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Two-body relativistic systems in external field

by

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ABSTRACT. — A system of two directly interacting particles is modified by an external field. In the framework of Predictive Relativistic Mechanics the main lines of a general treatment are sketched. Classical and quantum aspects of the problem are discussed.

More attention is devoted to the case of opposite charges in a weak constant electromagnetic field, especially when the binding force is harmonic. A classical relativistic analog of the Stark effect is computed. A first order perturbation of the relativistic harmonic oscillator shows that the size of the ellipse of relative motion is affected by a secular term. Quantum theory cannot take place anymore within the reduced Hilbert space in which it used to be restricted in the absence of perturbation. A direct sum over such reduced spaces provides the wider formalism which was needed.

RÉSUMÉ. — Un système de deux particules en interaction est modifié par la présence d'un champ extérieur. Dans le cadre de la mécanique relativiste prédictive nous présentons les principales étapes d'un traitement général. Les aspects classiques et quantiques du problème sont discutés.

Nous étudions plus en détail le cas de deux charges opposées dans un champ électromagnétique constant faible, en particulier avec une force harmonique. Nous calculons un analogue classique relativiste de l'effet Stark. Un calcul perturbatif au premier ordre de l'oscillateur harmonique relativiste montre que la taille de l'ellipse du mouvement relatif est affectée par un terme séculaire. La théorie quantique ne peut plus être développée dans l'espace de Hilbert réduit utilisé en l'absence de perturbations. Une somme directe de tels espaces fournit le formalisme général nécessaire.

I. INTRODUCTION

Relativistic particle dynamics does not seem as fundamental as quantum field theory but it is however a useful theory. Indeed it allows for the construction of covariant effective potentials which may provide an approximation of the underlaying field theory (valid within some limitations) [*1*].

This procedure is especially relevant when the field theory involved in the system is not completely known, or too complicated.

For this reason, two-body relativistic systems in which the constituent particles directly interact have been considered by many authors [*2-6*]. Good fits with experimental data have been obtained in the domain of hadron spectroscopy [*7*].

Nevertheless we would remain frustrated if relativistic particle mechanics where only a covariant receipe for the calculation of mass spectra. Indeed the particle aspect of dynamics can be developed from a very general point of view in harmony with relativistic invariance.

In particular relativistic quantum mechanics should also give predictions about scattering processes and transition probabilities.

The question of scattering has been treated in theoretical papers [*8-10*] but, to our knowledge, little phenomenological application has been made so far.

Perhaps more than scattering (which, after all, can receive a stationary formulation), a theory of *transitions* basically requires a complete « time » depending description of the system.

In contrast, almost all the works published in the field (Refs. [*8-10*] are exceptions. See also [*11*]) are concerned with a stationary formalism. (By this we mean that coupled Klein-Gordon equations—alternatively Dirac or Feynman-Gell-Mann equations when fermions are present—are the relativistic counterpart of a *stationary* (two-body) Schrödinger equation.) Now that the first advances performed along this line are well-understood, time has come for the next step.

An *evolutive* description of the system should now be undertaken in view of dealing with many interesting problems.

Among them, and as the continuation of research devoted so far to spectrum calculations, it is natural to consider the perturbation of a two-body bound state by some external field.

Of course, coupled Klein-Gordon equations can be written also in the presence of external fields (at the price of solving first the algebraic problem of compatibility) but the non conservation of the total linear momentum automatically leads us to ask, soon or later, about transitions. Thus some « time » depending description is desirable anyway. This problem is by no means trivial.

In the present article we intent to pose it as clearly as possible and to provide a solid ground for future discussions on the subject.

Therefore, in view of a better understanding, we shall start with a classical covariant analysis. Even at this stage it will be relevant to point out a list of open questions.

As a prequisite for the difficult question of transitions, we shall study the variation of P^2 in the classical version.

As a toy-model we exhibit a simple case where the compatibility problem is already solved.

Then a general discussion is sketched in terms of quantum mechanics and an appropriate Hilbert space is proposed as the theater of further investigation.

II. CLASSICAL THEORY

Since an external field breaks Poincaré invariance, the total linear momentum P^α is not exactly conserved and a lot of questions arise:

i) Does P^2 remains positive?

ii) Is it possible to define intrinsically the trajectory of the center of mass?

iii) Do the world-lines remain timelike?

iv) Are bound states stable under perturbation?

Questions *iii*) and *iv*) make sense only if a prescription is implicitly assumed for the solving of the position equations $\{ H_a, x_b \} = 0$, $a \neq b$, which determine the physical positions in terms of the canonical variables [*2*], [*12*].

