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Functional integration for Euclidean Dirac fields

by

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ABSTRACT. — An anticommutative integration is defined with a Gaussian measure and it is applied to the Euclidean quantum field theory of Dirac spinors. The Euclidean Dirac field is realized as random spinor with eight components, four of which are independent. This spinor is an element of a Grassmann algebra and the measure is defined on the Hilbert space which generates the field algebra. The Schwinger functions are integrals of polynomials on this space.

RÉSUMÉ. — Une intégration anticommutative est définie à l'aide d'une mesure gaussienne et est appliquée à la théorie quantique des champs euclidéens des spineurs de Dirac. Le champ euclidéen de Dirac est représenté comme un spineur aléatoire à huit composantes dont quatre sont indépendantes. Ce spineur est un élément d'une algèbre extérieure, et la mesure gaussienne est définie sur l'espace de Hilbert qui génère cette algèbre. Les fonctions de Schwinger peuvent être calculées comme intégrales de polynômes sur cette mesure.

1. INTRODUCTION

A Euclidean quantum field theory of fermions is essentially given by the exterior algebra of its Schwinger functions. In the case of free fields the symplectic form of the two-point functions is the basic building block, its exponential in the form algebra generates all n-point functions. To study a Euclidean theory of Fermi fields it is therefore sufficient to investigate this form algebra, see e. g. [1], [2], and an anticommutative integration has been developed on an algebraic basis as an effective tool [3], [4]. But due to the success of probabilistic methods for boson theories there have been many attempts to incorporate measure theoretic ideas also in a Euclidean theory for fermions, see e. g. [1], [5], [6] which use the gage space approach of Segal [7].

In this article an alternative and simpler form of anticommutative integration with a positive Gaussian measure is presented. In [8] the measure space was the Fock space of antisymmetric tensors. This measure is now reduced to a Gaussian measure on the « one particle » space. The anticommutativity originates from the structure of the algebra of integrable functions (and not from a formal anticommutativity of integration variables). The methods used here have some similarity with the representation of fermionic stochastic processes in [9] and [10].

In the fermionic case the Schwinger functions are not directly given by the moments of the measure, and in this paper no calculatorial results are derived which cannot be obtained by the algebraic methods as used in [1]-[4]. The consequences of the integral representations need further investigation. The main emphasis of this paper is the proof that a probabilistic interpretation of the Euclidean quantum field theory of fermions exists which is simpler than the gage space approach (and which is more closely related to the Berezin integral). This interpretation allows for a natural definition of Euclidean fields as the elements of the measure space (with the correct degrees of freedom) and it gives a better understanding of some problems of Euclidean field theory. E. g. the doubling of fields in [11] corresponds to correlated Gaussian variables. For vanishing mass the measure for Dirac fermions factorizes and Euclidean Weyl spinors can be defined unambigously.

In the first part of this paper the essential properties of the fermionic integration are presented. The Sects. 2 and 3 introduce the basic notations and normalizations of skew symmetric forms and exterior algebras (Grassmann algebras) on an infinitely dimensional Hilbert space, thereby an error of Ref. [8] is corrected. This theory is extended to a triplet of exterior algebras with weaker continuity conditions in Sect. 4. The measure theoretic formulation of the fermionic integral is then given in Sect. 5.

To apply this integration to the Euclidean quantum field theory of Dirac spinors one needs a positive quadratic form which is not immediately given by the Schwinger functions. To supplement the symplectic form of the two-point function with a positive form causes problems which are well known from the construction of Euclidean quantum field operators on a Fock space [11], [12], and which make the whole theory technically more involved. The algebraic part of this problem is already presented in Sect. 2.

The Euclidean quantum field theory of free Dirac fermions is then given in Sect. 6. The Euclidean Dirac field is an eight-component Gaussian random spinor in an exterior algebra. Only four of the spinor components are independent variables. An equivalent formulation is given by a pair of correlated four-component spinors. In Sect. 7 the existence of Euclidean Weyl spinors is derived, and in Sect. 8 it is shown that the probabilistic approach is meaningful also in the case of interactions with bosonic fields.

Finally a few remarks about the notations used in this paper. For any complex quantity the bar $\xi \rightarrow \bar{\xi}$ indicates complex conjugation, including spinors, etc. The positive definite inner product of a (complex) Hilbert space $\mathscr{H}$ is written as $(f \mid g)$. An operator A on $\mathscr{H}$ is denoted as positive operator of $(f \mid \mathrm{A}f) \geq 0$ for all $f \in \mathscr{H}$, the eigenvalue zero is allowed. Likewise a sesquilinear form $\gamma(f, g)$ is denoted as positive if $\gamma(f, f) \geq 0$ for all $f \in \mathscr{H}$, the degenerate case $\gamma(f, f) = 0$ for $f \neq 0$ admitted.

2. SKEW SYMMETRIC FORMS

The two point Schwinger functions of a fermionic field theory define a bilinear skew symmetric form on a test function space which can be chosen as Hilbert space. In theories of Weyl or Dirac spinors the test functions have to be complex valued. The starting point is therefore a complex Hilbert space endowed with a skew symmetric form.

Let $\mathscr{H}$ be a complex separable Hilbert space with the inner product $(f \mid g)$. The following structure of $\mathscr{H}$ is assumed.

i) There exists an antiunitary involution $f \in \mathscr{H} \rightarrow f^* \in \mathscr{H}$ with $(f^* \mid g^*) = \overline{(f \mid g)}$ and $f^{**} = f$.

ii) The space $\mathscr{H}$ can be split into two orthogonal isomorphic subspaces $\mathscr{H} = \mathscr{E} \oplus \mathscr{E}^*$ such that $f \in \mathscr{E} \curvearrowright f^* \in \mathscr{E}^*$ and vice versa.

Then

$$\langle f \mid g \rangle = (f^* \mid g) \tag{2.1}$$

is a bilinear symmetric form on $\mathscr{H}$.

Any continuous bilinear skew symmetric form $\omega(f, g)$ on $\mathscr{H}$ admits a decomposition of $\mathscr{H}$ into isotropic subspaces which after a unitary transformation may be identified with $\mathscr{E}$ and $\mathscr{E}^*$ (or with subspaces of $\mathscr{E}$ and $\mathscr{E}^*$ if ω is degenerate). Therefore any continuous bilinear skew symmetric form on $\mathscr{H}$ is equivalent to a form ω_{M} which is constructed as follows. If M is a bounded linear operator on $\mathscr{E}$ then there exist unique extensions $\mathrm{M}_{\pm}$ on $\mathscr{H}$ such that M_{+} is symmetric and M_{-} is skew symmetric with respect to the form (2.1)

$$\begin{cases} \mathrm{M}_{\pm} f = \mathrm{M}f & \text{if} \quad f \in \mathscr{E} \\ \mathrm{M}_{\pm} f = \pm (\mathrm{M}^{+} f^*)^* & \text{if} \quad f \in \mathscr{E}^* . \end{cases} \tag{2.2}$$

A continuous bilinear skew symmetric form is then defined by

$$\omega_M(f, g) = \langle f \mid M_- g \rangle \tag{2.3}$$

for $f, g \in \mathscr{H}$. This form is uniquely determined by $\omega(f^*, g) = (f \mid Mg)$ if $f, g \in \mathscr{E}$, since $\mathscr{E}$ and $\mathscr{E}^*$ are isotropic subspaces.

