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Analysis of a nontrivial, explicitly solvable multichannel scattering system *

by

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ABSTRACT. — We study the scattering theory associated with the one dimensional time dependent quantum Hamiltonian

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\lambda_1\,\delta(x-v_1\,t)-\lambda_2\,\delta(x-v_2\,t),$$

with $v_1 \neq v_2$. This system has nontrivial scattering between channels if λ_1 and λ_2 are both positive. We calculate the Faddeev series for the wave operators of this system explicitly. From this calculation we directly prove asymptotic completeness and study the entire S-matrix. The Faddeev series for the "charge transfer" matrix elements of the S-matrix exhibit rather surprizing behavior for large values of $|v_1-v_2|$.

RÉSUMÉ. — Nous étudions la diffusion par l'hamiltonien quantique unidimensionnel dépendant du temps.

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\lambda_1\,\delta(x-v_1\,t)-\lambda_2\,\delta(x-v_2\,t),$$

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avec $v_1 \neq v_2$. Ce système a une diffusion non triviale entre canaux si λ_1 et λ_2 sont positifs. Nous calculons explicitement la série de Faddeev de ce système. Comme conséquence de ce calcul nous montrons la complétude asymptotique et nous étudions la matrice S. La série de Faddeev des éléments de la matrice de « transfer de charge » de la matrice S présente un comportement intéressant pour les grandes valeurs de $|v_1 - v_2|$.

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1. INTRODUCTION

We wish to discuss an explicitly solvable nontrivial quantum mechanical multichannel scattering model. Its channel wave operators are given by convergent infinite series, all of whose terms are very simple. These series can be summed in terms of well-known special functions.

The model is motivated by a one dimensional three body problem in which particles 1 and 2 are infinitely massive and do not interact with one another. They move with constant velocities v_1 and v_2, respectively. Particle 3 has unit mass and interacts with particles 1 and 2 via Dirac delta functions. Because the motion of particles 1 and 2 is trivial, we only consider the quantum mechanical motion of particle 3. Its evolution is governed by the time dependent Hamiltonian

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2} - \lambda_1 \delta(x - v_1 t) - \lambda_2 \delta(x - v_2 t).$$

This Hamiltonian function is well defined and self-adjoint as a sum of quadratic forms. We will prove by explicit construction that it generates a strongly continuous unitary propagator.

In order to ensure a rich multichannel structure for the scattering, we assume $v_1 \neq v_2$, and that both λ_1 and λ_2 are positive. With these assumptions, the model has three scattering channels. Particle 3 can bind to particle 1, bind to particle 2, or be unbound as $t \to \pm\infty$. All three channels are nontrivially coupled to one another. Thus, the system exhibits "ionization," "charge transfer," and "capture" phenomena, as well as elastic scattering.

The time dependent analogs of the Faddeev series give rise to convergent series for the channel wave operators. We use these series to directly prove asymptotic completeness and to study the S-matrix in detail. The series for the S-matrix elements that describe charge transfer (e. g., for 3 bound to 1 in the past and 3 bound to 2 in the future) exhibit some peculiar behavior in the limit of large $|v_1 - v_2|$. The first Faddeev term for this amplitude is $O(|v_1 - v_2|^{-2})$. However, there is a cancellation with the second Faddeev term, and the charge transfer matrix elements are actually $O(|v_1 - v_2|^{-3})$. Although this cancellation is surprizing, there are physical systems in which strange behavior is conjectured to occur [1].

After obtaining our results, we were surprized to learn of other papers that discuss closely related models ([2]-[6]). Papers ([2]-[4]) were evidently simultaneously published by members of the same group. They studied a model in which the infinitely massive particles 1 and 2 move with piecewise constant velocities that are constant except at time 0, when they bounce off one another at a non-zero distance of closest approach. Their motivation was to test the accuracy of an approximation, and their analysis relied on a method of images technique. This technique applies to essentially no other models, and we found some comments in [2]-[4] to be misleading. A footnote on page 404 of [3] was particularly puzzling. Papers [5], [6] studied the same model we are studying. In [6] some particular solutions of the Schrödinger equation were represented as integrals. In [5] the channel wave operators corresponding to particle 3 being bound to one of the other particles were computed explicitly in terms of Bessel functions of complex order and complex argument. The authors appears to have summed the series computed in [2]-[4] for these channels. This explicit solution facilitated the analysis of the small $|v_1 - v_2|$ limit of some elements of the S-matrix, where the series converge very slowly.

In contrast to these papers, our motivation is to apply the general ideas of Faddeev series to this particular model, and to study the high energy behavior of the scattering. We hope it will shed some more light on the behavior of Faddeev series in other contexts. In addition, we compute the Faddeev series for all the channel wave operators. This allows us to study the entire S-matrix and prove asymptotic completeness. The earlier authors consider only certain channels.

The earlier papers did not point out the curious behavior of the charge transfer scattering amplitudes at high impact velocities. In the physics literature, there is some controversy [1] concerning the high energy behavior of charge transfer amplitudes in three dimensional scattering. As far as we know, there are no mathematically rigorous results that deal with these high energy asymptotics, but a computation of the high energy behavior of Faddeev series for those models would almost certainly resolve the controversy [7]. Our model shows that the leading term in the Faddeev series need not agree with the leading high energy asymptotics. Thus, high energy analysis using Faddeev series can be much more subtle than one might naïvely expect.

Since our model was explicitly solvable, we had hoped that it might provide some insight into the problem of multichannel asymptotic completeness. For this reason, we directly proved asymptotic completeness, using the explicit solutions. This turned out to be technically difficult, and it did not seem to provide any new insights. Other proofs of asymptotic completeness for impact parameter models in various dimensions can be found in [7], [12], [13].

The paper is organized as follows: In the next section we set up notation, precisely define the wave operators, and formally describe the Faddeev series for this model. In section 3, we formally calculate the action of the wave operators on certain states explicitly. In Section 4, we prove that the formal calculations are rigorously correct, and we prove asymptotic completeness. We study the charge transfer scattering amplitude in Section 5.

The way we obtained our results was rather amusing. We were trying to learn the numerical analysis of time dependent Schrödinger equations when we first put this model on the computer as an exercise. We observed that the part of the graphs of the numerical solution that intuitively corresponded to charge transfer had unexpected time dependence. It became large for times near 0, and then decayed considerably as the delta functions moved away from one another. This behavior prompted us to study the behavior of the Faddeev series numerically. After seeing how the individual terms looked, we simply guessed the form of each term in the series.

The numerical computations taught us one other surprizing fact that does not show up in our final analysis. We analytically solve from time $-\infty$ to time 0 for the past wave operators and solve backwards from time ∞ to time 0 for the future wave operators. We do not explicitly calculate the time evolution explicitly through time 0. In the numerical computations, time 0 is not particularly special, and there is no problem numerically propagating each Faddeev series term all the way from some large negative time to a large positive time. The curious behavior of the high velocity charge transfer amplitude exhibits itself in this context through the interference of the first and third order Faddeev terms for the propagator.

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2. KINEMATICAL CONSIDERATIONS

To simplify certain calculations, it is advantageous to work in frames of reference in which either particle 1 or particle 2 is at rest. We let $v=v_1-v_2$, and assume without loss that $v>0$. In the frame of reference in which particle 1 is at rest, the Schrödinger equation for our model is

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}-\lambda_1\,\delta(x)\,\psi-\lambda_2\,\delta(x-vt)\,\psi. \qquad (2.1)$$

We always denote the wave function in this representation by $\psi(x, t)$.

