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Time decay of solutions to the Schrödinger equation in exterior domains. I

by

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ABSTRACT. — We study the time decay of solutions for the following Schrödinger equation:

$$(*)\quad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, & (t,x)\in(0,\infty)\times \mathrm{D}, \\ u(0,x) = \phi(x), & x\in \mathrm{D}, \\ u(t,x) = 0, & (t,x)\in(0,\infty)\times \partial\mathrm{D}, \end{cases}$$

where D is the complement of a star-shaped, bounded domain in $\mathbb{R}^n$, $n \geqq 3$, and the boundary ∂D is smooth. We give upper bounds for decay rates of $\mathrm{L}^p(\mathrm{D})$-norm for the solution u of $(*)$, for example,

$$\| u(t) \|_p \leqq \begin{cases} \mathrm{C}\mathrm{I}^{1/2}(1+t)^{-2}(1+\log(1+t)), & n \geqq 5, \quad p = 2n/(n-4), \\ \mathrm{C}\mathrm{I}^{1/2}(1+t)^{-2(1-2\varepsilon)+\varepsilon_1}, & n = 4, \quad p = 1/\varepsilon, \\ \mathrm{C}\mathrm{I}^{1/2}(1+t)^{-11/10+\varepsilon}, & n = 3, \quad p = \infty, \end{cases}$$

where ε and ε_1 are sufficiently small positive constants,

$$\mathrm{I} = \mathrm{I}(\phi) = \|\, |x|^2\phi \,\|_{1,2}^2 + \| x\Delta\phi \|^2 + \| \phi \|_{2,2}^2 .$$

RÉSUMÉ. — Nous étudions la décroissance temporelle des solutions de l'équation de Schrödinger :

$$(*)\quad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, & (t,x)\in(0,\infty)\times \mathrm{D}, \\ u(0,x) = \phi(x), & x\in \mathrm{D}, \\ u(t,x) = 0, & (t,x)\in(0,\infty)\times \partial\mathrm{D}, \end{cases}$$

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où D est le complément d'un domaine étoilé borné de $\mathbb{R}^n$, $n \geq 3$, et de bord régulier. Nous prouvons une borne supérieure pour le taux de décroissance dans la norme de $L^p(D)$ des solutions u de $(*)$:

$$\| u(t) \|_p \leqq \begin{cases} CI^{1/2}(1+t)^{-2}(1+\log(1+t)), & n \geqq 5, \quad p = 2n/(n-4), \\ CI^{1/2}(1+t)^{-2(1-2\varepsilon)+\varepsilon_1}, & n = 4, \quad p = 1/\varepsilon, \\ CI^{1/2}(1+t)^{-11/10+\varepsilon}, & n = 3, \quad p = \infty, \end{cases}$$

où ε et ε_1 sont des constantes suffisamment petites et

$$I = I(\phi) = \| \, |x|^2 \phi \|_{1,2}^2 + \| x \Delta \phi \|^2 + \| \phi \|_{2,2}^2 .$$

1. INTRODUCTION AND MAIN RESULT

We consider the exterior boundary value problem for the following Schrödinger equation:

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \qquad (t, x) \in (0, \infty) \times D, \tag{1.1}$$

$$u(0, x) = \phi(x), \qquad x \in D, \tag{1.2}$$

$$u(t, x) = 0, \qquad (t, x) \in (0, \infty) \times \partial D, \tag{1.3}$$

where D is the complement of a star-shaped, bounded domain in $\mathbb{R}^n$, $n \geqq 3$, and the boundary ∂D is smooth. Our main purpose in this paper is to study L^p-time decay for solutions of (1.1)-(1.3). In this paper we use the following notations:

