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Post-Newtonian generation of gravitational waves

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ABSTRACT. — This paper derives a new multipole-moment gravitational wave generation formalism for slowly moving, weakly stressed, weakly self-gravitating systems. In this formalism the first-post-Newtonian ($O(v^2/c^2)$) corrections to the standard Einstein-Landau-Lifshitz quadrupole equation are given as explicit integrals over the stress-energy distribution of the material source. This is in contrast with the previous Epstein-Wagoner-Thorne formalism which involved badly defined integrals over the infinite support of the stress-energy distribution of the gravitational field (we show however that the latter integrals can be transformed, via formal manipulations, into our well-defined compact-support results). The method we use is a combination of a multipolar post-Minkowskian expansion for the metric in the weak-field region outside the system, and of a post-Newtonian expansion for the metric in the near zone, the two expansions being then "matched" in the weak-field-near-zone overlap region. Our method never making use of ill-justified retardation expansions, and giving the next term in a slow-motion expansion as a well-controlled correction, provides also a solid confirmation of the lowest-order (Einstein-Landau-Lifshitz) quadrupole equation.

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RÉSUMÉ. — Cet article présente un nouveau formalisme multipolaire de génération d'ondes gravitationnelles par des systèmes en mouvement lent ayant une faible auto-gravitation et de faibles tensions internes. Dans ce formalisme les premières corrections post-newtoniennes $(O(v^2/c^2))$ à l'équation du quadrupole standard d'Einstein-Landau-Lifchitz sont données explicitement comme des intégrales sur la distribution d'énergie-impulsion de la source matérielle. Ceci contraste avec le formalisme précédent d'Epstein-Wagoner-Thorne qui contenait des intégrales mal définies portant sur la distribution effective d'énergie-impulsion du champ gravitationnel, laquelle n'est pas à support compact (nous montrons cependant comment transformer formellement ces dernières intégrales en nos résultats bien définis à support compact). La méthode utilisée consiste à raccorder un développement multipolaire-post-minkowskien pour la métrique en dehors de la source avec un développement post-newtonien pour la métrique en zone proche. Notre méthode n'utilisant jamais des développements mal justifiés en temps de propagation, et donnant le terme suivant du développement en puissances de v/c comme une correction explicitement contrôlable, fournit une confirmation solide du terme le plus bas, c'est-à-dire de l'équation du quadrupole d'Einstein-Landau-Lifchitz.

I. INTRODUCTION

A central problem in gravitational wave research is the *generation problem*, *i. e.* the problem of relating the outgoing gravitational wave field to the structure and motion of the material source. A good understanding of this problem is essential for the analysis of the gravitational wave signals that will, hopefully, be detected before the turn of the century [1]. Although for many sources it will be necessary to resort to numerical methods, it is also important to refine the analytical methods of computing the generation of gravitational waves. Indeed, not only are analytical methods appropriate for dealing with some of the most promising sources of gravitational radiation, especially inspiralling neutron star binaries [2], but they can also be very useful for complementing [3] (and/or testing) numerical methods. In this paper we shall consider the generation problem only within the framework of Einstein's theory of gravity, because this theory has been strongly confirmed by the excellent agreement between the observations [4] and the theory [5] of gravitational radiation damping effects in a system of two condensed bodies.

The generation problem in general relativity was solved at *lowest order* ("Newtonian order"), and for the restrictive case of slowly moving, weakly stressed and negligibly self-gravitating sources, by Einstein [6] himself. His result can be expressed by saying that the asymptotic gravitational radiation amplitude, $h_{ij}^{\rm rad}=(g_{ij}-\delta_{ij})^{\rm TT}$, in a suitable ("radiative") coordinate system, $X^\mu=(c\,T,\vec{X})$ reads [7]

$$h_{ij}^{\rm rad}(X^\mu)=\frac{2G}{c^4 R}P_{ijkl}(\vec{N})\times\left\{\left(\frac{d^2 Q_{kl}}{dt^2}\right)(T-R/c)+O\left(\frac{1}{c}\right)\right\}+O\left(\frac{1}{R^2}\right), \quad (1.1\,a)$$

where the function $Q_{ij}(t)$ is the usual "Newtonian" trace-free quadrupole moment of the mass-energy distribution of the source:

$$Q_{ij}(t)=\int d^3\vec{x}\,c^{-2}\,T^{00}(\vec{x},t)\left(x^i x^j-\frac{1}{3}\delta^{ij}\vec{x}^2\right). \quad (1.1\,b)$$

In equation (1.1 *a*) $R=|\vec{X}|\equiv(\delta_{ij}X^i X^j)^{1/2}$, $\vec{N}=\vec{X}/R$, and $P_{ijkl}(\vec{N})$ denotes the transverse-traceless (TT) projection operator onto the plane orthogonal to $\vec{N}$, namely

$$P_{ijkl}(\vec{N})=(\delta_{ik}-N_i N_k)(\delta_{jl}-N_j N_l)-\frac{1}{2}(\delta_{ij}-N_i N_j)(\delta_{kl}-N_k N_l). \quad (1.2)$$

In equation (1.1 *b*) $T^{\mu\nu}(\vec{x},t)$ denotes the stress-energy tensor of the source expressed in some quasi-Minkowskian coordinate system $x^\mu=(ct,\vec{x})$ which covers the source.

The set of equations (1.1 *a*)-(1.1 *b*) constitutes what we shall call the *Einstein far-field quadrupole equation* [8]. As we said above, its original derivation [6] was restricted to the case of *negligibly* self-gravitating sources, *i. e.* sources whose motion is governed by non gravitational forces. Later Landau and Lifshitz [9] proposed a new derivation, based on the use of a gravitational stress-energy pseudo-tensor, which extended the applicability of equations (1.1) to (slowly moving, weakly stressed, and) *weakly* self-gravitating sources, thereby allowing one to compute the radiation generated by some types of gravitationally bound systems (e. g. an oscillating star, or a binary star). However the proof of Landau and Lifshitz is not quite satisfactory because it makes use of ill-justified retardation expansions in integrals extending over the infinite support of the gravitational stress-energy distribution. A different approach to the generation problem was proposed by Fock [10]. The main idea of his approach was to separate the problem into two sub-problems (one dealing with the gravitational field in the "near zone" of the source, and the other one dealing with the "wave zone" field) and then to "match" the results of both sub-problems. The implementation [10] at lowest order of this method for slowly moving, weakly stressed, weakly self-gravitating sources (that we shall henceforth call for short "non relativistic" sources) yielded again equations (1.1). While Fock's proof never made use of ill-justified retardation expansions, the implementation of the matching between the solutions of the two sub-problems had not been clearly defined. Several recent investigations [11] have tried either to fill the gaps in the Landau-Lifshitz and Fock derivations, or to set up new derivations. Although none of these works provides a mathematically fully satisfactory proof of the Einstein far-field quadrupole equation (1.1) (*see* the discussion in Reference [8]) they have made it practically certain that equations (1.1) are appropriate for computing the lowest order gravitational wave emission by non relativistic sources.

The purpose of this paper is to generalize the quadrupole equations (1.1), valid for non relativistic sources, to *semi-relativistic* ones. By this we mean sources for which the dimensionless parameter, $\varepsilon \sim v/c$, measuring the non relativistic character of the source [*see* equation (2.1) below] is not very small compared to one, so that it becomes necessary to keep in the gravitational wave amplitude the terms which are ε and ε^2 smaller than the lowest order term (1.1 *a*), while still neglecting the terms of order ε^3 and higher. Physical examples where this generalization might be important are: the collapse of a star to a neutron star, and the late stages of inspiralling binary neutron stars, in cases where, say, $\varepsilon \lesssim 0.2$. By analogy with the terminology used in the problem of motion, such an improved gravitational wave formalism, going one $(v/c)^2$-step beyond the lowest order formalism (1.1) which contains the Newtonian quadrupole (1.1 *b*), is called "first post-Newtonian" (1 PN) wave generation formalism.

The first attempt at deriving a post-Newtonian generation formalism is due to Epstein and Wagoner [12] (1 PN level). Later, Thorne [13] extended this formalism by including higher post-Newtonian corrections and higher multipoles. Several applications of these formalisms have been worked out [14]. These formalisms can be considered as direct generalizations of the Landau-Lifshitz derivation of equations (1.1) in that they make an essential use of gravitational stress-energy pseudo-tensors. This causes the appearance, during their derivations, of many divergent integrals due to the slow fall-off of the gravitational stress-energy distribution. These divergent integrals are a signal that the formal retardation expansions performed by Epstein and Wagoner, and Thorne, are mathematically unjustified. Even worse, the fact that their end results are still given in terms of formally divergent integrals [15] casts *a priori* a doubt on their physical soundness (we shall however prove *a posteriori* that they are formally correct by relating them to our results, see Appendix A). This restricts also their practical use as they must be handled with care to get a finite answer when applying them to particular systems [14]. This makes them also ill-adapted to numerical computations. This situation has not been improved by a recent work [16] on the post-Newtonian generation of gravitational waves which can be seen to be incorrect.

In view of this unsatisfactory situation, our aim has been to derive a new first post-Newtonian generation formalism which bypasses the difficulties linked to the slow fall-off of the gravitational stress-energy distribution by expressing the outgoing wave amplitude in terms of integrals extending only over the *material* stress-energy distribution. This result will be achieved by generalizing the Fock derivation [10] of equations (1.1) rather than the Landau-Lifshitz one [9]. This generalization has been made possible by a recent algorithmic implementation [17]-[19] of the main idea in Fock's derivation, namely the matching between a wave-zone expansion, and a near-zone expansion, of the gravitational field. Our algorithm has elucidated and perfected previous works by Bonnor, Burke, Thorne, Anderson, Kates and other authors. Let us now present for the reader's convenience the generalization to post-Newtonian order, that we shall derive below, of the two "Newtonian" equations (1.1 *a*) and (1.1 *b*). First, the gravitational wave amplitude $h_{ij}^{\rm rad}$ in some suitable coordinate system $X^\mu=(cT,\vec{X})$ is given, at the 1 PN accuracy, by a standard multipole decomposition [13].

$$h_{ij}^{\rm rad}(X^\mu)=\frac{2G}{c^4R}P_{ijkl}(\vec{N})\left\{\overset{(2)}{I}{}^{\rm rad}_{kl}+\frac{1}{3c}N_a\overset{(3)}{I}{}^{\rm rad}_{akl}+\frac{4}{3c}\varepsilon_{ab(k}\overset{(2)}{J}{}^{\rm rad}_{l)a}N_b+\frac{1}{12c^2}N_aN_b\overset{(4)}{I}{}^{\rm rad}_{abkl}+\frac{1}{2c^2}\varepsilon_{ab(k}\overset{(3)}{J}{}^{\rm rad}_{l)ac}N_bN_c\right\}(T-R/c)+O\left(\frac{1}{R^2}\right),\quad(1.3\,a)$$

where the $I^{\rm rad}_{i_1\ldots i_l}$ (resp. $J^{\rm rad}_{i_1\ldots i_l}$) are some 2^l-"radiative" mass multipole moments (resp. current multipole moments), where $\overset{(n)}{I}(t)\equiv d^nI/dt^n$ and where $T_{(ij)}\equiv\frac{1}{2}(T_{ij}+T_{ji})$ [*see* equation (3.37) below for the corresponding 1 PN energy-loss expression]. Second, the radiative multipole moments are explicitly given, at the 1 PN accuracy, by integrals over the material stress-energy distribution. Let us quote here only the radiative quadrupole moment for which it is crucial to have 1 PN accuracy:

$$I_{ij}^{\rm rad}(t)=\int d^3\vec{x}\,\hat{x}^{ij}c^{-2}[T^{00}(\vec{x},t)+T^{ss}(\vec{x},t)]+\frac{1}{14c^2}\frac{d^2}{dt^2}\int d^3\vec{x}\,\hat{x}^{ij}\vec{x}^2c^{-2}[T^{00}(\vec{x},t)+T^{ss}(\vec{x},t)]$$

$$-\frac{20}{21\,c^2}\,\frac{d}{dt}\int d^3\vec{x}\,\hat{x}^{ijk}\,c^{-1}\,\mathrm{T}^{0k}(\vec{x},t)+O\left(\frac{1}{c^3}\right). \qquad (1.3\,b)$$

In equation (1.3 *b*) $\mathrm{T}^{\mu\nu}(\vec{x},t)$ denotes the stress-energy tensor of the material source expressed in harmonic coordinates, and $\hat{x}^{i_1\ldots i_n}$ denotes the symmetric trace-free part of $x^{i_1}\ldots x^{i_n}$ (*see* reference [7] for our notation). *See* equation (3.34) below for the 1 PN expressions of the other radiative mass moments (in fact the Newtonian expressions for all moments besides $\mathrm{I}^{\mathrm{rad}}_{ij}$ are sufficiently accurate).

