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Commuting normal operators in partial Op*-algebras

by

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ABSTRACT. — As a first step towards a theory of Abelian partial Op*-algebras, we study the commutant of a family $\mathfrak{N}$ of commuting normal operators in a Hilbert space, that is, normal operators with the same spectral measure E(.). More precisely, we derive conditions under which a closed operator commuting (in the sense of strong partial Op*-algebras) with every operator in $\mathfrak{N}$ necessarily commutes with the spectral projections $E(\Delta)$. Under these conditions, which amount essentially to the existence of a dense domain of common analytic vectors, we show that the partial Op*-algebra generated by $\mathfrak{N}$ is of polynomial type, Abelian and standard.

RÉSUMÉ. — Comme première étape vers une théorie des Op*-algèbres partielles abéliennes, nous étudions le commutant d'une famille commutative $\mathfrak{N}$ d'opérateurs normaux dans un espace de Hilbert, c'est-à-dire des opérateurs normaux de même mesure spectrale E(.). Plus précisément, nous donnons des conditions pour qu'un opérateur normal commutant (au sens des Op*-algèbres partielles fortes) avec tout opérateur de $\mathfrak{N}$ commute nécessairement avec les projecteurs spectraux $E(\Delta)$. Sous ces

We dedicate this paper to the memory of our late friend Michel Sirugue.

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conditions, qui sont essentiellement équivalentes à l'existence d'un domaine dense de vecteurs analytiques communs, nous montrons que l'Op*-algèbre partielle engendrée par $\mathfrak{N}$ est de type polynomial, abélienne et standard.

I. INTRODUCTION

Algebras of unbounded operators have been around for a long time, since the pioneering work of Powers [*1*], Lassner [*2*], and many others. They are defined as follows. Given a Hilbert space $\mathcal{H}$ and a dense domain $\mathcal{D}$ in $\mathcal{H}$, one considers closable operators A which, together with their adjoint A*, are defined on $\mathcal{D}$ and leave it invariant. An Op*-algebra is a set of such operators which is stable under addition, involution (A $\leftrightarrow$ A*) and multiplication, thus a *-algebra.

For many reasons, both mathematical and physical, the requirement that the operators leave the domain $\mathcal{D}$ invariant is too restrictive. However, if one relaxes it, the product of two operators need not be defined any longer, i. e. the multiplication becomes *partial*. These circumstances have led us to introduce [*3*], [*4*] the concept of *partial *-algebra of closed operators* or, more concisely, *partial Op*-algebra*. Further developments include the beginnings of a representation theory [*5*], [*6*] and a systematic analysis of commutants and bicommutants [*7*]. Now, in the corresponding theory of algebras of bounded operators (C*- and W*-algebras), Abelian ones play a prominent role. Thus the question arises: what is an Abelian partial Op*-algebra and what are its properties?

In the bounded case, an algebra $\mathfrak{A}$ is Abelian iff it is contained in its commutant $\mathfrak{A}'$. For a partial Op*-algebra, however, several notions of unbounded commutants have been defined [*7*], and it is not clear which one will give a good notion of Abelianness. The aim of this paper is to analyze this question, in the restricted case where the partial Op*-algebra is generated by a family of normal operators with the *same* spectral measure, hence commuting in the strong sense. However limited as it is, this case already exhibits the pathologies that may happen and suggests the right choice to make.

Given a general partial Op*-algebra $\mathfrak{N}$, three different unbounded commutants have been defined [*7*], denoted respectively $\mathfrak{N}'_\bullet$, $\mathfrak{N}'_\square$, $\mathfrak{N}'_\sigma$; the first two (called natural) refer to the two possible partial multiplications $\bullet$ and $\square$, the third one is the weak unbounded commutant introduced by Epifanio and Trapani [*8*], and the three are ordered:

$$\mathfrak{N}'_\bullet \subset \mathfrak{N}'_\square \subset \mathfrak{N}'_\sigma. \tag{1.1}$$

In addition one may consider also the bounded parts of each of those sets:

$$\mathfrak{N}'_{\bullet b} \subset \mathfrak{N}'_{\Box b} = \mathfrak{N}'_{w} \tag{1.2}$$

where $\mathfrak{N}'_w \equiv \mathfrak{N}'_{\sigma b}$ (the equality in (1.2) was not noticed in Ref. [7] for a general partial Op*-algebra). Of course bounded commutants are of no use for defining an Abelian set that contains many unbounded operators! So we have to use the unbounded commutants (1.1) themselves. However there are indications that those are too big. For instance, a representation of $\mathfrak{N}$, as defined in Ref. 6, is irreducible iff the *bounded* part $\mathfrak{N}'_{\Box b} = \mathfrak{N}'_w$ of $\mathfrak{N}'_\Box$ is trivial, whereas the full commutants $\mathfrak{N}'$, $\mathfrak{N}'_\sigma$ need not be so. More seriously, the unbounded commutant $\mathfrak{N}'_\bullet$ does not exclude the Nelson pathology [9], [10]: as we will see in Section II, two essentially self-adjoint operators may commute in the • sense without having (strongly) commuting closures. A solution to this difficulty is to introduce a more restrictive commutant. If $\mathfrak{N}$ is a family of commuting normal operators with the same spectral measure E(.), we define the *spectral commutant* $\mathfrak{N}'_{sp} = (\mathfrak{N} \cup \{ E(\Delta) \}, \Delta \text{ Borel})'_\bullet$. But then we face two questions. Under what conditions does one have $\mathfrak{N}'_{sp} = \mathfrak{N}'_\bullet$?. And on the other hand, what is the relationship between usual Hilbert space notions and those introduced in the context of partial Op*-algebras?

In Section II below we analyze the concept of spectral commutant. Then in Sections III and IV we examine the first of the two questions asked above, first for a set $\mathfrak{N}$ consisting of a single normal operator (Sec. III), then for a finite family of normal operators with the same spectral measure (Sec. IV). In both cases, the key point is to build a suitable dense set of (jointly) analytic vectors, contained in $\mathscr{D}$. This of course is not a surprise. Since Nelson's original work [9], it is well-known that a large supply of analytic vectors eliminates most of the difficulties associated with commutation properties of unbounded operators. Many authors have discussed those problems, notably Schmüdgen [11] and Jörgensen [12].

Our main result, in Sections III and IV, is to provide a *sufficient* condition for the equality $\mathfrak{N}'_{sp} = \mathfrak{N}'_\bullet$ (to find a necessary *and* sufficient condition seems hopeless). However limited it is, that result is quite instructive. Indeed the condition we propose, although it may look *ad hoc* at first sight, is in fact very natural: it simply says that the domain $\mathscr{D}$ must be well-adapted to the family $\mathfrak{N}$ of operators, if one is to avoid pathologies. Moreover, the condition is easy to verify in concrete examples, as we shall see.

Finally (Section V) we discuss the partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$ generated by $\mathfrak{N} = \{ A_1, \ldots, A_N \}$, a concept introduced in Ref. [3]. In the present case, it turns out that $\mathfrak{M}[\mathfrak{N}]$ simply consists of all polynomials in $\{ A_1, \ldots, A_N \}$ that are allowed by the • multiplication. A similar result had been obtained in Ref. 3. The resulting partial Op*-algebra $\mathfrak{M} = \mathfrak{M}[\mathfrak{N}]$ is in fact *standard* (i. e. all symmetric operators are self-adjoint) and Abelian, in the sense that $\mathfrak{M} \subset \mathfrak{M}'_\sigma$. This definition is the one used already for the so-called V*-algebras [8], [13], but it is still too strong for a general partial Op*-algebra. However, the natural definition cannot be expressed as a simple inclusion relation. We discuss it at the end of the paper. Further work along this line is in progress and will be reported elsewhere [14].

At this point, however, the situation looks unnecessarily complicated. Indeed, it has been shown in Ref. [13] how to build from the operators $\{ A_1, \ldots, A_N \}$ a genuine Op*-algebra on the domain $\mathscr{D}^\infty(A_1, \ldots, A_N) = \cap_{k=1}^{N} \mathscr{D}^\infty(A_k)$. Similarly the restriction of $\mathfrak{M}$ to the domain of jointly analytic vectors contained in $\mathscr{D}$ is another Op*-algebra. So why should we still consider the domain $\mathscr{D}$ on which we have only a *partial* Op*-algebra? A possible answer lies in the analysis [13] of Dirac's approach to Quantum Mechanics. Schematically, a given physical system is characterized by a set $\mathfrak{O}$ of fundamental or labeled observables, represented by self-adjoint operators with a common dense domain $\mathscr{D}$. Contrary to the assumption made in the familiar Rigged Hilbert Space approach [15], [16], [17], this domain need *not* be invariant. Then the natural assumption is to take $\mathfrak{O}$ as a *partial* Op*-algebra on $\mathscr{D}$. Within $\mathfrak{O}$, one may choose a *complete set* $\mathfrak{N}$ *of commuting observables*, which by a result of von Neumann [18], [19] may be taken as having the same spectral measure (of course the choice of $\mathfrak{N}$ is not unique in general). Thus the domain $\mathscr{D}$ is fixed by the set $\mathfrak{O}$, and not by $\mathfrak{N}$. Hence we cannot assume that $\mathfrak{N}$ is an algebra, although it may have an extension or a restriction that is one. Those considerations will be discussed further at the end of the paper.

II. SPECTRAL (BI)COMMUTANTS OF NORMAL OPERATORS

We fix once and for all a separable Hilbert space $\mathscr{H}$ and a spectral family $\{ E(\lambda), \lambda \in \mathbb{R} \}$ in $\mathscr{H}$. The latter induces [10], [20-22] a unitary map U from $\mathscr{H}$ onto a direct integral Hilbert space $\tilde{\mathscr{H}}$:

$$U : \mathscr{H} \to \tilde{\mathscr{H}} = \int_{\mathbb{R}}^{\oplus} \mathscr{H}(\lambda) dp(\lambda), \tag{2.1}$$

In the sequel we will simply identify $\mathscr{H}$ and $\tilde{\mathscr{H}}$. As usual [23], [24], a closed operator B on $\mathscr{H} \equiv \tilde{\mathscr{H}}$ is called *decomposable* if

$$B = \int^{\oplus} B(\lambda) d\rho(\lambda) \tag{2.2}$$

with $B(\lambda)$ a closed operator in $\mathscr{H}(\lambda)$, and *diagonal* if

$$B = \int^{\oplus} \varphi_B(\lambda) \mathbf{1}_{\mathscr{H}(\lambda)} d\rho(\lambda). \tag{2.3}$$

A function $\varphi: \mathbb{R} \to \mathbb{C}$ is called E-*measurable* (resp. *E-essentially bounded*) iff it is ρ-measurable (resp. ρ-essentially bounded), a subset $\Delta \subset \mathbb{R}$ is E-measurable iff its characteristic function is. We denote by Σ the σ-algebra of all E-measurable subsets of $\mathbb{R}$.