NOTATION. — $\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$, $\nabla = (\partial_1, \ldots, \partial_n)$, $x = (x_1, \ldots, x_n)$, $|x| = r$, $\Delta = \sum_{k=1}^{n} \partial_k^2$; $S = S(t) = \exp\ (i|x|^2/2t)$, $t \in \mathbb{R} \setminus \{0\}$; $\partial_r = \partial/\partial r$; $J_k = J_k(t) = x_k + it\partial_k$, $J = J(t) = (J_1, \ldots, J_n)$, $K = r^2 + nit + 2itr\partial_r + 2it^2\partial_t$, $J^2 = r^2 + nit + 2itr\partial_r - t^2\Delta$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, J^α, $J_1^{\alpha_1} \ldots J_n^{\alpha_n}$, $\alpha \in (\mathbb{N} \cup \{0\})^n$, $\partial^0 = x^0 = J^0 = I$; $\mathscr{S}$ denotes the space of rapidly decreasing $C^\infty(D)$-functions from D to $\mathbb{C}$, $\mathscr{S}'$ is the dual space of $\mathscr{S}$; L^p denotes the Lebesgue space $L^p(D)$ or $L^p(D) \otimes \mathbb{C}^n$, with the norm $\| \cdot \|_p$, $1 \leqq p \leqq \infty$; $\| \cdot \| = \| \cdot \|_2$; $(.,.)$ denotes the L^2-scalar product; $H^{m,p} = H^{m,p}(D) = \Big\{ \psi \in \mathscr{S}'$; $\| \psi \|_{m,p} = \sum_{|\alpha| \leqq m} \| \partial^\alpha \psi \|_p < \infty \Big\}$, $H_0^{m,p} = H_0^{m,p}(D)$ denotes the completion of $C_0^\infty(D)$ in $H^{m,p}$;

$$\int \cdot \, dx = \int_D \cdot \, dx ; \qquad \| \cdot \|_b^2 = -\sum_{j=1}^{n} \int_D \partial_j(x_j | \cdot |^2) dx = -\int \partial_j(x_j | \cdot |^2) dx$$

when D is the complement of a star-shaped, bounded domain with smooth boundary ∂D.

The following relations will be used in the sequel:

$$J_k(t) = S(t)(it\partial_k)S(-t), \qquad J(t) = S(t)(it\nabla)S(-t),$$

$$J^2(t) = S(t)(-t^2\Delta)S(-t), \qquad L = i\partial_t + \frac{1}{2}\Delta, \qquad [L, J] = LJ - JL = 0,$$

$$[L, J^2] = LJ^2 - J^2L = 0, \qquad [L, K] = LK - KL = 4itL.$$

Different positive constants might be denoted by the same letter C. If necessary, by $C(*, \dots, *)$ we denote constants depending only on the quantities appearing in parentheses.

With these notations we state our main result.

THEOREM 1. — Let D be the complement of a star-shaped, bounded domain in $\mathbb{R}^n$ $(n \geqq 3)$, with smooth boundary ∂D. Let u be the solution of (1.1)-(1.3) with $\phi \in H = \{ \psi \in \mathscr{S}'$;

$$I = I(\psi) = \| \, |x|^2\psi \|^2 + \| x\Delta\psi \|^2 + \| \psi \|_{2,2}^2 < \infty \}.$$

Then u satisfies the following decay estimates

$$\| u(t) \|_p \leqq CI^{1/2}(\phi)(1 + t)^{-1-\gamma}Q(t, \beta, \gamma),$$

where
$$p = 2n/(n - 2 - 2\gamma),$$

and
$$Q(t, \beta, \gamma) = (1 + t)^{2(2-\beta)\gamma/(3-\beta)}(1 + \log(1 + t))^{\gamma/(3-\beta)}$$

where $0 \leqq \beta < 4/3$, $0 < \gamma \leqq 1/2$ if $n = 3$, $0 \leqq \beta < 2$, $0 < \gamma < 1$ if $n = 4$, $0 \leqq \beta \leqq 2$, $0 < \gamma \leqq 1$, if $n \geqq 5$.

More precise L^p-time decay for solutions of (1.1)-(1.3) has been studied by Y. Tsutsumi (lemma 3.1 in [5]).

However his assumptions on the initial data and the domain are different from ours, and his methods are also different from ours.

