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**Exponential estimates
on the one-dimensional
Schroedinger equation
with bounded analytic potential**

by

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ABSTRACT. — Using a Nekhoroshev-like perturbation technique, we investigate the solutions of the one-dimensional stationary Schroedinger equation, with bounded analytic potential. For sufficiently high energy E , we construct (in principle, up to any order in $1/\sqrt{E}$) approximate solutions, which resemble free waves, and are “very close” to the true solutions over very large distances, growing exponentially with \sqrt{E} . For potentials which decay sufficiently rapidly at infinity, we find that the scattering matrix differs from a trivial one by a quantity exponentially small in \sqrt{E} (in particular, the reflection coefficient is exponentially small in \sqrt{E}). Special attention is also devoted to the case of quasi-periodic potentials. These results unify, extend and make quantitative previous results by Fedoryuk, Neishtadt and Delyon and Foulon. Some numerical

and analytic tests show that our perturbative construction is, at least in the most relevant point, almost optimal.

RÉSUMÉ. — Nous étudions les solutions de l'équation de Schrödinger à une dimension avec un potentiel analytique en utilisant une technique de perturbation inspirée de Nekhoroshev. Pour une énergie E suffisamment grande nous construisons des solutions approchées (en principe à tous les ordres en $1/\sqrt{E}$) qui ressemblent à des ondes planes et sont très proches des vraies solutions sur des grandes distances qui croissent exponentiellement vite avec \sqrt{E} . Pour des potentiels qui décroissent suffisamment vite à l'infini nous montrons que la matrice de diffusion ne diffère de la matrice triviale que par une quantité exponentiellement petite en \sqrt{E} (en particulier, le coefficient de réflexion est exponentiellement petit en \sqrt{E}). Nous étudions tout particulièrement le cas des potentiels quasi périodiques. Les résultats unifient, étendent et rendent quantitatifs des résultats antérieurs de Fedoryuk, Neishtadt, Delyon et Foulon. Des tests numériques et analytiques montrent que notre construction perturbative est presque optimale, au moins pour les points les plus importants.

1. INTRODUCTION

We consider here the Schroedinger equation

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \psi = E \psi, \quad x \in \mathbb{R}, \quad (1.1)$$

where V is a bounded real-analytic function, and E a positive parameter. When E is large with respect to $\max |V(x)|$ one might naively expect that the solutions of (1.1) resemble, in some sense, the free solutions of the trivial equation $-[d^2/dx^2] \psi = E \psi$. Of course, as is well known, there are cases, e. g. when V is periodic [1], for which there are arbitrarily large values of E (corresponding to spectral gaps of the operator $L = -\frac{d^2}{dx^2} + V$), such that solutions of (1.1) are instead substantially different (in fact, L^2 at one side, and unbounded at the other side).

Nevertheless, we will construct a perturbation theory (for which the "zero-order" is the free problem), that shows that indeed *all solutions* ψ

of (1.1) are very similar to free waves over a very large distance, growing exponentially with \sqrt{E} , more precisely can be expressed as linear combinations of functions of the form

$$\psi_{\pm} = e^{\pm i\sqrt{E}x[1+\mathcal{O}(1/E)]} [1 + \mathcal{O}(1/E)], \quad \text{for } |x| \leq e^a \sqrt{E}, \quad (1.2)$$

where a is some positive constant. Evaluating the constant a is also one of our tasks: we will prove that (1.2) holds with $a = \alpha\rho$, where ρ is the width of a complex strip of analyticity of V (around \mathbb{R}), and α is any number smaller than $2/e$. On the other hand, in general, α cannot be taken to be larger than 2 , as will be shown below on some examples.

For our analysis, the hypothesis of analyticity of the potential is deeply essential. On the other hand, relation (1.2) is not true, in general, for non-analytic potentials: indeed, this relation implies that the length of the possible spectral gaps decays with E as $\exp - a\sqrt{E}$, while it is well known [2] that, for periodic V , the spectral gaps are exponentially small with \sqrt{E} if and only if V is analytic.

For relatively simple potentials, e. g. if V has a finite number of stationary points, or is periodic, WKB methods have turned out to be very powerful ([3], [4]) in investigating the structure of (1.1): however, these methods have not been extended yet to more difficult situations, like quasi-periodic or stochastic V , to which our technique instead applies.

Our method is based on a classical mechanics interpretation of the Schroedinger equation. In fact, (1.1) is equivalent to the Hamilton equations for a harmonic oscillator with time-dependent frequency $[E - V(x)]^{1/2}$ (here the role of time is played by x). To such a system we apply a Nekhoroshev-like technique ([5], [6]) (see also refs. [7]-[10]), carrying out a simple but very careful quantitative analysis. The same mechanical system has been studied by Neishtad [8], and an application to the Schroedinger equation has already been made by Delyon and Foulon [11], in order to prove the exponential decay of the Lyapunov exponents. However, these studies do not carry out any quantitative analysis, nor report, at least explicitly, direct estimates on the solution, say of the form (1.2). A novelty of our scheme is also the use of "local norms" ([12], [13]), which allow one to take explicitly into account the presence of regions where V is particularly small. For example, if V is L^1 (in a suitable sense) we construct approximate solutions, which differ from the actual solutions of (1.1) by $\exp - a\sqrt{E}$ over the whole real axis. In particular, as a byproduct of our local estimates, we get an exponentially small bound for the reflection coefficient R , of the form $|R| \simeq \exp - a\sqrt{E}$. A similar bound has been obtained by Fedoryuk [14], by means of asymptotic expansions; unfortunately, a precise comparison cannot be given, since in reference [14] the constants are not explicitly evaluated.

Our results cover, in particular, the case of a quasi-periodic potential. Such a case has been widely studied by means of KAM-like perturbative

techniques ([15]-[18], [24]), with the result that there exist solutions which are also quasi-periodic, for sufficiently high values of E belonging to a set of large measure, but with a Cantor-like structure. By means of our Nekhoroshev-like perturbation theory we obtain (as usual) a complementary result, namely that, for any (sufficiently high) value of E , the solutions are very close to quasi-periodic waves, over an exponentially large length scale, as in (1.1).

In the next Section we formulate precisely our main result (Proposition 1), and illustrate it on some relevant classes of potentials. In Section 3 we prove Proposition 1, on the basis of the result of our perturbative technique (Proposition 2), which in turn is proved in Section 4. Finally, Section 5 is devoted to the comparison of our estimate for the constant a with an exact result (in the particular case of a periodic potential), and also with some numerical results (for scattering problems).

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2. STATEMENT OF RESULTS

2.1. The main Proposition

We are concerned with the one-dimensional Schroedinger equation (1.1), with analytic potential. More precisely, denoting by \mathcal{S}_ρ the complex strip

$$\mathcal{S}_\rho = \{x \in \mathbb{C}; |\operatorname{Im} x| < \rho\}, \quad (2.1)$$

we assume that V is analytic in \mathcal{S}_ρ , and real for real x (similar functions, possibly vector- or matrix-valued, will be called, throughout the paper, real analytic). For any analytic function $f: \mathcal{S}_\rho \rightarrow \mathbb{C}$, it is convenient to introduce, besides the usual supremum norm, denoted $|f|_\rho$, the following family of “local norms”:

$$|f|_{x,\rho} = \sup_{|x'-x| < \rho} |f(x')|, \quad x \in \mathbb{R}. \quad (2.2)$$

For the solution of the Schroedinger equation (1.1) we shall use the vector notation $w = \left(\psi, \frac{1}{\sqrt{E}} \psi' \right)$, with $\psi' = \frac{d\psi}{dx}$; for vectors $w = (w_1, w_2) \in \mathbb{C}^2$ we adopt the norm $\|w\|^2 = |w_1|^2 + |w_2|^2$, while for linear operators acting on \mathbb{C}^2 we use the corresponding norm $\|A\| = \sup_w \|Aw\|/\|w\|$.