We occasionally change to the frame of reference in which particle 2 is at rest. To avoid confusion, we denote the wave function in that frame by $\varphi(y, t)$, where $y=x-vt$. The functions φ and ψ are related by

$$\varphi(y, t)=e^{-itv^2/2}\,e^{-ivy}\,\psi(y+vt, t),$$

and

$$\psi(x, t)=e^{-itv^2/2}\,e^{ivx}\,\varphi(x-vt, t).$$

The equation satisfied by φ is

$$i\frac{\partial\varphi}{\partial t}=-\frac{1}{2}\frac{\partial^2\varphi}{\partial y^2}-\lambda_1\,\delta(y+vt)\,\varphi-\lambda_2\,\delta(y)\,\varphi. \qquad (2.2)$$

In Section 4 of this paper, we prove that equation (2.1) generates a strongly continuous propagator $U(t, s)$ on $\mathscr{H}=L^2(\mathbb{R})$ for all times t and s. The proof follows from the explicit formal calculations of Section 3.

The scattering theory for our model involves three other unitary propagators. We set $U_0(t, s)=e^{-i(t-s)H_0}$, where $H_0=-\frac{1}{2}\frac{\partial^2}{\partial x^2}$. We define $U_1(t, s)$ and $U_2(t, s)$ to be the unitary propagators associated with the equations

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}-\lambda_1\,\delta(x)\,\psi$$

and

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}-\lambda_2\,\delta(x-vt)\,\psi,$$

respectively. In the ψ representation, U_0 and U_1 have time independent self adjoint generators. In the φ representation, U_2 has a time independent self adjoint generator. These facts assure the existence of the propagators.

The propagator U_1 has exactly one bound state of energy $-\lambda_1^2/2$ in the ψ representation. It is $\psi(x, t)=\lambda_1^{1/2}\,e^{it\lambda_1^2/2}\,e^{-\lambda_1|x|}$. Similarly, U_2 has only one bound state of energy $-\lambda_2^2$ in the φ representation. In the ψ representation, it has the form $\psi(x, t)=\lambda_2^{1/2}\,e^{-it(v^2-\lambda_2^2)/2}\,e^{ivx}\,e^{-\lambda_2|x-vt|}$.

We define wave operators $\Omega_0^\pm\{s\}:\mathscr{H}\to\mathscr{H}$, $\Omega_1^\pm\{s\}:\mathbb{C}\to\mathscr{H}$, and $\Omega_1^\pm\{s\}:\mathbb{C}\to\mathscr{H}$ for our system by

$$\Omega_0^\pm\{s\}\psi=\lim_{t\to\mp\infty}\mathrm{U}(s,t)\mathrm{U}_0(t,0)\psi, \tag{2.2}$$

$$\Omega_1^\pm\{s\}z=\lim_{t\to\mp\infty}\mathrm{U}(s,t)\mathrm{U}_1(t,0)z\lambda_1^{1/2}e^{-\lambda_1|x|}, \tag{2.3}$$

and

$$\Omega_2^\pm\{s\}z=\lim_{t\to\mp\infty}\mathrm{U}(s,t)\mathrm{U}_2(t,0)z\lambda_2^{1/2}e^{ivx}e^{-\lambda_2|x|}. \tag{2.4}$$

Since $t=0$ plays a special role, we let $\Omega_j^\pm=\Omega_j^\pm\{0\}$.

In Section 3 we explicitly calculate $\Omega_1^\pm\{s\}$ and $\Omega_2^\pm\{s\}$. The existence of $\Omega_0^\pm\{s\}$ is established in Section 4. From abstract considerations [8], the wave operators are partial isometries with trivial kernels. The ranges of $\Omega_0^+\{s\}$, $\Omega_1^+\{s\}$, and $\Omega_2^+\{s\}$ are mutually orthogonal, as are the ranges of $\Omega_0^-\{s\}$, $\Omega_1^-\{s\}$, and $\Omega_2^-\{s\}$. In Section 4 we prove asymptotic completeness, *i. e.*,

$$\begin{aligned}\mathrm{Ran}\,\Omega_0^+\{s\}\oplus\mathrm{Ran}\,\Omega_1^+\{s\}\oplus\mathrm{Ran}\,\Omega_2^+\{s\}\\ =\mathrm{Ran}\,\Omega_0^-\{s\}\oplus\mathrm{Ran}\,\Omega_1^-\{s\}\oplus\mathrm{Ran}\,\Omega_2^-\{s\}=\mathscr{H}.\end{aligned}$$

We define the asymptotic Hilbert space $\mathscr{H}_{\mathrm{asy}}=\mathscr{H}\oplus\mathbb{C}\oplus\mathbb{C}$, and define $\Omega^\pm:\mathscr{H}_{\mathrm{asy}}\to\mathscr{H}$ by

$$\Omega^\pm:\ \begin{pmatrix}\psi\\ z_1\\ z_2\end{pmatrix}\mapsto\Omega_0^\pm\psi+\Omega_1^\pm z_1+\Omega_2^\pm z_2.$$

We define the S-matrix $\mathrm{S}:\mathscr{H}_{\mathrm{asy}}\to\mathscr{H}_{\mathrm{asy}}$ by $\mathrm{S}=(\Omega^-)^*\Omega^+$. The unitary of $\Omega^\pm$ and S follows from asymptotic completeness.

The Faddeev series for $\Omega_0^\pm\{s\}$, $\Omega_1^\pm\{s\}$, and $\Omega_2^\pm\{s\}$ arise from series expansions for the propagators $\mathrm{U}(s,t)$ in equations (2.2), (2.3), and (2.4), respectively. We formally derive these series here by algebraic manipulation; for a more intuitive derivation, see [7].

We begin the derivation by recalling three formal expressions for $\mathrm{U}(s,t)$:

$$\begin{aligned}\mathrm{U}(s,t)=\mathrm{U}_0(s,t)-i\int_t^s\mathrm{U}(s,r)\mathrm{V}_1(r)\mathrm{U}_0(r,t)\,dr\\ -i\int_t^s\mathrm{U}(s,r)\mathrm{V}_2(r)\mathrm{U}_0(r,t)\,dr,\end{aligned} \tag{2.5}$$

$$\mathrm{U}(s,t)=\mathrm{U}_1(s,t)-i\int_t^s\mathrm{U}(s,r)\mathrm{V}_2(r)\mathrm{U}_1(r,t)\,dr, \tag{2.6}$$

and

$$\mathrm{U}(s,t)=\mathrm{U}_2(s,t)-i\int_t^s\mathrm{U}(s,r)\mathrm{V}_1(r)\mathrm{U}_2(r,t)\,dr, \tag{2.7}$$

where $V_1(r)$ denotes the potential $-\lambda_1 \delta(x)$ and $V_2(r)$ denotes the potential $-\lambda_2 \delta(x-vr)$.