It is usually assumed that x_a coincides with q_a on a submanifold of phase-space defined by

$$P \cdot (q_1 - q_2) = 0 .$$

As this condition refers to the center-of-mass frame *which is now accelerated* one may perhaps expect some extra complications.

Fortunately, as it was pointed out to occur in the absence of perturbation [*13*], some properties of two-body relativistic systems are independent of this prescription. For instance knowing the behaviour of P^α requires only to carry out the *abstract integration* of the system, in the sense of reference [*2*].

At least it is natural to look for the determination of the evolution of P^2, which will give an answer to question *i*).

In view of investigating anyone of the important questions listed above, the first difficulty is to write down explicitly the hamiltonian equations describing two point-particles which interact mutually *and also* with an external field.

J. Martin and J. L. Sanz have computed a perturbation expansion of the *predictive forces* exerted on a couple of particles which undergo interaction in the presence of an external field [*14*].

Here we mean to work out a hamiltonian formalism.

The problem is that relativistic interactions cannot be arbitrarily choosen because they are required to satisfy the predictivity conditions [*2*] (or alternatively mass-shell constraints) which insure the existence of world lines.

Therefore the superposition of distinct interactions generally requires the addition of extra terms [*15*], [*16*], of which the determination is a preliminary *algebraic problem.*

This difficulty has been met first in the construction of mutual N-body interactions.

It has received various perturbative solutions [*17-20*].

Here we simply consider a two-body system.

But, to the mutual interaction of the constituents, one adds an external field, so we face a similar problem, plus this complication that Poincaré invariance is broken.

Naturally it is possible to look for perturbative solutions of the algebraic problem also in this situation.

But we shall be especially interested with the cases where a solution can be written down *in closed form.*

So doing we mean to select the cases which can be discussed in more details, with the hope that understanding their properties will enlighten the general theory.

Once the superposition of interactions has been carried out exactly, we can still investigate perturbatively a lot of remaining problems which have a nonrelativistic analog: determination of the orbits, etc.

Let us assume that we know exactly how to describe.

a) The mutual interaction in the absence of external field.

b) The motion of each constituant particle *alone* in the external field.

In the framework of Predictive Mechanics [*21*] our goal is to find the hamiltonian generators H_1, H_2 which define the evolution equations in terms of two independent parameters which generalize the proper times. (The bridge to an alternative formulation in terms of constraints theory is available [*22*], [*23*]).

Let $\overline{H}_a = \frac{1}{2} p_a^2$ be the free-particle hamiltonians, $a, b = 1, 2$. If $G(q, p)$ is the contribution of the field to the *single particle* motion, the hamiltonian of particle *a*) alone is

$$K_a = \overline{H}_a + G_a(q_a, p_a)$$

and does not depend on q_b, p_b with $a \neq b$.

When the mutual interaction is exerted in the absence of external field we have

$$H_a^{(0)} = \overline{H}_a + U_a^{(0)}$$

where $U_a^{(0)}$ are Poincaré invariant and allow $\{ H_a^{(0)}, H_b^{(0)} \}$ to vanish.

Note that in contrast to G_a, $U_a^{(0)}$ involves the coordinates of *both* particles.

Throughout this paper the label (0) refers to the unterturbed system. In presence of the external field we have now

$$H_a = K_a + U_a$$

where U_a are suitable modifications of $U_a^{(0)}$ which must allow to satisfy the Poisson bracket condition

$$\{ H_a, H_b \} = 0 . \tag{2-1}$$

Eq. (2-1) will be developed as an equation for the determination of U_a. The quantum mechanical counterpart of this problem has been considered by J. Bijtebier who gave a solution in terms of *formal series* [*24*].

For simplicity we shall assume

$$U_1^{(0)} = U_2^{(0)} = U^{(0)}$$

and naturally

$$U_1 = U_2 = U .$$

Now eq. (2-1) reduces to

$$\{ \overline{H}_1 - \overline{H}_2 + G_1 - G_2, U \} = 0 \tag{2-2}$$

which is a *linear* partial differential equation for U. Still it has infinitely many solutions, but G_1 and G_2 in practice involve coupling constants and it is reasonable to demand that U reduces to $U^{(0)}$ when these constants are turned off. (When G_a vanish either in some domain or asymptotically it is also natural to expect that $U \to U^{(0)}$ under the same conditions).

Constant field.

It is remarkable that, in a special case, an exact solution of eq. (2-2) is obvious: Noticing that $\{ \overline{H}_1 - \overline{H}_2, U^{(0)} \} = 0$ by assumption, we see that provided

$$\{ U^{(0)}, G_1 - G_2 \} = 0 \tag{2-3}$$

then a solution of (2-2) is simply

$$U = U^{(0)} .$$

This solution is evidently well-behaved when the coupling constants are turned off.