Our results can be formulated in the following general statement:

PROPOSITION 1. — Let V be real analytic and bounded in \mathcal{S}_ρ , $\rho > 0$. For any E satisfying

$$E \geq 5 |V|_\rho \tag{2.3}$$

there exist a 2×2 real analytic matrix $T(x)$, and a real analytic function $\tilde{V}(x)$, such that:

(i) T is close to the identity, precisely

$$\|T(x) - 1\| \leq \exp \frac{|V|_{x,\rho}}{2E} - 1, \quad \forall x \in \mathbb{R}; \tag{2.4}$$

(ii) \tilde{V} is close to V , precisely

$$|\tilde{V}(x) - V(x)| \leq \frac{1}{2} \frac{|V|_{x,\rho}^2}{E}, \quad \forall x \in \mathbb{R}; \tag{2.5}$$

(iii) Denote $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$, and $\Phi(x) = \begin{pmatrix} -\varphi(x) & 0 \\ 0 & \varphi(x) \end{pmatrix}$, with

$$\varphi(x) = \sqrt{E} \left(x - \frac{1}{2E} \int_0^x \tilde{V}(t) dt \right); \tag{2.6}$$

then for any constant $u \in \mathbb{C}^2$ the solution $w(x)$ of (1.1), with $w(0) = MT(0)u$, is "exponentially close" to the function

$$\hat{w}(x) = MT(x) e^{i\Phi(x)} u; \tag{2.7}$$

precisely, denoting by N_E the integer part of $\rho \sqrt{E}/2$, one has

$$\|w(x) - \hat{w}(x)\| \leq \|T(x)\| \|u\| \beta(x), \quad x \in \mathbb{R}, \tag{2.8}$$

with

$$\beta(x) = \exp \left(\frac{2}{\sqrt{E}} \left| \int_0^x |V|_{t,\rho} dt \right| \exp -N_E \right) - 1. \tag{2.9}$$

Remarks. — (i) Both T and \tilde{V} also depend on E ; although they are not analytic in E , they are, in a sense, polynomials of degree N_E in $1/\sqrt{E}$: for

instance, one has $\tilde{V}(x) = \sum_{s=0}^{N_E} \tilde{V}_s(x) E^{-s/2}$, with \tilde{V}_s independent of E . As

will be clear in Section 4, both T and \tilde{V} could be explicitly written up to any order in $1/\sqrt{E}$, while (2.5) could also be replaced by inequalities accurate to any order in $1/\sqrt{E}$.

(ii) As remarked in the Introduction, one could replace the expression for N_E by $N_E = \alpha \rho \sqrt{E}$, α being any number smaller than $2/e$, with a corresponding worsening of condition (2.3) (see Section 4.5).

Proposition 1 is clearly interesting only for sufficiently high E ; in this case the expression (2.9) for β essentially reads

$$\frac{2}{\sqrt{E}} \exp -N_E \left| \int_0^x |V|_{t,\rho} dt \right| \ll 1.$$

In the next subsections we illustrate some consequences which can be drawn from Proposition 1.

2.2. General consequences

Let us discuss the basic estimate (2.8). The quantity $\|T(x)\|$ there appearing is uniformly bounded: for example, by (2.4) and (2.3) one has $\|T(x)\| \leq 2$. The function β remains instead small over a quite large interval, growing exponentially with \sqrt{E} : one has for instance $\beta(x) \leq \beta_0$, with

$$\beta_0 = \exp \frac{2|V|_\rho}{E} - 1 = \mathcal{O}(|V|_\rho/E), \quad (2.10)$$

as far as

$$|x| \leq \frac{1}{\sqrt{E}} \exp N_E. \quad (2.11)$$

Consequently, for x in this interval, the estimate (2.8) gives $\|w(x) - \hat{w}(x)\| < 2\beta_0 \|u\| = \mathcal{O}(|V|_\rho/E)$. On the other hand, from (2.4) and (2.7), using also $\|M\| = 1$, one obtains

$$\|\hat{w} - M e^{i\Phi(x)} u\| \leq \|T(x) - \mathbf{1}\| \|u\| \leq \beta_0 \|u\|.$$

In conclusion, for $|x|$ satisfying (2.11) one gets $\|w(x) - M e^{i\Phi(x)} u\| < 3\beta_0 u$, i. e.

$$\begin{aligned} \left| \psi(x) - \frac{1}{\sqrt{2}} (u_1 e^{-i\Phi(x)} + iu_2 e^{i\Phi(x)}) \right| &< 3\beta_0 \|u\| \\ \left| \frac{\psi'(x)}{\sqrt{E}} - \frac{i}{\sqrt{2}} (-u_1 e^{-i\Phi(x)} + iu_2 e^{i\Phi(x)}) \right| &< 3\beta_0 \|u\|. \end{aligned} \quad (2.12)$$

In turn, the phase $\varphi(x)$ satisfies

$$\varphi(x) = \sqrt{E} x \left[1 - \frac{1}{2Ex} \int_0^x V(t) dt + \mathcal{O}(|V|_\rho^2/E^2) \right]. \quad (2.13)$$

By the way, we also notice that in intervals shorter than (2.11), for instance

$$|x| \leq \left(\frac{E}{|V|_\rho} \right)^{m/2} \frac{1}{\sqrt{E}} \exp N_E, \quad m \geq 1, \quad (2.14)$$

one could replace (2.12) by an expression accurate up to order $1+m/2$ in $|V|_p/E$ (one needs approximations for T accurate to order $1+m/2$, see the above remark).

Another general consequence of Proposition 1 is an exponential estimate on the Lyapunov exponent (when it is defined) of the Schrodinger equation [11]. Indeed, from (2.7) and (2.8) one gets, for any solution w ,

$$\|w(x)\| \leq \|\hat{w}(x)\| + \|w(x) - \hat{w}(x)\| \leq \|T(x)\| \|u\| (1 + \beta), \quad (2.15)$$

and thus

$$\limsup_{x \rightarrow \pm \infty} \frac{1}{|x|} \ln \|w(x)\| \leq \limsup_{x \rightarrow \pm \infty} \frac{1}{|x|} \ln (1 + \beta); \quad (2.16)$$

this gives, for the (maximal) Lyapunov exponent λ_{\pm} of the Schrodinger equation, the exponential bound

$$\lambda_{\pm} \leq \frac{2 \exp - N_E}{\sqrt{E}} \limsup_{x \rightarrow \pm \infty} \frac{1}{x} \int_0^x |V|_{t,\rho} dt \leq \frac{2|V|_p}{\sqrt{E}} \exp - N_E. \quad (2.17)$$

2.3. The scattering problem

We shall now discuss the case of a potential $V(x)$ which decays sufficiently rapidly for $x \rightarrow \pm \infty$; precisely, we need $|V|_{x,\rho} \rightarrow 0$ for $x \rightarrow \pm \infty$, with $\int_{-\infty}^{+\infty} |V|_{x,\rho} dx = C < \infty$. From (2.8), (2.9) one has then, for any $x \in \mathbb{R}$,

$$\begin{aligned} \|w(x) - \hat{w}(x)\| &\leq \|T(x)\| \|u\| \left(\exp \frac{2C \exp - N_E}{\sqrt{E}} - 1 \right) \\ &= \mathcal{O} \left(\frac{C}{\sqrt{E}} \exp - N_E \right); \end{aligned} \quad (2.18)$$

this means that the approximate solution \hat{w} is now close to w for any $x \in \mathbb{R}$, and moreover, the error is exponentially small in \sqrt{E} .