We have found the new 1 PN formalism (1.3) especially convenient for applications of both analytical and numerical nature [20]. We can also consider our explicit 1 PN results (1.3) as providing a further, solid, confirmation of the Einstein far-field quadrupole equation (1.1) because, on the one hand, our derivation never makes use of ill-justified retardation expansions of non-compactly supported retarded integrals, and, on the other hand, our final results yield, for the first time, an explicit estimate of the errors brought about when using the "Newtonian" formulas (1.1).

The method we shall use for tackling the generation problem consists in splitting this problem into two sub-problems. The first sub-problem (treated in Section II) considers the gravitational field only in an *inner domain*, D_i, which comprises the source but does not extend beyond its near zone. We shall write this inner gravitational field in a very compact form, and adjust the coordinate system in the exterior part of D_i in anticipation of the matching to be performed later. Then in Section III we treat the second sub-problem which considers the gravitational field in an *external domain*, D_e, comprising the whole weak-field region outside the source (extending up to infinity). Using some results from our previous work on the multipolar post-Minkowskian (MPM) expansion of vacuum gravitational fields, we compute the 1 PN near-zone reexpansion of the MPM external field. We then match the latter 1 PN external field to the previously derived 1 PN near-zone field. This determines uniquely the unknown parameters of the external MPM field [which are themselves easily related to the "radiative" moments of equation (1.3 *a*)] as functionals of the material source, thereby yielding in a very direct way the sought for 1 PN generation formalism. Finally, in Appendix A we relate our results to the Epstein-Wagoner-Thorne formalism, while the Appendix B is devoted to the relativistic multipole analysis of the retarded potential generated by a source of spatially compact support.

II. THE INNER GRAVITATIONAL FIELD

Let us consider an isolated material system, S, which is, at once, *weakly* *self-gravitating, slowly moving* and *weakly stressed.* This means that the dimensionless parameter,

$$\varepsilon \equiv \sup\left[\left(\frac{Gm}{c^2 r_0}\right)^{1/2}, \left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \frac{r_0}{\hat{\lambda}}\right], \tag{2.1}$$

is required to be small (though maybe not very small) compared to one. In equation (2.1) m denotes a characteristic mass and r_0 a characteristic size of the system S (we shall assume that r_0 is strictly greater than the radius of a sphere in which S can be completely enclosed), and $T^{\mu\nu}$ denote the components of the stress-energy tensor describing S in, say, a harmonic coordinate system, $x^\mu = (ct, \vec{x})$, regularly covering S. We have also explicitly added in (2.1) the ratio $r_0/\hat{\lambda}$ between the size of S and a characteristic reduced wavelength, $\hat{\lambda} = \lambda/2\pi$, of its gravitational radiation so as to exclude any rapid internal vibrations that would not necessarily show up in the bulk velocity $v^{\text{bulk}} \sim c\,|T^{0i}/T^{00}|$ (evidently this addition is not necessary if one assumes that only the bulk motion is the source of the gravitational radiation). This aspect ("S being well within its near zone", $r_0 \leqslant \varepsilon\hat{\lambda}$) of the general slow-motion hypothesis will be of central importance in the following. As one can think of $v \equiv \varepsilon c$ as a characteristic slow velocity in S, we shall have $\varepsilon = c^{-1}$ in source-adapted units where $v = 1$, so that we can, as usual, take c^{-1} as ordering parameter in the combined weak-field-slow-motion expansion that we shall consider.

Let $D_i = \{(\vec{x}, t) \,|\, |\vec{x}| = r < r_1\}$, where $r_0 < r_1 \ll \hat{\lambda}$, be an "inner domain", including S, which define for our purpose the near zone of S. We assume for the inner gravitational field, *i. e.* for the space-time metric within D_i a weak-field-slow-motion (or post-Newtonian) expansion. Let $g^{\text{in}}_{\mu\nu}(x^\lambda)$ denote the components of the inner metric in a harmonic coordinate system $x^\mu = (\vec{x}, ct)$ regularly covering D_i. We use as starting point the following simple explicit expression for the Einstein equations in harmonic coordinates:

$$-g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma}\big(g_{\alpha\mu,\,\rho} g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho} g_{\beta\sigma,\,\nu} + g_{\alpha\mu,\,\rho} g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho} g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha} g_{\nu\sigma,\,\beta}\big) = \frac{16\pi G}{c^4}\left(g_{\alpha\mu} g_{\beta\nu} - \frac{1}{2} g_{\alpha\beta} g_{\mu\nu}\right) T^{\mu\nu}. \tag{2.2}$$

Following Fock [10], rather than the usual way of writing post-Newtonian (PN) expansions, we shall simplify the first post-Newtonian (1 PN) metric by using the near-zone assumption, $\partial_0 g^{\text{in}}_{\mu\nu}/\partial_i g^{\text{in}}_{\mu\nu} = O(1/c)$, for the variation of the metric only in the already algebraically small nonlinear terms in equation (2.2) (only the quadratic nonlinearities of the time-time component need to be considered at the 1 PN level). The insertion of the linearized results,

$$g_{00}+1=O(1/c^2),\ g_{0i}=O(1/c^3),\ g_{ij}-(g_{00}+2)\delta_{ij}=O(1/c^4)$$

[consequences of (2.2)] in the latter nonlinear terms leads, after the introduction of the *logarithm* of $-g_{00}$ as new variable, to the following *linear* equations:

$$\Box \ln(-g_{00}^{\mathrm{in}})=\frac{8\pi G}{c^4}(T^{00}+T^{ss}) \qquad +O\left(\frac{1}{c^6}\right), \tag{2.3a}$$

$$\Box g_{0i}^{\mathrm{in}} = \frac{16\pi G}{c^4}T^{0i} \qquad +O\left(\frac{1}{c^5}\right), \tag{2.3b}$$

$$\Box g_{ij}^{\mathrm{in}} = -\frac{8\pi G}{c^4}\delta_{ij}(T^{00}+T^{ss})+O\left(\frac{1}{c^4}\right), \tag{2.3c}$$

in which $\Box\equiv f^{\mu\nu}\partial_{\mu\nu}\equiv\Delta-c^{-2}\partial_t^2$, $T^{00}=O(c^2)$, $T^{0i}=O(c^1)$ and $T^{ij}=O(c^0)$.

It is appropriate to introduce some new notation. Let us introduce an "active gravitational mass density",

$$\sigma\equiv c^{-2}(T^{00}+T^{ss}), \tag{2.4}$$

and a corresponding "active mass current density",

$$\sigma_i\equiv c^{-1}T^{0i}. \tag{2.5}$$

Note that in the right-hand sides of equations (2.4)-(2.5) appear the *contravariant* components of $T^{\mu\nu}$ in a harmonic coordinate system. Through 1 PN accuracy, one can also express σ in terms of the mixed components:

$$\sigma=c^{-2}(-\det g_{\mu\nu})^{1/2}(-T_0^0+T_s^s)\,[1+O(c^{-4})]. \tag{2.6}$$

One recognizes in equation (2.6) the integrand of the Tolman mass formula valid for stationary systems [9]. The local energy conservation law, $T^{0\mu}{}_{;\mu}=0$, ensures the conservation of "active gravitational mass" at *Newtonian* order only:

$$\partial_t\sigma+\partial_i\sigma_i=O(c^{-2}). \tag{2.7}$$

See equation (3.38) below for the corresponding post-Newtonian conservation law. We introduce now the retarded "scalar" and "vector" potentials generated by σ and σ_i:

$$V^{\mathrm{in}}(\vec{x},t)=G\int\frac{d^3\vec{y}}{|\vec{x}-\vec{y}|}\sigma(\vec{y},t-|\vec{x}-\vec{y}|/c), \tag{2.8}$$

$$V_i^{\mathrm{in}}(\vec{x},t)=G\int\frac{d^3\vec{y}}{|\vec{x}-\vec{y}|}\sigma_i(\vec{y},t-|\vec{x}-\vec{y}|/c). \tag{2.9}$$

This leads to the following solution for the 1 PN equations (2.3):

$$g_{00}^{\mathrm{in}}=-\exp\left[-\frac{2V^{\mathrm{in}}}{c^2}\right]+O\left(\frac{1}{c^6}\right)$$

$$= -1 + \frac{2}{c^2} \mathrm{V}^{\mathrm{in}} - \frac{2}{c^4} (\mathrm{V}^{\mathrm{in}})^2 + O\left(\frac{1}{c^6}\right), \quad (2.10a)$$

$$g^{\mathrm{in}}_{0i} = -\frac{4}{c^3} \mathrm{V}^{\mathrm{in}}_i + O\left(\frac{1}{c^5}\right), \quad (2.10b)$$

$$g^{\mathrm{in}}_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2} \mathrm{V}^{\mathrm{in}}\right) + O\left(\frac{1}{c^4}\right). \quad (2.10c)$$

A priori it would have been possible to add in equations (2.10) arbitrary solutions, regular in D_i, of the wave operator. However all the previous work ([10], [21], [22], [19]) on the matching between the near zone and the wave zone has already proven that such terms did not appear until well after the 1 PN level. It can be noted that our form (2.8)-(2.10) (which generalizes some of Fock's results [10]) of the 1 PN inner metric is much more compact than the usual formulations of the first post-Newtonian approximation. The simplifications come both from a convenient choice of "mass density", equation (2.4), and from the use of a mixed post-Newtonian-post-Minkowskian approach (*i. e.* from keeping together the d'Alembert operators). The 1 PN metric (2.8)-(2.10) is also more general than its usual counterparts in that we have made no assumptions concerning the structure of the material source (we have not restricted ourselves e. g. to considering a perfect fluid model).

Let us now consider the 1 PN harmonic inner metric (2.10) in the exterior near zone, *i. e.* in that part of D_i which is outside the system $(r_0 < r < r_1)$. In order to relate there $g^{\mathrm{in}}_{\mu\nu}$ to the external metric, $g^{\mathrm{ext}}_{\mu\nu}$, that we shall construct below by means of a multipolar post-Minkowskian expansion, we need to perform the *relativistic multipole* analysis of the retarded potentials V^{in} and $\mathrm{V}^{\mathrm{in}}_i$, considered outside the support of σ and σ_i. This is easily done by means of a general expansion formula, worked out in the language of spherical harmonics ($\mathrm{Y}_{lm}(\theta, \varphi)$) by Campbell *et al.* [23], that we derive directly in the language of symmetric trace-free tensors in Appendix B. Equations (B.2)-(B.3) yield immediately in the exterior of S [7]:

$$\mathrm{V}^{\mathrm{in}}(\vec{x}, t) = \mathrm{G} \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \partial_{\mathrm{L}} (r^{-1} \mathrm{F}_{\mathrm{L}}(t - r/c)), \quad (2.11a)$$

$$\mathrm{V}^{\mathrm{in}}_i(\vec{x}, t) = \mathrm{G} \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \partial_{\mathrm{L}} (r^{-1} \mathrm{G}_{i\mathrm{L}}(t - r/c)), \quad (2.11b)$$

in which appear the following *Minkowskian* (instead of Newtonian) multipole moments of the sources σ and σ_i:

$$\mathrm{F}_{\mathrm{L}}(u) = \int d^3\vec{y}\, \hat{y}_{\mathrm{L}} \int_{-1}^{+1} dz\, \delta_l(z)\, \sigma(\vec{y}, u + |\vec{y}|\, z/c), \quad (2.12a)$$

$$\mathrm{G}_{i\mathrm{L}}(u)=\int d^3\vec{y}\,\hat{y}_\mathrm{L}\int_{-1}^{+1} dz\,\delta_l(z)\,\sigma_i(\vec{y}, u+|\vec{y}|\,z/c). \qquad (2.12\,b)$$

Note the presence in equations (2.12) of an average over the time variable with weight function

$$\delta_l(z)=\frac{(2l+1)!!}{2^{l+1}\,l!}(1-z^2)^l, \qquad \int_{-1}^{+1} dz\,\delta_l(z)=1. \qquad (2.13)$$