For the identity mapping $M = I$ the form $\omega_I(f, g)$ is the canonical symplectic form of $\mathscr{H} = \mathscr{E} \oplus \mathscr{E}^*$. If M is invertible then ω_M is non-degenerate and $\mathscr{H} = \mathscr{E} \oplus \mathscr{E}^*$ is a maximal isotropic decomposition of $\mathscr{H}$ with respect to ω_M.

Given a continuous non-degenerate skew symmetric form $\omega(f, g)$ on $\mathscr{H}$ then there exist uniquely a positive form $\beta(f, g)$ and an antiunitary operator j with $j^2 = -$ id, in the following denoted as conjugation, such that

$$\omega(f, g) = \beta(jf, g). \tag{2.4}$$

This statement is a modification of a theorem of Chernoff and Marsden (derived in [13], § 1.2 for real Hilbert spaces). The conjugation j is antiunitary with respect to the norm of $\mathscr{H}$ and with respect to β.

If the skew symmetric form $\omega_S(f, g)$ is defined by (2.3) with an invertible bounded operator S the related positive form β can be easily calculated using the polar decomposition of S on ε

$$S = UA_1 = A_2U \tag{2.5}$$

where U is unitary and A_1 and A_2 are positive operators. The positive form is then

$$\beta_S(f, g) = (x \mid A_1 y) + \overline{(u^* \mid A_2 v^*)} \tag{2.6}$$

if $f = x + u$ and $g = y + v$ with $x, y \in \mathscr{E}$ and $u, v \in \mathscr{E}^*$

and the conjugation is given by

$$j_S f = (U^+)_- f^*. \tag{2.7}$$

Only if $S = A$ is a positive operator the relation between ω_A and β_A or j_A is simple and j_A is independent of A and canonically related to the involution of $\mathscr{H}$. With these definitions the identity (2.4) remains valid also if Ker S is non trivial and ω_S is degenerate.

In the case of Euclidean Dirac fields the skew symmetric form ω_S is derived from a non-hermitean operator S and the conjugation (2.7) depends on the details of the operator S. One can circumvent this problem by an extension of $\mathscr{H}$ to $\hat{\mathscr{H}} = \mathscr{H}^{(1)} \oplus \mathscr{H}^{(2)}$ formed by two copies of $\mathscr{H}$.

Let $\hat{\mathscr{E}} = \mathscr{E}^{(1)} \oplus \mathscr{E}^{(2)}$ be the direct sum of two copies of $\mathscr{E}$ and $\hat{\mathscr{E}}^* = \mathscr{E}^{(1)*} \oplus \mathscr{E}^{(2)*}$ be the dual space, then $\hat{\mathscr{H}} = \hat{\mathscr{E}} \oplus \hat{\mathscr{E}}^*$ has again the structure assumed at the beginning of this section with a canonical involution $\hat{f} \in \hat{\mathcal{H}} \to \hat{f}^* \in \hat{\mathcal{H}}$ which maps $\hat{\mathcal{E}}$ into $\hat{\mathcal{E}}^*$ and vice versa. The operator (2.5) S on $\mathcal{E}$ is now extended to the positive operator

$$M = \begin{pmatrix} A_2 & S \\ S^+ & A_1 \end{pmatrix} \tag{2.8}$$

on $\hat{\mathcal{E}}$ with the non trivial kernel

$$\mathcal{F} = \ker M = \left\{ \hat{f} = \begin{pmatrix} Uf \\ -f \end{pmatrix}, f \in \mathcal{E} \right\} \subset \hat{\mathcal{E}} \tag{2.9}$$

which is isomorphic to $\mathcal{E}$.

The skew symmetric form $\omega_M(\hat{f}, \hat{g})$ defined on $\hat{\mathcal{H}}$ following the definition (2.3) is degenerate, but on the factor space $\hat{\mathcal{H}}/\mathcal{F} \oplus \mathcal{F}^*$ it is equivalent to the form $\omega_S(f, g)$. The advantage of using ω_M comes from the positivity of M. The relations (2.6) and (2.7) yield

$$\omega_M(\hat{f}, \hat{g}) = \beta_M(\mathcal{J}\hat{f}, \hat{g}) \tag{2.10}$$

with the positive form

$$\beta_M(\hat{f}, \hat{g}) = (\hat{f} \mid M_+ \hat{g}) \tag{2.11}$$

and the simple conjugation

$$\mathcal{J}\hat{f} = \begin{cases} -\hat{f}^* & \text{if } \hat{f} \in \hat{\mathcal{E}} \\ \hat{f}^* & \text{if } \hat{f} \in \hat{\mathcal{E}}^* . \end{cases} \tag{2.12}$$

3. EXTERIOR ALGEBRA AND GAUSSIAN FORMS

The Fock space of skew symmetric tensors of the Hilbert space $\mathcal{H}$ is written as

$$\mathcal{A}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n(\mathcal{H}) \tag{3.1}$$

where $\mathcal{A}_n(\mathcal{H})$ is the Hilbert space of skew symmetric tensors of degree n. On $\mathcal{A}_n(\mathcal{H})$ the exterior product of vectors is normalized to

$$(f_1 \wedge \ldots \wedge f_n | g_1 \wedge \ldots \wedge g_n) = \det((f_i | g_j)) \quad \text{if} \quad f_i, g_j \in \mathcal{H}, \quad i, j = 1, \ldots, n . \tag{3.2}$$

By linearity the exterior product is extended to the subspace of all tensors of finite rank

$$\mathcal{A}^0(\mathcal{H}) = \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^{N} \mathcal{A}_n(\mathcal{H}) \tag{3.3}$$

which is an algebra with respect to the exterior product and it is a dense subset of the Fock space $\mathscr{A}(\mathscr{H})$. The exterior product cannot be extended to a continuous operation on the whole space $\mathscr{A}(\mathscr{H})$, but nevertheless $\mathscr{A}(\mathscr{H})$ will be denoted as the exterior algebra of the Hilbert space $\mathscr{H}$ (1).