Let us then investigate the properties of \hat{w} . Clearly, for $x \rightarrow \pm \infty$ one has $T(x) \rightarrow \mathbf{1}$, while the limits

$$\gamma_{\pm} = \lim_{x \rightarrow \pm \infty} [\varphi(x) - \sqrt{E} x] = \frac{1}{2\sqrt{E}} \int_0^{\pm \infty} \tilde{V} dx \quad (2.19)$$

are finite (actually, $\gamma_{\pm} \simeq \frac{1}{2\sqrt{E}} \int_0^{\pm \infty} V dx$). By choosing

$$u = (\sqrt{2} b e^{i\gamma_+}, i\sqrt{2} a e^{-i\gamma_-})$$

in (2.7), and denoting $\gamma = \gamma_+ - \gamma_-$, one immediately deduces, for $\hat{w} = \left(\hat{\psi}, \frac{1}{\sqrt{E}} \hat{\psi}' \right)$, the asymptotic behavior

$$\hat{\psi} \rightarrow \begin{cases} ae^{i\sqrt{E}x} + be^{-i\sqrt{E}x - i\gamma}, & \text{for } x \rightarrow +\infty \\ ae^{i\sqrt{E}x + i\gamma} + be^{-i\sqrt{E}x}, & \text{for } x \rightarrow -\infty, \end{cases} \quad (2.20)$$

with a corresponding expression for $\hat{\psi}'$ ⁽¹⁾. In particular, for either a or b vanishing, (2.20) represents a purely left- or right-travelling wave, with no reflected wave, and small phase-shift $\gamma \simeq \frac{1}{\sqrt{E}} \int_{-\infty}^{+\infty} \tilde{V}(x) dx$.

From these approximations, we shall now derive exponential estimates for the scattering matrix and the reflection coefficient [14]. The scattering matrix corresponding to (2.20) is the trivial matrix $\hat{S} = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$; using (2.18) one can then say that the true scattering matrix S (which is known to exist essentially in our assumptions, see for example [19]) differs from \hat{S} by a quantity exponentially small in \sqrt{E} : for example, it is not difficult to get

$$\|S - \hat{S}\| \leq 4 \left(\exp \frac{2C \exp - N_E}{\sqrt{E}} - 1 \right); \quad (2.21)$$

in particular, the reflection coefficient R turns out to be bounded by $|R| \leq \frac{16C}{\sqrt{E}} \exp - N_E$.

2.4. Slowly decaying potential

Let us here consider the case of a potential which decays to zero, for $x \rightarrow \pm\infty$, in a non integrable way; to be definite, assume $|V|_{x,\rho} \sim (1 + |x|)^{-1}$ for large $|x|$. One can see, from (2.8) and (2.9), that, although \hat{w} now is not uniformly close to w , nevertheless the error remains small, $\|w - \hat{w}\| = \mathcal{O}(|V|_{\rho}/E)$, for an extremely large interval, growing with E essentially as $\exp \sqrt{E}$. In a shorter, still exponentially large interval, one gets instead, as in the scattering problem, $\|w - \hat{w}\| = \mathcal{O}(\exp - N_E)$.

⁽¹⁾ Be aware that $\hat{\psi}'$ is just a notation for the second component of \hat{w} : although $\hat{\psi}'$, according to (2.8), approximates $\psi' = \frac{d\psi}{dx}$, in general it does not coincide with $\frac{d\hat{\psi}}{dx}$.

Concerning \hat{w} , one can notice that it does not properly reduce, asymptotically, to a pair of plane waves, because the phase shift now diverges logarithmically with $|x|$. However, if one accepts to restrict the attention to a finite but large interval, growing exponentially with \sqrt{E} , then the phase shift is finite, and the "asymptotic" behavior is like in (2.20). On this finite but large scale, the overall picture is essentially the same as in the scattering problem.

2.5. Quasi-periodic potential

To treat this case, we assume that the potential fulfills the following two conditions (which are also needed in the KAM approach [15]-[18]):

(i) V is quasi-periodic in x , with real frequencies $\omega_1, \dots, \omega_n$; by this, we mean that one has $V(x) = \mathcal{V}(\omega_1 x, \dots, \omega_n x)$ for all $x \in \mathcal{S}_\rho$, the function $\mathcal{V}: \mathcal{S}_{\omega_1 \rho} \times \dots \times \mathcal{S}_{\omega_n \rho} \rightarrow \mathbb{C}$ being analytic and 2π -periodic in each of its arguments;

(ii) the frequencies $\omega_1, \dots, \omega_n$ satisfy the Diophantine condition

$$|k \cdot \omega| \geq c |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad |k| = \max_j |k_j|, \quad (2.22)$$

for some positive constants c and τ .

We shall denote by $V^{(k)}$ the Fourier coefficients of V , defined through $\mathcal{V}(\varphi_1, \dots, \varphi_n) = \sum_{k \in \mathbb{Z}^n} V^{(k)} e^{ik \cdot \varphi}$; similar notations will be used for the other quasi-periodic functions.

From the general theory of Section 2.2 we know that the actual solution w of the Schroedinger equation differs by quantities $\mathcal{O}(1/E)$ from the approximate solution \hat{w} defined by (2.7), over distances

$$|x| \leq \frac{1}{\sqrt{E}} \exp N_E.$$

We now prove that, in the present case, \hat{w} is a quasi-periodic function of x , with $n+1$ frequencies $\omega_0, \omega_1, \dots, \omega_n$, where $\omega_0(E) = \sqrt{E} + \mathcal{O}(1/\sqrt{E})$.

To this purpose, we first observe that, under condition (i), the functions $\tilde{V}(x)$ and $T(x)$ of Proposition 1 turn out to be quasi-periodic, with the same frequencies as $V(x)$ [indeed, this immediately follows from the corresponding statement in Lemma 2 of Section 4, if one takes into account the very construction of $\tilde{V}(x)$ and $T(x)$]. Furthermore, by (2.6) the phase φ entering the expression (2.7) for \hat{w} can be written as

$$\varphi(x) = \sqrt{E} \left(1 - \frac{\tilde{V}^{(0)}}{2E} \right) x + \theta(x), \quad (2.23)$$

with

$$\theta(x) = -\frac{1}{2\sqrt{E}} \int_0^x [\tilde{V}(t) - \tilde{V}^{(0)}] dt. \quad (2.24)$$

Now, using the quasi-periodicity of \tilde{V} , and also the Diophantine condition (2.22), one easily proves that θ is quasi-periodic too, with the same frequencies as V : indeed, one has clearly

$$\theta(x) = -\frac{1}{2\sqrt{E}} \sum_{k \neq 0} \frac{\tilde{V}^{(k)}}{i\omega \cdot k} (1 - e^{ik \cdot \omega x}), \quad (2.25)$$

and, as is well known, the Diophantine condition assures the convergence of the series. Moreover, the expression (2.23) for φ shows that the additional frequency ω_0 , which enters \hat{w} besides the frequencies $\omega_1, \dots, \omega_n$ of the potential, is given by $\omega_0(E) = \sqrt{E} [1 - \tilde{V}^{(0)}/2E]$, in agreement with the above statement.