Given an E-measurable, everywhere finite, function $\varphi_A: \mathbb{R} \to \mathbb{C}$, denote by Σ_A the σ-algebra generated by φ_A, that is, the smallest σ-algebra for which φ_A is measurable:

$$\Sigma_A = \{ \Delta \in \Sigma \mid \Delta = \varphi_A^{-1}(\delta), \delta \text{ a Borel subset of } \mathbb{C} \}. \tag{2.4}$$

Clearly $\delta_1 \cap \delta_2 = \emptyset$ implies $\varphi_A^{-1}(\delta_1) \cap \varphi_A^{-1}(\delta_2) = \emptyset$, but not conversely (take e. g. for φ_A a step function). However, if $\varphi_A^{-1}(\delta_1) \cap \varphi_A^{-1}(\delta_2) = \varphi_A^{-1}(\delta_1 \cap \delta_2) = \emptyset$, define $\delta'_j = \delta_j \backslash (\delta_1 \cap \delta_2)$ for $j = 1, 2$. Then $\varphi_A^{-1}(\delta'_j) = \varphi_A^{-1}(\delta_j)$, $\varphi_A^{-1}(\delta'_1) \cap \varphi_A^{-1}(\delta'_2) = \emptyset$ and $\delta'_1 \cap \delta'_2 = \emptyset$. On the other hand, if $\Delta \in \Sigma_A$ and $\lambda \in \Delta$, one has $\Delta_\lambda \equiv \varphi_A^{-1}(\{ \varphi_A(\lambda) \}) \subset \Delta$. That set Δ_λ may be reduced to one or several isolated points, but it contains every segment Δ_1 that contains λ and over which the function φ_A is constant with value $\varphi_A(\lambda)$.

The function φ_A defines a unique normal diagonal operator A on $\mathscr{H}$, namely:

$$\begin{aligned} A &= \int^{\oplus} \varphi_A(\lambda) \mathbf{1}_{\mathscr{H}(\lambda)} d\rho(\lambda) \\ &= \int \varphi_A(\lambda) dE(\lambda) \equiv \hat{E}(\varphi_A) \end{aligned} \tag{2.5}$$

with domain:

$$D(A) = \left\{ f \in \mathscr{H} \;\middle|\; \int | \varphi_A(\lambda) |^2 d \| E(\lambda) f \|^2 < \infty \right\}. \tag{2.6}$$

One has $\hat{E}(\varphi_A)^* = \hat{E}(\overline{\varphi}_A)$ and thus $A \equiv \hat{E}(\varphi_A)$ is self-adjoint iff φ_A is real. Notice that $E(\Delta_\lambda) \neq 0$ iff $\varphi_A(\lambda)$ is an eigenvalue of A, with corresponding eigenspace $E(\Delta_\lambda)\mathscr{H}$.

Of course one may also associate to the spectral measure E(.) a canonical self-adjoint operator, namely $A_0 = \int \lambda dE(\lambda)$ which is much easier to handle. But more singular functions are unavoidable when we treat several operators simultaneously. Thus we consider a general operator in Eq. (2.5).

We emphasize that our spectral measure E(.) is always defined on $\mathbb{R}$, following Riesz-Nagy [*18*], and *not* on the complex plane, as it is often the case in the spectral theory of normal operators [*20*].

Let now $\tilde{\mathfrak{N}}$ be a family of E-measurable, everywhere finite functions; we associate to it:

i) a family $\mathfrak{N}$ of strongly commuting normal operators, with common spectral measure E(.):

$$\tilde{\mathfrak{N}} \supset \varphi_A \leftrightarrow A = \hat{E}(\varphi_A) \in \mathfrak{N}$$

ii) the σ-algebra $\Sigma_{\mathfrak{N}}$ generated by $\tilde{\mathfrak{N}}$ (or $\mathfrak{N}$), i. e. the smallest σ-algebra containing all the sets $\Delta = \varphi_A^{-1}(\delta)$, for some $\varphi_A \in \mathfrak{N}$ and some Borel set $\delta \subset \mathbb{C}$.

Thus one has, for each $A \in \mathfrak{N}$:

$$\Sigma_A \subset \Sigma_{\mathfrak{N}} \subset \Sigma. \tag{2.7}$$

For convenience, we may simply assume without loss of generality (and shall do so unless otherwise indicated) that $\Sigma_{\mathfrak{N}} = \Sigma$. For eigenvalues, this means the following. If $E(\Delta)\mathscr{H}$ is a *common* eigenspace for all $A \in \mathfrak{N}$, and $\Delta' \subset \Delta$, $\Delta' \neq \Delta$, then Δ' is not E-measurable: the spectral measure E(.) « sees » only *maximal* common eigenspaces.

We return now to partial Op*-algebras. Let $\mathscr{D}$ be a fixed dense domain in $\mathscr{H}$, and $\mathfrak{C}(\mathscr{D})$ the set of closed, $\mathscr{D}$-minimal operators [3], [4]:

$A \in \mathfrak{C}(\mathscr{D}) \Leftrightarrow$ A closed, $\mathscr{D} \subset D(A) \cap D(A^*)$, $\mathscr{D}$ is a core for A: $A = \overline{A \upharpoonright \mathscr{D}}$.

On the set $\mathfrak{C}(\mathscr{D})$ we will consider the following operations:

i) vector space structure:

$$A \hat{+} B \equiv \overline{(A + B) \upharpoonright \mathscr{D}}, \qquad \lambda A \equiv \overline{(\lambda A) \upharpoonright \mathscr{D}}$$

ii) involution:

$$A^+ \equiv \overline{(A^*) \upharpoonright \mathscr{D}}$$

iii) strong partial multiplication:

$$A \bullet B \equiv \overline{A(B \upharpoonright \mathscr{D})}.$$

The product $A \bullet B$ is defined only for those pairs A, B which satisfy the two conditions $B\,\mathscr{D} \subset D(A)$, $A^+\,\mathscr{D} \subset D(B^+)$. In that case, we say that A is a strong left multiplier of B and B a strong right multiplier of A, and we write $A \in L^s(B)$, $B \in R^s(A)$. Let now $\mathfrak{N}$ be any subset of $\mathfrak{C}(\mathscr{D})$. We call *multipliers* of $\mathfrak{N}$ the elements of the set:

$$M^s\mathfrak{N} = \cap_{A \in \mathfrak{N}} L^s(A) \cap R^s(A). \tag{2.8}$$

The space of multipliers $M^s\mathfrak{N}$ carries two natural topologies, the so-called *quasi-uniform* topologies $\tau_*(\mathfrak{N})$ and $\tau_f(\mathfrak{N})$, described in detail in Refs. [3], [4].

Remark. — $\mathfrak{C}(\mathscr{D})$ also carries the so-called weak partial multiplication [3], [4] denoted $\square$, but we will not consider it here, except briefly in Section IV. The corresponding structure, called a weak partial Op*-algebra, will be studied systematically in other publications [14], [25]. Here we stick to the definitions and terminology of Refs. [3], [4].

Given a $+$-invariant subset $\mathfrak{N} = \mathfrak{N}^+$ of $\mathfrak{C}(\mathscr{D})$, we will consider two types of unbounded commutants:

i) its strong natural commutant:

$$\mathfrak{N}'_\bullet = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X \in M^s\mathfrak{N}, X \bullet A = A \bullet X, \forall A \in \mathfrak{N} \} \tag{2.9}$$

ii) its weak unbounded commutant:

$$\mathfrak{N}'_\sigma = \{ X \in \mathfrak{C}(\mathscr{D}) \mid \langle X^+ f, Ag \rangle = \langle A^+ f, Xg \rangle, \forall f, g \in \mathscr{D}, \forall A \in \mathfrak{N} \} \quad (2.10)$$

Obviously one has:

$$\mathfrak{N}'_\bullet \subset \mathfrak{N}'_\sigma. \quad (2.11)$$

In the sequel, we will consider a subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$ consisting of normal operators with the *same* spectral measure E(.), as described above. For such a set $\mathfrak{N}$, we want to characterize the commutant $\mathfrak{N}'_\bullet$ and the bicommutant $\mathfrak{N}'_\bullet{}'_\bullet$ and compare them with the familiar notion of commutant in the sense of unbounded operators.

First we observe that any *normal* operator $A \in \mathfrak{C}(\mathscr{D})$ is standard i. e. $A^* = A^+$. Indeed, if A is normal, $D(A^*) = D(A)$, and $\| A^* f \| = \| Af \|$, $\forall f \in D(A)$; hence $\mathscr{D}$ is dense in $D(A^*)$ with its A^*-graph norm, i. e. $A^* = \overline{A^* \upharpoonright \mathscr{D}} = A^+$.

We begin with bounded operators.

PROPOSITION 2.1. — Let $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathscr{D})$ consist of normal operators with the same spectral measure E(.). Then any *bounded* element B of $\mathfrak{N}'_\bullet$ commutes with $E(\Delta)$, for all $\Delta \in \Sigma$.

Proof. — Since A and $B \bullet A$ are closed and minimal, we have $D(B \bullet A) \supset D(A)$ and $(B \bullet A) \upharpoonright D(A) = B(A \upharpoonright D(A))$. Indeed, for $f \in D(A)$, take $f_n \in \mathscr{D}$ such that $f_n \to f, Af_n \to Af$; hence $(B \bullet A) f_n = BAf_n \to BAf$, so that $f \in D(B \bullet A)$. In the same way $Bf_n \to Bf$ and $ABf_n = BAf_n \to BAf$, so that $Bf \in D(A)$ and $ABf_n \to ABf$. Thus we get:

$$ABf = BAf, \qquad \forall f \in D(A),$$

i. e. B commutes with A in the sense of unbounded operators. It follows, as in the proof of Fuglede's theorem (see Ref. [*26*], § 1.6), that B commutes also with $A^* = A^+$ and with the spectral projections of each $A \in \mathfrak{N}$, i. e. with the spectral measure $E(.) = E_{\mathfrak{N}}(.)$. ■

The argument above, however, does not extend to unbounded operators $B \in \mathfrak{N}'_\bullet$. Take for instance the example of Nelson [*9*], [*10*]. Given a dense domain $\mathscr{D}$, he exhibits two operators X, Y, essentially self-adjoint on $\mathscr{D}$, such that $X\mathscr{D} \subset \mathscr{D}$, $Y\mathscr{D} \subset \mathscr{D}$, $XYf = YXf$, $\forall f \in \mathscr{D}$, and therefore $\overline{X} \bullet \overline{Y} = \overline{Y} \bullet \overline{X}$, i.e. $\overline{Y} \in \{ \overline{X} \}'_\bullet$. Yet $e^{it\overline{X}}$ and $e^{it\overline{Y}}$ do not commute, so that, in particular, $\overline{Y}$ does not commute with the spectral projections of $\overline{X}$.