By using (2.6) and (2.7), we obtain two more formal expressions for $U(s,t)$:

$$U(s,t)=U_1(s,t)-i\int_t^s U_2(s,r)V_2(r)U_1(r,t)\,dr \\ -\int_t^s dr\int_r^s dq\,U(s,q)V_1(q)U_2(q,r)V_2(r)U_1(r,t), \quad (2.8)$$

and

$$U(s,t)=U_2(s,t)-i\int_t^s U_1(s,r)V_1(r)U_2(r,t)\,dr \\ -\int_t^s dr\int_r^s dq\,U(s,q)V_2(q)U_1(q,r)V_1(r)U_2(r,t). \quad (2.9)$$

The Faddeev series for $\Omega_1^\pm\{s\}$ and $\Omega_2^\pm\{s\}$ are obtained by substituting the iterates of (2.8) and (2.9) into (2.3) and (2.4), respectively. The Faddeev series for $\Omega_0^\pm\{s\}$ are obtained by substituting (2.5) into (2.2), and then substituting the iterates of (2.8) and (2.9) into the resulting first and second integral terms, respectively.

3. EXPLICIT FORMAL SOLUTIONS

In this section we explicitly calculate the formal scattering solutions to equation (2.1) using the Faddeev series. We describe the calculation of $\Omega_1^\pm(1)$ in detail, and simply write down the results of the other similar calculations.

The vectors $\Omega_1^\pm(1)$ and $\Omega_2^\pm(1)$ belong to $\mathcal{H}$. The other formal scattering solutions $\psi^\pm(k,x,t)$ do not. Intuitively, they should be thought of as $\psi^\pm(k,x,t)=\Omega_0^\pm\{t\}(e^{ikx})$. Since $e^{ikx}\notin\mathcal{H}$, $\Omega_0^\pm\{t\}(e^{ikx})$ does not make sense, but we prove in Section 4 that if $\hat{f}$ is the Fourier transform of $f\in\mathcal{H}$, then

$$\Omega_0^\pm\{t\}\left((2\pi)^{-1/2}\int\hat{f}(k)\,e^{ikx}\,dk\right)=(2\pi)^{-1/2}\int\hat{f}(k)\,\psi^\pm(k,x,t)\,dk, \quad (3.1)$$

where both integrals are understood in an L^2 sense.

To calculate $\Omega_1^+(1)$ we calculate the solution $\Omega_1^+\{t\}(1)$ to the Schrödinger equation (2.1) for $t\leqq 0$. From the Faddeev series for the propagator, it can be expanded as

$$\Omega_1^+\{t\}(1)=\psi_0(x,t)+\psi_1(x,t)+\psi_2(x,t)+\dots$$

We restrict this series to time $t=0$ to obtain the Faddeev series for $\Omega_1^+(1)=\psi_0(x,0)+\psi_1(x,0)+\psi_2(x,0)+\dots$

The leading term in this series is

$$\psi_0(x,t)=\lambda_1^{1/2}\,e^{it\lambda_1^2/2}\,e^{-\lambda_1|x|}.$$

The next term, ψ_1, comes from the next term in the Faddeev series. It is the improper integral

$$\psi_1(.,t)=-i\int_{-\infty}^{t} U_2(t,r)\,V_2(r)\,\psi_0(.,r)\,dr,$$

which is more convenient to evaluate in the φ representation. In the φ representation, the analog of ψ_0 is

$$\varphi_0(y,t)=\lambda_1^{1/2}\,e^{-it(v^2-\lambda_1^2)/2}\,e^{-ivy}\,e^{-\lambda_1|y+vt|}.$$

The φ representation analog of ψ_1 is

$$\varphi_1(y,t)=-i\int_{-\infty}^{t}\tilde{U}_2(t,r)(-\lambda_2\,\delta(y))\,\varphi_0(y,r)\,dr$$
$$=i\lambda_2\int_{-\infty}^{t}\tilde{U}_2(t,r)\,\delta(y)\,\varphi_0(0,r)\,dr,$$

where $\tilde{U}_2$ is the φ representation analog of U_2. For $t\leqq 0$, the r dependence of the last integrand involves only exponential functions. The integral can be evaluated in terms of the Green's function for the generator of $\tilde{U}_2$. That Green's function is explicitly known, and the result for $\varphi_1(y,t)$ is

$$\varphi_1(y,t)=\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,e^{-it(v+i\lambda_1)^2/2}\,e^{-(\lambda_1-iv)|y|}.$$

Thus,

$$\psi_1(x,t)=\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,e^{-it(v^2-iv\lambda_1-(\lambda_1^2/2))}\,e^{ivx}\,e^{-(\lambda_1-iv)|x-vt|}.$$

The next term in the Faddeev series is easier to compute in the ψ representation. It is

$$\psi_2(.,t)=-i\int_{-\infty}^{t}U_1(t,r)\,V_1(r)\,\psi_1(.,r)\,dr$$
$$=i\lambda_1\int_{-\infty}^{t}U_1(t,r)\,\delta(x)\,\psi_1(0,r)\,dr.$$

Once again, the r dependence of the integrand is purely exponential, and the integral can be calculated in terms of an explicitly known Green's function. We obtain

$$\psi_2(x,t)=\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,\frac{\lambda_1}{-2iv}\,e^{-it(2v+i\lambda_1)^2/2}\,e^{-(\lambda_1-2iv)|x|}.$$

The next term is

$$\psi_3(.,t)=-i\int_{-\infty}^{t}U_2(t,r)\,V_2(r)\,\psi_2(.,r)\,dr.$$

The procedure for calculating this integral is the same as that used to find $\psi_1(x,t)$. The result is

$$\psi_3(x,t)=\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,\frac{\lambda_1}{-2iv}\,\frac{\lambda_2}{\lambda_1-\lambda_2-3iv}\times e^{-it(5v^2+3iv\lambda_1-(\lambda_1^2/2))}\,e^{-(\lambda_1-3iv)|x-vt|}.$$

An induction shows that for $n\geqq 1$, we have for $t\leqq 0$,

$$\left.\begin{aligned}\varphi_n(y,t)&=C_n e^{-it(nv+i\lambda_1)^2/2}\,e^{-(\lambda_1-inv)|y|}\\ C_n&=C_{n-1}\frac{\lambda_2}{\lambda_1-\lambda_2-inv}\end{aligned}\right\}\quad n\ \text{odd}$$

$$\left.\begin{aligned}\psi_n(x,t)&=C_n e^{-it(nv+i\lambda_1)^2/2}\,e^{-(\lambda_1-inv)|x|}\\ C_n&=C_{n-1}\frac{\lambda_1}{-inv}\end{aligned}\right\}\quad n\ \text{even}$$

Thus, if we restrict to time 0,

$$\begin{aligned}\Omega_1^+(1)=\lambda_1^{1/2}e^{-\lambda_1|x|}&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}e^{-(\lambda_1-iv)|x|}e^{ivx}\\&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,\frac{\lambda_1}{-2iv}e^{-(\lambda_1-2iv)|x|}\\&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\,\frac{\lambda_1}{-2iv}\,\frac{\lambda_2}{\lambda_1-\lambda_2-3iv}e^{-(\lambda_1-3iv)|x|}e^{ivx}+\ldots\end{aligned}\tag{3.2}$$

The calculation of $\Omega_1^-(1)$ is very similar, except that one calculates for $t\geqq 0$ rather than $t\leqq 0$. The result is

$$\begin{aligned}\Omega_1^-(1)=\lambda_1^{1/2}e^{-\lambda_1|x|}&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2+iv}e^{-(\lambda_1+iv)|x|}e^{ivx}\\&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2+iv}\,\frac{\lambda_1}{2iv}e^{-(\lambda_1+2iv)|x|}\\&+\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2+iv}\,\frac{\lambda_1}{2iv}\,\frac{\lambda_2}{\lambda_1-\lambda_2+3iv}e^{-(\lambda_1+3iv)|x|}e^{ivx}+\ldots\end{aligned}\tag{3.3}$$