Let us now consider more closely the approximate solution \hat{w} . First, one can produce a more accurate expression for the frequency ω_0 , precisely

$$\omega_0(E) = \sqrt{E} \left[1 - \frac{V^{(0)}}{2E} + \mathcal{O} \left(\frac{|V|_p^2}{E^2} \right) \right]. \quad (2.26)$$

This expression is obtained from the one above by using the relation $\tilde{V}^{(0)} = V^{(0)} + \mathcal{O}(1/E)$, which follows from

$$\begin{aligned} \tilde{V}^{(0)} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{V}(\omega_1 x, \dots, \omega_n x) dx = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{V}(x) dx \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T V(x) dx + \mathcal{O}(1/E) = V^{(0)} + \mathcal{O}(1/E) \end{aligned} \quad (2.27)$$

(use is made of the fact that the flow $x \mapsto (\varphi_1 + \omega_1 x, \dots, \varphi_n + \omega_n x)$ on \mathbb{T}^n is ergodic).

Furthermore, one can easily verify that $\hat{w} = \left(\Psi, \frac{1}{\sqrt{E}} \Psi' \right)$ is a linear combination of “Bloch waves” with frequencies $\omega_0, \omega_1, \dots, \omega_n$, *i. e.*

$$\hat{\Psi} = c_+ e^{+i\omega_0 x} \psi_+(\omega_1 x, \dots, \omega_n x) + c_- e^{-i\omega_0 x} \psi_-(\omega_1 x, \dots, \omega_n x), \quad (2.28)$$

the functions ψ_{\pm} being 2π -periodic in each argument. It is also possible to show that $\psi_{\pm} = 1 + \mathcal{O}(1/E)$. Indeed, from the expression (2.25) for θ , using standard techniques and the relation (2.5), one can estimate $|\theta(x)|$ as

$$|\theta(x)| \leq C \frac{|V|_p}{\sqrt{E}}, \quad x \in \mathbb{R}, \quad (2.29)$$

$C = C(c, \tau, n, \rho)$ being a suitable constant (for instance, using the estimates of refs. [20], [21], one gets $C = c^{-1} \rho^{-\tau} 2^{n-\tau+1} \sqrt{(2\tau)!}$). One then easily finds, using (2.7), (2.4), (2.23), (2.26) and (2.29):

$$\left\| \hat{w}(x) - M \exp \begin{pmatrix} -i\omega_0 x & 0 \\ 0 & i\omega_0 x \end{pmatrix} \right\| \leq [\exp |\theta(x)| - 1] \|u\| \\ = \mathcal{O} \left(\frac{1}{\sqrt{E}} \right), \quad x \in \mathbb{R}. \quad (2.30)$$

From this analysis, one concludes that the solutions of the Schroedinger equation with quasi-periodic potentials have the following structure, over distances $\mathcal{O} \left(\frac{1}{\sqrt{E}} \exp N_E \right)$: up to quantities of order $1/E$ they resemble (linear combinations of) Bloch waves (2.28), which in turn differ from the free solutions $e^{\pm i\sqrt{E}x}$ by quantities of order $1/\sqrt{E}$.

3. THE HAMILTONIAN THEOREM, AND PROOF OF PROPOSITION 1

3.1. Statement of the Hamiltonian theorem

The proof of Proposition 1 is obtained by regarding the Schroedinger equation as a dynamical system, namely an harmonic oscillator with frequency $[E - V(x)]^{1/2}$ depending on the "time" x . To this system we apply a Nekhoroshev-like perturbation theory, leading to the exponential estimates.

It is convenient, in our perturbative approach, to use canonical coordinates p, q related to $\psi, \psi' = \frac{d\psi}{dx}$ by

$$\begin{pmatrix} \psi \\ \frac{1}{\sqrt{E}} \psi' \end{pmatrix} = M \begin{pmatrix} p \\ q \end{pmatrix}, \quad (3.1)$$

M being the matrix introduced in Proposition 1; the Schroedinger equation (1.1) is then immediately seen to be equivalent to the canonical equations corresponding to the Hamilton function

$$H(p, q, x) = i\sqrt{E} \left(1 - \frac{V(x)}{2E} \right) pq - \frac{V(x)}{4\sqrt{E}} (p^2 - q^2), \quad (3.2)$$

precisely

$$\begin{aligned} \frac{dp}{dx} &= -i\sqrt{E}\left(1 - \frac{V(x)}{2E}\right)p - \frac{V(x)}{2\sqrt{E}}q, \\ \frac{dq}{dx} &= i\sqrt{E}\left(1 - \frac{V(x)}{2E}\right)q - \frac{V(x)}{2\sqrt{E}}p. \end{aligned} \quad (3.3)$$

The Hamiltonian H , as well as all functions we will deal with, is defined for $(p, q, x) \in \mathbb{C}^2 \times \mathcal{S}_p$, and is real if x is real and $q = -i\bar{p}$. The corresponding reality condition for linear transformations of the form $\begin{pmatrix} p \\ q \end{pmatrix} = T(x) \begin{pmatrix} p' \\ q' \end{pmatrix}$, $T(x)$ being an analytic 2×2 matrix, is that the relation $q = -i\bar{p}$ is preserved for any real x ; in particular, this condition guarantees that, after the substitution $(p, q) \mapsto (p', q')$, the new Hamiltonian is still real for real x and $q' = -i\bar{p}'$. Such reality conditions will be implicitly assumed, whenever working with the p, q coordinates.