This situation suggests the introduction of a new, more restrictive, type of commutant. Let $A \in \mathfrak{C}(\mathscr{D})$ be a normal operator, with spectral measure E(.). Its *spectral commutant* is defined as follows:

$$\{ A \}'_{sp} = \{ A \}'_\bullet \cap \{ E(\Delta), \Delta \in \Sigma_A \}'_\bullet . \quad (2.12)$$

Thus, when $B \in \{ A \}'_{sp}$, one has both $B \bullet A = A \bullet B$ and $B \bullet E(\Delta) = E(\Delta) \bullet B$

for all $\Delta \in \Sigma_A$. Similarly, for a family $\mathfrak{N}$ of normal operators with the *same* spectral measure E, we define:

$$\begin{aligned} \mathfrak{N}'_{sp} &= \cap_{A\in\mathfrak{N}} \{ A \}'_{sp} \\ &= \mathfrak{N}'_\bullet \cap \{ E(\Delta), \Delta \in \Sigma_{\mathfrak{N}} \}'_\bullet \\ &= (\mathfrak{N} \cup \{ E(\Delta) \}'_\bullet . \end{aligned} \tag{2.13}$$

In this language, the statement of Proposition 2.1 is that $\mathfrak{N}'_{sp}$ and $\mathfrak{N}'_\bullet$ contain the same bounded elements, or in the notation of Ref. [4], that $(\mathfrak{N}'_{sp})_b = (\mathfrak{N}'_\bullet)_b$. The properties of the spectral commutant are summarized in the next lemma.

LEMMA 2.2. — Let $\mathfrak{N} \subset \mathfrak{C}(\mathscr{D})$ be a family of normal operators with the same spectral measure E(.). Then:

i) every $B \in \mathfrak{N}'_{sp}$ is decomposable;

ii) $\mathfrak{N}'_{sp}$ is closed in the space of multipliers of $\mathfrak{N} \cup \{ E(\Delta) \}$ for the quasi-uniform topologies $\tau_{*,f}(\mathfrak{N} \cup \{ 1 \})$.

Proof. — *i*) Denote by $\langle E(\Delta) \rangle$ the vector space generated by the $\{ E(\Delta) \}$, by $\mathfrak{M} = \langle E(\Delta) \rangle''$ the Abelian von Neumann algebra generated by $\{ E(\Delta) \}$. Then we observe that:

$$\{ E(\Delta) \}'_\sigma = \langle E(\Delta) \rangle'_\sigma = \mathfrak{M}'_\sigma .$$

The first equality is trivial, the second follows from the fact that $\langle E(\Delta) \rangle$ is strongly dense in $\mathfrak{M}$. Indeed given $C \in \mathfrak{M}$, there is a net $\{ C_\alpha \} \in \langle E(\Delta) \rangle$ converging strongly to C. Hence, for $B \in \langle E(\Delta) \rangle'_\sigma$, we have $(f, g \in \mathscr{D})$:

$$\begin{aligned} \langle B^+ f, C^+ g \rangle &= \lim_\alpha \langle B^+ f, C_\alpha^+ g \rangle \\ &= \lim_\alpha \langle C_\alpha f, Bg \rangle = \langle Cf, Bg \rangle , \end{aligned}$$

i. e. $B \in \mathfrak{M}'_\sigma$. Now, as shown in Ref. [27], every element of $\mathfrak{M}'_\sigma$ is a closed decomposable operator in the direct integral (2.1), which proves the assertion.

ii) This follows from eq. (2.13) if one remarks [3], [4] that every strong natural commutant $\mathfrak{O}'_\bullet$ is closed in the space of its strong multipliers $M^s\, \mathfrak{O}$ for the quasi-uniform topologies $\tau_{*,f}(\mathfrak{N})$, and that $\tau_{*,f}(\mathfrak{N} \cup \{ E(\Delta) \})$ is equivalent to $\tau_{*,f}(\mathfrak{N} \cup \{ 1 \})$. ■

We notice that elements of the natural commutant $\mathfrak{N}'_\bullet$ need *not* be decomposable. This, together with the (Nelson) counter-example above, suggests that $\mathfrak{N}'_\bullet$ is actually too big. Furthermore, we can characterize explicitly the spectral bicommutant of $\mathfrak{N}$, $(\mathfrak{N}'_{sp})'_{sp} = (\mathfrak{N}'_{sp})'_\bullet$ (equality holds because $E(\Delta) \in \mathfrak{N}'_{sp}$).

We have seen that every $B \in \mathfrak{N}'_{sp}$ is decomposable, i. e. $B = \int^\oplus B(\lambda) d\rho(\lambda)$; $B(\lambda)$ a closed operator in $\mathscr{H}(\lambda)$. Conversely every bounded decomposable operator in $M^s\mathfrak{N}$ (that is, $B^{(+)}\mathscr{D} \subset D(A)$, $\forall A \in \mathfrak{N}$) belongs to $\mathfrak{N}'_{sp}$. Let now $C \in (\mathfrak{N}'_{sp})'_\bullet$. Since C commutes with every $E(\Delta)$, $\Delta \in \Sigma_\mathfrak{N}$, it is also decomposable, $C = \int^\oplus C(\lambda)d\rho(\lambda)$. In addition C must commute with every $B \in \mathfrak{N}'_{sp}$, i. e. $C(\lambda)$ must commute with every $B(\lambda)$, for a. e. λ, on an appropriate domain $\mathscr{D}(\lambda)$ (see Ref. [27]), hence $C(\lambda) = \varphi_C(\lambda)\mathbf{1}_{\mathscr{H}[\lambda)}$ a. e., that is, C is diagonal. For eigensubspaces, we can see it directly. Let $\mathscr{H}_1$ be a maximal common eigensubspace of $\mathfrak{N}$:

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2, \qquad \mathscr{H}_1 = E(\Delta)\mathscr{H}, \tag{2.14}$$

where $E(\Delta)\mathscr{H}$ contains no eigensubspace $E(\Delta')\mathscr{H}$ with $\Delta' \subset \Delta$, by the assumption $\Sigma_\mathfrak{N} = \Sigma$. Let now B be any bounded symmetric operator such that $\mathscr{H}_2$ is an eigenspace for B. Then, the decomposition (2.14) reduces both B and every $A \in \mathfrak{N}$ (B is arbitrary on $\mathscr{H}_1$) and $B \in \mathfrak{N}'_{sp}$. Let now $C \in (\mathfrak{N}'_{sp})'_\bullet$, in particular $B \bullet C = C \bullet B$. For $g \in \mathscr{H}_1 \cap \mathscr{D}$, $CBg = BCg$, i. e. $CE(\Delta)Bg = BCE(\Delta)g$. Thus $CE(\Delta)$ commutes with B on $\mathscr{H}_1$, hence $C \upharpoonright E(\Delta)\mathscr{H}$ is a multiple of the identity, i. e. $E(\Delta)\mathscr{H} = \mathscr{H}_1$ is an eigenspace for C as well, and of course $\mathscr{H}_1 \subset D(C)$.

We have seen that C is diagonal, $C = \int \varphi_C(\lambda)dE(\lambda) \equiv \hat{E}(\varphi_C)$. Then the diagonal operator C belongs to $(\mathfrak{N}'_{sp})'_\bullet$ if and only if $C \in R^s\mathfrak{N}'_{sp} \cup L^s\mathfrak{N}'_{sp} \subset \mathfrak{C}(\mathscr{D})$. Necessity is clear. To see that the condition is sufficient, observe that if $f \in \mathscr{D}$, then $E(\Delta)f \in D(C) \cap D(B)$, for every $\Delta \in \Sigma_\mathfrak{N}$ and $B \in \mathfrak{N}'_{sp}$. Then we have:

$$BE(\Delta)f = E(\Delta)Bf. \tag{2.15}$$

Let $C \in L^s\mathfrak{N}'_{sp}$; then $f \in \mathscr{D}$, $B \in \mathfrak{N}'_{sp}$ imply strong convergence of the integral

$$\int \varphi_C(\lambda)d(E(\lambda)Bf) = CBf. \tag{2.16}$$

But by (2.15), the l.h.s. of (2.16) equals:

$$B\int \varphi_C(\lambda)d(E(\lambda)f) = BCf. \tag{2.17}$$

If $C \in R^s\mathfrak{N}'_{sp}$, the same argument works, interchanging the two integrals. So in either case, we get $BCf = CBf$, $\forall f \in \mathscr{D}$, hence $B \bullet C = C \bullet B$, that is, $C \in (\mathfrak{N}'_{sp})'_\bullet$.

Thus, if we want to characterize $(\mathfrak{N}'_{sp})'_\bullet$, it remains to translate into the function φ_C the condition $C \in R^s\mathfrak{N}'_{sp} \cup L^s\mathfrak{N}'_{sp}$. This gives the following conditions, which state, respectively, that $Bf \in D(C)$ and $C^+ f \in D(B^+)$:

$$\int | \varphi_C(\lambda) |^2 d \| E(\lambda)Bf \|^2 < \infty, \tag{2.18a}$$

$$\int | \varphi_C(\lambda) |^2 \| B^+(\lambda)f(\lambda) \|^2 d\rho(\lambda) < \infty. \tag{2.18b}$$

We collect all those results in a proposition.

PROPOSITION 2.3. — Let $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathscr{D})$ be a family of normal operators with common spectral measure $E(.)$. Then:

i) $\mathfrak{N}'_{sp}$ is the family of all decomposable operators $B = \int^{\oplus} B(\lambda)d\rho(\lambda)$, $B \in \mathfrak{C}(\mathscr{D})$, which verify the conditions (2.18) for all $C \in \mathfrak{N}$, $f \in \mathscr{D}$.

ii) $(\mathfrak{N}'_{sp})'_{sp} = (\mathfrak{N}'_{sp})'_{\bullet}$ is the set of all diagonal operators

$$C = \int \varphi_C(\lambda)dE(\lambda) = \int^{\oplus} \varphi_C(\lambda)\mathbf{1}_{\mathscr{H}(\lambda)}d\rho(\lambda)$$

whose function φ_C verifies the conditions (2.18) for all $B \in \mathfrak{N}'_{sp}$, $f \in \mathscr{D}$. ■

III. STRONG NATURAL COMMUTANT VS. SPECTRAL COMMUTANT : THE CASE OF A SINGLE OPERATOR

Let $A = \hat{E}(\varphi_A)$ be a normal operator in $\mathfrak{C}(\mathscr{D})$. In Section II we have introduced its spectral commutant $\{A\}'_{sp}$, which is a subset of $\{A\}'_{\bullet}$. In view of Proposition 2.1 and the Nelson counterexample, the natural question to ask is, when does one have $\{A\}'_{sp} = \{A\}'_{\bullet}$? In other words, when does $B \bullet A = A \bullet B$ imply $B \bullet E(\Delta) = E(\Delta) \bullet B$ for all $\Delta \in \Sigma_A$? Clearly this statement involves both the domain $\mathscr{D}$ and the spectral properties of A, e. g. the function φ_A. As it is well-known the standard way of circumventing the Nelson pathology is to consider analytic vectors, thus we shall be concerned now with selecting a proper set of analytic vectors.

To begin with, we examine the σ-algebra Σ_A and, for later convenience, we subdivide it into two subsets. First we consider the ring consisting of inverse images of *bounded* Borel subsets:

$$\Xi_A = \{ \Delta \in \Sigma_A \mid \Delta = \varphi_A^{-1}(\delta), \delta \text{ a bounded Borel subset of } \mathbb{C} \}. \tag{3.1}$$

Clearly a subset $\Delta \in \Sigma_A$ belongs to Ξ_A iff φ_A is bounded on Δ, i.e. $\varphi_A \in L^\infty(\Delta, \rho)$. In general Ξ_A is only a ring, since $\mathbb{R} \notin \Xi_A$. It is a σ-algebra iff φ_A is bounded, but then $\Xi_A = \Sigma_A$.