The calculations of $\Omega_2^+(1)$ and $\Omega_2^-(1)$ are also very similar. The results are

$$\begin{aligned}\Omega_2^+(1)=\lambda_2^{1/2}e^{-\lambda_2|x|}e^{ivx}+\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1-iv}e^{-(\lambda_2-iv)|x|}\\ +\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1-iv}\frac{\lambda_2}{-2iv}e^{-(\lambda_2-2iv)|x|}e^{ivx}\\ +\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1-iv}\frac{\lambda_2}{-2iv}\frac{\lambda_1}{\lambda_2-\lambda_1-3iv}e^{-(\lambda_2-3iv)|x|}+\ldots,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\Omega_2^-(1)=\lambda_2^{1/2}e^{-\lambda_2|x|}e^{ivx}+\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}e^{-(\lambda_2+iv)|x|}\\ +\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\frac{\lambda_2}{2iv}e^{-(\lambda_2+2iv)|x|}e^{ivx}\\ +\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\frac{\lambda_2}{2iv}\frac{\lambda_1}{\lambda_2-\lambda_1+3iv}e^{-(\lambda_2+3iv)|x|}+\ldots\end{aligned}\tag{3.5}$$

We now turn our attention to the calculation of $\psi^+(k,x,t)$. We calculate a series expansion for this solution to the Schrödinger equation (2.1) for $t\leqq 0$. [The analogous calculation can be done for $\psi^-(k,x,t)$ for $t\geqq 0$.] This solution is in $L^\infty(\mathbb{R})$ rather than $L^2(\mathbb{R})$. It is formally

$$\begin{aligned}\psi^+(k,x,t)&=\psi_0(k,x,t)+\psi_1^1(k,x,t)+\psi_2^1(k,x,t)+\psi_3^1(k,x,t)+\ldots\\ &\quad+\psi_1^2(k,x,t)+\psi_2^2(k,x,t)+\psi_3^2(k,x,t)+\ldots\\ &=[U(t,0)\psi^+(k,.,0)](x)=\lim_{r\to-\infty}U(t,r)U_0(r,0)e^{ikx}.\end{aligned}$$

The leading term is

$$\psi_0(k,x,t)=e^{-itk^2/2}e^{ikx}.$$

There are two first order Faddeev terms. They are

$$\psi_1^1(k,.,t)=-i\int_{-\infty}^{t}U_1(t,r)V_1(r)\psi_0(k,.,r)\,dr,$$

and

$$\psi_1^2(k,.,t)=-i\int_{-\infty}^{t}U_2(t,r)V_2(r)\psi_0(k,.,r)\,dr.$$

These improper integrals do not exist because of a rapidly oscillating boundary term at time $-\infty$. We interpret these integrals as Abelian limits

$$\int_{-\infty}^{t}f(r)\,dr=\lim_{\varepsilon\to 0}\int_{-\infty}^{t}f(r)e^{\varepsilon r}\,dr.$$

In Section 4 we prove that this Abelian limit gives the correct result in the sense that equation (3.1) is fulfilled. Furthermore, for the remainder of this section, we will abuse notation and interpret all improper integrals as such Abelian limits.

The integral for ψ_1^1 can be calculated in the ψ representation since the r dependence of its integrand is purely exponential. The result involves an explicitly known Green's function. Similarly, the integrand in the expression for ψ_1^2 is purely an exponential in the φ representation, so it, too, can be calculated. The results are

$$\psi_1^1(k,x,t) = -\frac{\lambda_1}{\lambda_1+i|k|} e^{-itk^2/2} e^{i|k|.|x|}$$

and

$$\psi_1^2(k,x,t) = -\frac{\lambda_2}{\lambda_2+i|k-v|} e^{-it((k-v)^2/2)+(v^2/2)} e^{i|k-v|.|x-vt|} e^{ivx}.$$

The second order Faddeev terms are

$$\psi_2^1(k,.,t) = -i\int_{-\infty}^{t} U_2(t,r)\,V_2(r)\,\psi_1^1(k,.,r)\,dr,$$

and

$$\psi_2^2(k,.,t) = -i\int_{-\infty}^{t} U_1(t,r)\,V_1(r)\,\psi_1^2(k,.,r)\,dr.$$

The first of these is the integral of an exponential in the φ representation. The second is the integral of an exponential in the ψ representation. The results are

$$\psi_2^1(k,x,t) = \frac{\lambda_1}{\lambda_1+i|k|}\,\frac{\lambda_2}{\lambda_2+i(|k|+v)} \times e^{-it(((|k|+v)^2/2)+(v^2/2))} e^{i(|k|+v)|x-vt|} e^{ivx}$$

and

$$\psi_1^2(k,x,t) = \frac{\lambda_2}{\lambda_2+i|k-v|}\,\frac{\lambda_1}{\lambda_1+i(|k-v|+v)} \times e^{-it(|k-v|+v)^2/2} e^{i(|k-v|+v)|x|}.$$

We calculate the higher order terms in $\psi^+(k,x,t)$ by the obvious inductive argument. To show the pattern of the terms in $\psi^+(k,x)=\psi^+(k,x,0)$, we exhibit a large number of them:

$$\psi^+(k,x) = e^{ikx} - \frac{\lambda_1}{\lambda_1+i|k|} e^{i|k|.|x|} - \frac{\lambda_2}{\lambda_2+i|k-v|} e^{i|k-v|.|x|} e^{ivx} + \frac{\lambda_1}{\lambda_1+i|k|}\,\frac{\lambda_2}{\lambda_2+i(|k|+v)} e^{i(|k|+v)|x|} e^{ivx}$$

$$+\frac{\lambda_2}{\lambda_2+i|k-v|}\,\frac{\lambda_1}{\lambda_1+i(|k-v|+v)}e^{i(|k-v|+v)|x|}$$
$$-\frac{\lambda_1}{\lambda_1+i|k|}\,\frac{\lambda_2}{\lambda_2+i(|k|+v)}\,\frac{\lambda_1}{\lambda_1+i(|k|+2v)}e^{i(|k|+2v)|x|}$$
$$-\frac{\lambda_2}{\lambda_2+i|k-v|}\,\frac{\lambda_1}{\lambda_1+i(|k-v|+v)}$$
$$\times\frac{\lambda_2}{\lambda_2+i(|k-v|+2v)}e^{i(|k-v|+2v)|x|}e^{ivx}$$
$$+\frac{\lambda_1}{\lambda_1+i|k|}\,\frac{\lambda_2}{\lambda_2+i(|k|+v)}\,\frac{\lambda_1}{\lambda_1+i(|k|+2v)}$$
$$\times\frac{\lambda_2}{\lambda_2+i(|k|+3v)}e^{i(|k|+3v)|x|}e^{ivx}$$
$$+\frac{\lambda_2}{\lambda_2+i|k-v|}\,\frac{\lambda_1}{\lambda_1+i(|k-v|+v)}$$
$$\times\frac{\lambda_2}{\lambda_2+i(|k-v|+2v)}\,\frac{\lambda_1}{\lambda_1+i(|k-v|+3v)}e^{i(|k-v|+3v)|x|}-\dots \qquad (3.6)$$