If the Hilbert space $\mathscr{H}$ has the structure as assumed in Sect. 2 the involution can be extended to an involution on $\mathscr{A}(\mathscr{H})$ such that $F^{**} = F$ and $\| F^* \| = \| F \|$ for all $F \in \mathscr{A}(\mathscr{H})$ and $(F \wedge G)^* = G^* \wedge F^*$ if the exterior product exists.

The form (2.1) can be generalized to a continuous bilinear symmetric and non degenerate form on $\mathscr{A}(\mathscr{H}) : \langle F \mid G \rangle = (F^* \mid G)$.

With the normalization (3.2) inner products of tensors $F \wedge G$, where F and G belong to orthogonal subalgebras, factorize, in particular

$$\langle F_1^* \wedge F_2 \mid G_1^* \wedge G_2 \rangle = \langle F_2 \mid G_1^* \rangle \langle F_1^* \mid G_2 \rangle \tag{3.4}$$

is valid for arbitrary F_1, F_2, G_1, $G_2 \in \mathscr{A}(\mathscr{E})$. The exterior algebra (3.1) therefore factorizes into $\mathscr{A}(\mathscr{H}) = \mathscr{A}(\mathscr{E}) \wedge \mathscr{A}(\mathscr{E}^*)$.

A continuous bilinear or sesquilinear form $\gamma(f, g) = \langle f \mid Ag \rangle$ or $(f \mid Ag)$ is denoted as a Hilbert-Schmidt form—HS form—if the operator A is a HS operator, it is denoted as nuclear form if A is a nuclear operator.

An important property of skew symmetric HS forms is

LEMMA 1. — The skew symmetric form ω is a HS form if and only if it can be represented as

$$\omega(f, g) = \langle \Omega \mid f \wedge g \rangle \tag{3.5}$$

with a tensor $\Omega \in \mathscr{A}_2(\mathscr{H})$.

The proof is obvious.

For any $\Omega \in \mathscr{A}_2(\mathscr{H})$ the exponential series

$$\exp \Omega = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n} \tag{3.6}$$

converges within $\mathscr{A}(\mathscr{H})$. The proof is given at the end of Sect. 4.

The linear functional

$$L(F) = \langle \exp \Omega \mid F \rangle \tag{3.7}$$

which is the exponential of a skew symmetric form (3.5) within the form algebra generates the moments of an anticommutative Gaussian distribution

$$L(f_1 \wedge \ldots \wedge f_{2n}) = \frac{1}{2^n n!} \sum_{I_{2n}} \varepsilon_I \omega(f_{i_1}, f_{i_2}) \ldots \omega(f_{i_{2n-1}}, f_{i_{2n}}) \tag{3.8}$$

(1) In Ref. [8] the estimate (2.4) is false, and also the statements that the spaces $\mathscr{T}(\mathscr{H})$ and $\mathscr{G}(\mathscr{H})$ are Banach or Hilbert algebras are not correct. But the results of Sect. 2.2-7 of Ref. [8] remain unaffected. I am indebted to P. A. Meyer for pointing out these errors to me.

with $f_k \in \mathscr{H}$, $k = 1, 2, \ldots, 2n$, the summation is extended over all permutations $I_{2n} = (i_1, \ldots, i_{2n})$ of $(1, 2, \ldots, 2n)$. This is exactly the representation of the $2n$-point function of a free Fermi field in terms of the two-point function. The relation of this functional to the formalism of Berezin [3] is explicated in Refs. [2] and [8].

Until the end of this section the skew symmetric forms are taken as HS forms (2.3) ω_M generated by HS operators M on $\mathscr{E}$. The tensor Ω is then denoted by $\Omega(M)$ and the functional (3.8) is written as L_M.

Since $\mathscr{E}$ and $\mathscr{E}^*$ are isotropic spaces of ω_M the functional (3.8) simplifies to

$$L_M(F^* \wedge G) = (F \mid \Gamma(M)G) \quad \text{if} \quad F, G \in \mathscr{A}(\mathscr{E}). \tag{3.9}$$

Here $\Gamma(M)$ is the continuous linear operator on $\mathscr{A}(\mathscr{E})$ which is generated by M: $\Gamma(M) f_1 \wedge \ldots \wedge f_n = (Mf_1) \wedge \ldots \wedge (Mf_n)$.

The operator $\Gamma(M)$ is selfadjoint/HS/nuclear on $\mathscr{A}(\mathscr{E})$ if M is selfadjoint/HS/nuclear on $\mathscr{E}$. If M is a positive operator then $\Gamma(M)$ is also positive and from (3.9) follows the inequality

$$L_M(F^* \wedge F) \geqslant 0 \quad \text{if} \quad F \in \mathscr{A}(\mathscr{E}). \tag{3.10}$$

This inequality cannot be extended to $F \in \mathscr{A}(\mathscr{H})$. There is even the more general result: if L is a continuous linear functional on $\mathscr{A}(\mathscr{H})$ such that $L(F^* \wedge F) \geq 0$ is valid for all $F \in \mathscr{A}(\mathscr{H})$ then L is the trivial functional $L(F) = \alpha \langle 1 \mid F \rangle$ with $\alpha \geq 0$.

4. TRIPLETS OF HILBERT SPACES

In this section the limitations due to the HS condition will be removed. The mathematical techniques used here are well known, an extensive presentation can be found in [14], see also [15]. A detailed theory of forms on triplets of symmetric or exterior algebras has been developed by Krée [4].

Let T be an invertible HS operator on $\mathscr{H}$ then the spaces $\mathscr{H}_\alpha = T^\alpha \mathscr{H}$, $\alpha \in \mathbb{R}$, are (after closure) Hilbert spaces with the norms $\| f \|_\alpha = \| T^{-\alpha} f \|$. The antidual space of $\mathscr{H}_\alpha$ with respect to the pairing given by the inner product of $\mathscr{H}$ is exactly $\mathscr{H}_{-\alpha}$.

The exterior algebras $\mathscr{A}(\mathscr{H}_\alpha) = \Gamma(T^\alpha)\mathscr{A}(\mathscr{H})$, $\alpha \in \mathbb{R}$, are defined as in Sect. 3 with the norm induced by $\| f \|_\alpha$. For $\alpha > 0$ the spaces

$$\mathscr{A}(\mathscr{H}_\alpha) \subset \mathscr{A}(\mathscr{H}) \subset \mathscr{A}(\mathscr{H}_{-\alpha})$$

are ordered by inclusion and $\mathscr{A}(\mathscr{H}_\alpha)$ and $\mathscr{A}(\mathscr{H}_{-\alpha})$ are antidual spaces with the pairing given by the inner product of $\mathscr{A}(\mathscr{H})$. The symplectic structure of the Hilbert space $\mathscr{H} = \mathscr{E} \oplus \mathscr{E}^*$ as assumed in Sect. 2 is naturally transferred to the exterior algebras $\mathscr{A}(\mathscr{H}_\alpha) = \mathscr{A}(\mathscr{E}_\alpha) \wedge \mathscr{A}(\mathscr{E}_\alpha^*)$.