This weighted time average has its physical origin in the time delays due to the propagation at the velocity of light within the extended source [*see* equations (2.8)-(2.9)]. If we now use our assumption of slow time variability of the matter distribution ($\omega r_0/c=r_0/\bar{\lambda}\leqq\varepsilon$) we can expand the integrands of equations (2.12) in Taylor series of $|\vec{y}|\,z/c$. Since we are considering the inner metric only at the 1 PN approximation, it is sufficient to expand $\mathrm{F}_\mathrm{L}(u)$ up to order $\varepsilon^2=c^{-2}$ included, and to keep only the dominant term in $\mathrm{G}_{i\mathrm{L}}(u)$ [*see* equations (B.14) for the result of a complete expansion]. This yields

$$\mathrm{F}_\mathrm{L}(u)=\lambda_\mathrm{L}(u)+O(c^{-4}), \qquad (2.14)$$

with

$$\lambda_\mathrm{L}(u)\equiv\int d^3\vec{y}\,\hat{y}_\mathrm{L}\,\sigma(\vec{y}, u)+\frac{1}{2(2l+3)c^2}\frac{d^2}{du^2}\int d^3\vec{y}\,\hat{y}_\mathrm{L}\,\vec{y}^2\,\sigma(\vec{y}, u), \qquad (2.15)$$

and

$$\mathrm{G}_{i\mathrm{L}}(u)=\int d^3\vec{y}\,\hat{y}_\mathrm{L}\,\sigma_i(\vec{y}, u)+O(c^{-2}). \qquad (2.16)$$

In equation (2.15) we have taken out of the second integral the time differentiation to better display its structure (contrarily to the usual Landau-Lifshitz-type derivations we are allowed to do so because σ has a spatially compact support). The l-index spatial tensor λ_L, equation (2.15), is symmetric and trace-free (STF), and therefore belongs to an irreducible representation of the rotation group. We need now to decompose the right-hand side of equation (2.16), which is STF only with respect to the multi-index [7] $\mathrm{L}=i_1 i_2\ldots i_l$, into irreducible parts (*i.e.* in STF tensors). This is easily achieved by using equation (A.3) of paper I [17] [for short equation I (A.3)]:

$$\hat{y}_\mathrm{L}\,\sigma_i=\hat{\mathrm{A}}_{i\mathrm{L}}-\frac{l}{l+1}\varepsilon_{ai\langle i_l}\hat{\mathrm{B}}_{\mathrm{L}-1\rangle a}+\frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{\mathrm{C}}_{\mathrm{L}-1\rangle}, \qquad (2.17)$$

where $\hat{\mathrm{A}}_{\mathrm{L}+1}$, $\hat{\mathrm{B}}_\mathrm{L}$ and $\hat{\mathrm{C}}_{\mathrm{L}-1}$ are the following STF tensors (hence the carets on them)

$$\hat{A}_{i_1 \ldots i_{l+1}} = \hat{y}_{\langle i_1 \ldots i_l} \sigma_{i_{l+1} \rangle}, \quad (2.18\,a)$$

$$\hat{B}_{i_1 \ldots i_l} = \varepsilon_{ab \langle i_1} \hat{y}_{i_2 \ldots i_l \rangle a} \sigma_b, \quad (2.18\,b)$$

$$\hat{C}_{i_1 \ldots i_{l-1}} = \hat{y}_{a i_1 \ldots i_{l-1}} \sigma_a, \quad (2.18\,c)$$

and where the brackets $\langle\ \rangle$ mean taking the symmetric-trace-free part. Let us now put

$$K_L(u) \equiv l \int d^3\vec{y}\, \hat{y}_{\langle L-1} \sigma_{i_l \rangle} (\vec{y}, u), \quad (2.19\,a)$$

$$J_L(u) \equiv \int d^3\vec{y}\, \varepsilon_{ab \langle i_l} \hat{y}_{L-1 \rangle a} \sigma_b (\vec{y}, u), \quad (2.19\,b)$$

and

$$\mu_L(u) \equiv \int d^3\vec{y}\, \hat{y}_{iL} \sigma_i (\vec{y}, u). \quad (2.19\,c)$$

With this notation, the moments $G_{iL}(u)$ [equation (2.16)] admit the following irreducible decomposition

$$G_{iL}(u) = \frac{1}{l+1} K_{iL}(u) - \frac{l}{l+1} \varepsilon_{ai \langle i_l} J_{L-1 \rangle a}(u) + \frac{2l-1}{2l+1} \delta_{i \langle i_l} \mu_{L-1 \rangle}(u) + O\left(\frac{1}{c^2}\right). \quad (2.20)$$

Inserting equations (2.14) and (2.20) into equations (2.11), and using in the computation of the last term of (2.20) the fact that $\Delta (r^{-1} F(t-r/c)) = c^{-2} r^{-1} \partial_t^2 F = O(c^{-2})$, we find for the scalar and vector potentials, considered in the exterior region $r_0 < r < r_1$, the following forms:

$$V^{in}(\vec{x}, t) = G \sum_{l \geq 0} \frac{(-)^l}{l!} \partial_L \left(r^{-1} \lambda_L \left(t - \frac{r}{c} \right) \right) + O(c^{-4}), \quad (2.21\,a)$$

$$V_i^{in}(\vec{x}, t) = -G \sum_{l \geq 1} \frac{(-)^l}{l!} \left\{ \partial_{L-1} \left(r^{-1} K_{iL-1} \left(t - \frac{r}{c} \right) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left(r^{-1} J_{bL-1} \left(t - \frac{r}{c} \right) \right) \right\} + G \sum_{l \geq 1} \frac{(-)^l}{l!} \frac{2l-1}{2l+1} \partial_{iL-1} \left(r^{-1} \mu_{L-1} \left(t - \frac{r}{c} \right) \right) + O\left(\frac{1}{c^2}\right). \quad (2.21\,b)$$

Let us now perform a coordinate transformation in the exterior region $r_0 < r < r_1$ to go from the source-covering harmonic coordinate system x^α to a new harmonic coordinate system x'^α (*a priori* valid only in the region

$r_0<r<r_1$). Namely, let us put

$$x'^0=x^0-\frac{4G}{c^3}\sum_{l\geq 0}\frac{(-)^l}{(l+1)!}\frac{2l+1}{2l+3}\partial_L\left(r^{-1}\mu_L\left(t-\frac{r}{c}\right)\right), \quad (2.22a)$$

$$x'^i=x^i, \quad (2.22b)$$

where the STF tensors $\mu_L(u)$ are given by equation (2.19 *c*). Then, the new inner metric $g'^{\text{in}}_{\alpha\beta}(x')$ in the new coordinates x'^α will have exactly the same form as the old one, namely

$$g'_{00}=-\exp\left[-\frac{2}{c^2}V'^{\text{in}}\right]+O\left(\frac{1}{c^6}\right), \quad (2.23a)$$

$$g'_{0i}=-\frac{4}{c^3}V'^{\text{in}}_i+O\left(\frac{1}{c^5}\right), \quad (2.23b)$$

$$g'_{ij}=\delta_{ij}\left(1+\frac{2}{c^2}V'^{\text{in}}\right)+O\left(\frac{1}{c^4}\right), \quad (2.23c)$$

with some new potentials V'^{in} and V'^{in}_i related to the old ones V^{in} and V^{in}_i by

$$V'^{\text{in}}=V^{\text{in}}-\frac{4G}{c^2}\sum_{l\geq 0}\frac{(-)^l}{(l+1)!}\frac{2l+1}{2l+3}\partial_L\left(r^{-1}\overset{(1)}{\mu}_L\left(t-\frac{r}{c}\right)\right), \quad (2.24a)$$

$$V'^{\text{in}}_i=V^{\text{in}}_i-G\sum_{l\geq 1}\frac{(-)^l}{l!}\frac{2l-1}{2l+1}\partial_{iL-1}\left(r^{-1}\mu_{L-1}\left(t-\frac{r}{c}\right)\right) \quad (2.24b)$$

(where $\overset{(1)}{\mu}_L(u)=d\mu_L(u)/du$). In equations (2.24) we have suppressed for simplicity's sake the primes on the new coordinates and think of them as dummy variables.

The latter coordinate transformation has been designed to transfer the terms involving the functions $\mu_L(u)$ from the vector potential [third term in the right-hand side of (2.21 *b*)] to the scalar potential. In these new coordinates, the new scalar potential reads [*see* equations (2.21 *a*) and (2.24 *a*)]

$$V'^{\text{in}}=G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(r^{-1}I_L\left(t-\frac{r}{c}\right)\right)+O\left(\frac{1}{c^4}\right), \quad (2.25)$$

where

$$I_L(u)\equiv\lambda_L(u)-\frac{4(2l+1)}{c^2(l+1)(2l+3)}\overset{(1)}{\mu}_L(u). \quad (2.26)$$

Taking into account the expressions (2.15) and (2.19*c*) of $\lambda_L(u)$ and $\mu_L(u)$ we thus have

$$\mathrm{I}_L(u)=\int d^3\vec{y}\,\hat{y}_L\,\sigma(\vec{y},u) + \frac{1}{2(2l+3)c^2}\frac{d^2}{du^2}\left[\int d^3\vec{y}\,\hat{y}_L\,\vec{y}^2\,\sigma(\vec{y},u)\right] - \frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{du}\left[\int d^3\vec{y}\,\hat{y}_{iL}\,\sigma_i(\vec{y},u)\right]. \quad (2.27)$$

As for the vector potential $V_i'^{\mathrm{in}}$, it is given, from equations (2.21*b*) and (2.24*b*), by the first two terms in the right-hand side of equations (2.21*b*). But now, recalling the definition (2.19*a*) of $\mathrm{K}_L(u)$, and using the Newtonian "continuity equation" (2.7), we find easily that $\mathrm{K}_L(u)$ can be rewritten as

$$\mathrm{K}_L(u)=\frac{d}{du}\left[\int d^3\vec{y}\,\hat{y}_L\,\sigma(\vec{y},u)\right]+O\left(\frac{1}{c^2}\right), \quad (2.28)$$

so that $\mathrm{K}_L(u)$ is equal, modulo post-Newtonian corrections, to the time-derivative of the moment (2.27):

$$\mathrm{K}_L(u)=\overset{(1)}{\mathrm{I}}_L(u)+O\left(\frac{1}{c^2}\right). \quad (2.29)$$

Hence the following expression for the vector potential $V_i'^{\mathrm{in}}$,

$$V_i'^{\mathrm{in}}=-G\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(r^{-1}\overset{(1)}{\mathrm{I}}_{iL-1}\left(t-\frac{r}{c}\right)\right) + \frac{l}{l+1}\varepsilon_{iab}\,\partial_{aL-1}\left(r^{-1}\mathrm{J}_{bL-1}\left(t-\frac{r}{c}\right)\right)\right\}+O\left(\frac{1}{c^2}\right), \quad (2.30)$$

where $\mathrm{I}_L(u)$ is given by equation (2.27) and where we recall that $\mathrm{J}_L(u)$ is given by equation (2.19*b*).

Summarizing this section, the inner metric, as viewed in the exterior near zone $r_0<r<r_1$, can be put into the form (2.23) where the "scalar" and "vector" potentials V'^{in} and $V_i'^{\mathrm{in}}$ admit (through the required accuracy) the multipole decompositions (2.25) and (2.30) in terms of some "source" moments $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$ given explicitly by equations (2.27) and (2.19*b*). In the next section we are going to relate the "source" moments, $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$, to the "radiative" moments, $\mathrm{I}_L^{\mathrm{rad}}(u)$, $\mathrm{J}_L^{\mathrm{rad}}(u)$, which can be read off the asymptotic gravitational wave field.

III. EXTERNAL GRAVITATIONAL FIELD AND MATCHING

In order to relate the "source" moments, $I_L(u)$ and $J_L(u)$, which have been computed in the last section [equations (2.27) and (2.19 *b*)] to the "radiative" moments, $I_L^{rad}(u)$ and $J_L^{rad}(u)$, which appear in the asymptotic metric [*see* equation (1.3 *a*)), we must fill the gap between the inner domain $D_i = \{(\vec{x}, t) \,|\, r < r_1\}$ and the regions at infinity.