By the analogous procedure, we obtain

$$\psi^-(k,x)=e^{ikx}-\frac{\lambda_1}{\lambda_1-i|k|}e^{-i|k|.|x|}$$
$$-\frac{\lambda_2}{\lambda_2-i|k-v|}e^{-i|k-v|.|x|}e^{ivx}+\frac{\lambda_1}{\lambda_1-i|k|}\,\frac{\lambda_2}{\lambda_2-i(|k|+v)}e^{-i(|k|+v)|x|}e^{ivx}$$
$$+\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1}{\lambda_1-i(|k-v|+v)}e^{-i(|k-v|+v)|x|}$$
$$-\frac{\lambda_1}{\lambda_1-i|k|}\,\frac{\lambda_2}{\lambda_2-i(|k|+v)}\,\frac{\lambda_1}{\lambda_1-i(|k|+2v)}e^{-i(|k|+2v)|x|}$$
$$-\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1}{\lambda_1-i(|k-v|+v)}$$
$$\times\frac{\lambda_2}{\lambda_2-i(|k-v|+2v)}e^{-i(|k-v|+2v)|x|}e^{ivx}+\dots \qquad (3.7)$$

In the next section we prove that all solutions to the Schrödinger equation (2.1) are superpositions of the solutions we have computed above. That result is asymptotic completeness. We conclude this section by noting that our series solutions can be summed in terms of special functions. This allows one to study the small ν limit, in which the series converge slowly.

As noted in [5], $\Omega_1^{\pm}(1)$ and $\Omega_2^{\pm}(1)$ can be computed explicitly in terms of Bessel functions. For example, we have

$$[\Omega_1^{\pm}(1)](x)=\Gamma\left(\frac{1}{2}+i\gamma\right)e^{-\lambda_1|x|}\left(\frac{1}{2}(\gamma_1\gamma_2)^{1/2}e^{iv|x|}\right)^{(1/2)-i\gamma}$$
$$\times\{\lambda_1^{1/2}J_{-(1/2)+i\gamma}((\gamma_1\gamma_2)^{1/2}e^{iv|x|})+i\lambda_2^{1/2}e^{ivx}J_{(1/2)+i\gamma}((\gamma_1\gamma_2)^{1/2}e^{iv|x|})\},$$

where $\gamma_1=\dfrac{\lambda_1}{v}$, $\gamma_2=\dfrac{\lambda_2}{v}$, and $\gamma=\dfrac{\lambda_1-\lambda_2}{2v}$. If $\lambda_1=\lambda_2$, this can be written more simply in terms of sines and cosines since $\gamma=0$ in that case.

Similarly, we can sum the series for $\psi^{\pm}(k,x)$ in terms of generalized hypergeometric functions. For example,

$$\psi^{+}(k,x)=e^{ikx}-\frac{\lambda_1 e^{i|k|.|x|}}{\lambda_1-i|k|}$$
$$\times{}_1F_2\left(1;\frac{-i\lambda_2+|k|+v}{2v},\frac{-i\lambda_1+|k|+2v}{2v};-\frac{\lambda_1\lambda_2}{4v^2}e^{2iv|x|}\right)$$
$$-\frac{\lambda_2 e^{i|k-v|.|x|}e^{ivx}}{\lambda_2-i|k-v|}\,{}_1F_2\left(1;\frac{-i\lambda_1+|k-v|+v}{2v},\right.$$
$$\left.\frac{-i\lambda_1+|k-v|+2v}{2v};-\frac{\lambda_1\lambda_2}{4v^2}e^{2iv|x|}\right)$$
$$+\frac{\lambda_1}{\lambda_1-i|k|}\,\frac{\lambda_2 e^{i(|k|+v)|x|}e^{ivx}}{\lambda_2+i(|k|+v)}$$
$$\times{}_1F_2\left(1;\frac{-i\lambda_1+|k|+2v}{2v},\frac{-i\lambda_2+|k|+3v}{2v};-\frac{\lambda_1\lambda_2}{4v^2}e^{2iv|x|}\right)$$
$$+\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1 e^{i(|k-v|+v)|x|}}{\lambda_1+i(|k-v|+v)}$$
$$\times{}_1F_2\left(1;\frac{-i\lambda_2+|k-v|+2v}{2v},\frac{-i\lambda_1+|k|+3v}{2v};-\frac{\lambda_1\lambda_2}{4v^2}e^{2iv|x|}\right).$$

4. MATHEMATICAL TECHNICALITIES

In this section we prove existence of the propagator $U(s,t)$, existence and asymptotic completeness of the wave operators, and the validity of equation (3.1).

If t and s have the same sign, then the existence of the unitary propagator $U(s,t)$ can be proved by the techniques of [9]. Thus, we need only prove existence of $U(t,0)$ and verify the properties that must be satisfied by a unitary propagator.

We begin by defining transforms that are appropriate in the study of propagation for $t<0$. If $f \in \mathcal{S}(\mathbb{R})$, we define

$$\mathcal{A}_t f(k) = (2\pi)^{-1/2} \int \overline{\psi^+(k, x, t)} f(x)\, dx.$$

The formal adjoint of this transform acts on $g \in \mathcal{S}(\mathbb{R})$, by the transform

$$\mathcal{B}_t g(x) = (2\pi)^{-1/2} \int \psi^+(k, x, t) g(k)\, dk.$$

Using the explicit convergent series representations of $\psi^+(k, x, t)$ and properties of Fourier transforms, one can easily show that $\mathcal{A}_t$ and B_t extend to bounded operators on $\mathcal{H}$. Furthermore, the series for $\mathcal{A}_t f$ and $\mathrm{B}_t g$ are norm convergent in $\mathcal{H}$.

For $g \in \mathcal{S}(\mathbb{R})$, $\mathcal{B}_t g$ is a solution to the Schrödinger equation (2.1). If g has compact support away from integer multiples of v, then we can apply stationary phase techniques [8] to each term in the series for $\mathrm{B}_t g$ to study the $t \to -\infty$ asymptotics. By using the norm convergence of the series we conclude that

$$\lim_{t \to -\infty} \| \mathcal{B}_t g - e^{-it\mathrm{H}_0} (2\pi)^{-1/2} \int g(k) e^{ikx} dk \| = 0.$$

By the unitary of $\mathrm{U}(s, t)$ for both s and t negative, we conclude that

$$\| \mathcal{B}_t g \| = \| g \|,$$

for $t<0$. However, each term in the series for $\mathcal{B}_t g$ has a limit in $\mathcal{H}$ as $t \to 0$, and the series is uniformly norm convergent for $t \leqq 0$. It follows that $\mathcal{B}_t$ extends to a strongly continuous family of isometries for $t \leqq 0$.

On the range of $\mathcal{B}_0$ we define $\mathrm{U}(t, 0)$ for $t \leqq 0$ by

$$\mathrm{U}(t, 0)[\mathcal{B}_0 g] = \mathcal{B}_t g.$$

Since the kernel of $\mathcal{B}_0$ is trivial, this is well defined. With this definition and the stationary phase calculation described above, it follows by a density argument that Ω_0^+ exists, and that

$$\Omega_0^+(\psi) = \mathcal{B}_0 \hat{\psi},$$

for all $\psi \in \mathcal{H}$. So, $\operatorname{Ran} \Omega_0^+ = \operatorname{Ran} \mathcal{B}_0$, and equation (3.1) is valid.

