at $s = 4\mu^2$) in a domain that contains the part $[0, 16\mu^2 - \varepsilon[$ of the real s-axis and satisfy in particular bounds of the form (27) with (different) functions $C_1(k)$, $C_2(k)$ which are bounded as $s \to 4\mu^2$. At small coupling, the series (28) is thus convergent and defines U as a corresponding analytic function. For $s \notin V$, Eq. (7) follows from the expansions of F and U, the relation $o = \nabla + \frac{1}{2} *$, suitable regroupings of terms and convergence properties. Announced results then follows as in Sect. 1 of Part I. The same explicit formula expressing F in terms of U follows from Eq. (7), and the analyticity of U allows the analytic continuation of F to values of s in V.

(iii) From Eq. (7): same proof as that indicated in the remark that concludes Sect. 1.1 in Part I.

Alternatively, at least (if $d=2$) for $s \notin V$, from discontinuity formulae on individual terms $(G o \ldots o G)_{\text{ren}}$ according to the method proposed in [1] (*see* Sect. 3.1 in Part I). The latter ([1], [2]) are established in a rigorous way by the same methods as above. Subsequent algebraic arguments are made rigorous whenever convergence properties hold.

3. Subtracted Bethe-Salpeter equations and applications

In this section, we outline the second method based on subtracted BS equations. In Sect. 3.1, the first subtracted BS equation, which expresses F in terms of $\mathring{F}$ and of $B-\mathring{B}=G-\mathring{G}$, is introduced and the application [3] to the GN model in recalled. The second equation, used to first reconstruct $\mathring{F}$ in the axiomatic framework, is introduced in Sect. 3.2.

3.1. *First subtracted BS equation and application* ([2], [3])

It reads in terms of B ([6], [3]):

$$F=[1-\mathring{F}(\omega-\mathring{\omega})-(1+\mathring{F}\mathring{\omega})(B-\mathring{B})\,\omega]^{-1}[\mathring{F}+(1+\mathring{F}\mathring{\omega})(B-\mathring{B})] \quad (29)$$
$$=\sum\ldots\mathring{F}(\omega-\mathring{\omega})\,\mathring{F}(\omega-\mathring{\omega})\ldots\mathring{F}\,\omega(B-\mathring{B})\,\omega(B-\mathring{B})\ldots \quad (30)$$

where the sum in (30) runs over all convolution products of factors $\mathring{F}$, $B-\mathring{B}$, in arbitrary number and order; successive factors are separated by ω's except that two factors $\mathring{F}$ are separated by $\omega-\mathring{\omega}$.

Replacing $B-\mathring{B}$ by $G-\mathring{G}$ in (29) (30) yields corresponding expressions of F in terms of G. E. g. (30) gives:

$$F=\sum\ldots\mathring{F}(\omega-\mathring{\omega})\,\mathring{F}(\omega-\mathring{\omega})\ldots\mathring{F}\,\omega(G-\mathring{G})\,\omega(G-\mathring{G})\ldots \quad (31)$$

These expressions in terms of G [2] have been on the other hand derived formally in [2] from Eq. (3), as can be reobtained (in a similar way) from Lemma 2 [Eq. (25)] of Sect. 3.1 and the relation $\mathring{F}=\sum\mathring{F}^{(p)}$.

In the GN model, treated in [3], F is well defined and controlled from the outset in euclidean space and $\mathring{F}$ satisfies there bounds in cst λ. From the properties of $B-\mathring{B}=G-\mathring{G}$ and H_2 established in the model, the expansion (30) is shown to be convergent at small coupling in (bounded regions in) euclidean space and remains convergent when k is taken away from euclidean space in a 2-sheeted region around $s=4\mu^2$, for values of s away from a neighborhood V of $s=4\mu^2$. It defines the analytic continuation of F. U can be defined in a region that includes V by replacing in (30) the operation o (associated with ω) by ∇, $\mathring{\omega}$ unchanged. (An equivalent, somewhat more complicated definition of U has been given in [3].) Results follow as at the end of Sect. 2.2.

3.2. *Reconstruction of* $\mathring{F}$ *in the axiomatic framework*

$\mathring{F}$ will be defined below in terms of B and ${}_0\mathring{F}_0$ from the equation [6]:

$$\mathring{F}(z, z')=\mathring{R}_{(r)}(z, z')+{}_{(l)}\mathring{R}(z, o)\,\mathring{\Lambda}_{(r)}(z')+{}_{(l)}\mathring{\Lambda}(z)\,({}_0\mathring{F}_0)\,\mathring{\Lambda}_{(r)}(z') \quad (32)$$

where:

$$\mathring{R}_{(r)}(z, z') = (1 - (\mathring{B} - \mathring{B}_0)\,\mathring{\omega})^{-1}(\mathring{B} - \mathring{B}_0)$$
$$\equiv (\mathring{B} - \mathring{B}_0) + \sum_{n \geq 2} (\mathring{B} - \mathring{B}_0)\,\mathring{\omega}\,(\mathring{B} - \mathring{B}_0) \ldots \mathring{\omega}\,(\mathring{B} - \mathring{B}_0)^{(n)} \quad (33)$$
$$\mathring{\Lambda}_{(r)}(z') = 1 + 1\,\mathring{\omega}\,\mathring{R}_{(r)} \quad (34)$$

${}_{(l)}\mathring{R}$ is given by the expansion analogous to (33) with $\mathring{B}_0$ replaced by ${}_0\mathring{B}$, and ${}_{(l)}\mathring{\Lambda} = 1 + {}_{(l)}\mathring{R}\,\mathring{\omega}\,1$. (An expression analogous to (32) in which the first two terms are replaced by ${}_{(l)}\mathring{R} + {}_{(l)}\mathring{\Lambda}(z)\,\mathring{R}_{(r)}(z, z')$ can alternatively be used).

The analogue of (32) [2] in terms of G is obtained by replacing $\mathring{B} - \mathring{B}_0$, $\mathring{B} - {}_0\mathring{B}$, by $\mathring{G} - \mathring{G}_0$, $\mathring{G} - {}_0G$ and ${}_0\mathring{F}_0$ by ${}_0\mathring{G}_0$ [as a particular aspect of Eq. (3): at $k = z = z' = 0$ all terms vanish except ${}_0\mathring{G}_0$]. It follows also from Eq. (3), by (formal) methods of [2] or by Lemma 1 of Sect. 2.1: to see this, replace the factor $\mathring{G}_0$ by $(\mathring{G} - {}_0\mathring{G})_{\mathring{0}} + {}_0\mathring{G}_0$.

If one starts from the latter equation, the same methods as in [3] and in Sect. 2.2 allow one to conclude that F exists at weak coupling and does satisfy bounds in cst λ. If one starts from B, the analysis is completely analogous if ${}_0\mathring{F}_0$ is by definition equal to λ (as is the case in models where λ is the renormalized coupling). Otherwise a supplementary assumption is needed to control it.

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