The consequences for bilinear forms are special versions of the Kernel Theorem [14], [15]. The Lemma 1 of Sect. 3 has the obvious generalization in

LEMMA 2. — The bilinear skew symmetric form $\omega(f, g)$ is a HS form in the norm of $\mathscr{H}_\alpha$ if and only if there exists a tensor $\Omega \in \mathscr{A}_2(\mathscr{H}_{-\alpha})$ such that

$$\omega(f, g) = \langle \Omega \mid f \wedge g \rangle \quad \text{for} \quad f, g \in \mathscr{H}_\alpha . \tag{4.1}$$

A form ω_M derived from a bounded operator M on $\mathscr{E}_\alpha$ is continuous on $\mathscr{H}_\alpha$ and consequently HS on $\mathscr{H}_{\alpha+\frac{1}{2}}$. The representation (4.1) is therefore valid with $\Omega(M) \in \mathscr{A}_2(\mathscr{H}_{-\alpha-\frac{1}{2}})$ and the exponential series exp Ω converges within $\mathscr{A}(\mathscr{H}_{-\alpha-\frac{1}{2}})$. The linear functional (3.7) is then defined on $\mathscr{A}(\mathscr{H}_{\alpha+\frac{1}{2}})$

$$L_M(H) = \langle \exp \Omega(M) \mid H \rangle = (\exp \Omega(M^+) \mid H) \quad \text{with} \quad H \in \mathscr{A}(\mathscr{H}_{\alpha+\frac{1}{2}}). \tag{4.2}$$

The relation (3.9) can be derived on $\mathscr{A}(\mathscr{H}_{\alpha+\frac{1}{2}})$ and continuity allows to extend (3.10) again to $\mathscr{A}(\mathscr{E}_\alpha)$.

The tensors $\Omega(M_1) \in \mathscr{A}_2(\mathscr{H}_\alpha)$ and $\Omega(M_2) \in \mathscr{A}_2(\mathscr{H}_\beta)$ are elements of dual spaces, i. e. $\beta = -\alpha$, if and only if the product $M_1 M_2$ is a nuclear operator on $\mathscr{E}$. The calculation of

$$L_{M_1}(\exp \Omega(M_2)) = \langle \exp \Omega(M_1) \mid \exp \Omega(M_2) \rangle = \\ = \mathrm{Tr}_{\mathscr{A}(\mathscr{E})} \Gamma(M_1) \Gamma(M_2) = \det_{\mathscr{E}}(I + M_1 M_2) \tag{4.3}$$

see e. g. [8] Sect. 4, yields the well known fermionic determinant. If $\Omega(M) \in \mathscr{A}_2(\mathscr{H})$ then M is a HS operator on $\mathscr{E}$ and

$$\| \exp \Omega(M) \|^2 = \det (I + M^+ M)$$

gives a proof of the convergence of the exponential series (3.6).

5. MEASURES AND INTEGRALS

In this and the following Sections it is assumed that M is a positive bounded operator on $\mathscr{E}$. Then the sesquilinear form $(f \mid Mg)$ is nuclear on $\mathscr{E}_1$ and $(F \mid \Gamma(M) G)$ is a positive nuclear form on $\mathscr{A}(\mathscr{E}_1)$. The Theorem of Minlos [14], [15], [16] then guarantees the existence of a Gaussian measure on $\mathscr{A}(\mathscr{E}_{-1})$ such that

$$\int_{\mathscr{A}(\mathscr{E}_{-1})} e^{i(F|\phi) + i\overline{(G|\phi)}} d\rho(\phi) = e^{-(F|\Gamma(M)G)} \tag{5.1}$$

if $F, G \in \mathscr{A}(\mathscr{E}_1)$. More precisely, this measure is a Gaussian measure on the underlying real space $\mathscr{A}(\mathscr{E}_{-1})_{\mathbb{R}} \cong \mathscr{A}(\mathscr{E}_{-1\mathbb{R}})$.

The relation (5.1) implies that

$$\int_{\mathscr{A}(\mathscr{E}_{-1})} (\mathrm{F} \mid \phi)\overline{(\mathrm{G} \mid \phi)}d\rho(\phi) = (\mathrm{F} \mid \Gamma(\mathrm{M})\mathrm{G}) \tag{5.2}$$

if $\mathrm{F}, \mathrm{G} \in \mathscr{A}(\mathscr{E}_1)$. The integrant can be written as, cf. Eq. (3.4), $\langle \mathrm{F}^* \wedge \mathrm{G} \mid \phi^* \wedge \phi \rangle$. Since any element of $\mathrm{H} \in \mathscr{A}(\mathscr{H}_1)$ can be represented as $\mathrm{H} = \sum_n \mathrm{F}_n^* \wedge \mathrm{G}_n$ with $\mathrm{F}_n, \mathrm{G}_n \in \mathscr{A}(\mathscr{E}_1)$ the relations (3.9) and (5.2) yield that the linear functional (4.3) coincides with the integral

$$\mathscr{I}_{\mathrm{M}}(\mathrm{H}) = \int_{\mathscr{A}(\mathscr{E}_{-1})} \langle \mathrm{H} \mid \phi^* \wedge \phi \rangle \, d\rho(\phi) \tag{5.3}$$

for any $\mathrm{H} \in \mathscr{A}(\mathscr{H}_1)$. This identity is equivalent to

$$\int_{\mathscr{A}(\mathscr{E}_{-1})} \phi^* \wedge \phi d\rho(\phi) = \exp \Omega(\mathrm{M}) \,. \tag{5.4}$$

A necessary and sufficient condition to identify a linear functional (4.3) with an integral (5.3) is the positivity of M, i. e. the positivity condition (3.10) has to be fulfilled for all $\mathrm{F} \in \mathscr{A}(\mathscr{E}_1)$.

If $\mathrm{H} \in \mathscr{A}(\mathscr{H}_1)$ the integrant in (5.3) is a continuous function on the measure space $\mathscr{A}(\mathscr{E}_{-1})$. The class of integrable functionals in larger, it includes the algébra of functions $\{ \langle \mathrm{H} \mid \phi^* \wedge \phi \rangle \mid \mathrm{H} \in \mathscr{A}(\mathscr{H}_{\frac{1}{2}}) \}$ which corresponds to the argument in (4.2) for $\alpha = 0$.