The definition of $\mathrm{U}(t, 0)$ on the ranges of Ω_1^+ and Ω_2^+ is

$$\mathrm{U}(t, 0) \Omega_j^+ z = \Omega_j^+ \{t\} z.$$

Since the ranges of the wave operators are necessarily mutually orthogonal, $\mathrm{U}(t, 0)$ is thus well defined on the range of the total wave operator, Ω^+. Furthermore, the arguments used to show $\mathrm{U}(t, 0)$ is a partial isometry from $\operatorname{Ran} \Omega_0^+$ into $\mathcal{H}$ can be employed again to show that $\mathrm{U}(t, 0)$ is an isometry from $\operatorname{Ran} \Omega^+$ to $\mathcal{H}$. So, we have defined $\mathrm{U}(t, 0)$ on all of $\mathcal{H}$ if the range of Ω^+ is $\mathscr{H}$, *i. e.*, if asymptotic completeness holds. To see that $U(t, 0)$ is actually unitary and not just an isometry, we employ Proposition 4.1, below, which shows that for $t \leqq 0$,

$$\operatorname{Ran} \Omega_0^+ \{t\} \oplus \operatorname{Ran} \Omega_1^+ \{t\} \oplus \operatorname{Ran} \Omega_2^+ \{t\} = \mathscr{H}.$$

We repeat this argument to construct $U(t, 0)$ for $t \geqq 0$. We then define $U(s, t)$ for all s and t by $U(s, t) = U(s, 0) U(0, t)$. It is then easy to verify that the resulting object is a strongly continuous unitary propagator.

Thus, all the claims of this section follow from the following proposition on asymptotic completeness:

PROPOSITION 4.1. — *If* $t \leqq 0$, *then*

$$\operatorname{Ran} \Omega_0^+ \{t\} \oplus \operatorname{Ran} \Omega_1^+ \{t\} \oplus \operatorname{Ran} \Omega_2^+ \{t\} = \mathscr{H}.$$

If $t \geqq 0$, *then*

$$\operatorname{Ran} \Omega_0^- \{t\} \oplus \operatorname{Ran} \Omega_1^- \{t\} \oplus \operatorname{Ran} \Omega_2^- \{t\} = \mathscr{H}.$$

Proof. — We only prove the first statement for $t = 0$. The result for $t < 0$ is proved by the same argument applied to $e^{-itk^2/2} \hat{f}$ rather than $\hat{f}$. The formulas are more cumbersome in that case. The proof of the second statement of the proposition is similar.

Our approach is to show that if $f \in \mathscr{H}$ is orthogonal to $\operatorname{Ran} \Omega_0^+$, $\operatorname{Ran} \Omega_1^+$, and $\operatorname{Ran} \Omega_2^+$, then it must be zero. This is equivalent to showing that the following three conditions,

$$\mathscr{B}_0 f = 0, \tag{4.1}$$

$$\langle \Omega_1^+ \{0\}(1), f \rangle = 0, \tag{4.2}$$

and

$$\langle \Omega_2^+ \{0\}(1), f \rangle = 0, \tag{4.2}$$

guarantee $f = 0$.

We let $f_+(x) = f(x) \chi_{[0, \infty)}(x)$ and $f_-(x) = f(x) \chi_{(-\infty, 0]}(x)$ denote the projections of f onto the functions of support on the right and left half-lines, respectively. We denote their Fourier transforms by $\hat{f}_+$ and $\hat{f}_-$, respectively.

With this notation, condition (4.1) can be rewritten in terms of Fourier transforms:

$$\begin{aligned}
\hat{f}(k) = & \frac{\lambda_1}{\lambda_1 - i|k|} (\hat{f}_+(|k|) + \hat{f}_-(-|k|)) \\
& + \frac{\lambda_2}{\lambda_2 - i|k - v|} (\hat{f}_+(|k - v| + v) + \hat{f}_-(-|k - v| + v)) \\
& - \frac{\lambda_1}{\lambda_1 - i|k|} \frac{\lambda_2}{\lambda_2 - i(|k| + v)} (\hat{f}_+(|k| + 2v) + \hat{f}_-(-|k|))
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1}{\lambda_1-i(|k-v|+v)}\\
&\quad\times(\hat{f}_+(|k-v|+v)+\hat{f}_-(-|k-v|-v))\\
&+\frac{\lambda_1}{\lambda_1-i|k|}\,\frac{\lambda_2}{\lambda_2-i(|k|+v)}\,\frac{\lambda_1}{\lambda_1-i(|k|+2v)}\\
&\quad\times(\hat{f}_+(|k|+2v)+\hat{f}_-(-|k|-2v))\\
&+\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1}{\lambda_1-i(|k-v|+v)}\\
&\times\frac{\lambda_2}{\lambda_2-i(|k-v|+2v)}(\hat{f}_+(|k-v|+3v)+\hat{f}_-(-|k-v|-v))\\
&-\frac{\lambda_1}{\lambda_1-i|k|}\,\frac{\lambda_2}{\lambda_2-i(|k|+v)}\,\frac{\lambda_1}{\lambda_1-i(|k|+2v)}\\
&\times\frac{\lambda_2}{\lambda_2-i(|k|+3v)}(\hat{f}_+(|k|+4v)+\hat{f}_-(-|k|-2v))\\
&-\frac{\lambda_2}{\lambda_2-i|k-v|}\,\frac{\lambda_1}{\lambda_1-i(|k-v|+v)}\,\frac{\lambda_2}{\lambda_2-i(|k-v|+2v)}\\
&\times\frac{\lambda_1}{\lambda_1-i(|k-v|+3v)}(\hat{f}_+(|k-v|+3v)+\hat{f}_-(-|k-v|-3v))\\
&\qquad\qquad+\ldots=\hat{g}(k)+\hat{h}(k), \qquad (4.4)
\end{aligned}
$$

where $\hat{g}$ is the sum of the terms that involve only $|k|$, and $\hat{h}$ is the sum of the terms that involve only $|k-v|$.

From $\hat{g}$ *and* $\hat{h}$ we construct an even function $\hat{V}(k)$ that takes values in $\mathbb{C}^2$:

$$\hat{V}(k)=\begin{pmatrix}\hat{g}(k)\\ \hat{h}(k+v)\end{pmatrix}.$$

For $k\geqq 0$, $\hat{V}(k)$ is a norm convergent series. Each term in this series is the product of a rational function and the Fourier transform of a function in $L^2([0,\infty))$. By a Payley-Wiener theorem and the particular rational functions involved, it has a continuation to a meromorphic function in the lower half plane with non-tangential L^2 boundary values on the whole real axis in k. The singularities of the continuation are either removable singularities or simple poles. They can occur only at the points $-i\lambda_1-nv$ and $-i\lambda_2-nv$, where $n=0,1,2,\ldots$ Similarly, $\hat{V}(k)$ has a meromorphic continuation from $\{k\in\mathbb{R}:k\leqq 0\}$ to the upper half plane, with L^2 boundary values on the whole real axis. Its singularities are also either removable singularities or simple poles at the points $i\lambda_1+nv$ and $i\lambda_2+nv$, where $n=0,1,2,\ldots$

We now show that the boundary values of $\hat{V}$ on the real k axis from above and below agree with one another. Since $\hat{V}$ is in $L^2(\mathbb{R})$, its inverse

Fourier transform, V is in L^2, also. We denote its first and second components by V_1 and V_2, respectively.