For every $\lambda \in \mathbb{R}$, the set $\Delta_\lambda \equiv \varphi_A^{-1}(\{ \varphi_A(\lambda) \})$ belongs to Ξ_A, since the function φ_A is everywhere finite. It follows that every λ is contained in a measurable subset belonging to Ξ_A. In other words:

LEMMA 3.1. — The family Ξ_A covers the whole real line, i.e. :

$$\bigcup_{\Delta \in \Xi_A} \Delta = \mathbb{R}. \tag{3.2}$$

■

Next we consider spectral subspaces of A.

LEMMA 3.2. — Let $A = \hat{E}(\varphi_A)$ and $\Delta \in \Sigma_A$. Then:

i) $E(\Delta)\mathcal{H} \subset D(A)$ iff $\Delta \in \Xi_A$;

ii) For any closed decomposable operator B, $E(\Delta)\mathcal{H} \subset D(B)$ iff $E(\Delta)B = BE(\Delta)$ is bounded.

Proof. — Clearly $\varphi_A \in L^\infty(\Delta, \rho)$ implies $E(\Delta)\mathcal{H} \subset D(A)$, since one has:

$$\| AE(\Delta)f \|^2 = \int_\Delta | \varphi_A(\lambda) |^2 d \| E(\lambda)f \|^2$$
$$= \int_\Delta | \varphi_A(\lambda)f(\lambda) |^2 d\rho(\lambda) < \infty$$

Conversely, let $E(\Delta)\mathcal{H} \subset D(A)$. Since A is closed, its restriction to $E(\Delta)\mathcal{H}$ is a closed operator from the Hilbert space $E(\Delta)\mathcal{H}$ into $\mathcal{H}$, hence it must be bounded. But then the diagonal operator

$$AE(\Delta) = \int_\Delta \varphi_A(\lambda) dE(\lambda)$$

is bounded, hence φ_A must be essentially bounded on Δ. The argument is the same for *ii*), if one notices that the condition that $E(\Delta)B = \int_\Delta^\oplus B(\lambda) d\rho(\lambda)$ be bounded means that $B(\lambda)$ is a bounded operator in $\mathcal{H}(\lambda)$ for a.e. $\lambda \in \Delta$.

■

It follows from Lemma 3.2 *i*) that, if $\Delta \in \Xi_A$, then $E(\Delta)\mathcal{H} = E(\Delta)D(A) = E(\Delta)D^\infty(A)$, where $D^\infty(A) = \cap_{n \geq 1} D(A^n)$. Indeed since φ_A is bounded on Δ, it follows that $E(\Delta)f \in D(A^n)$ for $n = 1, 2, \ldots$, and hence $E(\Delta)f \in D^\infty(A)$.

It may be useful to consider also those subsets $\Delta \in \Sigma_A$ on which φ_A is *not* bounded and to define the *singular set* of A as

$$S_A = \bigcap_{\Delta \in \Sigma_A \setminus \Xi_A} \bar{\Delta} \quad (\bar{\Delta} \text{ is the closure of } \Delta) \tag{3.5}$$

For instance, if ρ is the Lebesgue measure and $\varphi_A(\lambda)$ is the function:

$$\begin{aligned} \varphi_A(\lambda) &= \tan \lambda\pi/2, \quad \lambda \neq 2k+1, \quad k \in \mathbb{Z} \\ &= 1, \qquad\qquad \lambda = 2k+1, \end{aligned}$$

then $S_A = \{ 2k + 1, k \in \mathbb{Z} \}$. Intuitively S_A is the set of points in the neighborhood of which the function φ_A is unbounded. Actually S_A need not be of E-measure 0. For instance, if we take the same function φ_A as above, but with $d\rho = d\lambda + \Sigma_{k\in\mathbb{Z}}\delta(\lambda - (2k + 1))$, then $S_A = \varphi_A^{-1}(\{ 1 \})$ has positive measure.

As can be seen from those examples, the singular set S_A may be used to characterize Ξ_A itself.

PROPOSITION 3.3. — A subset $\Delta \in \Sigma_A$ belongs to Ξ_A iff

. either $\Delta \cap S_A \neq \emptyset$ and then $\Delta^0 \cap S_A = \emptyset$ (Δ^0 the interior of Δ)

. or $\Delta \cap S_A = \emptyset$ and then $\overline{\Delta} \cap S_A = \emptyset$. ■

We omit the easy proof.

After these preliminaries, we come back to our search for the set of analytic vectors.

PROPOSITION 3.4. — Let $A = \hat{E}(\varphi_A)$, Ξ_A the corresponding ring. Then:

i) For every $\Delta \in \Xi_A$ and $f \in \mathscr{H}$, the vector $E(\Delta)f$ is an analytic vector for A.

ii) The set $\mathscr{D}_{an}(A) \equiv \cup_{\Delta\in\Xi_A} E(\Delta)\mathscr{H}$ is a dense subspace of $\mathscr{H}$ and a core for A; it consists of analytic vectors and contains all eigenspaces of A.

Proof. — Statement *i)* results from the following trivial estimate ($n = 1, 2, \ldots$):

$$\begin{aligned} \| A^n E(\Delta)f \|^2 &= \int_\Delta | \varphi_A(\lambda) |^{2n} d \| E(\lambda)f \|^2 \\ &\leq [\text{ess. sup}_{\lambda\in\Delta} | \varphi_A(\lambda) |^2]^n \| E(\Delta)f \|^2 \\ &\equiv [d_A(\Delta)]^{2n} \| E(\Delta)f \|^2 . \end{aligned}$$

Thus the set $\mathscr{D}_{an}(A)$ of *ii)* consists of analytic vectors for A. It is a vector space because Ξ_A is a ring, so that $\Delta_1, \Delta_2 \in \Xi_A$ implies $\Delta_1 \cup \Delta_2 \in \Xi_A$ and $E(\Delta_1)\mathscr{H} + E(\Delta_2)\mathscr{H} \subset E(\Delta_1 \cup \Delta_2)\mathscr{H}$. Clearly $\mathscr{D}_{an}(A)$ contains all eigenspaces of A, which are all of the form $E(\Delta_\lambda)\mathscr{H}$, with $\Delta_\lambda \equiv \varphi_A^{-1}(\{ \varphi_A(\lambda) \}) \in \Xi_A$ and $E(\Delta_\lambda) \neq 0$. In that case the argument of *ii)* holds with $d_A(\Delta) = | \varphi_A(\lambda) |$.

Next we show that $\mathscr{D}_{an}(A)$ is dense in $\mathscr{H}$. We cover $\mathbb{C}$ by an increasing sequence of bounded Borel sets δ_n, $n = 0, 1, 2, \ldots$, with $\delta_0 = \emptyset$, for instance open squares of side $2n$ centered at the origin, or open disks of radius n. Then $\mathbb{C} = \cup_{n=0}^{\infty}(\delta_{n+1}\backslash\delta_n)$ is a partition by (disjoint) bounded Borel sets. Hence $\Delta_n = \varphi_A^{-1}(\delta_n)$ and $\Delta_{n+1}\backslash\Delta_n = \varphi_A^{-1}(\delta_{n+1}\backslash\delta_n)$ both belong to Ξ_A. Now any $f \in \mathscr{H}$ may be written as:

$$f = \Sigma_{n=0}^{\infty} E(\Delta_{n+1}\backslash\Delta_n)f = \Sigma_{n=0}^{\infty} [E(\Delta_{n+1}) - E(\Delta_n)]f,$$

where the sum converges strongly since:

$$\| f \|^2 = \Sigma_{n=0}^{\infty} \| E(\Delta_{n+1}\backslash\Delta_n)f \|^2 < \infty .$$

This means that $\{E(\Delta_n)f\}$ is a Cauchy sequence. Therefore $f = \lim_{n\to\infty} E(\Delta_n)f$, which implies that $\mathscr{D}_{an}(A)$ is dense in $\mathscr{H}$. Finally $\mathscr{D}_{an}(A)$ is a dense set of analytic vectors, and it is invariant under A, hence it is a core for A [*10*]. ■

In fact, the vectors in $\mathscr{D}_{an}(A)$ are not only analytic, but even *bounded*, in the terminology of Faris [*28*]. Since the distinction between the two notions will not be used in the sequel, we will continue to call our vectors « analytic ».

Still the set $\mathscr{D}_{an}(A)$ is too big for our purposes. In view of Lemma 3.2 *ii*) the assumption $\mathscr{D}_{an}(A) \subset D(B)$ would drastically reduce the family of B's under consideration. To give an example, let $\mathscr{H} = L^2(\mathbb{R}^2, dxdy)$, A and B the self-adjoint operators of multiplication by x and y, respectively, $\mathscr{D} = D(A) \cap D(B)$. Then $B \bullet A = A \bullet B$ and $E_A(\Delta) \bullet B = B \bullet E_A(\Delta)$; yet $\mathscr{D}_{an}(A) \equiv \cup_{\Delta\in\Xi_A} E_A(\Delta)\mathscr{H}$ is *not* contained in D(B). Clearly one does not want to exclude *this* pair A, B !

But in fact we don't need the whole of $\mathscr{D}_{an}(A)$. Indeed, we want to consider the operator A only in the context of the partial *-algebra $\mathfrak{C}(\mathscr{D})$ of minimal operators. Thus $\mathscr{D}$ is necessarily a core for A. Put

$$\mathscr{D}(\Xi_A) \equiv \cup_{\Delta\in\Xi_A} E(\Delta)\mathscr{D} \tag{3.4}$$

Then $\mathscr{D}(\Xi_A) \subset \mathscr{D}_{an}(A)$ and hence $\mathscr{D}(\Xi_A)$ is a set of analytic vectors for A. It is also dense in $\mathscr{H}$, but not necessarily A-invariant. To remedy this, we consider the set $\mathscr{P}$ of all polynomials of one real variable and define:

$$\mathscr{D}^{\mathscr{P}}(\Xi_A) = \{ f \in \mathscr{H} \mid f = P(A)g,\ g \in \mathscr{D}(\Xi_A),\ P \in \mathscr{P} \}. \tag{3.5}$$

This set is dense in $\mathscr{H}$ since it contains $\mathscr{D}(\Xi_A)$. It is clearly A-invariant and its elements are analytic vectors for A. Thus $\mathscr{D}^{\mathscr{P}}(\Xi_A)$ is a core of analytic (in fact, bounded) vectors for A.

Now we are ready to formulate the main result of this Section:

THEOREM 3.5. — Let $A = \hat{E}(\varphi_A)$, $B \in \mathfrak{C}(\mathscr{D})$ and $\mathscr{D}(\Xi_A)$, $\mathscr{D}^{\mathscr{P}}(\Xi_A)$ the sets defined by (3.4) and (3.5) respectively. Assume that

i) $\mathscr{D}^{\mathscr{P}}(\Xi_A) \subset D(B)$,

ii) $\mathscr{D}(\Xi_A) \subset D(B^*)$

iii) $B\mathscr{D}^{\mathscr{P}}(\Xi_A) \subset D(A)$

iv) $ABf = BAf, \forall f \in \mathscr{D}^{\mathscr{P}}(\Xi_A)$.