The projection operator $\mathscr{A}(\mathscr{E}_{-1}) \to \mathscr{E}_{-1}$ induces a measure μ on $\mathscr{E}_{-1}$ with the Fourier-Laplace transform

$$\int_{\mathscr{E}_{-1}} e^{i(f\mid\varphi) + i\overline{(g\mid\varphi)}} d\mu(\varphi) = e^{-(f\mid \mathrm{M}g)} \,. \tag{5.5}$$

The main properties of the measure ρ can already be derived from the simpler measure μ. But one can go an essential step further and transfer the integration on the Fock space to an integration on the « one particle » space $\mathscr{E}_{-1}$. The following construction has some similarity with the representation of fermionic stochastic processes in refs. [*9*] and [*10*].

Let $\{ e_i^\alpha \}_{i\in\mathbb{N}}$ be a CONS of $\mathscr{E}_\alpha$ such that $(e_i^\alpha \mid e_j^{-\alpha}) = \delta_{ij}$. Then a multilinear mapping $\mathscr{E}_\alpha \times \ldots \times \mathscr{E}_\alpha \to \mathscr{A}_n(\mathscr{E}_\alpha)$ is defined by

$$f_1 \circ \ldots \circ f_n = n! \sum_{i_1 < \ldots < i_n} (e_{i_1}^{-\alpha} \mid f_1) \ldots (e_{i_n}^{-\alpha} \mid f_n) e_{i_1}^\alpha \wedge \ldots \wedge e_{i_n}^\alpha \tag{5.6}$$

for $f_i \in \mathscr{E}_\alpha$, $i = 1, \ldots, \mathrm{N}$. This mapping is continuous,

$$\| f_1 \circ \ldots \circ f_n \|_\alpha \leqslant n! \prod_i \| f_i \|_\alpha \,.$$

The essential difference compared to the exterior product is that (5.6) does not necessarily vanish if $f_1 = \ldots = f_n$. In that case it satisfies the stronger norm estimate

$$\| f \circ \ldots \circ f \|_\alpha \leqslant \sqrt{n!}\, \| f \|_\alpha^n . \tag{5.7}$$

The product (5.6) depends on the basis. In the following the basis $e_i^1 \in \mathscr{E}_1$ is chosen such that $(e_i^1 \mid \mathrm{M} e_j^1) = \lambda_i \delta_{ij}$, i. e. the basis diagonalizes the operator M.

For $\mathrm{F} \in \mathscr{A}_m(\mathscr{E}_1)$ and $\mathrm{G} \in \mathscr{A}_n(\mathscr{E}_1)$ the function $(\mathrm{F} \mid \psi \circ \ldots \circ \psi)\overline{(\mathrm{G} \mid \psi \circ \ldots \circ \psi)}$ is a homogeneous polynomial on $\mathscr{E}_{-1}$. An easy calculation with decomposable tensors yields the integral

$$\int_{\mathscr{E}_{-1}} (\mathrm{F} \mid \psi \circ \ldots \circ \psi)\overline{(\mathrm{G} \mid \psi \circ \ldots \circ \psi)}\, d\mu(\psi) = \delta_{mn}(n!)^2 (\mathrm{F} \mid \Gamma(\mathrm{M})\mathrm{G}) . \tag{5.8}$$

To write a closed expression it is convenient to defined the exponential of ψ with respect to the product (5.6)

$$e^\psi = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\psi \circ \ldots \circ \psi}_{n} . \tag{5.9}$$

Due to (5.7) this series converges in $\mathscr{A}(\mathscr{E}_1)$. If $\psi^* \in \mathscr{E}_{-1}^* \to e^{\psi^*} = (e^\psi)^*$ is defined in the same way, then $\langle \mathrm{H} \mid e^{\psi^*} \wedge e^\psi \rangle$, $\mathrm{H} \in \mathscr{A}(\mathscr{H}_1)$, is an entire function on $\mathscr{E}_{-1}$, which can be integrated. The integral

$$\mathscr{I}_\mathrm{M}(\mathrm{H}) = \int_{\mathscr{E}_1} \langle \mathrm{H} \mid e^{\psi^*} \wedge e^\psi \rangle \, d\mu\,(\psi) \tag{5.10}$$

coincides with the integral (5.3) as a comparison of (5.2) with (5.8) shows. This integral is the anticommutative counterpart of the usual bosonic (commutative) integral (which can be recovered from (5.10) in substituting the algebraic products $\wedge$ and $\circ$ by the symmetric tensor product). In both cases the Gaussian functionals evaluated for (symmetric or antisymmetric) tensors of rank n are given by Gaussian integrals over homogeneous polynomicals of degree n. For bosonic integration this yields the moments of the measure, whereas in the fermionic case the polynomials are rather complicated to account for the anticommutativity.

6. THE EUCLIDEAN DIRAC FIELD

The family of Schwinger functions of a Fermi field corresponds to a linear functional on an exterior algebra of test functions. In this Section the generating functional of the Schwinger functions of a free Dirac field will be constructed as integral on a measure space.

6.1. Eight-component spinors.

The notations for the Euclidean variables are as follows

$$\begin{aligned}
& x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \\
& px = \sum_{\mu=1}^{4} p_\mu x_\mu \\
& \gamma_\mu = \gamma_\mu^+ \quad \text{with} \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4 \\
& \gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \gamma_5^+ \\
& \not{p} = \sum_{\mu=1}^{4} p_\mu \gamma_\mu .
\end{aligned} \tag{6.1}$$

The Fourier transform is defined as

$$\mathscr{F}[f(x)] = \int f(x) e^{ipx} d^4x = g(p)$$

$$\mathscr{F}^{-1}[g(p)] = (2\pi)^{-4} \int g(p) e^{-ipx} d^4p = f(x) . \tag{6.2}$$

The Schwinger function related to the Feynman propagator is then $S_{\alpha\beta}(x - y)$, $\alpha, \beta = 1, 2, 3, 4$, with the matrix

$$S(x) = \mathscr{F}^{-1}\left[\frac{m - i\not{p}}{m^2 + p^2}\right]. \tag{6.3}$$

It is assumed that the mass m is positive. The mass zero case will be considered in Sect. 7.