Let us then introduce an *external domain*, $D_e = \{(\vec{x}, t) \,|\, r > r_0\}$, overlapping the inner domain D_i in the external near zone $D_i \cap D_e = \{(\vec{x}, t) \,|\, r_0 < r < r_1\}$. By our assumption of weak self-gravity we know a fortiori that the field is weak in D_e. We can now draw on our previous work (papers I-III) for setting up a multipolar post-Minkowskian (MPM) expansion scheme for the external metric $g^{ext}_{\alpha\beta}$. This consists of combining a post-Minkowskian (or weak-field) expansion (written for the "gothic" metric),

$$\mathfrak{g}^{\alpha\beta}_{ext} \equiv \sqrt{g_{ext}}\, g^{\alpha\beta}_{ext} = f^{\alpha\beta} + G\, h_1^{\alpha\beta} + G^2\, h_2^{\alpha\beta} + \ldots + G^n\, h_n^{\alpha\beta} + \ldots, \qquad (3.1)$$

where $f_{\alpha\beta}$ denotes the flat (Minkowski) metric [7], with separate multipole expansions (with respect to the group of rotations of the spatial coordinates) for $h_1^{\alpha\beta}(\vec{x}, t)$, $h_2^{\alpha\beta}(\vec{x}, t)$, etc. This type of expansion scheme, first investigated by Bonnor *et al.* [24], and Thorne [13], has already found several applications in gravitational wave research [25]. The usefulness of this expansion scheme comes from the fact that it allows one to control analytically the structure of vacuum radiative gravitational fields all over the weak-field region. As, in the case of weakly self-gravitating slow moving systems, the vacuum weak-field region extends from the exterior near zone up to the asymptotic wave zone, the MPM expansion scheme provides a bridge between the exterior near-zone metric (2.23) and the sought for outgoing gravitational radiation.

It has been shown in paper I [Theorem I (4.5)] that the most general MPM vacuum metric [satisfying the assumptions I (1.1)-I(1.6)] was isometric to some "canonical" MPM metric defined as a particular functional of two sets of symmetric and trace-free (STF) tensorial functions of one variable: $M_L(u)$ and $S_L(u)$. The construction of the canonical metric is achieved by means of a well-defined *algorithm* (*see* Theorem 4.1 of paper I, or Section III of paper III [19]). The functions $M_L(u)$ and $S_L(u)$ play the role of functional parameters for a general external metric, and we shall refer to them as the *algorithmic moments* of $g^{ext}_{\alpha\beta}$ (M_L, and S_L, have, respectively, the dimensions of mass, and mass current, multipole moments). These algorithmic moments have, *a priori*, no direct physical meaning, we shall see however below that they are useful go-betweens in the problem of relating the "source" moments, $I_L(u)$, $J_L(u)$, of the previous section, to the "radiative" moments of equation (1.3 *a*).

Let us first relate, at 1 PN order, the algorithmic moments to the source moments. To do this we shall compute the 1 PN expansion of the algorithmic external metric (3.1) considered in the exterior near zone $D_i \cap D_e$. As $h_n^{\mu\nu}$, in equation (3.1), is at least of order $O(c^{-2n})$ in $D_e \cap D_i$, it is sufficient to consider only the first two steps of the algorithm: $h_1^{\mu\nu}$ and $h_2^{\mu\nu}$. The *linearized* external metric, $G h_1^{\mu\nu}$, reads

$$G\,h_1^{00} = -\frac{4}{c^2}V^{\rm ext}, \tag{3.2a}$$

$$G\,h_1^{0i} = -\frac{4}{c^3}V_i^{\rm ext}, \tag{3.2b}$$

$$G\,h_1^{ij} = -\frac{4}{c^4}V_{ij}^{\rm ext}, \tag{3.2c}$$

where the "scalar", "vector" and "tensor" external potentials, $V^{\rm ext}$, $V_i^{\rm ext}$ and $V_{ij}^{\rm ext}$, are given in terms of M_L and S_L by

$$V^{\rm ext} = G\sum_{l \geq 0}\frac{(-)^l}{l!}\partial_L\left(r^{-1}M_L\left(t-\frac{r}{c}\right)\right), \tag{3.3a}$$

$$V_i^{\rm ext} = -G\sum_{l \geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(r^{-1}\overset{(1)}{M}_{iL-1}\left(t-\frac{r}{c}\right)\right) + \frac{l}{l+1}\varepsilon_{iab}\,\partial_{aL-1}\left(r^{-1}S_{bL-1}\left(t-\frac{r}{c}\right)\right)\right\}. \tag{3.3b}$$

$$V_{ij}^{\rm ext} = G\sum_{l \geq 2}\frac{(-)^l}{l!}\left\{\partial_{L-2}\left(r^{-1}\overset{(2)}{M}_{ijL-2}\left(t-\frac{r}{c}\right)\right) + \frac{2l}{l+1}\partial_{aL-2}\left(r^{-1}\varepsilon_{ab(i}\overset{(1)}{S}_{j)bL-2}\left(t-\frac{r}{c}\right)\right)\right\}. \tag{3.3c}$$

The *quadratic* external metric, $G^2 h_2^{\mu\nu}$, is then constructed as

$$G^2\,h_2^{\alpha\beta} = p_2^{\alpha\beta} + q_2^{\alpha\beta}, \tag{3.4}$$

where (with $G=1$)

$$p_2^{\alpha\beta} = \text{Finite Part}_{B=0}\,\Box_R^{-1}\left[(r/\hat{\lambda})^B N_2^{\alpha\beta}(h_1)\right], \tag{3.5}$$

is generated by the quadratic "gravitational stress-energy pseudo-tensor" whose explicit expression in terms of h_1 will be found, e. g., in equation III (3.5), and where $q_2^{\alpha\beta}$ is a particular solution of the homogeneous wave equation which is also algorithmically defined from the knowledge of h_1 (*see* papers I or III). The operator acting on $N_2^{\alpha\beta}(h_1)$ in equation (3.5), henceforth called for short FP $\Box^{-1}$, is a particular inverse of the wave operator. Its explicit definition (based on complex analytic continuation) will be found in papers I and III.

The technical tools worked out in paper I [above all our integration formula I (6.9 *a*), *see* also equations (4.18 *a*), (4.21) and (4.24) in paper III] would allow us to obtain an explicit analytical form of the quadratic metric $h_2^{\mu\nu}$ all over the external domain. However, for our present purpose it is sufficient to compute the near-zone reexpansion of $h_2^{\mu\nu}$ in the exterior near zone $D_e \cap D_i$, *i. e.* the first post-Newtonian expansion of the (multipolar-) post-Minkowskian external metric. Taking into account the link between the "gothic" metric (3.1) and the usual covariant metric $g_{\alpha\beta}^{\text{ext}}$, the explicit expressions (3.2)-(3.3) for the linearized external metric, and the fact that h_n^{00} and h_n^{ij} are at least $O(c^{-2n})$ while h_n^{0i} is at least $O(c^{-2n-1})$ [as easily deduced from equations (3.10) and (3.15) below], we obtain after an easy calculation

$$g_{00}^{\text{ext}} = -1 + \frac{2}{c^2} \mathrm{V}^{\text{ext}} - \frac{6}{c^4} (\mathrm{V}^{\text{ext}})^2 - \frac{1}{2}(p_2^{00} + p_2^{ss}) - \frac{1}{2}(q_2^{00} + q_2^{ss}) + O\left(\frac{1}{c^6}\right), \quad (3.6\,a)$$

$$g_{0i}^{\text{ext}} = -\frac{4}{c^3} \mathrm{V}_i^{\text{ext}} + O\left(\frac{1}{c^5}\right), \quad (3.6\,b)$$

$$g_{ij}^{\text{ext}} = \delta_{ij}\left(1 + \frac{2}{c^2} \mathrm{V}^{\text{ext}}\right) + O\left(\frac{1}{c^4}\right), \quad (3.6\,c)$$

where the post-Newtonian orders $O(c^{-n})$ refer to a near-zone reexpansion in $D_i \cap D_e$ (however we keep for convenience V^{ext} and $\mathrm{V}_i^{\text{ext}}$ in post-Minkowskian form). We need then only to compute the combination $p_2^{00} + p_2^{ss} + q_2^{00} + q_2^{ss}$ of $h_2^{\mu\nu}$, equation (3.4). Let us first consider $q_2^{00} + q_2^{ss}$. The algorithmic definition of $q_2^{\mu\nu}$ [*see* equations I (4.13), or equations III (3.6)] together with the general "factorization result" I (5.3) [or III (5.2)], implies that

$$q_2^{00} + q_2^{ss} = \frac{1}{c^{5+\underline{l}_1+\underline{l}_2}} \frac{\mathrm{A}(t-r/c)}{r} + \frac{1}{c^{4+\underline{l}_1+\underline{l}_2}} \partial_i \left(\frac{\mathrm{B}_i(t-r/c)}{r}\right), \quad (3.7)$$

where $\mathrm{A}(u)$ and $\mathrm{B}_i(u)$ are given by a sum of (zeroth, first or second) antiderivatives of contractions of time derivatives of two algorithmic moments, say $\mathcal{M}_{\mathrm{L}_1}$ and $\mathcal{M}_{\mathrm{L}_2}$ (where $\mathcal{M}_{\mathrm{L}}$ denote either $\mathrm{M}_{i_1 \ldots i_l}$ or $\varepsilon_{i_l i_{l+1} a} \mathrm{S}_{a i_1 \ldots i_{l-1}}$, so that $\underline{l} = l$ for a mass moment and $\underline{l} = l + 1$ for a current moment). For instance, terms of the type.

$$b_i(u) = \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw \, \mathrm{K}_{i\underline{\mathrm{L}}_1 \underline{\mathrm{L}}_2} \mathcal{M}_{\underline{\mathrm{L}}_1}^{(a_1)}(w) \mathcal{M}_{\underline{\mathrm{L}}_2}^{(a_2)}(w), \quad (3.8)$$

where $\mathrm{K}_{i\underline{\mathrm{L}}_1 \underline{\mathrm{L}}_2}$ is a constant tensor made out only of Kronecker deltas, may contribute to $\mathrm{B}_i(u)$. Now, we know from the analysis of stationary gravitational fields (Appendix IC) that at least one member of each pair of algorithmic moments contributing to A or B_i [say $\mathcal{M}_{\underline{\mathrm{L}}_1}$ in equation

(3.8)] must be non stationary, and therefore must have at least two indices (indeed, M, M_i and S_i are necessarily stationary): $\underline{l}_1 \geqq 2$. Moreover, to make a non-zero scalar (resp. vector) by contracting $\mathcal{M}_{\mathrm{L}_1}$ and $\mathcal{M}_{\mathrm{L}_2}$ one must have $\underline{l}_2 = \underline{l}_1$ (resp. $\underline{l}_2 = \underline{l}_1 \pm 1$). Hence, from equation (3.7), one has at least:

$$q_2^{00} + q_2^{ss} = O(c^{-7}). \tag{3.9}$$

Let us now consider $p_2^{00} + p_2^{ss}$. From equation (3.5) $p_2^{\alpha\beta}$ is determined everywhere in D_e by the quadratic gravitational stress-energy tensor $N_2^{\alpha\beta}(h_1)$. We need only to know the near-zone expansion (valid in $D_e \cap D_i$) of $p_2^{00} + p_2^{ss}$. Thanks to the properties of the operator $\mathrm{FP}\,\square^{-1}$ proven in paper I, we can relate, for any nonlinear order n, the near-zone expansion of $p_n^{\alpha\beta}$ (denoted, say, $\bar{p}_n^{\alpha\beta}$) to the near-zone expansion, say $\bar{N}_n{}^{\alpha\beta}$, of its "source". Namely, using equations I (3.23), I (3.5), and I (5.4), we find [26]

$$\bar{p}_n{}^{\alpha\beta} = \text{Finite Part}_{B=0} \sum_{k \geqq 0} \frac{1}{c^{2k}} \frac{\partial^{2k}}{\partial t^{2k}} \Delta^{-k-1} \left[\left(\frac{r}{\lambda} \right)^{B} \bar{N}_n{}^{\alpha\beta} \right] + \bar{\Phi}_n{}^{\alpha\beta}, \tag{3.10}$$

with

$$\bar{\Phi}_n{}^{\alpha\beta} = \sum_{l \geqq 0} \frac{1}{c^{3n + \Sigma \underline{l}_i}} \hat{n}_{\mathrm{L}} \left(\frac{r}{c} \right)^{l} F_{\mathrm{L}}^{\alpha\beta}(t). \tag{3.11}$$

