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Matrix second-order differential equations and hamiltonian systems of quartic type

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ABSTRACT. — We show the existence of conserved quantities for matrix differential equations and Hamiltonian systems of quartic type.

RÉSUMÉ. — Nous montrons l'existence de quantités conservées pour des équations différentielles matricielles et des hamiltoniens quartiques.

I. INTRODUCTION

We study second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X) \tag{1}$$

where $h(X) = \lambda X + \mu X^2, (\lambda, \mu) \in \mathbb{R}^2$, and the unknown function X is a $m \times m$ symmetric real matrix.

The method we use is the same which has allowed to prove the existence of a symplectic action of the torus associated to the Olshanetsky-Perelomov systems of the class V ([F1], [F2]).

We associate to (1) a Hamiltonian system for a symplectic form introduced by Kazhdan-Kostant-Sternberg [K-K-S] and used by J. Moser [M]. For a quadratic h, we get a quartic anharmonic flow on the cotangent bundle of the Lie algebra $u(m)$. We show the existence for this system of a $m \times m$ Lax pair whose eigenvalues are in involution for the symplectic form.

The Calogero-Moser Hamiltonian describes an integrable system of m particles on the line interacting pairwise via an inverse-quadratic potential ([C], [M]).

It has been observed by M. Adler [A] that the system stays integrable under the influence of an external quadratic potential.

Following the work of Kazhdan-Kostant-Sternberg, the systems of Calogero-Moser type can be seen as symplectic reductions of harmonic flows on the cotangent bundle of a simple Lie algebra.

The second-order differential equations that we consider here give after a symplectic reduction the Calogero-Moser system with a quartic external potential. We find as a consequence of the main theorem that the reduced system is completely integrable in the Arnol'd-Liouville sense. This last result had been obtained previously by Wojciechowski [W] and independently Inozemtzev [I]. The novelty of our approach is in the use of the Kazhdan-Kostant-Sternberg machinery which makes all the subject more coherent.

Hamiltonian systems of quartic type have been also studied in [F-S-W] in relation with the non-linear Schrödinger equation.

II. SECOND-ORDER MATRIX DIFFERENTIAL EQUATIONS AND HAMILTONIAN SYSTEMS FOR THE KAZHDAN-KOSTANT-STERNBERG SYMPLECTIC FORM

We are interested in second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X)$$

where h is polynomial with scalar coefficients and the unknown X is a $m \times m$ matrix.

We assume that X varies in a vector space V of matrices so that the trace provides an identification with the dual V^* through the mapping $A \mapsto (B \mapsto \mathrm{Tr}(AB))$. The cotangent bundle $T^* V \simeq V \times V^*$ is equipped with the symplectic form $\omega = \mathrm{Tr}(dX \wedge dY)$ following [K-K-S]. The Hamiltonian

flow of the function $H : T^* V \mapsto \mathbb{R}$, defined by:
$H(X, Y) = (1/2)\operatorname{Tr}(h(X)^2 + Y^2)$ is given by

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -hh'(X)) \end{aligned} \tag{2}$$

and it coïncides with (1) for solutions X in V. Hereafter V will be the space of real symmetric matrices.

III. HAMILTONIAN SYSTEMS OF QUARTIC TYPE

We use the matrices

$$Z = \sqrt{-1}\, h(X) + L,\ Z^* = -\sqrt{-1}\, h(X) + L, \tag{3}$$

In the following, (X, Y) are assumed to be real symmetric matrices so that:

$$Z^* = {}^t\bar{Z} \qquad \text{(transposed of the complex conjugated of Z).}$$

We are concerned with the Hamiltonian system

$$H(X, Y) = (1/2)\operatorname{Tr}(h(X)^2 + Y^2),\ \omega = \operatorname{Tr}(dX \wedge dY) \tag{4}$$

for $h(X) = \lambda X + \mu X^2$.

We use the Hermitian matrix $P = ZZ^*$ that we consider as an element of the Lie algebra $u(m)$.

THEOREM 1. — *The matrix* P *defines a Lax pair for the Hamiltonian system* (4) *and its eigenvalues are in involution for the symplectic form* ω.

Proof. — We get from (4):

$$H = (1/2)\operatorname{Tr}(P).$$

Hamilton's equations (2) give:

$$\begin{aligned} \dot{Z} &= \sqrt{-1}(\lambda Y + \mu(XY + YX)) - hh'(X) \\ \dot{Z}^* &= -\sqrt{-1}(\lambda Y + \mu(XY + YX)) - hh'(X) \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{Z} &= \{\{(\sqrt{-1}/2)\, h'(X), Z\}\} \\ \dot{Z}^* &= -\{\{(\sqrt{-1}/2)\, h'(X), Z\}\} \end{aligned} \tag{6}$$

where the symbol $\{\{A, B\}\}$ means the anti-commutator of the two matrices A and B.