REMARK 1. — Let v be the solution of the initial value problem for the linear Schrödinger equation with the initial data ϕ. Then we have by well known decay estimates of free Schrödinger group and Sobolev's inequality

$$\begin{aligned} \| v(t) \|_{L^p(\mathbb{R}^n)} &\leqq C(\| \phi \|_{L^{p'}(\mathbb{R}^n)} + \| \phi \|_{H^{2,2}(\mathbb{R}^n)})(1 + t)^{-1-\gamma} \\ &\leqq C(\| r^2\phi \|_{L^2(\mathbb{R}^n)} + \| \phi \|_{H^{2,2}(\mathbb{R}^n)})(1 + t)^{-1-\gamma}, \end{aligned}$$

where $1/p + 1/p' = 1$ and $\gamma = \gamma(p)$ is the same one as that of theorem 1.

REMARK 2. — We can treat the nonlinear Schrödinger equations in exterior domains by using theorem 1, since the decay rates obtained in theorem 1 are larger than 1 (see [*5*], [*7*]).

Throughtout the paper we assume that the assumptions of theorem 1 are satisfied.

2. PROOF OF THEOREM 1

For the convenience of the reader we first give a sketch of the strategy of the proof. The main result follows from Sobolev's inequality

$$\| u(t) \|_p \leqq Ct^{-1-\gamma} \| Ju(t) \|^{1-\gamma} \cdot \sum_{|\alpha|=2} \| J^\alpha u(t) \|^\gamma, \qquad t > 0,$$

where p and γ are same as those in theorem 1. The first norm is estimated by lemma 2.1, the second norm is reduced basically to $\| J^2 u \|$ by lemma 2.5 (which does not use the equation), then $\| J^2 u \| = \| Ku \|$ for the solutions, $\| Ku \|$ is estimated in lemma 2.6 which requires *a priori* estimates of solutions on the boundary given in lemmas 2.2-2.4. We note that computation stated below is rather formal, but it can be justified by considering the solutions u_k of regularized equations such that

$$\begin{aligned} i\partial_t u + \frac{1}{2}\Delta u &= 0, \qquad (t, x) \in (0, \infty) \times D, \\ u(0, x) &= \phi_k(x), \qquad x \times D, \\ u(t, x) &= 0, \qquad (t, x) \in (0, \infty) \times \partial D, \end{aligned}$$

where $\phi_k \in X = \{ \psi \in H^{2N,2}(D) \cap H ; \Delta^l \psi \in H_0^{1,2}(D), 1 \leqq l \leqq N-1, N \in \mathbb{N} \}$ and $\phi_k \to \phi$ strongly in H. It is well known that for any k, there exists a unique smooth solution

$$u_k \in \bigcap_{l=0}^{N-1} C^l([0, \infty) ;$$

$$H^{2(N-l),2}(D) \cap H_0^{1,2}(D)) \cap C^N([0, \infty) ; L^2(D))$$

(see, e. g., K. Yosida [*6*]). This and a limiting procedure allow us to justify the formal calculation stated below.

LEMMA 2.1. — Let u be the solution of (1.1)-(1.3). Then we have

$$\| Ju(t) \|^2, \qquad \int_0^t s \| \nabla u(s) \|_b^2 ds \leqq C \| x\phi \|^2, \tag{2.1}$$

$$\| J\partial_t u(t) \|^2, \qquad \int_0^t s \| \nabla \partial_s u(s) \|_b^2 ds \leqq C \| x\Delta\phi \|^2. \tag{2.2}$$

Proof. — From (1.1) we have

$$i\partial_t \mathrm{J}v + \frac{1}{2}\Delta \mathrm{J}v = \mathrm{LJ}v = 0, \tag{2.3}$$

where $v = u$ or $\partial_t u$. We multiply (2.3) by $\overline{\mathrm{J}v}$ and take the imaginary part. This leads us to

$$\frac{d}{dt}\| \mathrm{J}v \|^2 + \operatorname{Im} \int \nabla(-\, it\nabla v \cdot r\partial_t \bar{v})dx = 0, \tag{2.4}$$

where Im f denotes the imaginary part of f. For any a, $b \in \mathcal{S}$, we have

$$\nabla(\nabla a \cdot r\partial_r b) = \nabla(x\Delta a \cdot b - (n-1)\nabla a \cdot b - r\partial_r \nabla a \cdot b) + \sum_{j=1}^{n} \partial_j(x_j \nabla a \cdot \nabla b). \tag{2.5}$$