Condition (2-3) is certainly satisfied when $G_1 - G_2$ is linear in the generators of the Poincaré algebra, with constant coefficients, and this *irrespectively of the form of* $U^{(0)}$ (provided it is admissible as a two-body potential for Poincaré invariant interaction between the constituent particles of the system). This situation occurs in particular when

$$(2\text{-}4)\qquad \begin{aligned} G_1 &= \frac{e}{2}F^{\alpha\beta}q_{1\alpha}p_{1\beta} \\ G_2 &= -\frac{e}{2}F^{\alpha\beta}q_{2\alpha}p_{2\beta} \end{aligned}$$

where e is a constant and F a skew-symmetric *constant* tensor. Then, if $\varepsilon_1 = -\varepsilon_2 = 1$,

$$K_a = \frac{1}{2}p_a^2 + \varepsilon_a\frac{e}{2}F^{\alpha\beta}q_{a\alpha}p_{a\beta}$$

approximates the coupling of the charge $\varepsilon_a e$ alone with a constant electromagnetic field, up to terms of second order in eF. In other words the hamiltonians

$$(2\text{-}5)\qquad H_a = \overline{H}_a + \varepsilon_a\frac{e}{4}F^{\alpha\beta}(q_a \wedge p_a)_{\alpha\beta} + U^{(0)}$$

exactly satisfy the compatibility condition (2-1) and therefore provide a mathematically well-defined model for investigating the effects of Poincaré invariance breaking by an external field (in practice, a *weak* field). Moreover the assumption that equations (2-4) hold yields a simplification of the multitime formalism:

Let τ_a be the evolution parameters (thus every phase-space function depends on τ_1, τ_2 through the evolution of the canonical coordinates). If f is a Poincaré scalar and, in addition, satisfies $\{\overline{H}_1 - \overline{H}_2, f\} = 0$, then f will depend on τ_1, τ_2 only through $(\tau_1 + \tau_2)$.

This property is shared by the *squares* of: the total momentum, the Pauli-Lubanski vector, and the relative *spatial* variables $\tilde{y}$, $\tilde{z}$, with the notations

$$P = p_1 + p_2, \qquad z = q_1 - q_2, \qquad y = \frac{1}{2}(p_1 - p_2)$$

$$\tilde{a} \equiv \Pi a, \qquad \Pi = \eta - P \otimes P/P^2$$

$$Q = \frac{1}{2}(q_1 + q_2).$$

The simplest effect to consider is the variation of P^2.

As an insight to question *i*) we can evaluate P^2 in terms of $\tau_1 + \tau_2$ at the first order.

Since P^2 is constant in the unperturbed motion, we simply have

$$(2\text{-}6)\qquad \frac{\partial}{\partial\tau_a}P^2 = \{P^2, G_a\}.$$

From (2-4) we find the exact formula

$$\left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)P^2 = -\,2eF^{\alpha\beta}P_\alpha\tilde{y}_\beta\,. \tag{2-7}$$

At the first order in e we can replace P and $\tilde{y}$ by their zeroth order expressions, furnished by integration of the unperturbed equations of motion. For instance we have

$$P^\alpha = C^\alpha + 0(e)$$

where C^α is a *constant* time-like vector (depending on initial conditions). At this stage we see that the right-hand side of (2-7) vanishes in the particular case where $F^{\alpha\beta}C_\alpha$ is zero, that is when the field is purely *magnetic* in the rest-frame of the unperturbed system. In this special situation, P^2 will be constant at first order, and this result does not depend on the form of $U^{(0)}$, with this reservation that the validity of a perturbative treatment is probably limited to a certain class of potentials.

For generic orbits, however, P^2 has a first order variation (due to the electric field) which can be in principle calculated by integration of eq. (2-7).

For an illustration let us assume that $U^{(0)}$ is the harmonic potential $k\tilde{z}^2$, and set $\omega^2 = 2k = \text{const.}$

Let us substitute C^α for P^α and $\tilde{y}^{(0)}_\alpha$ for $\tilde{y}_\alpha$. From the theory of the unperturbed harmonic oscillator [12], [25] we know that

$$2\tilde{y}^{(0)} = \left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)\tilde{z}^{(0)}\,. \tag{2-8}$$

After straightforward integration we have, setting $C \cdot C = M^2$,

$$P^2 = M^2 - eF^{\alpha\beta}C_\alpha\tilde{z}^{(0)}_\beta\,. \tag{2-9}$$

Thus, *at first order,* P^2 *oscillates around its average* M^2 *with the period of the unperturbed motion.*

The amplitude of this oscillation could be easily calculated since $\tilde{z}^{(0)}$ varies according to the law

$$\tilde{z}^{(0)} = A\sin\theta + B\cos\theta \tag{2-10}$$

where $\theta = \omega(\tau_1 + \tau_2) + C^{te}$, A and B constant vectors orthogonal to C and mutually [12].