The Hilbert spaces $\mathscr{E}_\lambda$ are taken as

$$\mathscr{E}_\lambda = W_\lambda \otimes \mathbb{C}^4, \qquad \lambda \in \mathbb{R} \tag{6.4}$$

with $W_0 = \mathscr{L}^2(\mathbb{R}^4)$ and $W_\lambda = T^\lambda W_0$ as defined in Sect. 4. The HS operator T may be defined on $\mathscr{L}^2(\mathbb{R}^4)$ as

$$T\varphi(x) = (1 - \Delta)^{-2}(1 + x^2)^{-2}\varphi(x) \tag{6.5}$$

$$\text{with the Laplacian } \Delta = \sum_{\mu=1}^{4} \frac{\partial^2}{\partial x_\mu^2} .$$

The operator $S : \mathscr{E}_\lambda \to \mathscr{E}_\lambda$

$$(Sf)_\alpha(x) = \sum_{\beta=1}^{4} \int S_{\alpha\beta}(x - y) f_\beta(y) d^4y \tag{6.6}$$

is continuous in the norm $\| f \|_\lambda$ for $\lambda \in \mathbb{R}$. According to Eq. (2.3) this operator generates a continuous symplectic form ω_S on $\mathcal{H}$ such that

$$\omega_S(f, g) = 0$$
$$\omega_S(f^*, g) = \sum_{\alpha,\beta=1}^{4} \int \overline{f_\alpha(x)} S_{\alpha\beta}(x - y) g_\beta(y) d^4x d^4y \tag{6.7}$$

if $f, g \in \mathcal{E}$.

The family of Schwinger functions of the free Dirac field is then reproduced by the functional (4.3) L_S. This functional is well defined on $\mathcal{A}(\mathcal{H}_1)$, but the operator S is not positive and, consequently, the positivity condition (3.10) is violated. Hence the functional does not allow the identification with an integral.

The non-positivity of the Schwinger functions of a Dirac field is an old problem of Euclidean quantum field theory which makes the construction of Euclidean Dirac fields much more involved than the construction of scalar fields. In his pioneering paper [*18*] Schwinger proposed an eight-component Euclidean Dirac field. In [*11*] Osterwalder and Schrader presented a rigorous construction with a doublet of four-component fields. The following solution restores positivity in using the doubling $\mathcal{H} \to \hat{\mathcal{H}}$ of Sect. 2.

On $\mathcal{E}_\lambda$ the operator S has the polar decomposition $S = AU = UA$ with the positive operator $A = \sqrt{S^+S}$ and a unitary operator U. The explicit representations of these operators are

$$Af = \mathcal{F}^{-1}[(m^2 + p^2)^{-\frac{1}{2}}(\mathcal{F}f)(p)](x) \tag{6.8}$$
$$Uf = \mathcal{F}^{-1}\left[\frac{m - i\not{p}}{\sqrt{m^2 + p^2}}(\mathcal{F}f)(p)\right](x). \tag{6.9}$$

The spaces $\mathcal{E}_\lambda$ are extended to spaces of eight-component spinors

$$\hat{\mathcal{E}}_\lambda = W_\lambda \otimes \mathbb{C}^8 = \mathcal{E}_\lambda^{(1)} \oplus \mathcal{E}_\lambda^{(2)} \tag{6.10}$$

where the subspaces $\mathcal{E}_\lambda^{(1)}$ and $\mathcal{E}_\lambda^{(2)}$ of four-component spinors are isomorphic to (6.4) $\mathcal{E}_\lambda$. The operator S is then extended to the positive operator (2.8) M on $\hat{\mathcal{E}}_\lambda$ (with $A_1 = A_2 = A$). This operator has the non trivial kernel (2.9) $\mathcal{F}_\lambda$. The kernel $\mathcal{F}_\lambda$ and the factor space $\hat{\mathcal{E}}_\lambda/\mathcal{F}_\lambda$ are isomorphic to $\mathcal{E}_\lambda$.

As indicated in Sect. 2 the skew-symmetric form ω_M is equivalent to ω_S. Consequently the linear functional L_M on $\mathcal{A}(\hat{\mathcal{H}}_1)$ is equivalent to the functional L_S on $\mathcal{A}(\mathcal{H}_1)$. The advantage of L_M is the positivity of M such that L_M has the integral representations (5.3) and (5.10).

Since M is a bounded positive operator on $\hat{\mathcal{E}}$ the Theorem of Minlos guarantees that there exists a unique Gaussian measure $\hat{\mu}$ on $\hat{\mathcal{E}}_{-1}$ with the Fourier-Laplace transform (5.5). The Schwinger functions of the Euclidean Dirac field can then be calculated from the integral, cf. (5.10),

$$\mathscr{I}_{\mathrm{M}}(\mathrm{H}) = \int_{\hat{\mathscr{E}}_{-1}} \langle \mathrm{H} \mid e^{\psi^*} \wedge e^{\psi} \rangle \, d\hat{\mu}(\psi) \quad \text{with} \quad \mathrm{H} \in \mathscr{A}(\hat{\mathscr{H}}_1). \qquad (6.11)$$

The non-trivial kernel of the operator M implies that the Gaussian measure is concentrated on a subspaces of $\hat{\mathscr{E}}_{-1}$. Since the quadratic form $(f \mid \mathrm{M}f)$, $f \in \mathscr{E}_1$, vanishes on $\mathscr{F}_1$, the measure $\hat{\mu}$ is concentrated on that subspace which annihilates $\mathscr{F}_1$, i. e.

$$\mathscr{F}^{\perp} = \left\{ \hat{\psi} = \begin{pmatrix} \psi \\ \mathrm{U}^{+}\psi \end{pmatrix}, \psi \in \mathscr{E}_{-1} \right\} \subset \hat{\mathscr{E}}_{-1}, \qquad (6.12)$$

cf. Ref. [*15*], p. 339 or Ref. [*16*], § 19.

If the integral (6.11) is written in the form (5.3) the corresponding measure on $\mathscr{A}(\hat{\mathscr{E}}_{-1})$ is concentrated on the subalgebra $\mathscr{A}(\mathscr{F}^{\perp})$ which is generated by the elements $\hat{\psi} \in \mathscr{F}^{\perp}$.

To obtain more explicit relations between the Schwinger functions and the measure $\hat{\mu}$ the test functions in the integral (6.11) have to be chosen in a specific way.

Let $\mathrm{I}^{(i)}$, $i = 1, 2$, be the isomorphisms between $\mathscr{A}(\mathscr{E}_1)$ and $\mathscr{A}(\mathscr{E}_1^{(i)}) \subset \mathscr{A}(\hat{\mathscr{E}}_1)$ which are defined by the identification of $\mathscr{E}_1^{(i)}$ with $\mathscr{E}_1$. For $f, g \in \mathscr{E}_1$ and $\hat{f} = \mathrm{I}^{(1)}f$ and $\hat{g} = \mathrm{I}^{(2)}g$ the form ω_{M} yields $\omega_{\mathrm{M}}(\hat{f}^* \mid \hat{g}) = (\hat{f} \mid \mathrm{M}\hat{g}) = (f \mid \mathrm{S}g)$ and all other matrix elements are not relevant for the calculation of the Schwinger function.