We get:

$$\begin{aligned} \dot{P} &= \dot{Z}Z^* + Z\dot{Z}^*, \\ \dot{P} &= \{\{(\sqrt{-1}/2)\, h'(X), Z\}\} Z^* - Z\{\{(\sqrt{-1}/2)\, h'(X), Z^*\}\}, \\ \dot{P} &= [\sqrt{-1}/2)\, h'(X), P]. \end{aligned} \tag{7}$$

So that P is a Lax matrix for the flow of H. As corollary, we obtain that the eigenvalues of P are constants of the motion.

Let Λ be one the eigenvalues of P and Ψ be the corresponding eigenvector. Let T be the projector on the subspace generated by Ψ. The matrix P being Hermitian, we can assume that Ψ is normalized for the standard Hermitian product $\langle\ ,\ \rangle$ on $\mathbb{C}^m$ preserved by P. We get:

$$d\Lambda = \langle d\mathrm{P}\Psi, \Psi \rangle = \mathrm{Tr}\,(d\mathrm{PT}). \tag{8}$$

Let us now consider the Hamiltonian flow generated by Λ and the symplectic form ω. The dot will designate the time-derivative along this flow.

Hamilton's equations give:

$$\mathrm{Tr}\,(\dot{\mathrm{X}}\, d\mathrm{Y} - \dot{\mathrm{Y}}\, d\mathrm{X}) = d\Lambda = \mathrm{Tr}\,(d\mathrm{ZZ}^*\,\mathrm{T} + \mathrm{Z}\, d\,\mathrm{Z}^*\,\mathrm{T}) \tag{9}$$

and so we get first,

$$\begin{aligned} \dot{\mathrm{X}} &= \mathrm{Z}^*\,\mathrm{T} + \mathrm{TZ} \\ \dot{\mathrm{Y}} &= -\{\{(\sqrt{-1}/2)\, h'(\mathrm{X}), \mathrm{Z}^*\,\mathrm{T} - \mathrm{TZ}\}\} \end{aligned} \tag{10}$$

then,

$$\begin{aligned} \dot{\mathrm{Z}} &= \{\{\sqrt{-1}\, h'(\mathrm{X}), \mathrm{TZ}\}\} \\ \dot{\mathrm{Z}}^* &= -\{\{\sqrt{-1}\, h'(\mathrm{X}), \mathrm{Z}^*\,\mathrm{T}\}\}. \end{aligned} \tag{11}$$

This allows to compute $\dot{\mathrm{P}}$,

$$\begin{aligned} \dot{\mathrm{P}} &= \dot{\mathrm{Z}}\mathrm{Z}^* + \mathrm{Z}\dot{\mathrm{Z}}^*, \\ \dot{\mathrm{P}} &= [\mathrm{T}, \mathrm{Z}(\sqrt{-1}\, h'(\mathrm{X})/2)\,\mathrm{Z}^*] + [(\sqrt{-1}\, h'(\mathrm{X})/2), \mathrm{PT}]. \end{aligned} \tag{12}$$

Now we observe that P and T can be codiagonalized so that

$$[\mathrm{P}, \mathrm{T}] = 0.$$

We deduce then that,

$$\dot{\mathrm{P}}\mathrm{P}^k = [\mathrm{T}, \mathrm{Z}(\sqrt{-1}\, h'(\mathrm{X})/2)\,\mathrm{Z}^*\,\mathrm{P}^k] + [(\sqrt{-1}\, h'(\mathrm{X})/2)\,\mathrm{P}^k, \mathrm{PT}] \tag{13}$$

for all integer k, and then,

$$\mathrm{Tr}\,(\dot{\mathrm{P}}\mathrm{P}^k) = 0 \tag{14}$$

which implies that the quantities $\mathrm{tr}\,(\mathrm{P}^k)$ are constants of the motion. Hence, the eigenvalues of P are constants of the motion. But this is true for the flow of any of the eigenvalues of P, so we deduce that the eigenvalues of P are in involution. This ends the proof. □

Example. — Let us consider the case where X is a 2×2 symmetric real matrix.

$$\mathrm{X} = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

The equation (1) gives:

$$\dot{x}_1 = -\lambda^2 x_1 - 3\lambda\mu(x_1^2+x_3^2) - 2\mu^2(x_1(x_1^2+x_3^2)+x_3^2(x_1+x_2))$$
$$\dot{x}_2 = -\lambda^2 x_2 - 3\lambda\mu(x_2^2+x_3^2) - 2\mu^2(x_2(x_2^2+x_3^2)+x_3^2(x_1+x_2))$$
$$\dot{x}_3 = -\lambda^2 x_3 - 3\lambda\mu_3(x_1+x_2) - 2\mu^2(x_1x_3(x_1+x_2)+x_3(x_2^2+x_3^2)).$$

From the theorem 1, we know that there are two conserved quantities. It would be interesting to decide in that case if the system is integrable.

Remarks. — 1. Let us consider the special case $\mu=0$ (quadratic case) in order to compare with the general case.

For $\mu=0$, $h'(X)=\lambda\,\mathrm{Id}$, and the equation (7) becomes $\dot{P}=0$ identically. This means that all the entries of P are constants of the motion.