In formula (2-9) we recognize the *electric dipolar momentum* and in fact we have here a covariant expression of the Stark effect exerted on the oscillator.

It is natural to introduce

$$E^\beta = M^{-1}F^{\alpha\beta}C_\alpha \tag{2-11}$$

so that eq. (2-9) becomes

$$P^2 = M(M - eE \cdot \tilde{z}^{(0)})\,. \tag{2-12}$$

A sufficient condition which insures the timelikeness of P is

$$|eE|\,|A_{max}| < M \tag{2-13}$$

where $|A_{max}|$ is $|A|$ if $|A^2| \geqslant |B^2|$ and is $|B|$ otherwise.

Indeed $|\tilde{z}^{(0)}|$ is bounded by $|A_{max}|$ in the elliptic motion, thus condition (2-13) implies

$$|eE|\,|\tilde{z}^{(0)}| < M\,.$$

Since E and $\tilde{z}^{(0)}$ are both spacelike vectors, we have $|E|\,|\tilde{z}^{(0)}| \geqslant |E\cdot\tilde{z}^{(0)}|$ and finally $M > |eE\cdot\tilde{z}^{(0)}|$.

Let us stress that the limitation implied by condition (2-13) is consistent with the remark made previously that our model is physically relevant only in so far as the strenght of the external field is not too large.

Remark. — Though $F^{\alpha\beta}C_\alpha$ is supposed nonzero, it may happen that it is accidentally parallel to the unperturbed Pauli-Lubanski vector, thus orthogonal to $\tilde{z}^{(0)}$.

Again in that case the effect would be of second order.

Let us go back to a general form of $U^{(0)}$.

The study made above can be generalized to all the first integrals of the unperturbed motion, in the spirit of the Lagrange method of variation of the constants.

If Z is constant in the unperturbed motion we have simply $\dfrac{\partial Z}{\partial \tau_a} = \{Z, G_a\}$.

When Z is a scalar under the Lorentz group, that is $\{Z, M_{\alpha\beta}\} = 0$, we have the simplification that Z will depend on τ_1 and τ_2 only through the combination $\tau_1 + \tau_2$.

Hence the importance of computing $\{Z, G_1 + G_2\}$.

Let us define

$$\Lambda = q_1 \wedge p_1 - q_2 \wedge p_2 \tag{2-14}$$

then we have from (2-4)

$$\left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)Z = \frac{e}{4}F^{\alpha\beta}\{Z, \Lambda_{\alpha\beta}\}\,. \tag{2-15}$$

The parameters which define the unperturbed worldlines undergo a variation which can be in principle calculated at first order.

Thus it may be possible to investigate if the modified motion will remain bounded in space, in view of answering question *iv*).

For simplicity let us assume again that $U^{(0)}$ is harmonic.

Since the constancy of

$$N = \tilde{y}^2 + 2k\tilde{z}^2$$

insures the boundness of the unperturbed motion, it is natural to consider now the variation of N.

So we have to compute the brackets $\{N, \Lambda\}$.

First we separate from Q and P the relative variables in Λ and obtain

$$\Lambda = 2Q \wedge y + \frac{1}{2} z \wedge P . \tag{2-16}$$

We remind that z, y, Q, P satisfy the canonical Poisson bracket relations $\{ z^\alpha, y_\beta \} = \{ Q^\alpha, P_\beta \} = \delta^\alpha_\beta$, etc.

Thus

$$\{ \tilde{y}^2, \Lambda \} = 2 \{ \tilde{y}^2, Q \} \wedge y + \frac{1}{2} \{ \tilde{y}^2, z \} \wedge P . \tag{2-17}$$

Then it is long but not difficult to find that

$$\{ \tilde{y}^2, Q^\alpha \} = 2 \frac{y \cdot P}{P^2} \tilde{y}^\alpha . \tag{2-18}$$

Moreover we have immediately

$$\{ \tilde{y}^2, z^\alpha \} = - 2 \tilde{y}^\alpha . \tag{2-19}$$

These two last formulas yield

$$\{ \tilde{y}^2, \Lambda \} = 4 \left(\frac{y \cdot P}{P^2} \right) \tilde{y} \wedge \left(\frac{y \cdot P}{P^2} \right) P - \tilde{y} \wedge P \tag{2-20}$$

which can be arranged as

$$\{ \tilde{y}^2, \Lambda \} = P \wedge \tilde{y} (1 - (2y \cdot P/P^2))^2 . \tag{2-21}$$