More generally, if $\mathrm{F}, \mathrm{G} \in \mathscr{A}(\mathscr{E}_1)$ and $\hat{\mathrm{F}} = \mathrm{I}^{(1)}\mathrm{F} \in \mathscr{A}(\mathscr{E}_1^{(1)})$ and $\hat{\mathrm{G}} = \mathrm{I}^{(2)}\mathrm{G} \in \mathscr{A}(\mathscr{E}_1^{(2)})$ then

$$\mathscr{I}_{\mathrm{M}}(\mathrm{F}^* \wedge \mathrm{G}) = (\hat{\mathrm{F}} \mid \Gamma(\mathrm{M})\hat{\mathrm{G}}) = (\mathrm{F} \mid \Gamma(\mathrm{S})\mathrm{G}) \qquad (6.13)$$

and the integral reduces to the functional $\mathrm{L_S}$. The subalgebra $\mathscr{A}(\mathscr{E}_1^{(1)*} \oplus \mathscr{E}_1^{(2)})$ of the test functions is therefore sufficient to determine the Schwinger functions of the Dirac field. This subalgebra corresponds by duality to the field algebra $\mathscr{A}(\mathscr{E}_{-1}^{(1)} \oplus \mathscr{E}_{-1}^{(2)*}) \subset \mathscr{A}(\hat{\mathscr{H}}_{-1})$.

6.2. A doublet of four-component spinors.

The Euclidean quantum field theory of a Dirac field presented in Sect. 6.1 corresponds to the construction of a degenerate measure on a space of « classical » eight-component spinor fields which are elements of a Grassmann algebra. Only four of these components are independent variables. This result can be formulated in a different way [*8*] which is more closely related to the Euclidean field operators of Osterwalder and Schrader [*11*].

The original symplectic form ω_{S} defines by the equation (2.4) a unique positive form (2.6) on $\mathscr{H}_1$. On the subspace $\mathscr{E}_1$ this form is given by $(f \mid \mathrm{A}g)$

where A is the positive operator (6.8). Then one can define a Gaussian measure μ on $\mathscr{E}_{-1}$ with the Fourier-Laplace transform $\exp[-(f \mid Ag)]$. The related measure ρ on $\mathscr{A}(\mathscr{E}_{-1})$ has the Fourier-Laplace transform $\exp[-(F \mid \Gamma(A)G)]$.

The operator (6.9) U is continuous in the norm $\| f \|_\lambda$ for any $\lambda \in \mathbb{R}$, and it satisfies $(U^+f, g) = (f, Ug)$ if $f \in \mathscr{E}_\lambda$ and $g \in \mathscr{E}_{-\lambda}$.

Given the Gaussian variable $\psi \in \mathscr{E}_{-1}$ one can define the linearly dependend (and *a fortiori* correlated) Gaussian variables

$$\psi_{(1)} = \psi \qquad \text{and} \qquad \psi_{(2)} = U^+\psi . \tag{6.14}$$

The Fock space variables are then given by the construction of Sect. 5 $\phi_{(1)} = e^{\psi_{(1)}}$ and $\phi_{(2)} = e^{\psi_{(2)}} = \Gamma(U^+)\phi_{(1)}$. For F, $G \in \mathscr{A}(\mathscr{E}_1)$ the identity

$$\langle F^* \wedge G \mid \phi^*_{(2)} \wedge \phi_{(1)} \rangle = (F \mid \phi_{(1)})\overline{(G \mid \phi_{(2)})} = (F \mid \phi)\overline{(\Gamma(U)G \mid \phi)}$$

allows to calculate the integral

$$\int \langle F^* \wedge G \mid \phi^*_{(2)} \wedge \phi_{(1)} \rangle \, d\mu(\psi) = \int (F \mid \phi)\overline{(\Gamma(U)G \mid \phi)} d\mu(\psi) =$$
$$= (F \mid \Gamma(A)\Gamma(U)G) = (F \mid \Gamma(S)G) \tag{6.15}$$

which coincides with the linear functional L_S. Therefore all Schwinger functions can be obtained from the integral

$$L_S(H) = \int_{\mathscr{E}_{-1}} \langle H \mid \phi^*_{(2)} \wedge \phi_{(1)} \rangle \, d\mu(\psi) . \tag{6.16}$$

If Eq. (6.15) is evaluated for $F = f \in \mathscr{E}_1$ and $G = g \in \mathscr{E}_2$ one obtains

$$\int \langle f^* \wedge g \mid \psi^*_{(2)} \wedge \psi_{(1)} \rangle \, d\mu(\psi) = (f \mid Sg)$$

or

$$\int_{\mathscr{E}_{-1}} \psi_{(1)\alpha}(x)\overline{\psi_{(2)\beta}(y)} d\mu(\psi) = S_{\alpha\beta}(x - y) . \tag{6.17}$$

Under the action of the Euclidean group the fields $\psi_{(i)}$, $i = 1, 2$, transform as

$$\psi_{(i)\alpha}(x) \to \sum_\beta S_{\alpha\beta}(u_1, u_2)\psi_{(i)\beta}(R(u_1, u_2)x + a) \tag{6.18}$$

where the notation of Ref. [*11*] Eq. (3.15) has been used for the group variables. Therefore $\sum_\alpha \overline{\psi_{(2)\alpha}(x)}\psi_{(1)\alpha}(y)$ is the bilocal scalar field which corresponds to the Minkowski space scalar $\sum_{\alpha,\beta} \Psi^+_\alpha(x)(\gamma_0)_{\alpha\beta}\Psi_\beta(y)$. The relation to the Euclidean field operators of Osterwalder and Schrader [11] is

$$\Psi^{(1)}_{0S_\alpha}(x) \leftrightarrow \psi_{(1)\alpha}(x) \quad \text{and} \quad \Psi^{(2)}_{0S_\alpha}(x) \leftrightarrow \overline{\psi_{(2)\alpha}(x)}\,.$$

The two fields $\psi_{(1)}$ and $\psi^*_{(2)}$ which show up in the integral (6.16) are related by the antilinear mapping $\psi = \psi_{(1)} \rightarrow \psi^*_{(2)} = (U^+\psi)^*$ which originates from the conjugation (2.7) of the symplectic form ω_S. If one does not care about the positivity (3.10) (which has not to be confused with Osterwalder-Schrader positivity) one can use $\psi \rightarrow (U^+\psi)^*$ as the basic involution of the theory. Exactly these fields ψ and $(U^+\psi)^*$ show up in the formal Berezin integral.

In a bosonic theory the Schwinger functions are given as moments of a positive measure which allows to derive correlation inequalities. For the free Euclidean Dirac theory the $2n$-point Schwinger function is obtained from the Gaussian integrals (6.11) or (6.16) over homogeneous polynomials of degree $2n$. But these polynomials have a rather complicated structure as has been seen in Sect. 5. Moreover, in the case of Dirac fields there is the additional complication due to the doubling (or, equivalently, due to the unitary operator U^+ in (6.14)). So far no calculatorial consequences from the existence of the positive measure can be presented. But the probabilistic approach gives a natural basis for the definition of Euclidean Dirac fields which have the correct degrees of freedom. As further application this method will be used in the following Section to define in a unique way Euclidean Weyl fields.