Proposition 1 turns out to be a direct consequence of the following

PROPOSITION 2. — Consider the Hamiltonian (3.2), V being real analytic in \mathcal{S}_p , and assume $E \geq 5|V|_p$. Then there exist a 2×2 analytic matrix $T(x)$, and two analytic functions $\tilde{V}(x)$ and $F(x)$, such that:

- (i) T is close to the identity, as in (2.4);
- (ii) \tilde{V} is close to V , as in (2.5), while F is bounded by $|F(x)| \leq |V|_{x, p}$, $x \in \mathbb{R}$;
- (iii) the transformation

$$\begin{pmatrix} p \\ q \end{pmatrix} = T(x) \begin{pmatrix} p' \\ q' \end{pmatrix} \quad (3.4)$$

is canonical and, for $x \in \mathbb{R}$, gives the new Hamiltonian H' the form

$$\begin{aligned} H'(p', q', x) &= i\sqrt{E}\left(1 - \frac{\tilde{V}(x)}{2E}\right)p'q' \\ &\quad - \frac{\exp - N_E}{4\sqrt{E}}(F(x)p'^2 - \bar{F}(x)q'^2). \end{aligned} \quad (3.5)$$

3.2. Proof of Proposition 1

We defer to the next Section the proof of Proposition 2, and show here how one can use it to prove Proposition 1. To this purpose, let us write the equations of motion deduced from (3.5) in the vector form

$$\frac{d\xi}{dx} = [iA(x) + B(x)]\xi, \quad (3.6)$$

where $\xi = (p', q')$, and

$$\begin{aligned} A &= \frac{d\Phi}{dx} = \sqrt{E} \left(1 - \frac{\tilde{V}(x)}{2E} \right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ B &= \frac{\exp - N_E}{2\sqrt{E}} \begin{pmatrix} 0 & F(x) \\ \bar{F}(x) & 0 \end{pmatrix}. \end{aligned} \tag{3.7}$$

From (3.6) one first works out the preliminary estimate

$$\|\xi(x)\| \leq \|\xi(0)\| \exp \left| \int_0^x \|B(t)\| dt \right|. \tag{3.8}$$

This inequality can be achieved by introducing a variable $\eta = e^{-i\Phi(x)}\xi$, whose equation of motion is immediately seen to be $\frac{d\eta}{dx} = e^{-i\Phi} B e^{i\Phi} \eta$ (one profits here of the fact that the matrices Φ and $\dot{\Phi}$ commute); from this equation one deduces $\frac{1}{2} \left| \frac{d}{dx} \|\eta\|^2 \right| \leq \|B\| \|\eta\|^2$, which in turn, using $\|\xi\| = \|\eta\|$, gives (3.8). Let us then write (3.6) in the integral form

$$\xi(x) = e^{i\Phi(x)} \left[\xi(0) + \int_0^x e^{-i\Phi(t)} B(t) \xi(t) dt \right]. \tag{3.9}$$

Using (3.8) inside the integral, one gets

$$\begin{aligned} &\|\xi(x) - e^{i\Phi(x)}\xi(0)\| \\ &\leq \|\xi(0)\| \left| \int_0^x \|B(t)\| \exp \left| \int_0^t \|B(s)\| ds \right| dt \right| \\ &= \|\xi(0)\| \left[\exp \left| \int_0^x \|B(t)\| dt \right| - 1 \right], \end{aligned} \tag{3.10}$$

and since $\|B(x)\| = \frac{\exp - N_E}{2\sqrt{E}} |F(x)| \leq \frac{\exp - N_E}{2\sqrt{E}} |V|_{x, p'}$ one obtains

$$\|\xi(x) - e^{i\Phi(x)}\xi(0)\| \leq \|\xi(0)\| \beta(x), \tag{3.11}$$

β being defined by (2.9). Finally, if one recalls

$$w(x) = M \begin{pmatrix} p \\ q \end{pmatrix} = MT(x) \xi(x),$$

and denotes $\hat{w}(x) = MT(x) e^{i\Phi(x)} \xi(0)$, one finds (2.7) and (2.8), with $u = \xi(0)$; $\|M\| = 1$ has also been used. The proof of Proposition 1 is complete.

4. PROOF OF PROPOSITION 2

4.1. An equivalent autonomous system

As a preliminary step in the proof, we replace the non-autonomous Hamiltonian (3.2) with an equivalent autonomous one, having one more degree of freedom. To this purpose, we consider x as a dependent variable, with equation of motion $\dot{x}=1$ and initial datum $x(0)=0$; one then immediately recognizes that the non-autonomous Hamiltonian (3.2) is equivalent, for what concerns the variables x , p and q , to the autonomous one $K(p, q, y, x)=y+H(p, q, x)$, y being the momentum conjugated to x . We shall work out, in the extended phase space $\mathbb{C}^3 \times \mathcal{S}_\rho$, a canonical transformation \mathcal{C} which preserves the property $\dot{x}=1$ and extends (3.4), namely of the form

$$\begin{aligned} x &= x', \\ y &= y' + h(p', q', x'), \end{aligned} \quad \begin{pmatrix} p \\ q \end{pmatrix} = T(x') \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad (4.1)$$

where one has denoted $(p, q, y, x) = \mathcal{C}(p', q', y', x')$; the matrix T is obviously required to have the properties stated in Proposition 2, while h is a suitable analytic function. The new Hamiltonian $K' = K \circ \mathcal{C}$ will be required to have the form $K'(p', q', y', x') = y' + H'(p', q', x')$, H' having the form and the properties stated in Proposition 2.

4.2. The algebraic framework

Let us denote by \mathcal{A}_ρ the space of all analytic functions: $\mathcal{S}_\rho \rightarrow \mathbb{C}$; for $F \in \mathcal{A}_\rho$, and any $\rho' \leq \rho$, we shall use the local norm $|F|_{x, \rho'}$ introduced in Section 2. If $F \in \mathcal{A}_\rho$, let us define $F^*(x) = \bar{F}(\bar{x})$; one has clearly $F^* \in \mathcal{A}_\rho$, and $\frac{dF^*}{dx} = \left(\frac{dF}{dx}\right)^*$. Throughout the proof we shall be primarily concerned with homogeneous polynomials of degree two in p and q , with coefficients in \mathcal{A}_ρ ; in particular, a basic role will be played by the spaces

$$\begin{aligned} \Lambda_\rho^p &= \left\{ f = \frac{1}{4} F(x) p^2 - \frac{1}{4} F^*(x) q^2, F \in \mathcal{A}_\rho \right\} \\ \Lambda_\rho^m &= \left\{ f = \frac{1}{2} i F(x) pq, F = F^* \in \mathcal{A}_\rho \right\} \end{aligned} \quad (4.2)$$

(“ p ” and “ m ” are abbreviations for “pure” and “mixed”). We shall also consider the space $\Lambda_\rho = \Lambda_\rho^p \oplus \Lambda_\rho^m$, and denote by Π^p, Π^m the projections from Λ_ρ to Λ_ρ^p and respectively Λ_ρ^m . For f belonging to either Λ_ρ^p or Λ_ρ^m , and any $\rho' \leq \rho$, we shall use the norm $|f|_{x, \rho'} = |F|_{x, \rho'}$, while for $f \in \Lambda_\rho$ we denote $|f|_{x, \rho'} = |\Pi^p f|_{x, \rho'} + |\Pi^m f|_{x, \rho'}$.

Functions $f \in \Lambda_\rho$ will be also regarded as y -independent functions defined in the whole phase space $\mathbb{C}^3 \times \mathcal{S}_\rho$. Denoting by $\{, \}$ the usual Poisson bracket, one immediately deduces the following elementary algebraic relations:

$$\{\Lambda_\rho^p, \Lambda_\rho^p\} \subset \Lambda_\rho^m, \quad \{\Lambda_\rho^p, \Lambda_\rho^m\} \subset \Lambda_\rho^p, \quad \{\Lambda_\rho^p, \Lambda_\rho^p\} = 0, \quad (4.3)$$

and also gets the basic estimate

$$|\{f, g\}|_{x, \rho'} \leq |f|_{x, \rho'} |g|_{x, \rho'} \quad (4.4)$$

for any $f, g \in \Lambda_\rho$, any $\rho' \leq \rho$ and any $x \in \mathbb{R}$.