Now we have to compute

$$\{ \tilde{z}^2, \Lambda \} = 2 \{ \tilde{z}^2, Q \} \wedge y + 2Q \wedge \{ \tilde{z}^2, y \} \tag{2-22}$$

where $\{ \tilde{z}^2, Q \}$ has been already computed in a previous article [*12*],

$$\{ \tilde{z}^2, Q^\alpha \} = 2 \left(\frac{P \cdot z}{P^2} \right) \tilde{z}^\alpha \tag{2-23}$$

whereas obviously

$$\{ \tilde{z}^2, y^\alpha \} = 2 \tilde{z}^\alpha . \tag{2-24}$$

So eq. (2-22) becomes

$$\{ \tilde{z}^2, \Lambda \} = 4 \tilde{z} \wedge \left(\frac{P \cdot z}{P^2} y - Q \right) . \tag{2-25}$$

Using the identity

$$P \cdot M = (P \cdot Q) P - P^2 Q + (P \cdot z) y - (P \cdot y) z \tag{2-26}$$

we can also write

$$\frac{1}{4} \{ \tilde{z}^2, \Lambda \} = \tilde{z} \wedge \frac{P \cdot M}{P^2} - \left(\frac{P \cdot Q}{P^2} \right) \tilde{z} \wedge P + \left(\frac{P \cdot y}{P^2} \right) \tilde{z} \wedge z . \tag{2-27}$$

But $\tilde{z} \wedge z = \tilde{z} \wedge \left(\frac{z \cdot P}{P^2}\right)P$ so we have

$$(2\text{-}28) \quad \{ \tilde{z}^2, \Lambda \} = 4\tilde{z} \wedge \frac{P \cdot M}{P^2} + 4\left(\frac{P \cdot Q}{P^2} - \left(\frac{P \cdot y}{P^2}\right)\left(\frac{P \cdot z}{P^2}\right)\right)P \wedge \tilde{z}\,.$$

According to formula (2-15) our goal was to calculate,

$$(2\text{-}29) \qquad \left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)N = \frac{e}{4}F^{\alpha\beta}\{ \tilde{y}^2, \Lambda_{\alpha\beta} \} + k\frac{e}{2}F^{\alpha\beta}\{ \tilde{z}^2, \Lambda_{\alpha\beta} \}\,.$$

This can be carried out by insertion of (2-21) and (2-28).

Up to quadratric quantities in e, we see that the first term is periodic in $\tau_1 + \tau_2$, whereas the second one in general countains a *secular* contribution which comes from the second term in the right-hand side of (2-28).

To summarize we can write (with notation $F \cdot (a \wedge b) = F^{\alpha\beta}(a \wedge b)_{\alpha\beta}$)

$$(2\text{-}30) \quad \left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)N = 2ke(P^2)^{-1}\left(P \cdot Q - \frac{(P \cdot y)}{P^2}(P \cdot z)\right)F \cdot (P \wedge \tilde{z}) \\ + \text{periodic terms} + 0(e^2)\,.$$

Obviously, we can in this formula replace y, z, Q and P by their zeroth order approximations.

Using notations (2-11) we have

$$(2\text{-}31) \qquad F \cdot (P \wedge \tilde{z})^{(0)} = 2ME \cdot \tilde{z}^{(0)}\,.$$

Note that we have from a previous work [*12*]

$$(P \cdot Q)^{(0)} = \frac{P^2}{4}(\tau_1 + \tau_2) + \text{const.}$$

and

$$(P \cdot z)^{(0)} = \frac{P^2}{2}(\tau_1 - \tau_2) + y \cdot P(\tau_1 + \tau_2) + \text{const.}$$

Thus

$$(2\text{-}32) \quad \left(P \cdot Q - \frac{y \cdot P}{P^2} P \cdot z\right)^{(0)} = \frac{1}{4}(\tau_1 + \tau_2)M^{-2}[M^4 - (m_1^2 - m_2^2)^2] + \text{Const.}$$

Inserting (2-31), (2-32) into (2-30) we obtain

$$(2\text{-}33) \quad \left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)N = keM^{-3}(\tau_1 + \tau_2)[M^4 - (m_1^2 - m_2^2)^2]E \cdot \tilde{z}^{(0)} \\ + \text{periodic terms} + 0(e^2)\,.$$

Here $\tilde{z}^{(0)}$ is periodic with pulsation $\omega = \sqrt{2k}$.