For $k \geqq 0$ (and hence in the lower half plane), we obtain a formula, equivalent to (4.4), for $\hat{V}(k)$ by the convergent iteration of the equation

$$\hat{V}(k)=\begin{pmatrix} \dfrac{\lambda_1}{\lambda_1-ik}(\hat{f}_+(k)+\hat{f}_-(-k)) \\ \dfrac{\lambda_2}{\lambda_2-ik}(\hat{f}_+(k+v)+\hat{f}_-(-k+v)) \end{pmatrix} - \begin{pmatrix} 0 & \dfrac{\lambda_1}{\lambda_1-ik} \\ \dfrac{\lambda_2}{\lambda_2-ik} & 0 \end{pmatrix} \hat{V}(k+v). \quad (4.5)$$

By exploiting the evenness in k of $\hat{g}(k)$ and $\hat{h}(k+v)$, we obtain the following two identities:

$$\hat{f}_+(k)+\hat{f}_-(-k)=\hat{g}_+(k)+\hat{h}_+(k)+\hat{g}_-(-k)+\hat{h}_-(-k) = 2\hat{g}_+(k)+\hat{h}_+(k)+\hat{h}_+(k+2v)$$

and

$$\hat{f}_+(k+v)+\hat{f}_-(-k+v) = \hat{g}_+(k+v)+\hat{h}_+(k+v)+\hat{g}_-(-k+v)+\hat{h}_-(-k+v) = 2\hat{h}_+(k+v)+\hat{g}_+(k+v)+\hat{g}_+(k-v).$$

By substituting these in equation (4.5), we rewrite (4.5) as

$$\hat{V}(k)=\begin{pmatrix} \dfrac{\lambda_1}{\lambda_1-ik}(2\hat{V}_{1+}(k)+\hat{V}_{2+}(k-v)+\hat{V}_{2+}(k+v)) \\ \dfrac{\lambda_2}{\lambda_2-ik}(2\hat{V}_{2+}(k)+\hat{V}_{1+}(k-v)+\hat{V}_{1+}(k+v)) \end{pmatrix} - \begin{pmatrix} 0 & \dfrac{\lambda_1}{\lambda_1-ik} \\ \dfrac{\lambda_2}{\lambda_2-ik} & 0 \end{pmatrix} \hat{V}(k+v)$$
$$= \begin{pmatrix} \dfrac{\lambda_1}{\lambda_1-ik}(2\hat{V}_{i+}(k)+\hat{V}_{2+}(k-v)-\hat{V}_{2-}(k+v)) \\ \dfrac{\lambda_2}{\lambda_2-ik}(2\hat{V}_{2+}(k)+\hat{V}_{1+}(k-v)-\hat{V}_{1-}(k+v)) \end{pmatrix}$$

In the j-th component of the last equation, we replace $V_{j+}(k)$ by the equivalent expression $V_j(k)-V_{j-}(k)$, and move the $V_j(k)$ term to the left hand side. We then multiply the j-th component on both sides of the resulting equation by the inverse of

$$\left(1-\frac{2\lambda_j}{\lambda_j-ik}\right)=-\frac{\lambda_j+ik}{\lambda_j-ik}.$$

We thus obtain

$$\hat{V}(k)=\begin{pmatrix} \dfrac{\lambda_1}{\lambda_1+ik}(2\hat{V}_{1-}(k)-\hat{V}_{2+}(k-v)+\hat{V}_{2-}(k+v)) \\ \dfrac{\lambda_2}{\lambda_2+ik}(2\hat{V}_{2-}(k)-\hat{V}_{1+}(k-v)+\hat{V}_{1-}(k+v)) \end{pmatrix}$$

In the $(3-j)$-th component of the right had side, we replace $V_{j+}(k-v)$ by the equivalent expression $V_j(k-v)-V_{j-}(k-v)$. The result is

$$\hat{V}(k)=\begin{pmatrix} \dfrac{\lambda_1}{\lambda_1+ik}(2\hat{V}_{1-}(k)+\hat{V}_{2-}(k-v)+\hat{V}_{2-}(k+v)) \\ \dfrac{\lambda_2}{\lambda_2+ik}(2\hat{V}_{2-}(k)+\hat{V}_{1-}(k-v)+\hat{V}_{1-}(k+v)) \end{pmatrix} - \begin{pmatrix} 0 & \dfrac{\lambda_1}{\lambda_1+ik} \\ \dfrac{\lambda_2}{\lambda_2+ik} & 0 \end{pmatrix}\hat{V}(k-v).$$

By employing identities like those used above to rewrite equation (4.5), we can rewrite this equation as

$$\hat{V}(k)=\begin{pmatrix} \dfrac{\lambda_1}{\lambda_1+ik}(\hat{f}_+(-k)+\hat{f}_-(k)) \\ \dfrac{\lambda_2}{\lambda_2+ik}(\hat{f}_+(-k+v)+\hat{f}_-(k+v)) \end{pmatrix} - \begin{pmatrix} 0 & \dfrac{\lambda_1}{\lambda_1+ik} \\ \dfrac{\lambda_2}{\lambda_2+ik} & 0 \end{pmatrix}\hat{V}(k-v). \quad (4.6)$$

This is precisely the $k \leqq 0$ analog of equation (4.5), and it, too may be solved by iteration. So, if one starts with positive real k, meromorphically continues $V(k)$ to the lower half plane, and then computes the boundary values on the negative real k axis, one obtains the same values as the original of $V(k)$ on the negative k axis. One may similarly start on the negative real k axis, meromorphically continue to the upper half plane, and then reproduce the original values of $V(k)$ on the positive k axis by taking boundary values. It then follows ([10], Lemma 6.6, p. 223) that $\hat{V}$ is meromorphic in the whole complex plane.

We next employ conditions (4.2) and (4.3) to show that the singularities of $\hat{V}$ are all removable. Condition (4.2) is equivalent to

$$\begin{aligned}0=&\hat{f}_+(-i\lambda_1)+\hat{f}_-(i\lambda_1)\\&+\frac{\lambda_2}{\lambda_1-\lambda_2+iv}(\hat{f}_+(-i\lambda_1+2v)+\hat{f}_-(i\lambda_1))\\&+\frac{\lambda_2}{\lambda_1-\lambda_2+iv}\,\frac{\lambda_1}{2iv}(\hat{f}_+(-i\lambda_1+2v)+\hat{f}_-(i\lambda_1-2v))\\&+\frac{\lambda_2}{\lambda_1-\lambda_2+iv}\,\frac{\lambda_1}{2iv}\,\frac{\lambda_2}{\lambda_1-\lambda_2+3iv}\\&\qquad\times(\hat{f}_+(-i\lambda_1+4v)+\hat{f}_-(i\lambda_1-2v))+\dots\end{aligned}$$

This is precisely the statement that $(\lambda_1-ik)\,\hat{g}(k)$ has a zero at $k=-i\lambda_1$. Thus, the singularity in $\hat{g}$ at $k=-i\lambda_1$ is removable. Since $\hat{g}$ is even, the singularity at $k=+i\lambda_1$ is also removable.

Similarly, condition (4.3) requires

$$\begin{aligned}0=&\hat{f}_+(-i\lambda_2+v)+\hat{f}_-(i\lambda_2+v)\\&+\frac{\lambda_1}{\lambda_2-\lambda_1+iv}(\hat{f}_+(-i\lambda_2+v)+\hat{f}_-(i\lambda_2-v))\\&+\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\,\frac{\lambda_2}{2iv}(\hat{f}_+(-i\lambda_2+3v)+\hat{f}_-(i\lambda_2-v))\\&+\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\,\frac{\lambda_2}{2iv}\,\frac{\lambda_1}{\lambda_2-\lambda_1+3iv}(\hat{f}_+(-i\lambda_2+3v)+\hat{f}_-(i\lambda_2-3v))+\dots\end{aligned}$$

By reasoning as above this implies that the singularities in k of $\hat{h}(k+v)$ at $k=\pm i\lambda_2$ are removable.