7. ZERO MASS DIRAC SPINORS AND WEYL SPINORS

If $m = 0$ the operator (6.6) S is unbounded in the norm of $\mathscr{E}$ and (6.7) ω_S is an unbounded form on $\mathscr{H}$. For the construction of a measure it is essential to define $\mathscr{E}_1 \subset \mathscr{E}$ in such a way that ω_S is nuclear on $\overset{\circ}{\mathscr{H}}_1$. That is achieved if the imbedding operator (6.5) is modified to $\overset{\circ}{T} = (-\Delta)^{\frac{1}{4}}T$. This operator is a HS operator and $\overset{\circ}{T}{}^+S\overset{\circ}{T}$ is a nuclear operator on $\mathscr{E}$. With $\mathscr{E}_1 = \overset{\circ}{T}\mathscr{E}$ and $\| f \|_1 = \| \overset{\circ}{T}{}^{-1}f \|$ the form ω_S is nuclear on $\mathscr{E}_1$ and the measures can be constructed as in Sect. 6.1. All results of Sects. 6.1-6.2 remain valid.

For $m = 0$ the inverse Dirac operator (6.6) anticommutes with γ_5, whereas $A = \sqrt{S^+S}$ commutes with γ_5

$$S\gamma_5 = -\gamma_5 S, \qquad A\gamma_5 = \gamma_5 A\,. \tag{7.1}$$

The operators

$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 \mp \gamma_5 & 0 \\ 0 & 1 \pm \gamma_5 \end{pmatrix} \tag{7.2}$$

are projection operators on orthogonal subspaces of $\hat{\mathscr{E}} = \mathscr{E}^{(1)} \oplus \mathscr{E}^{(2)}$. They can be extended to the whole space $\hat{\mathscr{H}}$ by $P_\pm \hat{f} = (P_\pm \hat{f}^*)^*$ if $\hat{f} \in \hat{\mathscr{E}}^*$. As a consequence of (7.1) the operator M commutes with $P_\pm$

$$P_\pm M = MP_\pm = M^\pm . \tag{7.3}$$

Hence for $m = 0$ the operator M separates into two positive mappings $M = M^+ + M^-$ which operate nontrivially on the orthogonal subspaces $\hat{\mathscr{E}}_\pm = P_\pm \hat{\mathscr{E}}$. Consequently, the measure $\hat{\mu}$ on $\hat{\mathscr{E}}_{-1}$ factorizes into the Gaussian measures $\mu_\pm$ on $\hat{\mathscr{E}}_{\pm\,-1}$ with the quadratic forms $(f \mid M^\pm g)$. Moreover the skew symmetric form ω_M separates into two forms

$$\omega_M(\hat{f}, \hat{g}) = \omega_+(\hat{f}, \hat{g}) + \omega_-(\hat{f}, \hat{g}) \tag{7.4}$$

where

$$\omega_\pm(\hat{f}, \hat{g}) = \omega_M(P_\pm \hat{f}, \hat{g}) = \omega_M(\hat{f}, P_\pm \hat{g})$$

are nontrivial forms on the orthogonal subspaces $\hat{\mathscr{H}}_\pm = P_\pm \hat{\mathscr{H}}$. Hence the theory of a massless Dirac field is split into the theory of two Weyl fields $\hat{\psi}^\pm = P_\pm \hat{\psi} \in \hat{\mathscr{E}}_{\pm\,-1}$.

8. INTERACTING DIRAC FIELDS

The results of Sect. 6 can be generalized to Dirac particles which interact with classical vector or axial-vector fields (which may originate in a bosonic system). The classical Euclidean action is then $(\psi \mid (\mathscr{D} + V)\psi)$ with the Dirac operator $\mathscr{D} = -\not{\partial} + m$ and an interaction term

$$V = i\not{A} + i\gamma_5 \not{B} \tag{8.1}$$

where $A_\mu(x)$ and $B_\mu(x)$ are real vector and axial-vector fields. Without investigation of the admitted potentials it is assumed that $S_v = (\mathscr{D} + V)^{-1}$ is a bounded operator on $\mathscr{E}$ such that S_v and S_v^+ map the subset $\mathscr{E}_1$ into itself. The polar decomposition of S_v on the space $\mathscr{E}$ leads to $S_v = A_{v1} U_v = U_v A_{v2}$. The extension of S_v to a positive operator M_v on $\hat{\mathscr{E}}$ is defined as in Eq. (2.8).

Then Gaussian measures $\hat{\rho}_v$ and $\hat{\mu}_v$ can be constructed on the spaces $\mathscr{A}(\hat{\mathscr{E}}_{-1})$ and $\hat{\mathscr{E}}_{-1}$, respectively, as in Sect. 6. The measure $\hat{\mu}_v$ is concentrated on the subspace

$$\mathscr{F}_v^\perp = \left\{ \hat{\psi} = \begin{pmatrix} \psi \\ U_v^+ \psi \end{pmatrix}, \psi \in \mathscr{E}_{-1} \right\}. \tag{8.2}$$

For a non vanishing but well behaved potential V (e. g. a nuclear operator) the operator U_v differs from the operator (6.9) of the free theory and the subspaces (8.2) and (6.12) do not coincide. Therefore the measures for an interacting theory and the measure for the free theory are mutually singular even if the potential is weak and regularized.

The situation is better if the fields of Sect. 6.2 are used. For a theory with the bosonic interaction (8.1), the generating functional for the Schwinger functions is still given by (4.2), i. e.

$$L(H) = \langle \exp \Omega(S_v) \mid H \rangle \tag{8.3}$$

which can be written as the integral (6.16) where the Gaussian measure is defined with the quadratic form $(f \mid A_v f)$ and the fields $\psi_{(1)}$ and $\psi_{(2)}$ are related by $\psi_{(2)} = U_v^+ \psi_{(1)}$. For a sufficiently regularized interaction $\mathscr{D}^{-1}V_{reg}$ is a HS operator on $\mathscr{E}$ and the measures μ_v and μ are mutually equivalent. Then (8.3) has an integral representation with the free measure.

If the mass vanishes Euclidean Weyl fermions can be defined as in the case of free fields. The relations (7.1)-(7.3) are still valid for the interaction (8.1). Hence the measure space of the interacting Dirac field factorizes into the measure spaces of two Euclidean Weyl fermions, and the form algebra of the Dirac field factorizes into the form algebras of the Weyl fermions. In the literature some problems have been seen to define Euclidean Weyl spinors, see e. g. [*19*], but the framework of this paper yields a unique solution.

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