Thus, unless E is accidentally zero, integration with respect to $\tau_1 + \tau_2$ yields a secular variation of N.

The rapidity of development of this term depends critically on the magnitude of the coefficient $M^4 - (m_1^2 - m_2^2)^2$ which tends to zero in the limit where $\frac{m_1}{m_2} \to 0$ and $\frac{N^{(0)}}{m_2} \to 0$.

This remark permits to consider the secular term as a genuine recoil effect.

If we accept the usual definition of the physical positions x_a in terms of the canonical variables, we have a secular variation of the relative orbit which affects the squares of the axes of the ellipse.

III. QUANTUM THEORY

Quantum theory is seriously more difficult. We can at least write wave equations.

The algebraic problem of finding U is the same as in the classical case, the Poisson bracket being replaced by a commutator in eq. (2-2).

For a weak constant electromagnetic field we can write eq. (2-4) without factor odering problems and have the solution $U = U^{(0)}$ like in the classical theory.

Spin can be introduced through Dirac spinors [*4*], [*7*], [*26*]. Then the mass-shell constraints are two coupled Dirac equations [*27*] and U is simply $U^{(0)}$ provided G_1 and G_2 are of the form

$$G_1 = \frac{e}{4} F^{\mu\nu}(q_1 \wedge p_1 + S_1)_{\mu\nu} \tag{3-1}$$

$$G_2 = -\frac{e}{4} F^{\mu\nu}(q_2 \wedge p_2 + S_2)_{\mu\nu} \tag{3-2}$$

where S_1 and S_2 are the individual intrinsic spin tensors, such that

$$M = \sum_1^2 q_a \wedge p_a + S_a .$$

In order to simplify the discussion we shall now go back to the case of spinless constituents, although this restriction is not essential.

We remark that hamiltonians of the form $H_a = K_a + U$ do not provide a unipotential system since $K_1 - K_2 \neq \overline{H}_1 - \overline{H}_2$.

Thus, following the notations of a previous work [*3*] we set

$$\begin{aligned} G_1 + U &= V + W \\ G_2 + U &= V - W \end{aligned} \tag{3-3}$$

(separating symmetric and antisymmetric parts with respect to particle exchange).

Elementary manipulations made in ref. [*3*] yield

$$4(\mathrm{H}_1 + \mathrm{H}_2) - \mathrm{P}^2 = 4y^2 + 8\mathrm{V}. \tag{3-4}$$

But in contrast to the case of isolated systems treated therein, the *diagonalisation of* P^2 *cannot be required together with the mass-shell constraints* $\mathrm{H}_a\Psi = \frac{1}{2}m_a^2\Psi$ imposed on the wave-function $\Psi(\mathrm{Q}, z)$. Recalling that $y^2 = -\square_z$ we have

$$\begin{aligned} \frac{1}{2}(m_1^2 + m_2^2)\Psi &= \left(-\square_z + \frac{\mathrm{P}^2}{4} + 2\mathrm{U} + \mathrm{G}_1 + \mathrm{G}_2\right)\Psi \\ \frac{1}{2}(m_1^2 - m_2^2)\Psi &= (y\cdot\mathrm{P} + \mathrm{G}_1 - \mathrm{G}_2)\Psi. \end{aligned} \tag{3-5}$$

But in spite of the appearance of $\square_z$ it is no more possible to separate the center-of-mass variables because P^α is not constant.

All the methods inspired from the quasi-potential approach [*28*] seem to break down.

It is clear that we are not in a situation where the spectrum is simply perturbed. Indeed P^2 cannot have anymore a sharp value.

We are led to consider transitions. But the original scheme which was sufficient to account for relativistic bound states in the absence of external field has to be generalized for two reasons:

a) In Ref. [*2*], [*3*] the spectrum of the unperturbed P^2 was obtained through the diagonalization of $\mathrm{H}_1^{(0)} + \mathrm{H}_2^{(0)}$ *within the Hilbert space of a reduced system* corresponding to the relative motion (essentially three-dimensional), more precisely, an eigenspace of P^α and $y\cdot\mathrm{P}$.

(Alternative choices of this reduced Hilbert space have been considered by other authors [*29*], [*11*]. For instance in Ref. [*11*], a space of eigenfunctions of $y\cdot\mathrm{P}$ and $(\mathrm{P}^2)^{-1}\mathrm{P}^\alpha$ is proposed.)

Now, under the influence of the external field, the state vector of our system cannot remain within one fixed Hilbert space of this kind.

In the course of its evolution, it will range over various spaces of this kind, corresponding to various values of P^α and $y\cdot\mathrm{P}$.