So, the singularities of $\hat{V}(k)$ at $\pm i\lambda_1$ and $\pm i\lambda_2$ are removable. However, by inductively using these facts and equation (4.5) with $k=-i\lambda_j-nv$, we see that the singularity at $k=-i\lambda_j-nv$ is removable. In the upper half plane, the analogous argument with equation (4.6) shows that the singularities at $k=i\lambda_j+nv$ are also removable. Thus, $\hat{V}$ is entire.

Our next goal is to show that $\hat{V}$ is bounded. We begin by noticing that equation (4.4) and the Payley-Wiener theorem imply that $\hat{V}(k)$ is L^2 on horizontal lines with $\operatorname{Im} k\neq\pm\lambda_1$ and $\operatorname{Im} k\neq\pm\lambda_2$. Thus, $V(x)\,e^{b|x|}$ is L^2 for some $b>0$, and consequently, $V(x)\,e^{(b-\varepsilon)|x|}$ is L^1 for any $\varepsilon>0$. It follows that $\hat{V}(k)$ tends to 0 as $|\operatorname{Re} k|$ tends to infinity with $|\lambda_1\pm\operatorname{Im} k|$ and $|\lambda_2\pm\operatorname{Im} k|$ kept bounded away from zero. Since the Payley-Wiener theorem and equation (4.4) show that $\hat{V}(k)$ tends to 0 as $|\operatorname{Im} k|$ tends to infinity, we see that $\hat{V}(k)$ tends to 0 as $|k|$ tends to infinity, except possibly if k is allowed to get arbitrarily close to a sequence of the removable singularities.

To see that $\hat{V}(k)$ tends to zero regardless of how k goes to infinity, we first note that equation (4.5) implies that if $\operatorname{Re} k = \left(n + \frac{1}{2}\right) v$ and $-\varepsilon \geqq \operatorname{Im} k$, then

$$\|\hat{V}(k)\| \leqq C_1 + C_2 \|\hat{V}(k+v)\|.$$

We choose $\varepsilon < \frac{1}{2} \operatorname{Min} \{\lambda_1, \lambda_2\}$ and choose $\alpha = 2 \operatorname{Max} \{\lambda_1, \lambda_2\}$. By the maximum modulus principle and some simple estimates, it follows that on the rectangle whose corners are $\frac{1}{2} v - i\varepsilon$, $\frac{1}{2} v - i\alpha$, $-\left(n + \frac{1}{2}\right) v - i\varepsilon$, and $-\left(n + \frac{1}{2}\right) v - i\alpha$, the maximum of $\|\hat{V}(k)\|$ grows at most exponentially with n. By the Phragman-Lindelöf principal ([11], p. 276) and simple estimates, we see that $\|\hat{V}(k)\|$ is actually bounded in the strip $-\alpha \leqq \operatorname{Im} k \leqq -\varepsilon$. By using equation (4.6) in the upper half plane, we can similarly establish the existence of a uniform bound on $\|\hat{V}(k)\|$ for $\varepsilon \leqq \operatorname{Im} k \leqq \alpha$.

Thus, we can apply Liouville's theorem to see that $\hat{V}$ is constant. Since it tends to zero as k goes to infinity in certain directions, it must be the zero function. ■

5. CHARGE TRANSFER AT HIGH VELOCITIES

With the results of the previous sections, one can compute the large v asymptotics of S-matrix elements easily. In this section we calculate the asymptotics of one of the "charge transfer" matrix elements to point out the strange cancellation in the first few terms of the Faddeev series. Other matrix elements can easily be calculated, but they do not exhibit the cancellations.

Specifically, we calculate the large v asymptotics of the S-matrix element

$$A(v) = \langle \Omega_2^-(1), \Omega_1^+(1) \rangle, \tag{5.1}$$

by using the series expressions (3.2) and (3.5). The zeroth order approximation to A (v) that results from taking only the zeroth order term from each of the series yields

$$A_0(v)=(\lambda_1\lambda_2)^{1/2}\int e^{-(\lambda_1+\lambda_2)|x|}e^{-ivx}dx$$
$$=(\lambda_1\lambda_2)^{1/2}\left(\frac{1}{\lambda_1+\lambda_2+iv}+\frac{1}{\lambda_1+\lambda_2-iv}\right)$$
$$=2(\lambda_1\lambda_2)^{1/2}\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_2)^2+v^2}.$$

For large v, this is

$$A_0(v)=2(\lambda_1\lambda_2)^{1/2}(\lambda_1+\lambda_2)v^{-2}+O(v^{-4}).$$

The first order term in the approximation to (5.1) is the sum of two inner products

$$A_1(v)=\left\langle\lambda_2^{1/2}e^{-\lambda_2|\cdot|}e^{iv\cdot},\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}e^{-(\lambda_1-iv)|\cdot|}e^{iv\cdot}\right\rangle$$
$$+\left\langle\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}e^{-(\lambda_2+iv)|\cdot|},\lambda_1^{1/2}e^{-\lambda_1|\cdot|}\right\rangle$$
$$=(\lambda_1\lambda_2)^{1/2}\left(\frac{\lambda_2}{\lambda_1-\lambda_2-iv}+\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\right)\int e^{-(\lambda_1+\lambda_2-iv)|x|}dx$$
$$=(\lambda_1\lambda_2)^{1/2}\left(\frac{\lambda_2}{\lambda_1-\lambda_2-iv}+\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\right)\frac{2}{\lambda_1+\lambda_2-iv}$$
$$=-2(\lambda_1\lambda_2)^{1/2}(\lambda_1+\lambda_2)v^{-2}+8i(\lambda_1\lambda_2)^{3/2}v^{-3}+O(v^{-4}).$$

The second order term is

$$A_2(v)=\left\langle\lambda_2^{1/2}e^{-\lambda_2|\cdot|}e^{iv\cdot},\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}\frac{\lambda_1}{-2iv}e^{-(\lambda_1-2iv)|\cdot|}e^{iv\cdot}\right\rangle$$
$$+\left\langle\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}e^{-(\lambda_2+iv)|\cdot|},\right.$$
$$\left.\lambda_1^{1/2}\frac{\lambda_2}{\lambda_1-\lambda_2-iv}e^{-(\lambda_1-iv)|\cdot|}e^{iv\cdot}\right\rangle$$
$$+\left\langle\lambda_2^{1/2}\frac{\lambda_1}{\lambda_2-\lambda_1+iv}\frac{\lambda_2}{2iv}e^{-(\lambda_2+2iv)|\cdot|}e^{iv\cdot},\lambda_1^{1/2}e^{-\lambda_1|\cdot|}\right\rangle$$
$$=-\frac{8}{3}i(\lambda_1\lambda_2)^{3/2}v^{-3}+O(v^{-4}).$$

The terms of higher order than this are all at most $O(v^{-4})$ by trivial estimates. Thus, the charge transfer amplitude is

$$A(v)=\frac{16}{3}i(\lambda_1\lambda_2)^{3/2}v^{-3}+O(v^{-4}),$$

contrary to what one would think from the first term of the expansion.

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