4.3. Elementary canonical transformations

The canonical transformation \mathcal{C} is obtained as the composition of a finite number of elementary canonical transformations generated by the Lie method, namely as the time-one map of a suitable auxiliary Hamiltonian flow; we shall use, in the phase space $\mathbb{C}^3 \times \mathcal{S}_\rho$, y -independent auxiliary Hamiltonians belonging to Λ_ρ^p . The properties of the elementary canonical transformations generated in this way are stated in the following

LEMMA 1 (on canonical transformations). — Consider, in the phase-space $\mathbb{C}^3 \times \mathcal{S}_\rho$, the Hamiltonian $\chi = \frac{1}{4} X(x) p^2 - \frac{1}{4} X^*(x) q^2 \in \Lambda_\rho^p$, and denote by Θ the corresponding time-one map. Then:

(i) Θ is an analytic canonical transformation of $\mathbb{C}^3 \times \mathcal{S}_\rho$ onto itself, of the form

$$\begin{aligned} x &= x' \\ y &= y' + \kappa(p', q', x'), \end{aligned} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \tau(x') \begin{pmatrix} p' \\ q' \end{pmatrix}. \quad (4.5)$$

(ii) The operator $L = \{\chi, \cdot\}$ is bounded by

$$|Lf|_{x, \rho'} \leq |\chi|_{x, \rho'} |f|_{x, \rho'} \quad (4.6)$$

for any $f \in \Lambda_\rho$, any $x \in \mathbb{R}$ and any $\rho' \leq \rho$. Furthermore, if $f \in \Lambda_\rho$, then one has $f \circ \Theta = (\exp L) f \in \Lambda_\rho$, the exponential being defined by its series, which is convergent.

(iii) The 2×2 matrix τ is analytic in \mathcal{S}_ρ , and for any $\rho' \leq \rho$ it satisfies the estimate

$$\|\tau - 1\|_{x, \rho'} \leq \exp \frac{1}{2} |\chi|_{x, \rho'} - 1, \quad (4.7)$$

while $\kappa \in \Lambda_\rho$ is given by

$$\kappa = - \sum_{k=1}^{\infty} \frac{L^{k-1}}{k!} \frac{\partial \chi}{\partial x}, \quad \frac{\partial \chi}{\partial x} = \frac{1}{4} \frac{dX}{dx} p^2 - \frac{1}{4} \frac{dX^*}{dx} q^2. \quad (4.8)$$

Proof. — The equations of motion associated to χ are

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -\frac{1}{4} \frac{\partial \chi}{\partial x}, \end{aligned} \quad \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & X^*(x) \\ X(x) & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}; \quad (4.9)$$

the flow is certainly defined and invertible for any initial datum in $\mathbb{C}^3 \times \mathcal{S}_\rho$ and any time, and in particular at $t=1$ it has the form (4.5). A trivial integration gives $\tau = \exp \frac{1}{2} \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$, and consequently (4.7). Consider now $f \in \Lambda_\rho$. The inequality (4.6) follows from (4.4); on the other hand, since $\dot{f} = Lf$, one has $f \circ \Theta = \sum_{k=0}^{\infty} \frac{1}{k!} L^k f$, and the series converges for any fixed x . Finally, (4.8) is an obvious consequence of $\{\chi, y\} = -\frac{\partial \chi}{\partial x}$. ■

4.4. The iterative lemma

It is convenient to use here the two parameters

$$\eta = \sqrt{|V|_\rho} \quad \text{and} \quad \varepsilon = \eta / \sqrt{E} = \sqrt{|V|_\rho / E}.$$

We aim to construct a sequence $\Theta_1, \dots, \Theta_N$ of elementary canonical transformations of $\mathbb{C}^3 \times \mathcal{S}_\rho$ onto itself, such that, denoting $K_0 = K$ and $K_s = K \circ \Theta_1 \circ \dots \circ \Theta_s$, $s = 1, \dots, N$, one has

$$K_s = y + i \frac{\eta}{\varepsilon} pq + \frac{\varepsilon}{\eta} g_s(p, q, x; \varepsilon) + \frac{\varepsilon}{\eta} f_s(p, q, x; \varepsilon), \quad (4.10)$$

where $g_s \in \Lambda_\rho^m$ is analytic in ε , while $f_s \in \Lambda_\rho^p$ is analytic in ε , and divisible by ε^s .

We proceed iteratively, and determine f_{s+1}, g_{s+1} in terms of f_s, g_s ; more precisely, denoting $\delta = \rho/N$, and

$$\rho_s = \rho - s\delta, \quad s = 0, \dots, N, \quad (4.11)$$

we work out iterative estimates for $\left| f_s \right|_{x, \rho_s}$ and $\left| g_s \right|_{x, \rho_s}$. Let us notice that K_0 has indeed the form (4.10), with $\left| f_0 \right|_{x, \rho_0} = \left| g_0 \right|_{x, \rho_0} = |V|_{x, \rho}$. Our estimates are stated in the following

LEMMA 2 (Iterative Lemma). — *Let K_s be as in (4.10), $s < N$, with $g_s \in \Lambda_\rho^m$ analytic in ε and $f_s \in \Lambda_\rho^p$ analytic in ε and divisible by ε^s . Let ρ_0, \dots, ρ_N be defined by (4.11). Then there exists a canonical transformation Θ_{s+1} of $\mathbb{C}^3 \times \mathcal{S}_\rho$ onto itself, such that:*

(i) *The new Hamiltonian $K_{s+1} = K_s \circ \Theta_{s+1}$ has the form (4.10), with $g_{s+1} \in \Lambda_\rho^m$ analytic in ε , and $f_{s+1} \in \Lambda_\rho^p$ analytic in ε and divisible by ε^{s+1} ,*

(ii) Denoting $\mathcal{F}_s(x) = \eta^{-2} |f_s|_{x, \rho_s}$, $\mathcal{G}_s(x) = \eta^{-2} |g_s - g_0|_{x, \rho_s}$, one has $\mathcal{F}_0 = 1$, $\mathcal{G}_0 = 0$, and

$$\begin{aligned} \mathcal{F}_{s+1} &\leq \mathcal{F}_s \left[\frac{\varepsilon N}{2\eta\rho} + \frac{\varepsilon^2}{2}(1 + \mathcal{G}_s) + \frac{\varepsilon^4}{12}\mathcal{F}_s^2 \right] \cosh\left(\frac{\varepsilon^2}{2}\mathcal{F}_s\right) \\ \mathcal{G}_{s+1} &\leq \mathcal{G}_s + \frac{1}{4}\varepsilon^2\mathcal{F}_s^2 \left[1 + \frac{\varepsilon N}{2\eta\rho} + \frac{\varepsilon^2}{2}(1 + \mathcal{G}_s) \right] \cosh\left(\frac{\varepsilon^2}{2}\mathcal{F}_s\right); \end{aligned} \tag{4.12}$$

(iii) Denoting $(p, q, y, x) = \Theta_{s+1}(p', q', y', x')$, one has $x' = x$ and $\begin{pmatrix} p \\ q \end{pmatrix} = \tau_{s+1}(x') \begin{pmatrix} p' \\ q' \end{pmatrix}$, with

$$\|\tau_{s+1} - 1\|_{x, \rho_{s+1}} \leq \exp\frac{1}{4}\varepsilon^2\mathcal{F}_s - 1. \tag{4.13}$$

Moreover, if g_s and f_s are quasi-periodic in x , with frequencies $\omega_1, \dots, \omega_n$ (in the sense of Section 2.5), then so are τ_{s+1} , g_{s+1} and f_{s+1} .