In equation (3.11), $\bar{\Phi}_n{}^{\alpha\beta}$ is the formal [26] near-zone expansion of a solution of the homogeneous wave equation which is regular at the origin. Each coefficient $F_{\mathrm{L}}^{\alpha\beta}(t)$ is a "retarded functional" [19] of the multipole moments whose index structure is a generalization of equation (3.8). Hence

$$\bar{\Phi}_n{}^{\mathrm{S}} \sim \hat{n}_{\mathrm{L}}\, K_{\mathrm{LS}\underline{\mathrm{L}}_1 \ldots \underline{\mathrm{L}}_n} \mathcal{M}_{\underline{\mathrm{L}}_1} \ldots \mathcal{M}_{\underline{\mathrm{L}}_n}, \tag{3.12}$$

where S denote the spatial indices among $(\alpha\beta)$ [e. g. $\mathrm{S} = i$ if $(\alpha\beta) = (0i)$]. As we know from the Appendix C of paper I that $\bar{\Phi}_n$ is zero when all the algorithmic multipole moments are stationary, we must necessarily have at least one time-dependent moment, say $\mathcal{M}_{\underline{\mathrm{L}}_1}$, among each n-plet of moments constituting $\bar{\Phi}_n$, equation (3.12). Let $c_{\mathrm{S}} = c_{i_1 \ldots i_s}$ be some constant spatial tensor, and let us consider the spatial scalar,

$$c_{\mathrm{S}} \bar{\Phi}_n{}^{\mathrm{S}} \sim K_{\mathrm{LS}\underline{\mathrm{L}}_1 \ldots \underline{\mathrm{L}}_n} \hat{n}_{\mathrm{L}}\, c_{\mathrm{S}} \mathcal{M}_{\underline{\mathrm{L}}_1} \ldots \mathcal{M}_{\underline{\mathrm{L}}_n}. \tag{3.13}$$

We recall that the tensor K in equation (3.13) is made only of Kronecker deltas, and represent therefore some operation of complete contraction of the indices $\mathrm{LS}\underline{\mathrm{L}}_1 \ldots \underline{\mathrm{L}}_n$. As the indices of the trace-free moment $\mathcal{M}_{\underline{\mathrm{L}}_1}$ cannot be contracted between themselves, they must be contracted among the $l + s + \underline{l}_2 + \ldots + \underline{l}_n$ other indices, hence $\underline{l}_1 \leqq l + s + \underline{l}_2 \ldots + \underline{l}_n$ which implies the inequality

$$\sum_{i=1}^{n} \underline{l}_i \geqq 2\underline{l}_1 - l - s \geqq 4 - l - s, \tag{3.14}$$

where we have used the fact that a time-dependent moment has necessarily $\underline{l}_1 \geqq 2$ (*see* reference [22] for a study of the case where the lowest-order time-dependent moment is of any order $l \geqq 2$). The use of the inequality (3.14) in equation (3.11) allows one to conclude very generally that

$$\bar{\Phi}_n{}^{\alpha\beta} = O\left(\frac{1}{c^{3n+4-s}}\right), \tag{3.15}$$

where s is the number of (non contracted) spatial indices among $(\alpha\beta)$, so that the combination of $\bar{\Phi}_n^{\alpha\beta}$ appearing in $p_2^{00} + p_2^{ss}$ (*i. e.* $n=2$, $s=0$) is at least,

$$\bar{\Phi}_2{}^{00} + \bar{\Phi}_2{}^{ss} = O\left(\frac{1}{c^{10}}\right). \tag{3.16}$$

From equations (3.6*a*), (3.10) and (3.16) we see now that the post-Newtonian expansion of the $p^{\mu\nu}$-contribution to g_{00}^{ext} is simply given by

$$-\frac{1}{2}(p_2^{00} + p_2^{ss}) = \text{Finite part}_{B=0}\, \Delta^{-1} \times \left[\left(\frac{r}{\bar{\lambda}}\right)^B \left\{-\frac{1}{2}(\bar{N}_2{}^{00} + \bar{N}_2{}^{ss})\right\}\right] + O\left(\frac{1}{c^6}\right), \tag{3.17}$$

in which it is sufficient to insert the lowest order term of the post-Newtonian expansion, $\bar{N}_2{}^{\alpha\beta}$, of $N_2^{\alpha\beta}(h_1)$. Using the explicit expression III (3.5) for $N_2^{\alpha\beta}(h_1)$, together with the expressions (3.2) for $h_1^{\mu\nu}$, one obtains after straightforward computations,

$$\bar{N}_2{}^{00} = -\frac{14}{c^4}\partial_i U^{\text{ext}}\, \partial_i U^{\text{ext}} + O\left(\frac{1}{c^6}\right), \tag{3.18a}$$

$$\bar{N}_2{}^{ij} = \frac{4}{c^4}\partial_i U^{\text{ext}}\, \partial_j U^{\text{ext}} - \frac{2}{c^4}\delta_{ij}\, \partial_k U^{\text{ext}}\, \partial_k U^{\text{ext}} + O\left(\frac{1}{c^6}\right), \tag{3.18b}$$

so that

$$-\frac{1}{2}(\bar{N}_2{}^{00} + \bar{N}_2{}^{ss}) = \frac{8}{c^4}\partial_i U^{\text{ext}}\, \partial_i U^{\text{ext}} + O\left(\frac{1}{c^6}\right) = \frac{4}{c^4}\Delta[(U^{\text{ext}})^2] + O\left(\frac{1}{c^6}\right), \tag{3.18c}$$

where we have introduced the Newtonian approximation of the Minkowskian exterior scalar potential (3.3 *a*):

$$U^{ext} = G \sum_{l \geq 0} \frac{(-)^l}{l!} (\partial_L r^{-1}) M_L(t) = V^{ext} + O\left(\frac{1}{c^2}\right). \tag{3.19}$$

The identity

$$\Delta(r^B U^2) \equiv r^B \Delta(U^2) + 2B r^{B-1} \frac{\partial}{\partial r} U^2 + B(B+1) r^{B-2} U^2, \tag{3.20}$$

together with the fact that the multipole expansion of $(U^{ext})^2$ is of the form $\sum \hat{n}_L r^{-(l+2+2k)}$ [so that, as easily seen from equation I (3.9), neither $r^{B-1}\, \partial/\partial r (U^{ext})^2$, nor $r^{B-2} (U^{ext})^2$, can generate any pole at B=0] shows that one has simply Finite part$_{B=0} \Delta^{-1} \left\{ \left(\frac{r}{\lambda}\right)^B \Delta[(U^{ext})^2] \right\} = (U^{ext})^2$, and therefore, from equations (3.17), (3.18 *c*) and (3.19),

$$-\frac{1}{2}(p_2^{00} + p_2^{ss}) = \frac{4}{c^4}(V^{ext})^2 + O\left(\frac{1}{c^6}\right). \tag{3.21}$$

Combining equations (3.6), (3.9) and (3.21) we conclude that the algorithmic external metric viewed in the exterior near-zone $D_e \cap D_i$ can be written at the 1 PN accuracy as:

$$g_{00}^{ext} = -1 + \frac{2}{c^2} V^{ext} - \frac{2}{c^4} (V^{ext})^2 + O\left(\frac{1}{c^6}\right), \tag{3.22 a}$$

$$g_{0i}^{ext} = -\frac{4}{c^3} V_i^{ext} + O\left(\frac{1}{c^5}\right), \tag{3.22 b}$$

$$g_{ij}^{ext} = \delta_{ij}\left(1 + \frac{2}{c^2} V^{ext}\right) + O\left(\frac{1}{c^4}\right), \tag{3.22 c}$$

where we recall that V^{ext} and V_i^{ext} are given by equations (3.3 *a*), (3.3 *b*).

Let us now apply our matching procedure [19] which is a variant of the method of "matched asymptotic expansions" [27]. It consists in requiring that in the overlap domain $D_e \cap D_i$ ($r_0 < r < r_1$) the inner metric $g_{\mu\nu}^{in}$ should be *isometric* to the external metric $g_{\mu\nu}^{ext}$, *i. e.* that there should exist a coordinate transformation, $x_{ext}^\mu = x_{in}^\mu - \xi^\mu(x_{in})$, such that the coordinate transform of the PN-reexpansion, equations (3.22), of $g_{\mu\nu}^{ext}$ should coincide with the PN-expansion, equations (2.10) or equation (2.23), of $g_{\mu\nu}^{in}$. In fact, we have already anticipated the need to perform such a coordinate transformation, *see* equations (2.22), and the resulting inner metric (2.23) has already been put in the same form as the algorithmic external metric

(3.22) [compare equations (2.25) and (2.30) respectively to equations (3.3 *a*) and (3.3 *b*)]. This shows clearly that any possible residual coordinate transformation can only be of small PN-order so that we can treat the effect of $\xi'^{\mu}=x'^{\mu}_{\text{in}}-x^{\mu}_{\text{ext}}$ in a linearized way:

$$g^{\text{ext}}_{\mu\nu}-g'^{\text{in}}_{\mu\nu}=\partial_{\mu}\xi'_{\nu}+\partial_{\nu}\xi'_{\mu}. \qquad (3.23)$$

We can eliminate the *a priori* unknown $\xi'_{\mu}\equiv f_{\mu\nu}\xi'^{\nu}$ by computing the linearized curvature, $\mathrm{R}^{\text{lin}}_{\alpha\beta\gamma\delta}(\partial_{(\mu}\xi'_{\nu)})\equiv 0$, and in particular its $(0i0j)$ component which is well suited to the 1 PN remainder terms:

$$\partial_{ij}(g^{\text{ext}}_{00}-g'^{\text{in}}_{00})-\frac{1}{c}\partial_t\partial_i(g^{\text{ext}}_{0j}-g'^{\text{in}}_{0j})$$
$$-\frac{1}{c}\partial_t\partial_j(g^{\text{ext}}_{0i}-g'^{\text{in}}_{0i})+\frac{1}{c^2}\partial_t^2(g^{\text{ext}}_{ij}-g'^{\text{in}}_{ij})=0. \qquad (3.24)$$

Equation (3.24) gives a relation between the still undetermined algorithmic moments $\mathrm{M_L}$ and $\mathrm{S_L}$, which parametrize $g^{\text{ext}}_{\mu\nu}$, equations (3.3), and the source moments, $\mathrm{I_L}$, equation (2.27), and $\mathrm{J_L}$, equation (2.19 *b*). Using the irreducibility of multipole expansions (and working iteratively in $1/c$), one can solve uniquely equation (3.24) for $\mathrm{M_L}$ and $\mathrm{S_L}$ with the result that (for all possible values of l)

$$\mathrm{M_L}(u)=\mathrm{I_L}(u)+O\left(\frac{1}{c^4}\right), \qquad (3.25\,a)$$

$$\mathrm{S_L}(u)=\mathrm{J_L}(u)+O\left(\frac{1}{c^2}\right), \qquad (3.25\,b)$$

where the PN error terms (1 PN relative precision for the mass moments, but only Newtonian precision for the current moments) come directly from the $O(1/c^{6-s})$ error terms in $g_{\mu\nu}$ [s denoting the number of spatial indices among $(\mu\nu)$]. Equations (3.25) give now some physical significance to the algorithmic moments. However, what we are interested in are not the algorithmic moments but the directly observable "radiative" multiple moments to which we now turn (*see* however below for the direct physical consequences of equations (3.25) for the non-radiative moments, $l\leqq 1$).