Then, for every $\Delta \in \Sigma_A$ and $h \in \mathscr{D}$, one has $BE(\Delta)h = E(\Delta)Bh$. ■

The proof of this theorem will be subdivided into a series of Lemmas and Propositions. We begin our discussion with $\Delta \in \Xi_A$. Take $a, b \in \mathbb{C}$, $a \neq b$, and denote by δ_a, δ_b compact subsets of $\mathbb{C}$ containing a and b, respectively, such that $\delta_a \cap \delta_b = \emptyset$. Let $\Delta_a = \varphi_A^{-1}(\delta_a)$, $\Delta_b = \varphi_A^{-1}(\delta_b)$.

Then $\Delta_a, \Delta_b \in \Xi_A$ and $\Delta_a \cap \Delta_b = \emptyset$. By Urysohn's lemma there exists two continuous functions $\psi^{(1)}, \psi^{(2)}$ such that:

i) $\psi^{(1)}(z) - \psi^{(2)}(z) = 0$ for $z \in \delta_a$,

ii) $|\psi^{(1)}(z) - \psi^{(2)}(z)| > \gamma > 0$ for $z \in \delta_b$.

For $i = 1, 2$, let $\{ P_n^{(i)}(z) \}$ be a sequence of polynomials converging uniformly to $\psi^{(i)}(z)$ on $\delta_a \cup \delta_b$. Then we claim:

LEMMA 3.6. — With the notations as above, we have:

i) For $i = 1, 2$, $P_n^{(i)}(\varphi_A(\lambda)) \to \psi^{(i)}(\varphi_A(\lambda))$ uniformly on $\Delta_a \cup \Delta_b$;

ii) $\psi^{(1)}(\varphi_A(\lambda)) - \psi^{(2)}(\varphi_A(\lambda)) = 0$ for $\lambda \in \Delta_a$;

iii) $|\psi^{(1)}(\varphi_A(\lambda)) - \psi^{(2)}(\varphi_A(\lambda))| > \gamma > 0$ for $\lambda \in \Delta_b$. ■

The proof is easy and will be omitted. Notice that, for $\delta_b = \{ b \}$, the function φ_A must be constant on Δ_b, so that the convergence indicated in *i)* is pointwise on Δ_b, and similarly for $\delta_a = \{ a \}$.

Next we show:

LEMMA 3.7. — Let $\Delta_a = \varphi_A^{-1}(\delta_a) \in \Xi_A$, and $\Delta_b = \varphi_A^{-1}(\delta_b) \in \Xi_A$ such that $\Delta_a \cap \Delta_b = \emptyset$. Then, under the assumptions of Theorem 3.5, one has for every $h \in \mathcal{D}$:

$$E(\Delta_b)BE(\Delta_a)h = 0 \tag{3.6}$$

or, equivalently:

$$BE(\Delta_a)h = E(\Delta_a)BE(\Delta_a)h. \tag{3.6a}$$

Proof. — Assume first that δ_a, δ_b are disjoint compact subsets, so that $\Delta_a \cap \Delta_b = \emptyset$. Then one has, for $i = 1, 2$, $f \in \mathcal{H}$ and $\Delta \subset \Delta_a \cup \Delta_b$:

$$\| (P_n^{(i)}(A) - \psi^{(i)}(A))E(\Delta)f \|^2 = \int_\Delta | P_n^{(i)}(\varphi_A(\lambda)) - \psi^{(i)}(\varphi_A(\lambda)) |^2 d\langle f, E(\lambda)f \rangle$$

$$\leq \sup_{\lambda\in\Delta} | P_n^{(i)}(\varphi_A(\lambda)) - \psi^{(i)}(\varphi_A(\lambda)) |^2 \int_\Delta d\langle f, E(\lambda)f \rangle \to 0 \text{ as } n \to \infty. \tag{3.7}$$

In particular we have

$$P_n^{(i)}(A)E(\Delta_a)f \to \psi^{(i)}(A)E(\Delta_a)f. \tag{3.8}$$

If $h \in \mathcal{D}$, then $E(\Delta_a)h \in \mathcal{D}(\Xi_A) \subset D(B)$, so that, similarly,

$$P_n^{(i)}(A)E(\Delta_b)BE(\Delta_a)h \to \psi^{(i)}(A)E(\Delta_b)BE(\Delta_a)h. \tag{3.9}$$

We also have by *iii)* and *iv)*:

$$P_n^{(i)}(A)E(\Delta_b)BE(\Delta_a)h = E(\Delta_b)BP_n^{(i)}(A)E(\Delta_a)h. \tag{3.10}$$

We claim that (3.10) converges to $\overline{E(\Delta_b)B}\psi^{(i)}(A)E(\Delta_a)h$, where $\overline{E(\Delta_b)B}$ is the closure of $E(\Delta_b)B$. This clearly follows from (3.8), (3.9) and (3.10) provided $E(\Delta_b)B$ is closable, but it must be so, since $(E(\Delta_b)B)^*$ contains $\mathscr{D}$ in its domain, as follows from *ii*). Thus we have:

$$\psi^{(i)}(A)E(\Delta_b)BE(\Delta_a)h = \overline{E(\Delta_b)B}\psi^{(i)}(A)E(\Delta_a)h . \tag{3.11}$$

The functions $\psi^{(1)}$, $\psi^{(2)}$ are equal on Δ_a, hence

$$\begin{aligned} 0 &= \| \overline{E(\Delta_b)B} \{ \psi^{(1)}(A) - \psi^{(2)}(A) \} E(\Delta_a)h \|^2 \\ &= \| \{ \psi^{(1)}(A) - \psi^{(2)}(A) \} E(\Delta_b)BE(\Delta_a)h \|^2 \\ &= \int_{\Delta_b} | \psi^{(1)}(\varphi_A(\lambda)) - \psi^{(2)}(\varphi_A(\lambda)) |^2 d \| E(\lambda)BE(\Delta_a)h \|^2 > \gamma^2 \int_{\Delta_b} d \| E(\lambda)BE(\Delta_a)h \|^2 \\ &= \gamma^2 \| E(\Delta_b)BE(\Delta_a)h \|^2 . \end{aligned}$$

This proves (3.6) for δ_a and δ_b compact.

For general disjoint subsets $\Delta_a, \Delta_b \in \Xi_A$, we may always suppose that δ_a and δ_b are disjoint. Indeed, if $\delta_a \cap \delta_b \neq \emptyset$, one may replace δ_a, δ_b by δ'_a, δ'_b such that $\Delta_a = \varphi_A^{-1}(\delta'_a)$, $\Delta_b = \varphi_A^{-1}(\delta'_b)$ and $\delta'_a \cap \delta'_b = \emptyset$. So if δ_a, δ_b are compact, we are done. If not, they may be approximated from inside by compact subsets, $\delta_{a,n} \uparrow \delta_a$, $\delta_{b,m} \uparrow \delta_b$, with $\delta_{a,n} \cap \delta_{b,m} = \emptyset$, $\forall n, m$, as in the proof of Proposition 3.4.

Assume that δ_b is compact and δ_a is not. Then $E(\Delta_{a,n})h \to E(\Delta_a)h \in D(B)$. But $E(\Delta_b)BE(\Delta_{a,n})h = 0$ for every n and hence the sequence converges. Since $E(\Delta_b)B$ is closable, the limit is $0 = \overline{E(\Delta_b)B}E(\Delta_a)h = E(\Delta_b)BE(\Delta_a)h$. Thus (3.6) holds for every $\Delta_a \in \Xi_A$, and every disjoint Δ_b with δ_b compact. Let now δ_b be bounded but not closed and $\delta_{b,n}$ as discussed above. We get:

$$0 = E(\Delta_{b,n})BE(\Delta_a)h \to E(\Delta_b)BE(\Delta_a)h = 0 ,$$

where δ_a can be compact or not. So (3.6) holds true in general.

Clearly (3.6*a*) implies (3.6). To prove the converse implication, take any $f \in \mathscr{H}$, $h \in \mathscr{D}$; then:

$$\langle E(\Delta_b)f, (1 - E(\Delta_a)BE(\Delta_a)h \rangle = \langle f, E(\Delta_b \backslash \Delta_a)BE(\Delta_a)h \rangle = 0 .$$

Since $\mathscr{D}_{an}(A)$ is dense, this proves the statement and completes the proof of Lemma 3.7. ■

A similar lemma is valid for B*.

Lemma 3.8. — Let $\Delta_a = \varphi_A^{-1}(\delta_a) \in \Xi_A$, $\Delta_b = \varphi_A^{-1}(\delta_b) \in \Xi_A$ such that $\Delta_a \cap \Delta_b = \emptyset$. Then, under the assumptions of Theorem 3.5, one has for every $h \in \mathscr{D}$:

$$E(\Delta_b)B^*E(\Delta_a)h = 0 \tag{3.12}$$

or, equivalently:

$$B^*E(\Delta_a)h = E(\Delta_a)B^*E(\Delta_a)h . \tag{3.12a}$$

Proof. — Let $h, f \in \mathscr{D}$. Then:

$$0 = \langle f, E(\Delta_b)BE(\Delta_a)h \rangle = \langle E(\Delta_b)B^*E(\Delta_a)f, h \rangle,$$

which implies (3.12) and (3.12*a*). ■

Now we go back to Theorem 3.5. Let again $h, f \in \mathscr{D}$ and $\Delta \in \Xi_A$. We have:

$$\begin{aligned}\langle f, E(\Delta)Bh \rangle &= \langle B^*E(\Delta)f, h \rangle = \langle E(\Delta)B^*E(\Delta)f, h \rangle \\ &= \langle f, E(\Delta)BE(\Delta)h \rangle = \langle f, BE(\Delta)h \rangle .\end{aligned}$$

Hence

$$E(\Delta)Bh = BE(\Delta)h. \tag{3.13}$$

This formula can be extended to any $\Delta \in \Sigma_A$. Indeed, let $\Delta = \varphi_A^{-1}(\delta)$ with δ an unbounded Borel subset of $\mathbb{C}$ and $f \in \mathscr{D}$. If $\delta_n \uparrow \delta$ with δ_n bounded and $\Delta_n = \varphi_A^{-1}(\delta_n)$, then

$$E(\Delta_n)f \to E(\Delta)f,$$
$$E(\Delta)Bf = \lim_n E(\Delta_n)Bf = \lim_n BE(\Delta_n)f = BE(\Delta)f.$$

The last equality follows because B is closed, thus (3.13) holds for any $f \in \mathscr{D}$ and $\Delta \in \Sigma_A$. This completes the proof of Theorem 3.5. ■

Suppose the assumptions of Theorem 3.5 hold. Does that imply $E(\Delta) \bullet B = B \bullet E(\Delta)$? The answer is affirmative if both products exist. In fact, $B \bullet E(\Delta)$ always exists, since the non-trivial condition $E(\Delta)\mathscr{D} \subset D(B)$ follows from (3.13). Exactly the same argument, using the closedness of B^*, yields the condition $E(\Delta)\mathscr{D} \subset D(B^*)$. However, existence of $E(\Delta) \bullet B$ is equivalent to the stronger statement $E(\Delta)\mathscr{D} \subset D(B^+)$. If this holds, then (3.13) implies $E(\Delta) \bullet B = B \bullet E(\Delta)$. Notice that the stronger statement holds true whenever B is normal, which implies $B^* = B^+$, and in particular self-adjoint. We state those results as a corollary.