Proof. — We use Lemma 1, choosing $\chi \in \Lambda_p^m$ such that f_{s+1} does not contain terms of order ε^2 . Let us denote $\hat{g} = ipq \in \Lambda_p^m$; one can write

$$\begin{aligned} K_{s+1} &= y + \kappa + \frac{\eta}{\varepsilon} \left(\hat{g} + L\hat{g} + \sum_{k=2}^{\infty} \frac{L^k}{k!} \hat{g} \right) \\ &\quad + \frac{\varepsilon}{\eta} \left(g_s + \sum_{k=1}^{\infty} \frac{L^k}{k!} g_s \right) + \frac{\varepsilon}{\eta} \left(f_s + \sum_{k=1}^{\infty} \frac{L^k}{k!} f_s \right). \end{aligned} \tag{4.14}$$

We determine χ in such a way that $\frac{\eta}{\varepsilon}L\hat{g} + \frac{\varepsilon}{\eta}f_s = 0$, i.e., denoting $f_s = \frac{1}{4}F(x)p^2 - \frac{1}{4}F^*(x)q^2$, we let $\chi = -\frac{1}{8}\left(\frac{\varepsilon}{\eta}\right)^2 (iF(x)p^2 - (iF)^*q^2)$, so that $|\chi|_{x, \rho_s} = \frac{1}{2}\left(\frac{\varepsilon}{\eta}\right)^2 |f_s|_{x, \rho_s}$. We then obtain K_{s+1} of the form (4.10), with

$$\begin{aligned} g_{s+1} - g_s &= \Pi^m \left[\frac{\eta}{\varepsilon}\kappa + \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) L^k f_s + \sum_{k=1}^{\infty} \frac{L^k}{k!} g_s \right] \\ f_{s+1} &= \Pi^p \left[\frac{\eta}{\varepsilon}\kappa + \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) L^k f_s + \sum_{k=1}^{\infty} \frac{L^k}{k!} g_s \right]. \end{aligned} \tag{4.15}$$

Let us now estimate $\Pi^m \kappa$ on the basis of its expression (4.8). First of all, since $\frac{\partial \chi}{\partial x} \in \Lambda_p^p$, using the algebraic relations (4.3) one obtains

$$\Pi^m \kappa = - \sum_{k=2, 4, \dots} \frac{1}{k!} L^{k-1} \frac{\partial \chi}{\partial x}. \text{ On the other hand, since } X \text{ is assumed to}$$

be analytic in \mathcal{S}_{ρ} , its derivative is also analytic in \mathcal{S}_{ρ} ; Cauchy inequality gives then $\left| \frac{dX}{dx} \right|_{x, \rho_{s+1}} \leq \delta^{-1} |X|_{x, \rho_s}$, and thus $\left| \frac{d\chi}{dx} \right|_{x, \rho_{s+1}} \leq \delta^{-1} |\chi|_{x, \rho_s}$ (this estimate is the only point where analyticity is ever used!). Using the basic estimate (4.4), one then immediately gets

$$\begin{aligned} |\Pi^m \kappa|_{x, \rho_{s+1}} &\leq \frac{1}{\delta} \sum_{k=2,4,\dots} \frac{1}{k!} |\chi|_{x, \rho_s}^k \leq \frac{1}{2\delta} |\chi|_{x, \rho_s}^2 \cosh |\chi|_{x, \rho_s} \quad (4.16) \\ &\leq \frac{\varepsilon^4 \mathcal{F}_s^2 N}{8\rho} \cosh |\chi|_{x, \rho_s}. \end{aligned}$$

In a very similar way one can estimate $\Pi^p \kappa$, as well as the sums entering (4.15); as a result, the inequalities (4.12) are found.

Concerning point (iii) of the statement, it is a trivial consequence of the inequality (4.7) of Lemma 1, together with the estimate $|\chi|_{x, \rho_s} = \frac{1}{2} \left(\frac{\varepsilon}{\eta} \right)^2 |f_s|_{x, \rho_s}$. To prove the last statement, one observes that χ is quasi-periodic whenever f_s is; as a consequence, the matrix τ_{s+1} is quasi-periodic too, and, moreover, since the Poisson brackets preserve the quasi-periodicity, the series (4.15) define quasi-periodic functions. ■

4.5. Conclusion of the proof

On the basis of Lemma 2, the proof of Proposition 2 is easily accomplished. Let us look at the recurrent estimates (4.12), and proceed, for a moment, heuristically, by considering the asymptotic behavior for $\varepsilon \ll 1$ ($E \gg |V|_{\rho}$). The first inequality reduces, in this approximation, to $\mathcal{F}_{s+1} < \frac{\varepsilon N}{2\eta\rho} \mathcal{F}_s$, which gives (recalling $\mathcal{F}_0 = 1$) $\mathcal{F}_N < \left(\frac{\varepsilon N}{2\eta\rho} \right)^N$. A simple computation shows that the optimal choice of N , namely the choice which makes minimal \mathcal{F}_N for each given ε , η and ρ , is $N = \left[\frac{2\eta\rho}{e\varepsilon} \right] = \left[\frac{2}{e} \rho \sqrt{E} \right]$, where $[\cdot]$ denotes the integer part; correspondingly, one has (still within this heuristic procedure) $\mathcal{F}_N < \exp[-\alpha\rho\sqrt{E}]$, with $\alpha = 2/e$. Passing now to a rigorous analysis, the simplest (although not completely optimal) procedure is to slightly lower α , taking for example $\alpha = \frac{1}{2}$, i. e. $N = N_E \equiv \left[\frac{1}{2} \rho \sqrt{E} \right]$; in fact, using this value of N in (4.12),

(²) and assuming $\varepsilon^2 \leq \frac{1}{5}$ (i. e., $E \geq 5|V|_\rho$), one easily verifies, by induction, the following recurrent inequalities:

$$\mathcal{F}_s < e^{-s}, \quad \mathcal{G}_s < \frac{1}{2} \varepsilon^2 \leq 0.1, \quad s \leq N. \tag{4.17}$$

If we denote $g_N = \frac{1}{2} i \tilde{V}(x) p q$, $f_N = \frac{1}{4} e^{-N} [F(x) p^2 - F^*(x) q^2]$, then we see that $K' \equiv K_N$ has indeed the form $K' = y + H'$, with H' as in (3.5); in particular, F and \tilde{V} satisfy part (ii) of the statement. Concerning the inequality in part (i), it is an easy consequence of (4.13). Indeed, one has $T(x) = \tau_1(x) \tau_2(x) \dots \tau_N(x)$; using (4.13), and the inequality $\|\tau_1 \tau_2 \dots \tau_N - 1\| \leq \prod_{s=0}^N (1 + \|\tau_s - 1\|) - 1$, which is immediately verified (by induction on N) for any set of matrices, one obtains

$$\|T(x) - 1\| \leq \exp\left(\frac{\varepsilon^2}{4} \sum_{s=0}^{N-1} \mathcal{F}_s\right) - 1 < \exp \frac{\varepsilon^2}{2} - 1 = \exp \frac{|V|_\rho}{2E} - 1 \tag{4.18}$$

for any real x . This concludes the proof of Proposition 2.