We obtain by (2.4), (2.5) and the fact that $v = 0$ on $\partial\mathrm{D}$,

$$\frac{d}{dt}\| \mathrm{J}v \|^2 - t\int \partial_j(x_j | \nabla v |^2)dx = \frac{d}{dt}\| \mathrm{J}v \|^2 + t\| \nabla v \|_b^2 = 0. \tag{2.6}$$

(2.1) and (2.2) follow from (2.6) and $i\partial_t u(0) = -\frac{1}{2}\Delta\phi$. Q. E. D.

LEMMA 2.2. — Let u be the solution of (1.1)-(1.3). Then we have for $t>0$

$$\| \partial_r \nabla u(t) \|_b \leqq \mathrm{C}t^{-1/2}(1+t)^{-1/2}(\| x\Delta\phi \| + \| x\phi \| + \| \phi \|_{2,2}).$$

Proof. — We put $\zeta = (1+r)^{-k}$, $k > 1$. We have by a simple calculation $\zeta\partial_r\nabla u = \partial_r\nabla(\zeta u) - \partial_r u \cdot \nabla\zeta - \nabla u \cdot \partial_r \zeta - u \cdot \partial_r \nabla\zeta$. From this and the fact that $\partial\mathrm{D}$ is bounded we have

$$\begin{aligned} \| \partial_r \nabla u \|_b &\leqq \mathrm{C} \| \zeta\partial_r\nabla u \|_b \\ &\leqq \mathrm{C}(\| \partial_r\nabla(\zeta u) \|_b + \| \partial_r u \cdot \nabla\zeta \|_b + \| \nabla u \cdot \partial_r\zeta \|_b + \| u \cdot \partial_r \nabla\zeta \|_b) \\ &\leqq \mathrm{C}(\| \partial_r\nabla(\zeta u) \|_b + \nabla(\zeta u) \|_b + \| \zeta u \|_b) \leqq \mathrm{C} \| \zeta u \|_{3,2}^{1/2} \| \zeta u \|_{2,2}^{1/2}, \end{aligned} \tag{2.7}$$

here we have used the Schwarz inequality. We multiply (1.1) by ζ to obtain

$$-\Delta\zeta u + \zeta u = 2i\partial_t(\zeta u) - 2\nabla\zeta \cdot \nabla u - (\Delta\zeta)u + \zeta u. \tag{2.8}$$

By the elliptic estimates (see, e. g., [*1*]) and (2.8) we get

$$\| \zeta u \|_{2,2} \leqq \mathrm{C}(\| \zeta\partial_t u \| + \| \zeta\nabla u \| + \| \zeta u \|), \tag{2.9}$$

$$\| \zeta u \|_{3,2} \leqq \mathrm{C}(\| \zeta\nabla\partial_t u \| + \| \zeta\partial_t u \| + \| \zeta\nabla u \| + \| \zeta u \|). \tag{2.10}$$

By Hölder's and Sobolev's inequalities we have

$$\| \zeta v \| \leqq \mathrm{C} \| v \|_{2n/(n-2)} \leqq \begin{cases} \mathrm{C}t^{-1} \| \mathrm{J}v \|, & t > 0, \\ \mathrm{C} \| \nabla v \|, \end{cases} \tag{2.11}$$

for any $v \in \mathrm{H}^{1,2}$ with $| x | v \in \mathrm{L}^2$.