Therefore we are obliged to consider a unified picture of all these various Hilbert spaces, labelled by the (physically relevant) eigenvalues of P and $y\cdot\mathrm{P}$.

The most natural way to set up this framework is to superpose these spaces by direct integration.

Fortunately it turns out that the structure obtained by this construction coincides with a natural restriction of $\mathrm{L}^2(\mathbb{R}^8, d^4q_1, d^4q_2)$ as we shall see below.

b) Calculating transition probabilities requires an evolution operator, that is to say a time-depending formulation.

The question of a time-depending formalism is generally avoided in a large part of the litterature devoted to relativistic quantum mechanics.

However such a formalism has been already proposed and used in scattering theory [*8-10*]. It will receive new applications in a theory of transitions.

Finally our program of setting a wider framework will be carried out as follows:

If Ψ and Ψ' are functions of Q and z, let us consider the eightfold integral $\int \Psi^* \Psi' d^4Q d^4z$ which defines $L^2(\mathbb{R}^8, d^4Q, d^4z)$.

Such a quantity has *a priori* no physical meaning but is tacitly involved at the foundations of relativistic two-body quantum mechanics, because it makes application of the correspondance principle straightforward.

For instance, formally hermitian operators are just hermitian with respect to this eightfold integral, and, as we pointed out earlier [*8*], the wave equations (3-5) are in fact eigenvalue equations in the space $L^2(\mathbb{R}^8)$. If we want to go beyond simple heuristic formalism, it is natural to impose restrictions on $L^2(\mathbb{R}^8)$. Indeed the total linear momentum should not have eigenvalues outside the future light cone. Thus we are led to accept only those wave functions which are of the form

$$\Psi(Q, z) = \frac{1}{(2\pi)^2} \int e^{iK.Q} \psi(K, z) d^4K \tag{3-6}$$

where ψ is *retarded* in K, that is, its support (in the variable K) vanishes outside the future light cone.

For a technical reason which will appear soon, let us assume that ψ also vanishes when K^2 is strictly zero.

We shall say that Ψ is in $L^2(\mathbb{R}^{8+}, d^4Q, d^4z)$.

Or equivalently $\psi(K, z) \in L^2(\mathbb{R}^8_+, \text{Ret } d^4K, d^4z)$.

We can perform again Fourier transformation, on the variable $z^{\langle 0 \rangle} = \dfrac{K.z}{|K|}$. Then

$$\psi(K, z) = \frac{1}{\sqrt{2\pi}} \int e^{iuz^{\langle 0 \rangle}} \Phi(K, u, z) du \tag{3-7}$$

where Φ satisfies the condition

$$K \cdot \frac{\partial}{\partial z} \Phi = 0 \tag{3-8}$$

which tells that Φ depends on z only through its transverse part with respect to K.

To summarize we have

$$\Psi = \frac{1}{(2\pi)^{5/2}} \int e^{iK.Q} e^{iuz^{\langle 0 \rangle}} \Phi(K, u, z) d^4K du \tag{3-9}$$

and a similar development of Ψ' in terms of Φ'.

If we consider K and u as parameters, functions like Φ can be normalized through the definition

$$(\Phi, \Phi')_{Ku} = \int \Phi^* \Phi' \delta(z^{\langle 0 \rangle}) d^4z \tag{3-10}$$

or equivalently, setting

$$\hat{z} = z - \frac{z \cdot K}{K^2} K, \qquad \Phi = \varphi\,(K, u, \hat{z}), \qquad \Phi' = \varphi'(K, u, \hat{z})$$

we have with obvious notations

$$(\Phi, \Phi')_{Ku} = \int \varphi^* \varphi' d^3 \hat{z}\,.$$

For the unperturbed system this trick allows to calculate mass spectra within the reduced space of states $L^2_{Ku}(\mathbb{R}^3, d^3 \hat{z})$ made of these functions Φ. With the help of defining

$$J(K) = \int \psi^* \psi' d^4z$$

it is not difficult to show the important formula

$$\langle \psi, \psi' \rangle = \int \psi^* \psi' d^4K\, d^4z = \int (\Phi, \Phi')_{Ku} d^4K du\,.$$

Looking at this development we see that we have simply

$$L^2(\mathbb{R}^8_+, \text{Ret}\ d^4K, d^4z) = \int^{\oplus} L^2_{Ku} d^4K du$$

which solves our superposition problem.

We stress that two different values of K^2 will always correspond to distinct spaces L^2_{Ku}.

A first step towards a theory of transitions will be to investigate asymptotic problems only.

Indeed a relativistic definition of transition probability « per unit time » requires a delicate conceptual discussion about the meaning of τ_1 and τ_2 in quantum mechanics, which is beyond the scope of this article.