5. TESTING THE OPTIMALITY OF OUR PERTURBATIVE APPROACH

As remarked in the Introduction, we devoted some attention to the evaluation of the constants appearing throughout the paper, with special care for the expression of N_E . In the statement of Propositions 1 and 2, one finds (forgetting here the integer part) $N_E = \frac{1}{2} \rho \sqrt{E}$; however, as commented after the statement of Proposition 1, one could obtain (for sufficiently large E) $N_E = \alpha \rho \sqrt{E}$, α being any number smaller than $2/e$. It is clearly interesting to know whether this value of α , which is obtained at the end of a rather long chain of estimates, is nevertheless, so to speak, reasonably good.

(²) Apparently, according to (4.11), one should impose $N \geq 1$, i. e., $E \geq \frac{1}{4} \rho^2$: but of course, for smaller E the statement of Proposition 2 is trivially true, with $T=1$, $\tilde{V}=V$ and $H'=H$.

To this purpose, we made two different tests: (i) in the case of the scattering, we compared our estimate for the reflection coefficient R (Sect. 2.3) with the results of rather accurate numerical computations; (ii) for a periodic potential, we compared our estimate from above for the Lyapunov exponents, which in turn implies an estimate from above for the length of the spectral gaps [22], with a corresponding estimate from below, taken from the theory of Hill's equation.

(i) *Results of the first test.* — We have performed some numerical computations of the reflection coefficient R which, according to our analysis of Section 2.3, is expected to decay, for large E , as

$$|R| \simeq (\text{const.}) e^{-\alpha\sqrt{E}}/\sqrt{E}. \quad (5.1)$$

We have considered the following four potentials:

$$\begin{aligned} \text{(i)} \quad V(x) &= \frac{e^{-x^2}}{1+x^2}; \\ \text{(ii)} \quad V(x) &= \frac{e^{-x^2}}{1+4x^2}; \\ \text{(iii)} \quad V(x) &= \frac{e^{-x^2}}{(1+x^2)(1+(x-1)^2)(1+(x+1)^2)}; \\ \text{(iv)} \quad V(x) &= \frac{e^{-x^2} \sin x}{1+x^2}. \end{aligned}$$

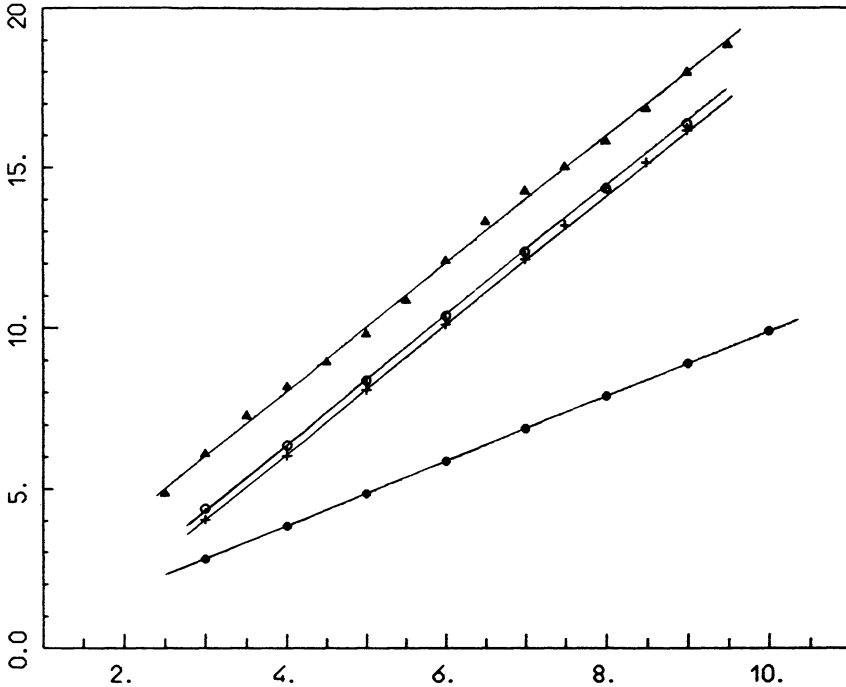
Potential (i) has two poles in $x = \pm i$, so that one has $\rho = 1$; for potential

(ii) one has instead $\rho = \frac{1}{2}$. Potential (iii) has more poles, with $\rho = 1$, and

finally, potential (iv) is like potential (i), but with additional oscillations.

The Figure reports the results of the numerical computations, namely $-\log(|R| \sqrt{E})$ versus \sqrt{E} . In agreement with (5.1), one finds straight lines, of slope $a = \alpha\rho$. The value of a , and thus of α , are rather well defined; as a remarkable fact, the three potentials with $\rho = 1$ show exactly the same slope $a \simeq 2.02$, while for potential (ii) (for which $\rho = \frac{1}{2}$) one has instead $a \simeq 1.01$. These values should be considered with some care; in particular, we are not able to perform any reasonable error analysis. Nevertheless, one has the strong impression that the law $a = \alpha\rho$ is correct, with the same α for all of the above potentials; in fact, one is even tempted to say that $\alpha = 2$ is a kind of universal constant, say for a convenient class of scattering potentials. This value must be compared with $\alpha = 2/e$, produced by perturbation theory.

(ii) *Results of the second test.* — Under assumptions weaker than ours, one can prove [22] that (for sufficiently large values of E) if the solution of the Schroedinger equation (1.1) has a Lyapunov exponent less than a



Illustrating the results of the numerical test for α .

number $\mu \leq 1$, then the distance d of E from the spectrum of the Schroedinger operator is bounded by $d \leq C_1 \mu \sqrt{E}$ (here and in the following, C_1, C_2, \dots denote convenient positive constants). Consider then the special case of the Mathieu equation, $V(x) = \cos(2x)$; from our estimate (2.17), taking (as is possible and convenient) ρ such that the condition $E \geq 5|V|_\rho$ (Proposition 1) is strictly fulfilled, one easily finds

$$d \leq (C_2 \sqrt{E})^{-\alpha \sqrt{E}}. \tag{5.2}$$

On the other hand, from the theory of the Hill's equation (see refs. [1], [23]), one knows that there exist gaps in the spectrum at arbitrarily high energy; more precisely, for any natural number n there is a gap around $E_n \geq n^2$, having length $D \geq (C_3 n)^{-2n} \geq (C_4 \sqrt{E})^{-2\sqrt{E}}$. The comparison between d and D shows that α cannot exceed 2; as a remarkable fact, this is exactly the same value obtained numerically in the case of scattering, *i.e.* with a substantially different potential. As before, this value must be compared with our limit value $\alpha = 2/e$.

These results suggest that $\alpha = 2$ is perhaps a kind of universal constant in the theory of the one-dimensional analytic Schroedinger equation.

Concerning the question posed at the beginning of this Section, one can say that, although the estimate for α coming from perturbation theory is not optimal, the error is nevertheless rather small, namely within a factor e . We are rather confident that this result could be improved by a more careful perturbative construction.

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