In a previous work (reference [18], paper II) one of us has studied the structure of the external multipolar post-Minkowskian metric $g^{\text{ext}}_{\alpha\beta}$ in the asymptotic wave zone. It has been proven (under the assumption of past stationarity) that it is always possible, by going from the harmonic coordinates, x^{μ}_{ext}, used in the original algorithm defining $g^{\text{ext}}_{\alpha\beta}$ [*see* e.g. equation (3.5)] to some "radiative" coordinates, $\mathrm{X}^{\mu}=(c\mathrm{T},\vec{\mathrm{X}})$, to put the external metric in a convenient "radiative" form, *i.e.* a form where the metric coefficients, say $G^{ext}_{\alpha\beta}(X^\gamma)$, admit an asymptotic expansion [28] in powers of R^{-1} at future null infinity ($R \equiv |\vec{X}| \to \infty$, with $T - R/c$ and $\vec{N} \equiv \vec{X}/R$ fixed):

$$G^{ext}_{\alpha\beta}(X^\gamma) = f_{\alpha\beta} + \sum_{n \geqq 1} R^{-n} G^{ext}_{n\,\alpha\beta}(\vec{N}, T - R/c). \tag{3.26}$$

One can now define, as usual ([13], [8]), the part of the metric (3.26) describing the outgoing radiation as the transverse-traceless (TT) projection of the leading R^{-1} term,

$$h^{rad}_{ij}(T, \vec{X}) \equiv R^{-1} P_{ijkl}(\vec{N})\, G^{ext}_{1\,kl}(\vec{N}, T - R/c), \tag{3.27}$$

where the TT projection operator onto the plane orthogonal to $\vec{N}$, $P_{ijkl}(\vec{N})$, is given by equation (1.2) [beware of the fact that h^{rad}_{ij} denote the components of the usual covariant metric, while h^{ij}_n in equations (3.1)-(3.3) denote the components of the gothic metric]. Then the "radiative" multipole moments, $I^{rad}_L(u)$ and $J^{rad}_L(u)$ (with $l \geqq 2$), are defined [13] as antiderivatives of the coefficients of the (irreducible) multipole expansion of $R h^{rad}_{ij}$, namely

$$h^{rad}_{ij} = + \frac{4G}{Rc^2} P_{ijkl}(\vec{N}) \sum_{l \geqq 2} \frac{1}{l!\, c^l} \Big\{ N_{L-2} \overset{(l)}{I}{}^{rad}_{klL-2}(T - R/c) - \frac{2l}{(l+1)c} N_{aL-2}\, \varepsilon_{ab(k} \overset{(l)}{J}{}^{rad}_{l)bL-2}(T - R/c) \Big\}. \tag{3.28}$$

The numerical coefficients in equation (3.28) have been chosen so that I^{rad}_L and J^{rad}_L can be made to agree, at linear order in G, with the algorithmic moments M_L and S_L respectively [as easily seen from equation (3.3 c) with due account taken of the sign-changing transformation between the gothic and the covariant metrics]. More precisely, as the MPM external metric is, by construction, a functional of the algorithmic moments expandable in powers of the gravitational constant, the radiative moments will admit a formal nonlinearity expansion of the type [29]

$$I^{rad}_L(u) = M_L(u) + \sum_{n \geqq 2} \frac{G^{n-1}}{c^{3(n-1) + \Sigma \underline{l_i} - l}} X_L(u), \tag{3.29 a}$$

$$J^{rad}_{L-1}(u) = S_{L-1}(u) + \sum_{n \geqq 2} \frac{G^{n-1}}{c^{3(n-1) + \Sigma \underline{l_i} - l}} Y_{L-1}(u), \tag{3.29 b}$$

where $X_L(u)$ and $Y_{L-1}(u)$ are some n-uply nonlinear (and in general "hereditary" [19]) functionals of the algorithmic moments of the form (endowing current moments with their natural Levi-Civita symbols):

$$\left.\begin{matrix} X_L(u) \\ \varepsilon_{ai_l i_{l-1}} Y_{aL-2}(u) \end{matrix}\right\} = \sum \int_{-\infty}^{u} \dots \int_{-\infty}^{u} du_1 \dots du_n \mathscr{K}_{L\underline{L}_1 \dots \underline{L}_n}(u, u_1, \dots, u_n, P) \cdot \mathscr{M}_{\underline{L}_1}(u_1) \dots \mathscr{M}_{\underline{L}_n}(u_n). \quad (3.30)$$

The powers of $1/c$ appearing in equations (3.29) are simply obtained from the fact that I_L^{rad} and J_{L-1}^{rad} must vary under a change of length units like $[\text{length}]^l$, and by taking into account the fact that the multi-kernel appearing in equation (3.30) contains only variables having the dimension of time [$P \equiv \hat{\lambda}/c$ being the time scale that is introduced in the algorithm through the factors $(r/\hat{\lambda})^B = (r/cP)^B$, *see* Section 5 of paper I]. Now, as the index structure of the kernel $\mathscr{K}_{L\underline{L}_1 \dots \underline{L}_n}$ is made out only of Kronecker deltas, and must be trace-free with respect to its first l indices, $L = i_1 \dots i_l$, the latter indices can come only from some uncontracted indices among the $\Sigma \underline{l}_i$ indices of $\mathscr{M}_{\underline{L}_1} \dots \mathscr{M}_{\underline{L}_n}$. Hence the inequality,

$$l \leqq \sum \underline{l}_i, \quad (3.31)$$

which shows immediately that the nonlinear corrections in equations (3.29), now considered according to their post-Newtonian order, are such that, when $1/c \to 0$,

$$I_L^{rad}(u) = M_L(u) + O\left(\frac{1}{c^3}\right), \quad (3.32\,a)$$

$$J_L^{rad}(u) = S_L(u) + O\left(\frac{1}{c^3}\right). \quad (3.32\,b)$$

The leading $O(c^{-3})$ terms in equations (3.32) will be studied in a separate paper, and related to the recently investigated "tail" effects [19]. The information contained in equations (3.32) is however sufficient for our present purpose. Indeed, combining equations (3.32) with equations (3.25), we see that one can eliminate the algorithmic moments (which have played the role of useful go-betweens) and get directly the following relation between the radiative moments and the source moments (when $l \geqslant 2$):

$$I_L^{rad}(u) = I_L(u) + O\left(\frac{1}{c^3}\right), \quad (3.33\,a)$$

$$J_L^{rad}(u) = J_L(u) + O\left(\frac{1}{c^2}\right). \quad (3.33\,b)$$

Equations (3.33) constitute the central result of the present work in that they solve the generation problem through the first post-Newtonian order.

Indeed, writing them explicitly from equations (2.19*b*) and (2.27) above, we have (for $l \geqq 2$)

$$\mathrm{I}_{\mathrm{L}}^{\mathrm{rad}}(u)=\int d^3\vec{y}\,\hat{y}_{\mathrm{L}}\,\sigma(\vec{y},u)+\frac{1}{2(2l+3)c^2}\frac{d^2}{du^2}\times\int d^3\vec{y}\,\hat{y}_{\mathrm{L}}\,\vec{y}^2\,\sigma(\vec{y},u)-\frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{du}\times\int d^3\vec{y}\,\hat{y}_{i\mathrm{L}}\,\sigma_i(\vec{y},u)+O\left(\frac{1}{c^3}\right),\qquad(3.34)$$

which give with 1 PN accuracy the radiative mass moments directly in terms of the stress-energy tensor of the material source [recalling the definitions (2.4)-(2.5) of σ and σ_i], while (for $l \geqq 2$)

$$\mathrm{J}_{\mathrm{L}}^{\mathrm{rad}}(u)=\int d^3\vec{y}\,\varepsilon_{ab\langle i_l}\hat{y}_{\mathrm{L}-1\rangle a}\,\sigma_b(\vec{y},u)+O\left(\frac{1}{c^2}\right),\qquad(3.35)$$

is determined in terms of the material source with only Newtonian accuracy. From equation (1.3*a*) [which represents the lowest order terms of equation (3.28)] we see that the respective accuracies of equations (3.34) and (3.35) are exactly what is needed to get all the relative corrections of order $O(c^{-1})$ and $O(c^{-2})$ in the outgoing gravitational wave amplitude. In fact the only moment for which it is crucial to have the $O(c^{-2})$ corrections is the quadrupole mass moment, which has been already explicitly written down in equation (1.3*b*). Let us also recall, to be precise, that the definitions (2.4)-(2.5) of the source variables σ and σ_i [and therefore the integrals (3.34)-(3.35)] must make use of an inner *harmonic* coordinate system in which the metric in the source region takes the form (2.10) with (2.8)-(2.9).

For the sake of completeness, let us also quote the 1 PN "quadrupole-octupole energy loss equation" giving the time derivative of the Bondi energy, $\mathrm{E_B}(u)$. The general exact relation,

$$\frac{d\mathrm{E_B}}{du}(u)=-\frac{c^3}{32\pi\mathrm{G}}\int\left(\frac{\partial h_{ij}^{\mathrm{rad}}}{\partial u}\right)^2\mathrm{R}^2\,d\Omega(\vec{\mathrm{N}}),\qquad(3.36)$$

gives at 1 PN order:

$$\frac{d\mathrm{E_B}}{du}(u)=-\frac{\mathrm{G}}{5c^5}\left\{\overset{(3)}{\mathrm{I}}{}_{ij}^{\mathrm{rad}}\overset{(3)}{\mathrm{I}}{}_{ij}^{\mathrm{rad}}+\frac{1}{c^2}\left[\frac{5}{189}\overset{(4)}{\mathrm{I}}{}_{ijk}^{\mathrm{rad}}\overset{(4)}{\mathrm{I}}{}_{ijk}^{\mathrm{rad}}+\frac{16}{9}\overset{(3)}{\mathrm{J}}{}_{ij}^{\mathrm{rad}}\overset{(3)}{\mathrm{J}}{}_{ij}^{\mathrm{rad}}\right]+O\left(\frac{1}{c^4}\right)\right\}.\qquad(3.37)$$

In view of the importance of the "active gravitational mass density" σ in the generation problem, it is interesting to discuss the (non-radiative) low order moments of σ. They are directly obtained from the intermediate results (3.25), remembering that the low order algorithmic moments ($l \leq 1$) have a direct physical significance. Namely, M is the Arnowitt-Deser-Misner mass of the system, M_{ADM}, while M_i and S_i can be respectively interpreted as the dipole moment and the spin of the system as read off from the asymptotic gravitational field (at least, in the past-stationary case for which there is no ambiguity in extracting M_i and S_i from the asymptotic fall-off in space of the metric). From the $l=0$ case of equation (3.25 *a*), with equation (2.27) and the continuity equation (2.7), one finds easily that

$$M_{ADM} = \int d^3\vec{y}\,\sigma(\vec{y}, t) - \frac{1}{c^2}\mathcal{V}(t) + O\left(\frac{1}{c^4}\right), \tag{3.38}$$

where $\mathcal{V}$ denotes the "virial" of the system,

$$\mathcal{V}(t) \equiv \frac{1}{2}\frac{d^2}{dt^2}\int d^3\vec{y}\,\vec{y}^2\,\sigma(\vec{y}, t), \tag{3.39 a}$$

which can be rewritten as ("virial theorem")

$$\mathcal{V}(t) = \int d^3\vec{y}\left[T^{ss} - \frac{1}{2}\sigma\,U^{in}\right] = \int d^3\vec{y}\,\tau_N^{ss}, \tag{3.39 b-c}$$

where

$$\tau_N^{ij} = T^{ij} + \frac{1}{4\pi G}\left[\partial_i U^{in}\,\partial_j U^{in} - \frac{1}{2}\delta_{ij}\,\partial_k U^{in}\,\partial_k U^{in}\right], \tag{3.40}$$

is the sum of the stress-tensor of the matter and of the Newtonian stress-tensor of the non relativistic gravitational potential U^{in}:

$$U^{in}(\vec{x}, t) = G\int\frac{d^3\vec{y}}{|\vec{x}-\vec{y}|}\,\sigma(\vec{y}, t) = V^{in} + O\left(\frac{1}{c^2}\right). \tag{3.41}$$

Consistently with the Tolman mass formula, one sees immediately from equation (3.39 *a*) that, in the case of stationary matter distribution, σ integrates up to the total ADM mass of the system. In the general non-stationary case it can be useful to decompose σ in a "conserved part", say

$$\mu \equiv \left(1 + \frac{1}{2c^2}U^{in}\right)\frac{T^{00}}{c^2}, \tag{3.42}$$

and a "virial part",

$$\sigma-\mu=\frac{1}{c^2}\left[T^{ss}-\frac{1}{2}\sigma U^{in}\right]+O\left(\frac{1}{c^4}\right)$$

$$=\frac{1}{c^2}\tau_N^{ss}+\Delta\left(\frac{(U^{in})^2}{16\pi G c^2}\right)+O\left(\frac{1}{c^4}\right). \quad (3.43)$$

The "conserved" mass-energy density satisfies, as immediately seen from equations (3.38), (3.39 *b*) and (3.42),

$$M_{ADM}=\int d^3\vec{y}\,\mu(\vec{y},t)+O\left(\frac{1}{c^4}\right). \quad (3.44)$$

Moreover, from the $l=1$ case of equation (3.25 *a*) together with the use of a vectorial generalization of the Newtonian virial theorem (3.39), one finds that μ satisfies

$$M_i=\int d^3\vec{y}\,y^i\mu(\vec{y},t)+O\left(\frac{1}{c^4}\right). \quad (3.45)$$

The left-hand sides of equations (3.44)-(3.45) satisfy very generally the "conservation laws",

$$0=\frac{dM_{ADM}}{dt}=\frac{d^2 M_i}{dt^2}. \quad (3.46)$$

In conclusion, let us emphasize that the somewhat heavy machinery we used in this section to deal with the nonlinearities of the gravitational field was the price we had to pay to be able to relate the outgoing gravitational wave amplitude to the stress-energy distribution of the matter alone. In appendix A we exhibit the formal link between our unambiguously defined compact-support generation formalism, and the formally divergent integrals over the total matter + gravitational stress-energy distribution that constitute the end results of the Epstein-Wagoner-Thorne approach.