In general case when $\mu\neq 0$, the equation (7) tells that only the eigenvalues of P are preserved by the flow. Also (12) becomes:

$$\dot{P}=[T, Z(\sqrt{-1}\,\lambda\,\mathrm{Id}/2)Z^*]+[(\sqrt{-1}\,\lambda\,\mathrm{Id}/2), PT]$$
$$\dot{P}=[T, (\sqrt{-1}\,\lambda\,\mathrm{Id}/2)P]$$

but $[P,T]=0$, so $\dot{P}=0$ identically also. Thus, all the entries of P are preserved by the flow of any eigenvalue.

Going back to (11), we get:

$$\begin{aligned}\dot{Z}&=\sqrt{-1}\,\lambda TZ\\ \dot{Z}^*&=-\sqrt{-1}\,\lambda Z^*T.\end{aligned}\tag{15}$$

One can show [F] that (15) defines a symplectic action of the torus T^m, this is no longer true for $\mu\neq 0$.

2. The proof does not extend as it stands to Hamiltonian systems of the sextic type.

IV. COMPLETE INTEGRABILITY OF THE CALOGERO-MOSER SYSTEM WITH AN EXTERNAL QUARTIC POTENTIAL

Let $\mathcal{U}$ be the vector space of $m\times m$ Hermitian matrices. The cotangent bundle $T^*\mathcal{U}$ may be identified with $\mathcal{U}\times\mathcal{U}^*\simeq\mathcal{U}\times\mathcal{U}$. As a cotangent bundle, it has a symplectic structure which can be written [K-K-S]:

$$\omega=\mathrm{Tr}\, dX\wedge dL,\qquad (X, L)\in\mathcal{U}\times\mathcal{U}.$$

We define on $T^*\mathcal{U}$ the Quartic Anharmonic Flow by the Hamiltonian:

$$H=(1/2)\,\mathrm{Tr}(h(X)^2+L^2).$$

The group $G=U(m)$ acts on $\mathcal{U}$ by the adjoint action; this action lifts into an Hamiltonian action on $T^*\mathcal{U}$. The corresponding moment map is given by:

$$T^*\mathcal{U}\cong\mathcal{U}\times\mathcal{U}\ni(X, L)\mapsto\sqrt{-1}\,[X, L]\in\mathcal{U}\cong\mathcal{U}^*.\tag{16}$$

The Hamiltonian H is invariant under this action.

Following [K-K-S] we proceed to a reduction of $T^* \mathcal{U}$ by the symplectic action of G. By using the identification (16), a fiber of the moment map can be seen as $\{(X, L) : [X, L] = -1 g C\}$.

We choose for C the element $(C_{ij} = 1 - \delta_{ij})$. Let G_c be the isotropy subgroup of C. The reduced manifold

$X_c = \{(X, L) : [X, L] = \sqrt{-1} g C\}/G_c$ can be parametrized (*see* [K-K-S]) by:

$$X_{ij} = x_i \delta_{ij}$$
$$L_{ij} = y_i \delta_{ij} + \sqrt{-1} g/(x_i - x_j)(1 - \delta_{ij}) \qquad (g > 0),$$

and is identified with $T^* W$, $W = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m / x_i - x_j \neq 0,\ i \neq j\}$ equipped with the standard symplectic form $\omega = \sum_{i=1}^{m} dx_i \wedge dy_i$ After the reduction, the Hamiltonian $H = (1/2) \operatorname{Tr}(h(X)^2 + L^2)$ becomes the function

$$H = (1/2) \sum_{i=1}^{m} y_i^2 + g^2 \sum_{i, j;\ i<j} (x_i - x_j)^{-2} + (1/2) \sum_{i=1}^{m} (\lambda x_i + \mu x_i^2)^2. \qquad (17)$$

This System can be viewed as a perturbation of the Calogero-Moser System by an external quartic potential.

Hamilton's equations define a vector field whose flow is a solution of

$$\dot{x}_i = \partial H / \partial y_i = y_i \qquad (18)$$
$$\dot{y}_i = -\partial H / \partial x_i = 2 g^2 \sum_{j=i} (x_i - x_j)^{-3} - \sum_{i=1}^{m} 2((\lambda + 2 \mu x_i)(\lambda x_i + \mu x_i^2).$$

The following result is a direct consequence of the theorem proved in the previous paragraph:

COROLLARY 1. — *The Calogero-Moser System with an external quartic potential defined by the Hamiltonian* (17) *is integrable in the Arnol'd-Liouville sense.*

Proof. — The theorem shows that the m quantities $F_k = tr(P^k)$ are in involution. They stay in involution after the reduction. Furthermore $dF_1 \wedge \ldots \wedge dF_k$ is not identically zero since for $\mu = 0$, we have the usual Calogero-Moser System with an external quadratic potential and we know that the functions $F_1, \ldots, F_k$ are generically independent in that case [A]. The symplectic manifold $T^* W$ is $2m$-dimensional, so we have shown the Arnol'd-Liouville integrability of the System (1). □

Note that the hypersurfaces of constant energy are compact and so by the Arnol'd-Liouville theorem, there are invariant tori and the solutions of (18) are quasi-periodic functions of the time.

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