COROLLARY 3.9. — Suppose that $A = \hat{E}(\varphi_A)$ and $B \in \mathfrak{C}(\mathscr{D})$ verify the assumptions of Theorem 3.5. Assume, in addition, that either *i*) $E(\Delta)\mathscr{D} \subset D(B^+)$, or *ii*) B is normal, in particular, self-adjoint. Then $E(\Delta) \bullet B = B \bullet E(\Delta)$. ■

If we assume now that $\mathscr{D}(\Xi_A) \subset \mathscr{D}$, then condition *ii*) of Theorem 3.5 automatically fulfilled, and in fact $\mathscr{D}(\Xi_A) \subset D(B^+)$. Hence, if $\Delta \in \Xi_A$, we have $E(\Delta)\mathscr{D} \subset \mathscr{D} \subset D(B^+)$. For an arbitrary $\Delta \in \Sigma_A$, the condition $E(\Delta)f \in D(B^+)$ follows from the closedness of B^+, exactly as in the proof of Theorem 3.5. Thus we have:

THEOREM 3.10. — Let $A = \hat{E}(\varphi_A)$, $B \in \mathfrak{C}(\mathscr{D})$, and the sets $\mathscr{D}(\Xi_A)$, $\mathscr{D}^{\mathscr{P}}(\Xi_A)$ defined by (3.4) and (3.5) respectively. Assume conditions *i*), *iii*) and *iv*) of Theorem 3.5, and in addition:

ii') $\mathscr{D}(\Xi_A) \subset \mathscr{D}$,

Then, for every $\Delta \in \Sigma_A$, one has $E(\Delta) \bullet B = B \bullet E(\Delta)$. ■

At this point one may ask the following questions:

1) Are the assumptions of Theorem 3.10 sufficient to conclude that $A \bullet B = B \bullet A$?

2) If assumption *iv*) in Theorems 3.5 and 3.10 is replaced by $A \bullet B = B \bullet A$, does the relation $E(\Delta) \bullet B = B \bullet E(\Delta)$ still hold?

As for the first question the answer is negative. Take for example $\mathcal{H} = L^2(\mathbb{R}, dx)$, $\mathcal{D} = \{ f \in \mathcal{H} \mid f(x) = (1 + |x|)^{-1} g(x), g \in \mathcal{H} \}$, $A : f(x) \mapsto xf(x)$. Consider $B = A$. Then all assumptions of Theorem 3.10 are fulfilled, but the product $A \bullet A$ is not even defined!

However, if $\mathcal{D}^{\mathcal{P}}(\Xi_A) \subset \mathcal{D}$ and $A \bullet B = B \bullet A$, then all assumptions of Theorem 3.10 are fulfilled and we have:

Corollary 3.11. — With the notations of Theorem 3.10, assume:

i) $\mathcal{D}^{\mathcal{P}}(\Xi_A) \subset \mathcal{D}$,

ii) $A \bullet B = B \bullet A$.

Then for every $\Delta \in \Sigma_A$, one has $E(\Delta) \bullet B = B \bullet E(\Delta)$, and thus $B \in \{ A \}'_{sp}$. ■

Putting all together, we may state finally:

Theorem 3.12. — Assume that $\mathcal{D}^{\mathcal{P}}(\Xi_A) \subset \mathcal{D}$. Then $\{ A \}'_{sp} = \{ A \}'_{\bullet}$. ■

This answers our original question for the case of a single operator A. In the next Section we will extend the analysis to the case of several operators. But, before that, we want to comment the main result, Theorem 3.12.

Of course, the condition $\mathcal{D}^{\mathcal{P}}(\Xi_A) \subset \mathcal{D}$ is only a *sufficient* condition for the equality $\{ A \}'_{sp} = \{ A \}'_{\bullet}$, but it is a natural one. In fact it must be seen as a consistency condition on the domain $\mathcal{D}$: the latter must be *well-adapted* to the operator A, lest all pathologies of Nelson type break loose. Let $f \in \mathcal{D}$. Then the condition says that the vector $P(A)E(\Delta)f$ must belong to $\mathcal{D}$ again, for any polynomial P and every $\Delta \in \Xi_A$ (remember that $P(A)$ is bounded on $E(\Delta)\mathcal{H}$, so that the first factor is rather harmless). Let us give some simple examples. Take $\mathcal{H} = L^2(\mathbb{R}, dx)$, A = multiplication by x. Then the condition is satisfied for each of the following domains: $D(A)$, $D(A^k)$ or the domain $\mathcal{D}_k = \{ f \in \mathcal{H} \mid f(x) = (1 + |x|)^{-k} g(x),\ g \in \mathcal{H} \}$, for any fixed $k > 1$. Notice that none of those domains is invariant under A.

IV. THE CASE OF SEVERAL OPERATORS

For a family $\mathfrak{N}$ containing more than one operator, the analysis proceeds essentially along the same lines. The key point in the argument of Section III is the introduction of the set Ξ_A for generating a dense subspace of analytic vectors. What we need now is a subset $\Xi_{\mathfrak{N}}$ of $\Sigma_{\mathfrak{N}}$ that will play the same role.

We start by considering a two element family $\mathfrak{N} = \{ A_1, A_2 \}$, corresponding to the functions $\varphi_j, j = 1, 2$. Each A_j generates the σ-algebra Σ_j, with Ξ_j the subset of Σ_j consisting of inverse images under φ_j of bounded Borel subsets of $\mathbb{C}$. Finally $\Sigma_{\mathfrak{N}}$ is the σ-algebra generated by $\{ \Sigma_1, \Sigma_2 \}$, and we assume that $\Sigma_{\mathfrak{N}} = \Sigma$, as indicated in Section II.

Let now Δ be an element of the « good » subset $\Xi_{\mathfrak{N}}$ we are looking for. A vector of the form $E(\Delta)f$ is surely jointly analytic for A_1, A_2 if both operators are bounded on the subspace $E(\Delta)\mathscr{H}$. This means that $\Delta \in \varphi_1^{-1}(\delta_1) \cap \varphi_2^{-1}(\delta_2)$ for some bounded Borel subsets $\delta_1, \delta_2 \in \mathbb{C}$. Notice that it may, and in fact often will, happen that φ_2 is not bounded on the whole of $\varphi_1^{-1}(\delta_1)$, or φ_1 on $\varphi_2^{-1}(\delta_2)$. Let us give an example of such a behavior.

Let the measure ρ defined by $E(.)$ be the Lebesgue measure for simplicity (see Eq. (2.1)), and consider the following functions:

$$
\begin{aligned}
\varphi_1(\lambda) &= (\sin \pi\lambda)^{-1} , & \lambda &\notin \mathbb{Z} \\
&= 1 , & \lambda &\in \mathbb{Z} \\
\varphi_2(\lambda) &= (\sin \pi\varepsilon\lambda)^{-1}, & \varepsilon\lambda &\notin \mathbb{Z} \\
&= 1 , & \varepsilon\lambda &\in \mathbb{Z}
\end{aligned}
\tag{4.1}
$$

where ε is a fixed *irrational* number, with $0 < \varepsilon < 1$. In the terminology of Section III, Eq. (3.5), the singular sets of φ_1, φ_2 are $\mathbb{Z}$ and $\mathbb{Z}/\varepsilon$, respectively. Given any two numbers a, b such that $0 < a < b < 1$, define $\delta_1 = [b^{-1}, a^{-1}]$. Then $\varphi_1^{-1}(\delta_1)$ is the union of infinitely many disjoint intervals, and for *any* such δ_1, $\varphi_1^{-1}(\delta_1)$ contains infinitely many points $\lambda_n^{(2)}$ of $S_2 \equiv \mathbb{Z}/\varepsilon$. This follows from Birkhoff's ergodic theorem [Ref. [*28*], Sec. 16, Th. C]: taking everything modulo one, the orbit $\{ n\varepsilon, n = 1, 2 \dots \}$ is dense in $[0, 1]$. In other words the set $\{ \varphi_1(\lambda_n^{(2)}), n = 1, 2 \dots \}$ is dense in δ_1. Yet there are plenty of intervals where both φ_1 and φ_2 are bounded: any closed interval contained in $\mathbb{R} \backslash (S_1 \cup S_2)$ will do. Take, for instance, any closed interval $\Delta_0 \subset (0, \varepsilon)$: both φ_1 and φ_2 are bounded on Δ_0, $\varphi_j(\Delta_0) \subset \delta_j$ $(j = 1, 2)$, with δ_j bounded Borel and $\Delta_0 \subset \varphi_1^{-1}(\delta_1) \cap \varphi_2^{-1}(\delta_2)$. Another example that looks even more pathological, but is in fact equivalent to the previous one, is given by the functions:

$$
\begin{aligned}
\varphi_1(\lambda) &= (\sin \pi/\lambda)^{-1} , & \lambda^{-1} &\notin \mathbb{Z} \\
&= 1 , & \lambda^{-1} &\in \mathbb{Z} \\
\varphi_2(\lambda) &= (\sin \varepsilon\pi/\lambda)^{-1}, & \varepsilon\lambda^{-1} &\notin \mathbb{Z} \\
&= 1 , & \varepsilon\lambda^{-1} &\in \mathbb{Z} .
\end{aligned}
\tag{4.2}
$$

Those examples suggest to choose as « good » subset the family:

$$
\Xi_{\mathfrak{N}} = \{ {}^1\Delta \cap {}^2\Delta, {}^1\Delta \in \Xi_1, {}^2\Delta \in \Xi_2 \} \tag{4.3}
$$

In general, $\Xi_{\mathfrak{N}}$ is not a ring and does not contain Ξ_1 or Ξ_2 (unless φ_1 and/or φ_2 is bounded). In fact, in the examples above, one has $\Xi_1 \cap \Xi_2 = \emptyset$.

LEMMA 4.1. — Let $\Xi_{\mathfrak{N}}$ be the set defined in (4.3). Then:

i) ordered by inclusion, $\Xi_{\mathfrak{N}}$ is stable under intersection and directed to the right;

ii) $\Xi_{\mathfrak{N}}$ covers the whole line: $\cup \{ \Delta : \Delta \in \Xi_{\mathfrak{N}} \} = \mathbb{R}$.