Asymptotic problems can be posed as follows:

If the external field vanishes in the remote past as well as in the remote future, let us assume that in the remote *past* our system is in some eigenstate of $H_1^{(0)}$, $H_2^{(0)}$ and P^2.

What is now the probability that the system be in some *other* eigenstate of these observables, in the remote *future?*

Even simplified in this way, the theory of transitions requires an underlaying *evolution* formalism just as much as does scattering theory.

This formulation will perhaps be debated a long time before all the researchers in this field reach a general agreement.

That is why, for the moment, we limit our developments to a general scheme which naturally comes out of our *multitime* description [*3*], [*8*], [*9*].

The asymptotic problem posed above has the structure of a scattering problem [*30*], the two-body system being obviously *free in a generalized sense* when no external field is present.

One is lead to introduce the evolution operators

$$\mathcal{U}^{(0)} = \exp\left(- i\tau_1 \mathrm{H}_1^{(0)} - i\tau_2 \mathrm{H}_2^{(0)}\right)$$
$$\mathcal{U} = \exp\left(- i\tau_1 \mathrm{H}_1 - i\tau_2 \mathrm{H}_2\right)$$

and the generalized Möller operators

$$\Omega_{\pm} = \lim \mathcal{U}^{-1}\mathcal{U}^{(0)}$$

for τ_1 *and* $\tau_2 \to \pm\infty$.

The question of the existence of the limits and independence of their order arises. Similar questions have received a satisfactory answer at least in the case of N-body scattering without external field [*10*].

They might have to be rechecked in the present case.

If we disregard this question of rigor, the scattering operator is then $\mathrm{S} = \Omega_-^*\Omega_+$.

In the general situation where the external field is not constant, the difficulty remains in writting down *explicitly* H_1 and H_2, which requires determination of U.

Conclusion.

For the *classical* treatment, we have sketched a general method, inspired from the perturbation theory which is commonly used in celestial mechanics [*31*].

As an application we have computed the variation of P^2 in a constant electromagnetic field.

This formula allows to consider experimental verifications provided one keeps in mind that its domain of applicability has the following limitations:

a) The « bound » state may have a finite lifetime.

b) The mutual electromagnetic interaction cannot be neglected for all values of the angular momentum.

A more accurate treatment would incorporate this electromagnetic contribution in the unperturbed potential $\mathrm{U}^{(0)}$.

This is possible since an effective potential describing (to a good approximation) the electromagnetic coupling between charges is available [*32*].

c) Our formula has been obtained under the assumption of a constant field. Utilization in other situations is questionable because an extra term might be present in eq. (2-9).

d) The approach developed in Section II was purely classical, and the spin of the constituent particles not taken into account.

Precise evaluation of these limitations obviously require numerical specifications of the system. It will be sensitive to the magnitude of many parameters: the constituent charges and masses, the strenght of the field, the mass of the unperturbed bound state, its angular momentum and the size of its relative orbit.

Another calculation performed about the classical relativistic harmonic oscillator exhibits a secular variation of the relative orbit which does not permit to consider the system as strictly bounded when the perturbation is due to a constant field. Interpretation of this phenomenon requires precautions.

First of all, this result was obtained in an idealized model. A *constant* field cannot be maintained in the whole space-time without an infinite source of energy.

Then, we have not taken into account the electromagnetic coupling *between* the charges. As this coupling is attractive it tends to stabilize the system.

In any case the concept of composite particle remains relevant if the relative motion undergoes a large number of revolutions before the size of the orbit is significantly increased.

And finally, we cannot completely exclude the possibility that our definition of the physical positions needs to be modified in the presence of external field.

Thus the question of the stability of confinement in this classical scheme remains open.

Assuming a more general form of the external field we have indicated a framework for the *quantum* mechanical picture: the overall Hilbert space in which wave equations are written is decomposed into a direct integral of various reduced spaces, each one corresponding to a description of the unperturbed motion. This procedure unifies the so-called « mathematical scalar product » [*8*], [*9*] with the norm which is currently used in mass spectrum calculations.

Although a lot of formal computations could be already carried out, just by introducing proper times in the spirit of our previous works on scattering, we prefer to reserve for a forthcoming paper all the matters related with a « time » depending formulation, especially the theory of transitions. Indeed many questions of physical interpretation remain open in this domain. For instance it is desirable to make the contact between the multitime formalism of particle relativistic mechanics and the Tomonaga-Schwinger formalism of quantum field theory.

We hope that the discussion presented here will stimulate new efforts towards a formulation of relativistic quantum mechanics in which the evolutive aspect will be fully taken into account.

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