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APPENDIX A
LINK WITH THE EPSTEIN-WAGONER-THORNE FORMALISM

In this appendix we investigate the relation between our result (3.34) for the 2^l-mass radiative moments and previous formal expressions derived by Epstein and Wagoner [12] and Thorne [13] by making use of a pseudo stress-energy tensor for the gravitational field. We shall make the comparison with the results of Thorne [13] which are more general than the ones of Epstein and Wagoner [12]. Let us introduce a total (matter + gravitational field) pseudo stress-energy tensor,

$$\tau^{\alpha\beta}=g\,T^{\alpha\beta}+\frac{c^4}{16\pi G}N^{\alpha\beta}, \qquad (A.1)$$

where $g=-\det g_{\alpha\beta}$ and where, using the notation of equations III (3.4), $N^{\alpha\beta}=\sum_{n=2}^{\infty} G^n N_n^{\alpha\beta}$ is the total effective nonlinear source for the gravitational field (*see* e.g. equation (5.3) of reference [13] for the closed form expression of $\tau^{\alpha\beta}$). $\tau^{\alpha\beta}$ is conserved in the usual sense

$$\partial_\beta \tau^{\alpha\beta}=0. \qquad (A.2)$$

Using the explicit expression III (3.5) for $N_2^{\alpha\beta}$, we find, in harmonic coordinates, the following 1 PN accurate equations where appears the scalar potential V^{in} of equation (2.8):

$$\tau^{00}=g\,T^{00}-\frac{7}{8\pi G}\partial_i V^{in}\,\partial_i V^{in}+O\left(\frac{1}{c^2}\right), \qquad (A.3\,a)$$

$$\tau^{0i}=g\,T^{0\,i}+O\left(\frac{1}{c}\right), \qquad (A.3\,b)$$

$$\tau^{ij}=g\,T^{ij}+\frac{1}{4\pi G}\left[\partial_i V^{in}\,\partial_j V^{in}-\frac{1}{2}\delta_{ij}\,\partial_k V^{in}\,\partial_k V^{in}\right]+O\left(\frac{1}{c^2}\right). \qquad (A.3\,c)$$

[Outside the material source equations (A.3) become equations (3.18).] From these relations, and also from

$$g=1+\frac{4}{c^2}V^{in}+O\left(\frac{1}{c^4}\right), \qquad (A.4)$$

we find the following 1 PN approximate equations relating $\tau^{\alpha\beta}$ with $\sigma=c^{-2}(T^{00}+T^{ss})$ and $\sigma_i=c^{-1}T^{0\,i}$ (where the $T^{\mu\nu}$'s are in contravariant form):

$$\tau^{00}+\tau^{ss}=\sigma c^2-\frac{1}{2\pi G}\Delta((V^{in})^2)+O\left(\frac{1}{c^2}\right), \qquad (A.5)$$

$$\tau^{0i} = \sigma_i c + O\left(\frac{1}{c}\right). \tag{A.6}$$

In equation (A.5), Δ denotes the usual Laplacian $\Delta = \partial_i \partial_i$. (Note also that we could equivalently use the Newtonian potential U^{in} instead of the Minkowskian one V^{in}.) Equations (A.5)-(A.6) allow one to transform our well-defined result (3.34), containing only compact-support integrals, into the following formal result

$$\begin{aligned} I_L^{\text{rad}}(t) \sim \int d^3\vec{y}\, \hat{y}_L c^{-2} [\tau^{00} + \tau^{ss}] & \\ + \frac{1}{2c^2(2l+3)} \frac{d^2}{dt^2} \left[\int d^3\vec{y}\, \hat{y}_L \vec{y}^2 c^{-2} \tau^{00} \right] & \\ - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \frac{d}{dt} \left[\int d^3\vec{y}\, \hat{y}_{iL} c^{-1} \tau^{0i} \right] & \\ + \frac{1}{2\pi G c^2} \int d^3\vec{y}\, \hat{y}_L \Delta((V^{\text{in}})^2) + O\left(\frac{1}{c^3}\right), & \end{aligned} \tag{A.7}$$

containing infinite-support spatial integrals of $\tau^{\alpha\beta}(\vec{y}, t)$ considered at a fixed time t [denoted u in equation (3.34)]. At spatial infinity, τ^{00}, τ^{ss}, $\tau^{00} + \tau^{ss}$ and $\Delta(V^{\text{in}})^2$ fall off only as the inverse fourth power of $|\vec{y}|$. Therefore many of the integrals appearing in the right-hand side of equation (A.7), taken separately, are divergent (for large enough l). To emphasize this formal manipulation of divergent integrals, we use in equation (A.7) (and in similarly formal equations derived below) the symbol $\sim$ instead of the equal sign. Now, as $\Delta\hat{y}_L = 0$, the last term in equation (A.7) (though divergent) is formally zero by Green's formula:

$$\int d^3\vec{y}\, \hat{y}_L \Delta[(V^{\text{in}})^2] \sim \int d^3\vec{y}\, \partial_i \{ \hat{y}_L \partial_i (V^{\text{in}})^2 - (V^{\text{in}})^2 \partial_i \hat{y}_L \} \sim 0, \tag{A.8}$$

modulo the formal discarding of all surface terms. Let us also introduce the time derivatives inside the integrals of the right-hand side of equation (A.7). Using the conservation of $\tau^{\alpha\beta}$, equation (A.2), and integrating (formally) by parts we get

$$\begin{aligned} I_L^{\text{rad}}(t) \sim \frac{1}{c^2} \int d^3\vec{y} \Bigg[(\tau^{00} + \tau^{ss}) \hat{y}_L + \frac{1}{2(2l+3)} \tau^{ij} \partial_{ij}(\hat{y}_L \vec{y}^2) & \\ - \frac{4(2l+1)}{(l+1)(2l+3)} \tau^{ij} \partial_i(\hat{y}_{jL}) \Bigg]. & \end{aligned} \tag{A.9}$$

Finally, we use the appendix A of paper I to decompose $\partial_{ij}(\hat{y}_L \vec{y}^2)$ and

$\partial_i(\hat{y}_{jL})$ into irreducible pieces. The result is

$$\mathrm{I}_{\mathrm{L}}^{\mathrm{rad}}(t) \sim \frac{1}{c^2}\int d^3\vec{y}\left[\tau^{00}\hat{y}_{\mathrm{L}} + \frac{2l(l-1)}{(l+1)(2l+3)}\tau^{ss}\hat{y}_{\mathrm{L}} - \frac{6l(l-1)}{(l+1)(2l+3)}\tau^{s\langle i_l}\hat{y}^{\mathrm{L}-1\rangle}y_s + \frac{l(l-1)(l+9)}{2(l+1)(2l+3)}\tau^{\langle i_l i_{l-1}}\hat{y}^{\mathrm{L}-2\rangle}\vec{y}^2\right], \qquad (\mathrm{A}.10)$$

which agrees with the (formally divergent) result of Thorne [13] [his equation (5.32 *a*)], for the radiative mass moments at the 1 PN level.

APPENDIX B
RELATIVISTIC TIME-DEPENDENT MULTIPOLE ANALYSIS OF A COMPACT SOURCE

The aim of this appendix is to derive, directly within the STF formalism, a result, originally worked out by Campbell *et al.* [23] within the spherical harmonics formalism, giving the multipole expansion of a retarded field in the region outside its spatially compact-supported source.

Let $\mathrm{S}(\vec{x}, t)$ be some "source" which is spatially compact-supported in the sense that $\mathrm{S}(\vec{x}, t)=0$ when $|\vec{x}|>r_0$ where r_0 is some finite fixed radius. Let V be the retarded solution of

$$\Box \mathrm{V} = -4\pi\mathrm{S}, \qquad (\mathrm{B}.1\,a)$$

i. e.

$$\mathrm{V}(\vec{x}, t) = -4\pi\,\Box_{\mathrm{R}}^{-1}\mathrm{S} = +\int\frac{d^3\vec{y}}{|\vec{x}-\vec{y}|}\mathrm{S}\left(\vec{y}, t-\frac{1}{c}|\vec{x}-\vec{y}|\right). \qquad (\mathrm{B}.1\,b)$$

Let us prove that, in the region exterior to the source S $(r>r_0)$, $\mathrm{V}(\vec{x}, t)$ admits the following *exact* multipole expansion

$$\mathrm{V}(\vec{x}, t) = \sum_{l\geq 0}\frac{(-)^l}{l!}\partial_{\mathrm{L}}\left(r^{-1}\mathrm{F}_{\mathrm{L}}\left(t-\frac{r}{c}\right)\right), \qquad (\mathrm{B}.2)$$

where the functions $\mathrm{F}_{\mathrm{L}}(u)$ are simply the usual STF multipole moments of a particular l-dependent weighted time average of the source $\mathrm{S}(\vec{y}, t')$, namely

$$\mathrm{F}_{\mathrm{L}}(u) = \int d^3\vec{y}\,\hat{y}_{\mathrm{L}}\,\bar{\mathrm{S}}_l(\vec{y}, u), \qquad (\mathrm{B}.3\,a)$$

with

$$\overline{\mathrm{S}}_l(\vec{y}, u) = \frac{(2l+1)!!}{2^{l+1}\, l!} \int_{-1}^{+1} dz\, (1-z^2)^l\, \mathrm{S}(\vec{y}, u + z\,|\vec{y}|/c). \qquad (\mathrm{B}.3\,b)$$

Note that the coefficient in the right-hand side of equation (B.3 *b*) is simply the inverse of $\int_{-1}^{+1} dz\,(1-z^2)^l$ (as is clearly necessary from the well-known limiting case of stationary sources).

To prove equations (B.2)-(B.3) we first expand the source $\mathrm{S}(\vec{x}, \mathrm{t})$ in spherical harmonics, which means, in the STF notation,

$$\mathrm{S}(\vec{x}, t) = \sum_{l \geqq 0} \hat{n}_{\mathrm{L}}(\theta, \varphi)\, \mathrm{S}_{\mathrm{L}}(r, t), \qquad (\mathrm{B}.4)$$

where the $\mathrm{S}_{\mathrm{L}}(r, t)$ are some (STF) tensorial functions of $r = |\vec{x}|$ and t. Then we apply to each one of the terms in the expansion (B.4) the following expression for the retarded integral of a multipolar source which has been derived in Section 6 of paper I [equation I (6.8)]

$$[\Box_{\mathrm{R}}^{-1}(\hat{n}_{\mathrm{L}}\,\mathrm{S}_{\mathrm{L}})](\vec{x}, t) = \hat{\partial}_{\mathrm{L}}\left\{\frac{1}{r}\int_{-\infty}^{t-r} d\tau\, \mathrm{R}_{\mathrm{L}}^{[a]}\left(\frac{t-r-\tau}{2}, \tau\right)\right\} - \int_{-\infty}^{t-r} d\tau\, \hat{\partial}_{\mathrm{L}}\left\{\frac{1}{r}\mathrm{R}_{\mathrm{L}}^{[a]}\left(\frac{t+r-\tau}{2}, \tau\right)\right\} \qquad (\mathrm{B}.5)$$

(with $c = 1$) where $\mathrm{R}_{\mathrm{L}}^{[a]}(\rho, \tau)$ is related to $\mathrm{S}_{\mathrm{L}}(r, t)$ by

$$\mathrm{R}_{\mathrm{L}}^{[a]}(\rho, \tau) = \rho^l \int_a^{\rho} dx \frac{(\rho - x)^l}{l!}\left(\frac{2}{x}\right)^{l-1} \mathrm{S}_{\mathrm{L}}(x, \tau + x). \qquad (\mathrm{B}.6)$$

This result is valid everywhere (inside and outside the source) and for any source (with compact or unbounded support), and it is also independent of the choice of a, the origin of integration in equation (B.6) (*see* Remark 1 in Section 6 of paper I). Now, in the particular case of a field point outside a compact source ($r > r_0$), the first argument of the second term in the right-hand side of equation (B.5) will satisfy $\frac{1}{2}(t + r - \tau) > r_0$. Therefore if we choose $a \equiv r_0$, it becomes evident from equation (B.6) that the latter second term will simply vanish when $r > r_0$. This gives immediately

$$-4\pi\, \Box_{\mathrm{R}}^{-1}(\hat{n}_{\mathrm{L}}\,\mathrm{S}_{\mathrm{L}}) = \frac{(-)^l}{l!}\hat{\partial}_{\mathrm{L}}\left\{\frac{\mathrm{F}_{\mathrm{L}}(t - r/c)}{r}\right\}, \qquad (\mathrm{B}.7)$$

with

$$F_L(u) = -4\pi(-)^l l! \int_{-\infty}^{u} d\tau \left(\frac{u-\tau}{2}\right)^l \times \int_{r_0}^{(u-\tau)/2} dx \frac{((u-\tau)/2-x)^l}{l!} \left(\frac{2}{x}\right)^{l-1} S_L(x, \tau+x). \quad (B.8)$$

By a simple change of variables ($\tau = u + (z-1)x$) equation (B.8) is transformed into

$$F_L(u) = \frac{4\pi}{2^{l+1}} \int_{-1}^{+1} dz (1-z^2)^l \int_0^{r_0} dx\, x^{l+2} S_L(x, u+zx). \quad (B.9)$$

Finally we use [*see* e. g. equation I (A.9 *b*)]

$$S_L(r, t) = \frac{(2l+1)!!}{4\pi l!} \int d\Omega\, \hat{n}_L S(\vec{x}, t), \quad (B.10)$$

which yields indeed equation (B.3).