Proof. — Statement *i*) is immediate; directedness means that every pair of elements has an upper bound:

$$({}^1\Delta \cap {}^2\Delta) \cup ({}^1\Delta' \cap {}^2\Delta') \subset ({}^1\Delta \cup {}^1\Delta') \cap ({}^2\Delta \cup {}^2\Delta') \tag{4.4}$$

where the r.h.s. belongs to $\Xi_{\mathfrak{N}}$ again since ${}^j\Delta \cup {}^j\Delta' \in \Xi_j$, $j=1,2$. As for *ii*), it follows, as in Lemma 3.1, from the fact that both functions φ_1, φ_2 are everywhere finite by assumption. ■

Now we proceed as in Proposition 3.4. Clearly, for every $\Delta \in \mathfrak{N}_{\mathfrak{N}}$ and $f \in \mathscr{H}$, the vector $\mathrm{E}(\Delta)f$ is jointly analytic for A_1 and A_2, since both operators are bounded on $\mathrm{E}(\Delta)\mathscr{H}$. Then we introduce the set $\mathscr{D}_{an}(\mathfrak{N}) = \cup_{\Delta \in \Xi_n} \mathrm{E}(\Delta)\mathscr{H}$. First it is a vector subspace of $\mathscr{H}$, by the inclusion (4.4). Second it is dense in $\mathscr{H}$. For proving this we write again $\mathbb{C} = \cup_{n=0}^{\infty} \delta_n$, $\delta_n \subset \delta_{n+1}$, $\delta_0 = \emptyset$, and define, for $j = 1, 2$, ${}^j\Delta_n = \varphi_j^{-1}(\delta_n)$. Then every $f \in \mathscr{H}$ may be written as follows:

$$f = \lim_{n \to \infty} \mathrm{E}({}^j\Delta_n)f, \qquad j = 1, 2, \tag{4.5}$$

and therefore

$$f = \lim_{n_1, n_2 \to \infty} \mathrm{E}({}^1\Delta_{n_1} \cap {}^2\Delta_{n_2})f, \tag{4.6}$$

where ${}^1\Delta_{n_1} \cap {}^2\Delta_{n_2} \in \Xi_{\mathfrak{N}}$, which shows that $\mathscr{D}_{an}(\mathfrak{N})$ is dense. Notice that some of those elements of $\Xi_{\mathfrak{N}}$ at least have positive measure, because of the equality:

$$\| f \|^2 = \Sigma_{n_1 n_2} \| \mathrm{E}(({}^1\Delta_{n_1+1} \backslash {}^1\Delta_{n_1}) \cap ({}^2\Delta_{n_2+1} \backslash {}^2\Delta_{n_2}))f \|^2 \neq 0 \tag{4.7}$$

Finally $\mathscr{D}_{an}(\mathfrak{N})$ is a dense invariant set of analytic vectors, hence it is a core for A_1 and A_2.

Clearly all those arguments hold true for any *finite* family $\mathfrak{N} = \{ \mathrm{A}_j, j = 1, 2, \ldots, \mathrm{N} \}$, $\mathrm{A}_j = \hat{\mathrm{E}}(\varphi_j)$. In particular, Eq. (4.4) becomes [*29*]:

$$(\cap_{j=1}^{\mathrm{N}} {}^j\Delta) \cup (\cap_{j=1}^{\mathrm{N}} {}^j\Delta') \subset \cap_{j=1}^{\mathrm{N}} ({}^j\Delta \cup {}^j\Delta') \tag{4.8}$$

We summarize the discussion above as a proposition.

PROPOSITION 4.2. — Let $\mathfrak{N} = \{ \mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{N} \}$ be a finite family of normal operators with the same spectral measure $\mathrm{E}(.)$, namely $\mathrm{A}_j = \hat{\mathrm{E}}(\varphi_j)$, $j = 1, \ldots, \mathrm{N}$. For each $j = 1, \ldots, \mathrm{N}$, define the ring

$$\Xi_j = \{ \varphi_j^{-1}(\delta), \delta \text{ a bounded Borel subset of } \mathbb{C} \} \tag{4.9}$$

and the set

$$\Xi_{\mathfrak{N}} = \{ \Delta = \cap_{j=1}^{\mathrm{N}} {}^j\Delta, {}^j\Delta \in \Xi_j \} \tag{4.10}$$

Then:

i) $\Xi_{\mathfrak{N}}$ is stable under intersection, directed to the right, i. e. verifies (4.8), and covers the whole line: $\cup \{ \Delta : \Delta \in \Xi_{\mathfrak{N}} \} = \mathbb{R}$.

ii) For every $\Delta \in \Xi_{\mathfrak{N}}$ and $f \in \mathscr{H}$, the vector $E(\Delta)f$ is jointly analytic for $A_1, A_2, \ldots, A_N$.

iii) The set $\mathscr{D}_{an}(\mathfrak{N}) = \cup_{\Delta \in \Xi_{\mathfrak{N}}} E(\Delta)\mathscr{H}$ is a dense subspace of jointly analytic vectors and a common core for $A_1, A_2, \ldots, A_N$. ■

The density of $\mathscr{D}_{an}(\mathfrak{N})$ rests on the following approximation, valid for any f:

$$f = \lim_{n_1, n_2, \ldots, n_N \to \infty} E(\cap_{j=1}^{N} {}^j\Delta_{n_j})f \tag{4.11}$$

where ${}^j\Delta_{n_j} = \varphi_j^{-1}(\delta_{n_j})$ and $\{ \delta_{n_j} \}$ is our standard increasing sequence of bounded Borel sets covering $\mathbb{C}$. Here again the subsets $\Delta = \cap_{j=1}^{N} {}^j\Delta_{n_j}$ cannot have all measure zero: this corresponds to the fact that there are sufficiently many sets of positive measure where $A_1, \ldots, A_N$ are simultaneously bounded; in other words, the joint singular set of the family $\mathfrak{N}$, $S_{\mathfrak{N}} = \cup_{j=1}^{N} S(A_j)$, is a discrete set. However, as one sees easily, this property does *not* extend to an infinite family $\mathfrak{N}$. To get a counterexample, one may generalize the examples (4.1) or (4.2). For every $j = 1, 2, \ldots,$ define the function:

$$\begin{aligned} \varphi_j(\lambda) &= (\sin j\pi\lambda)^{-1}, \quad && \lambda \notin \mathbb{Z}/j \\ &= 1 \quad , && \lambda \in \mathbb{Z}/j \end{aligned} \tag{4.12}$$

Then $S(A_j) = \mathbb{Z}/j$, which is discrete. So is the union of any *finite* number of similar sets $S(A_{j_k})$, $k = 1, 2, \ldots, N$. But if we take all $j \in \mathbb{N}$, we get:

$$S_{\mathfrak{N}} = \cup_{j \in \mathbb{N}} S(A_j) = \mathbb{Q},$$

the rational numbers, which are dense in $\mathbb{R}$. Thus the construction breaks down and so does the approximation (4.11): the corresponding set $\Xi_{\mathfrak{N}} = \{ \cap_{j=1}^{\infty} {}^j\Delta, {}^j\Delta \in \Xi_j \}$ contains no set of positive measure, only isolated points. Thus for an infinite family $\mathfrak{N}$, we need an assumption, for instance that $S_{\mathfrak{N}} = \cup_{j \in \mathbb{N}} S(A_j)$ is nowhere dense [*10*]. Then $\overline{\mathbb{R} \backslash S_{\mathfrak{N}}}$ contains a dense open set and the construction may be generalized. However, we shall not pursue that point here.

Having thus identified the « good » subset $\Xi_{\mathfrak{N}}$ of $\Sigma_{\mathfrak{N}}$, we may proceed exactly as in Section III. As in (3.4), (3.5), we define two sets of analytic vectors:

$$\mathscr{D}(\Xi_{\mathfrak{N}}) = \cup_{\Delta \in \Xi_{\mathfrak{N}}} E(\Delta)\mathscr{D} \tag{4.13}$$

$$\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) = \{ f \in \mathscr{H} \mid f = P(A_1, \ldots, A_N)g, \, g \in \mathscr{D}(\Xi_{\mathfrak{N}}), \, P \in \mathscr{P}_N \} \tag{4.14}$$

where $\mathscr{P}_N$ denotes the set of polynomials in N real variables, and $P(A_1, \ldots, A_N)$ is defined by the functional calculus. Then the reasoning of Lemma 3.7 goes through. Taking $N = 2$ for simplicity, let $\Delta = {}^1\Delta \cap {}^2\Delta$, $\Delta' = {}^1\Delta' \cap {}^2\Delta'$, with $\Delta, \Delta' \in \Xi_{\mathfrak{N}}$, $\Delta \cap \Delta' = \emptyset$, and consider the operator $E(\Delta)BE(\Delta')$, where B verifies the assumptions of Lemma 3.7. Using the decomposition $\Gamma = (\Gamma \cap \Gamma') \cup (\Gamma \backslash \Gamma')$ for the four sets ${}^j\Delta$, ${}^j\Delta'$, $j = 1, 2$, and applying

Lemma 3.7 for $j = 1$ and $j = 2$ successively, we obtain:

$$\begin{aligned} E(\Delta)BE(\Delta') &= E({}^2\Delta \cap {}^2\Delta')[E({}^1\Delta \cap {}^1\Delta')BE({}^1\Delta \cap {}^1\Delta')]E({}^2\Delta \cap {}^2\Delta') \\ &= E(\Delta \cap \Delta')BE(\Delta \cap \Delta') = 0. \end{aligned}$$

The approximation method defined in the proof of Lemma 3.7 works for several operators as well, so that we may extend our analysis to arbitrary subsets $\Delta, \Delta' \in \Sigma_{\mathfrak{N}}$. The same is true for Lemma 3.8.

Putting all together, we obtain the analogue of Theorem 3.5:

THEOREM 4.3. — Let $\mathfrak{N} = \{ A_1, \ldots, A_N \}$, $A_j = \hat{E}(\varphi_j)$ as before, $B \in \mathfrak{C}(\mathscr{D})$, $\mathscr{D}(\Xi_{\mathfrak{N}})$ and $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}})$ the sets defined in (4.13), (4.14) respectively. Assume that

i) $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset D(B)$,

ii) $\mathscr{D}(\Xi_{\mathfrak{N}}) \subset D(B^*)$,

iii) $B\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset D(A)$,

iv) $ABf = BAf, \forall f \in \mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}})$.

Then, for every $\Delta \in \Sigma_{\mathfrak{N}}$ and $h \in \mathscr{D}$, one has $BE(\Delta)h = E(\Delta)Bh$. ■

The rest of the discussion of Section III goes through as well, replacing everywhere $\{ A \}'_{\bullet}$, $\{ A \}'_{sp}$ by $\mathfrak{N}'_{\bullet}$, $\mathfrak{N}'_{sp}$ respectively. We simply reformulate the last two theorems.

THEOREM 4.4. — Let $\mathfrak{N}$, B, $\mathscr{D}(\Xi_{\mathfrak{N}})$ and $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}})$ be as above. Assume conditions *i*), *iii*) and *iv*) of Theorem 4.3, and in addition:

ii') $\mathscr{D}(\Xi_{\mathfrak{N}}) \subset \mathscr{D}$.

Then, for every $\Delta \in \Sigma_{\mathfrak{N}}$, one has $E(\Delta) \bullet B = B \bullet E(\Delta)$. ■

THEOREM 4.5. — Assume that $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset \mathscr{D}$. Then $\mathfrak{N}'_{sp} = \mathfrak{N}'_{\bullet}$. ■

The comments made at the end of Section III apply here too. Again the condition $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset \mathscr{D}$ is a natural one to impose on the domain $\mathscr{D}$, for the same reasons. Examples are easy to give, using for instance the concrete situations described in [*13*]. Besides the complete sets of commuting operators $\{ x_1, x_2, x_3 \}$ and $\{ p_1, p_2, p_3 \}$, which are obvious for a quantum mechanical system, the case of $\{ H, \mathbf{L}^2, L_Z \}$ is easy to treat also, especially when the Hamiltonian H has a purely discrete spectrum $\{ E_n \}$. For instance, the following domains verify our conditions, for any triple of positive integers r, s, t:

$$\mathscr{D}_{rst} = \{ \psi = \Sigma_{nlm} a_{nlm} \psi_{nlm} : \Sigma_{nlm} | a_{nlm} |^2 | E_n |^r [l(l+1)]^s m^t < \infty \}.$$

The philosophy is always the same: the domain $\mathscr{D}$ must be well-adapted to the set $\mathfrak{N}$.