Let us now consider the l-dependent weighted time average appearing in equation (B.3 *a*) (It is clear from the above demonstration that this time average has its physical origin in the time delays $t - |\vec{x}-\vec{y}|/c$ due to the finite velocity of propagation.) We can rewrite it more clearly as

$$\bar{S}_l(\vec{y}, u) = \int_{-1}^{+1} dz\, \delta_l(z) S(\vec{y}, u + z|\vec{y}|/c), \quad (B.11)$$

where

$$\delta_l(z) \equiv \frac{(2l+1)!!}{2^{l+1} l!} (1-z^2)^l, \quad (B.12)$$

satisfies

$$\int_{-1}^{+1} \delta_l(z)\, dz = 1, \quad (B.13a)$$

$$\lim_{l\to\infty} \delta_l(z) = \delta(z). \quad (B.13b)$$

Equation (B.13 *b*), in the right-hand side of which appears the usual Dirac distribution, shows that for large values of l one can neglect the time delays [23]. For practical applications it is very useful to consider the slow-motion (or long-wavelength) expansion of the time average (B.11). After easy computations (using for instance Euler's beta function) one finds

$$\bar{S}_l(\vec{y}, u) = \sum_{p=0}^{\infty} \frac{(2l+1)!!}{2^p p! (2l+2p+1)!!} \left(\frac{|\vec{y}|}{c}\right)^{2p} \frac{\partial^{2p}}{\partial u^{2p}} S(\vec{y}, u), \quad (B.14a)$$

so that

$$\mathrm{F_L}(u)=\int d^3\vec{y}\,\hat{y}_L\left\{\mathrm{S}(\vec{y},u)+\frac{1}{2(2l+3)}\frac{\vec{y}^2}{c^2}\overset{(2)}{\mathrm{S}}(\vec{y},u)+\frac{1}{8(2l+3)(2l+5)}\frac{(\vec{y}^2)^2}{c^4}\overset{(4)}{\mathrm{S}}(\vec{y},u)+\dots\right\}. \quad (\mathrm{B}.14\,b)$$

Note that only even powers of c^{-1} appear in the expansion (B.14), and that the relatively smaller influence of time delays for higher multipoles is clearly displayed in equation (B.14*b*). Note also that, while the multipole expansion (B.2) will be a *convergent* series under only weak hypotheses of differentiability (*see* e.g. Appendix B of paper I), the post-Newtonian expansions (B.14) are in general only *asymptotic* series for $c^{-1}\to 0$.

REFERENCES

[1] For a recent general introduction to gravitational wave research *see* K. S. THORNE, in 300 *Years of Gravitation*, S. W. HAWKING and W. ISRAEL Eds., Cambridge University Press, Cambridge, England, 1987, pp. 330-458.

[2] J. P. A. CLARK and D. M. EARDLEY, *Astrophys. J.*, Vol. **215**, 1977, p. 311; B. F. SCHUTZ, in *Gravitational Collapse and Relativity*, H. SATO and T. NAKAMURA Ed., World Scientific, Singapore, 1986, pp. 350-368; A. KROLAK and B. F. SCHUTZ, *Gen. Relativ. Gravit.*, Vol. **19**, p. 1163, 1987.

[3] J. L. ANDERSON and D. W. HOBILL, in *Dynamical Spacetimes and Numerical Relativity*, J. M. CENTRELLA, Ed., Cambridge University Press, Cambridge, England, 1986, p. 389; T. DAMOUR, in *Gravitational Collapse and Relativity*, H. SATO and T. NAKAMURA, Eds., World Scientific, Singapore, 1986, pp. 63-73.

[4] J. H. TAYLOR and J. M. WEISBERG, *Astrophys. J.*, Vol. **253**, 1982, p. 908; J. M. WEISBERG and J. H. TAYLOR, *Phys. Rev. Lett.*, Vol. **52**, 1984, p. 1348; J. H. TAYLOR, in *Timing Neutron Stars*, H. ÖGELMAN and E. P. J. VAN DEN HEUVEL, Eds., Kluwer, Dordrecht, 1989, p. 17.

[5] For a review and references, *see* T. DAMOUR, in 300 *Years of Gravitation*, S. W. HAWKING and W. ISRAEL Eds., Cambridge University Press, Cambridge, England, 1987, pp. 128-198.

[6] A. EINSTEIN, *Preuss. Akad. Wiss. Sitzber. (Berlin)*, 1918, p. 154.

[7] Our conventions and notation are the following: signature $-+++$; greek indices$=0$, 1, 2, 3; latin indices$=1$, 2, 3; $g=-\det(g_{\mu\nu})$; $f_{\alpha\beta}=f^{\alpha\beta}=$flat metric$=$diag $(-1, +1, +1, +1)$; $r=(x_1^2+x_2^2+x_3^2)^{1/2}$; $n_i=n^i=x^i/r$; $\partial_i=\partial/\partial x^i$; $n^L=n_L=n_{i_1}n_{i_2}\dots n_{i_l}$ and $\partial_L=\partial_{i_1}\partial_{i_2}\dots\partial_{i_l}$, where $L=i_1 i_2\dots i_l$ is a multi-index with l indices ($L-1$ denoting $i_1 i_2\dots i_{l-1}$, etc.); $\hat{n}_L\equiv n_{\langle L\rangle}$ and $\hat{\partial}_L\equiv\partial_{\langle L\rangle}$ are the (symmetric) trace-free parts of n_L and ∂_L (we freely raise or lower spatial indices by means of $f^{ij}=f_{ij}=\delta_{ij}$).

[8] We are following the terminology of T. DAMOUR, in *Gravitation in Astrophysics*, Cargèse, 1986, B. CARTER and J. B. HARTLE, Eds., Plenum, New York, 1987, pp. 3-62.

[9] L. D. LANDAU and E. M. LIFSHITZ, *Teoriya Polya*, Nauka, Moscow, 1941.

[10] V. A. FOCK, *Teoriya prostranstva. vremeni i tyagoteniya*, Fizmatgiz, Moscow, 1955.

[11] M. WALKER and C. M. WILL, *Phys. Rev. Lett.*, Vol. **45**, 1980, p. 1741; J. L. ANDERSON, *Phys. Rev. Lett.*, Vol. **45**, 1980, p. 1745; G. SCHÄFER, *Astrophys. J.*, Vol. **250**, 1981, p. L5; R. A. ISAACSON, J. S. WELLING and J. WINICOUR, *Phys. Rev. Lett.*, Vol. **53**, 1984, p. 1870; F. I. COOPERSTOCK and P. H. LIN, *Phys. Rev. Lett.*, Vol. **55**, 1985,

p. 265; T. FUTAMASE and B. F. SCHUTZ, *Phys. Rev. D.*, Vol. **32**, 1985, p. 2557; J. WINICOUR, *J. Math. Phys.*, Vol. **28**, 1987, p. 668.

[12] R. EPSTEIN and R. V. WAGONER, *Astrophys. J.*, Vol. **197**, 1975, p. 717; R. V. WAGONER, in *Gravitazione Sperimentale*, B. BERTOTTI Ed., Accademia Nazionale dei Lincei, Rome, 1977, p. 117.

[13] K. S. THORNE, *Rev. Mod. Phys.*, Vol. **52**, 1980, p. 299.

[14] R. V. WAGONER and C. M. WILL, *Astrophys. J.*, Vol. **210**, 1976, p. 764; M. TURNER and C. M. WILL, *Astrophys. J.*, Vol. **220**, 1978, p. 1107; M. TURNER and R. V. WAGONER, in *Sources of Gravitational Radiation*, L. SMARR Ed., Cambridge University Press, Cambridge, 1979, p. 383; R. V. WAGONER, *Phys. Rev. D*, Vol. **19**, 1979, p. 2897.

[15] We are aware of the fact that the convergence of these PN generation integrals is improved by keeping the time derivatives inside the integral signs, however this does not remove all the formal divergencies.

[16] V. P. TSETKOV, *Sov. Astron.*, Vol. **28**, 1984, p. 394.

[17] L. BLANCHET and T. DAMOUR, *Phil. Trans. R. Soc. Lond. A*, Vol. **320**, 1986, p. 379, hereafter referred to as paper I.

[18] L. BLANCHET, *Proc. R. Soc. Lond. A*, Vol. **409**, 1987, p. 383, hereafter referred to as paper II.

[19] L. BLANCHET and T. DAMOUR, *Phys. Rev. D*, Vol. **37**, 1988, p. 1410, hereafter referred to as paper III. See the introduction of paper III for references to previous contributions to the problem of the matching between the near-zone and the wave-zone gravitational fields.

[20] L. BLANCHET, T. DAMOUR and G. SCHÄFER, *Mon. Not. R. Astro. Soc.*, 1990 (in press); L. BLANCHET and G. SCHÄFER, *Mon. Not. R. Astro. Soc.*, Vol. **239**, 1989, p. 845.

[21] W. L. BURKE, *J. Math. Phys.*, Vol. **12**, 1971, p. 401; K. S. THORNE, *Astrophys. J.*, Vol. **158**, 1969, p. 997.

[22] L. BLANCHET and T. DAMOUR, *Phys. Lett.*, Vol. **104A**, 1984, p. 82.

[23] W. B. CAMPBELL, J. MACEK and T. A. MORGAN, *Phys. Rev. D*, Vol. **15**, 1977, p. 2156.

[24] W. B. BONNOR, *Phil. Trans. R. Soc. Lond. A*, Vol. **251**, 1959, p. 233; W. B. BONNOR and M. A. ROTENBERG, *Proc. R. Soc. Lond. A*, Vol. **289**, 1966, p. 247; A. J. HUNTER and M. A. ROTENBERG, *J. Phys. A*, Vol. **2**, 1969, p. 34.

[25] B. L. SCHUMAKER and K. S. THORNE, *Mon. Not. Roy. Astron. Soc.*, Vol. **203**, 1983, p. 457; L. S. FINN, *Mon. Not. R. Astr. Soc.*, Vol. **222**, 1986, p. 393.

[26] The right-hand sides of equations (3.10)-(3.11) are understood in the sense of formal series in $(\ln c)^p/c^n$. *See* paper I for a complete analysis which takes into account the remainder terms, $O^N((r/c)^N)$, when considering only a finite number of terms in the expansions.

[27] P. A. LAGERSTROM, *Singular Perturbations-Ideas and Techniques*, Springer-Verlag, New York, 1989 (in press).

[28] The right-hand side of equation (3.26) is here understood in the sense of a formal power series in R^{-1}. *See* paper II for a complete analysis which takes into account the remainder terms, $O^\infty(R^{-N})$, and which makes the link with the definition, by Penrose and other authors, of asymptotic simplicity.

[29] Note that this requirement (compatible with the assumptions of the MPM framework) fixes uniquely which antiderivative of the multipole moments of Rh_{ij}^{rad} must be taken.

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