Theorem 4.5 answers our original question for the case of a *finite* set $\mathfrak{N}$ of operators. The last step is to extend the analysis to the (partial) *-algebra generated by $\mathfrak{N}$, and this will be done in Section V.

V. OUTCOME : AN ABELIAN PARTIAL Op*-ALGEBRA

For simplicity, we restrict ourselves to $N = 2$ and real-valued functions φ_1, φ_2 such that $\hat{E}(\varphi_1)$, $\hat{E}(\varphi_2)$ are self-adjoint and belong to $\mathfrak{C}(\mathcal{D})$. We assume throughout that $\mathcal{D} \supset \mathcal{D}^{\mathcal{P}}(\Xi_{\mathfrak{N}})$, where $\mathfrak{N} = \{ \hat{E}(\varphi_1), \hat{E}(\varphi_2) \}$.

Take first the self-adjoint operator $\hat{E}(a\varphi_1 + b\varphi_2)$, $a, b \in \mathbb{R}$. By standard results [*18*], [*20*], we have $\hat{E}(a\varphi_1 + b\varphi_2) \supset a\hat{E}(\varphi_1) + b\hat{E}(\varphi_2)$ and $D(\hat{E}(\varphi_1)) \cap D(\hat{E}(\varphi_2)) \supset \mathcal{D} \supset \mathcal{D}^{\mathcal{P}}(\Xi_{\mathfrak{N}})$. Since $\mathcal{D}^{\mathcal{P}}(\Xi_{\mathfrak{N}})$ is an invariant domain of analytic vectors for $\hat{E}(a\varphi_1 + b\varphi_2)$, it is a core, hence $\mathcal{D}$ is also a core, and therefore $\hat{E}(a\varphi_1 + b\varphi_2) \in \mathfrak{C}(\mathcal{D})$, i. e.:

$$\hat{E}(a\varphi_1 + b\varphi_2) = a\hat{E}(\varphi_1) \dotplus b\hat{E}(\varphi_2) \tag{5.1}$$

Similarly, one has

$$\hat{E}(\varphi_1\varphi_2) \supset \hat{E}(\varphi_1)\hat{E}(\varphi_2) \text{ and } D(\hat{E}(\varphi_1)\hat{E}(\varphi_2)) = D(\hat{E}(\varphi_2)) \cap D(\hat{E}(\varphi_1\varphi_2)).$$

Observe that the following three conditions are equivalent:

i) $$\hat{E}(\varphi_2) \in L^s(\hat{E}(\varphi_1)) \tag{5.2a}$$

ii) $$\hat{E}(\varphi_1) \in L^s(\hat{E}(\varphi_2)) \tag{5.2b}$$

iii) $$\mathcal{D} \subset D(\hat{E}(\varphi_1\varphi_2)). \tag{5.2c}$$

If they hold, we get $D(\hat{E}(\varphi_1)\hat{E}(\varphi_2)) \supset \mathcal{D} \supset \mathcal{D}^{\mathcal{P}}(\Xi_{\mathfrak{N}})$ and as above this implies that $\mathcal{D}$ is a core for $\hat{E}(\varphi_1, \varphi_2)$, i. e. $\hat{E}(\varphi_1\varphi_2) \in \mathfrak{C}(\mathcal{D})$ and therefore:

$$\hat{E}(\varphi_1\varphi_2) = \hat{E}(\varphi_1) \bullet \hat{E}(\varphi_2) = \hat{E}(\varphi_2) \bullet \hat{E}(\varphi_1) \tag{5.3}$$

In particular, *if* $\mathcal{D} \subset D(\hat{E}(\varphi^2))$, with $\varphi = \varphi_1$ or φ_2, we get:

$$\hat{E}(\varphi^2) = [\hat{E}(\varphi)]^2 = \hat{E}(\varphi) \bullet \hat{E}(\varphi) \tag{5.4}$$

Notice that the first equality is always true. Similar relations hold for all powers: if the k-fold product $\hat{E}(\varphi) \bullet \ldots \bullet \hat{E}(\varphi)$ exists, it coincides with $[\hat{E}(\varphi)]^k = \hat{E}(\varphi)^k$. Such products are associative, as already noted in Ref. [*3*].

Clearly all those results extend to complex-valued functions φ_j $(j = 1, 2)$, which yield normal operators $\hat{E}(\varphi_j)$, as well as to any *finite* number N of operators $\hat{E}(\varphi_j)$.

As observed in Section 2, all self-adjoint and normal operators in $\mathfrak{C}(\mathcal{D})$ are standard. Furthermore, if A and B are standard, one has $A \bullet B = A \square B$, where $\square$ denotes the weak partial multiplication defined in Ref. [*3*]. Then the discussion above shows that all operators $\hat{E}(\varphi_j)$ are standard, and so are their $\dotplus$-sums and their $\bullet$- or $\square$-products. It follows that $\mathfrak{N} = \{ \hat{E}(\varphi_1), \ldots, \hat{E}(\varphi_N) \}$ generates a partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$, namely the smallest one containing $\mathfrak{N}$ (with the $\bullet$ multiplication only, difficulties may arise with distributivity, see Add./Err. to Ref. [*3*]). As in Example 3.5 of Ref. [*3*], $\mathfrak{M}[\mathfrak{N}]$ is a *polynomial* partial Op*-algebra, consisting of all polynomials in

$\hat{E}(\varphi_1), \ldots, \hat{E}(\varphi_N)$ which are well-defined with respect to the $\bullet$ multiplication. If P_j is such a polynomial, the existence condition is that $D(P_j(\varphi_1, \ldots, \varphi_N)) \supset \mathscr{D}$. Then we have:

$$\hat{P}_j(\hat{E}(\varphi_1), \ldots, \hat{E}(\varphi_N)) = \hat{E}(P_j(\varphi_1, \ldots, \varphi_N)) \tag{5.5}$$

where $\hat{P}_j$ denotes the polynomial P_j evaluated with the operations $\bullet$ and $\hat{+}$. Clearly the operator in (5.5) is normal, which means that the partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$ itself is *standard*, in the sense of Ref. 3.

The partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$ is also Abelian, in a sense that we will specify now. An abstract partial *-algebra $\mathfrak{A}$ is naturally called *Abelian* if the following two conditions hold:

i) $R(x) = L(x)$, for all $x \in \mathfrak{A}$;
ii) $x.y = y.x$, for all $x \in \mathfrak{A}$, $y \in R(x)$.

For a partial Op*-algebra as considered here, there is an alternative definition, namely $\mathfrak{A} \subset \mathfrak{A}'_\sigma$, or explicitly:

$$\langle B^+f, Ag \rangle = \langle A^+f, Bg \rangle, \qquad \forall f, g \in \mathscr{D}, \quad \forall A, B \in \mathfrak{U} \tag{5.6}$$

As it is easily seen, the inclusion $\mathfrak{A} \subset \mathfrak{A}'_\sigma$ implies that $\mathfrak{A}$ is Abelian, but not the converse in general. We refer to a further publication [*14*] for a systematic analysis of those questions.

For the partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$, the relation (5.6) is readily verified, namely:

$$\langle \hat{E}(\overline{\varphi_1})f, \hat{E}(\varphi_2)g \rangle = \langle \hat{E}(\overline{\varphi_2})f, \hat{E}(\varphi_1)g \rangle, \qquad \forall f, \quad g \in \mathscr{D}. \tag{5.7}$$

Hence we may state:

Proposition 5.1. — Let $\mathfrak{N} = \{ \hat{E}(\varphi_1), \ldots, \hat{E}(\varphi_N) \}$. Assume that $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset \mathscr{D}$. Then the partial Op*-algebra $\mathfrak{M}[\mathfrak{N}]$ generated by $\mathfrak{N}$ consists of all allowed polynomials in $\hat{E}(\varphi_1), \ldots, \hat{E}(\varphi_N)$. It is standard and Abelian, and furthermore $\mathfrak{M}[\mathfrak{N}] \subset \mathfrak{M}[\mathfrak{N}]'_\sigma$. A given polynomial P_j is allowed iff $\mathscr{D} \subset D(P_j(\varphi_1, \ldots, \varphi_N))$, and all relations (5.1)-(5.5) hold true. ■

Finally we come back to the comparison of the present results with those obtained previously, using the language of V*-algebras. Let again $\mathfrak{N} = \{ A_1, \ldots, A_N \}$, $A_j = \hat{E}(\varphi_j)$. From the results of Ref. 13, one infers that $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset D^\infty(\mathfrak{N}) = \cap_{j=1}^N D^\infty(A_j)$, both domains are dense in $\mathscr{H}$ and a common core for $A_1, \ldots, A_N$, and the restriction of $\mathfrak{N}$ to each of them is an Op*-algebra. In addition, $\mathfrak{N}^\infty \equiv \mathfrak{N} \upharpoonright D^\infty(\mathfrak{N})$ is standard, self-adjoint and completely regular (that is $(\mathfrak{N}^\infty)''_{w\sigma} \subset \mathscr{L}^+(\mathscr{D})$) and $\mathfrak{B} \equiv (\mathfrak{N}^\infty)''_{\sigma\sigma}$ is an SV*-algebra ($\mathfrak{B} = \mathfrak{B}''_{w\sigma}$).

If we take now $\mathfrak{N} \subset \mathfrak{C}(\mathscr{D})$ and make the natural assumption that $\mathscr{D} \subset D^\infty(\mathfrak{N})$, then *all* polynomials $P(A_1, \ldots, A_N)$ are well-defined on $\mathscr{D}$ and are normal; let $\mathfrak{P}[\mathfrak{N}]$ denote their set. It follows that $\mathfrak{P}[\mathfrak{N}]$ is a *-algebra for the weak multiplication $\square$, which coincides here with the ordinary operator product. However, this need not be the case for the $\bullet$ multiplication. $\mathscr{D}$ is not necessarily a core for every polynomial $P(A_1, \ldots, A_N)$, hence the $\bullet$ product of two such polynomials need not be always defined, i. e. $\mathfrak{P}[\mathfrak{N}]$ is only a partial Op*-algebra.

All those pathologies disappear if we add our standing assumption, thus getting $\mathscr{D}^{\mathscr{P}}(\Xi_{\mathfrak{N}}) \subset \mathscr{D} \subset D^{\infty}(\mathfrak{N})$: then $\mathscr{D}$ *is* indeed a core for every $P(A_1, \ldots, A_N)$, and therefore $\mathfrak{M}[\mathfrak{N}] = \mathfrak{P}[\mathfrak{N}]$ is an Abelian, standard Op*-algebra on $\mathscr{D}$ as well, for both $\bullet$ and $\square$, which now coincide.

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