By a simple calculation we obtain for any $v \in H^{1,2}$

$$\| \zeta \nabla v \| \leqq \begin{cases} Ct^{-1}(\| Jv \| + \| v \|), & t > 0, \\ C \| v \|_{1,2} \end{cases} \tag{2.12}$$

(2.9)-(2.12) and (2.7) give

$$\begin{aligned} & \| \partial_r \nabla u \|_b \\ & \leqq Ct^{-1/2}(\| J\partial_t u \| + \| Ju \|)^{1/2}(1+t)^{-1/2}(\| J\partial_t u \| + \| Ju \| + \| u \|_{2,2})^{1/2} \\ & \leqq Ct^{-1/2}(1+t)^{-1/2}(\| J\partial_t u \| + \| Ju \| + \| u \|_{2,2}) . \end{aligned} \tag{2.13}$$

Since $\| u \|_{2,2} \leqq C \| \phi \|_{2,2}$ by the energy estimates of (1.1)-(1.3), lemma 2.2 follows from lemma 2.1 and (2.13). Q. E. D.

LEMMA 2.3. — Let u be the solution of (1.1)-(1.3). Then we have

$$\| \nabla Ku(t) \|^2, \quad \int_0^t s^3 \| \nabla \partial_s u(s) \|_b^2 ds, \quad t^2 \| \nabla u(t) \|_b^2 \leqq C \cdot I(\phi)(1 + \log(1+t)) .$$

Proof. — From (1.1) we have

$$LKu = 0 . \tag{2.14}$$

We multiply (2.14) by $\partial_t \overline{(Ku)}$ and take the real part to obtain

$$\frac{d}{dt} \| \nabla Ku \|^2 + 2Re \int \nabla (\nabla Ku \cdot (2ir\partial_r \overline{u} + 2itr\partial_r \partial_t \overline{u}))dx = 0, \tag{2.15}$$

where Re f denotes the real part of f. By using (2.5) we have

$$\begin{aligned} & \int \nabla (\nabla Ku \cdot (2ir\partial_r \overline{u} + 2itr\partial_r \partial_t \overline{u}))dx = 2i \int \partial_j (x_j \nabla Ku \cdot ((1 + t\partial_t) \nabla \overline{u}))dx \\ & \quad = 2i \int \partial_j (x_j (r^2 \nabla u + nit \nabla u + 2itr\partial_r \nabla u + 2it(1 + t\partial_t) \nabla u)((1 + t\partial_t) \nabla \overline{u}))dx \\ & \quad = -4t \int \partial_j (x_j | (1 + t\partial_t) \nabla u |^2)dx - 2 \int \partial_j (x_j (nt + 2tr\partial_r) \nabla u)((1 + t\partial_t) \nabla \overline{u}))dx \\ & \quad = +2i \int \partial_j (x_j (r^2 | \nabla u |^2 + r^2 t \nabla u \cdot \partial_t \nabla \overline{u}))dx . \end{aligned} \tag{2.16}$$

We have by (2.15), (2.16) and the Schwarz inequality

$$\frac{d}{dt} \| \nabla Ku \|^2 + 4t \| (1 + t\partial_t) \nabla u \|_b^2 \leqq Ct(\| \nabla u \|_b^2 + \| \partial_r \nabla u \|_b^2 + \| \nabla u \|_b \| \nabla \partial_t u \|_b) .$$

From this we have

$$\frac{d}{dt}\left(\| \nabla \mathrm{K}u \|^2 + 4t^2 \| \nabla u \|_b^2 \right) + 4t^3 \| \nabla \partial_t u \|_b^2 \leqq \mathrm{C}t \left(\| \nabla u \|_b^2 + \| \partial_r \nabla u \|_b^2 + \| \nabla u \|_b \| \nabla \partial_t u \|_b \right), \tag{2.17}$$

since $t \| (1 + t\partial_t) \nabla u \|_b^2 = -t \| \nabla u \|_b^2 + t^3 \| \nabla \partial_t u \|_b^2 + \dfrac{d}{dt} t^2 \| \nabla u \|_b^2$.

Thus from (2.17), lemmas 2.1-2.2 and the Schwarz inequality it follows that

$$\| \nabla \mathrm{K}u \|^2 + 4t^2 \| \nabla u \|_b^2 + 4 \int_0^t s^3 \| \nabla \partial_s u \|_b^2 \leqq \mathrm{CI}(\phi)(1 + \log(1+t)).$$

This completes the proof of lemma 2.3. Q. E. D.

Lemma 2.4. — Let $w \in \mathrm{H}_0^{1,2} \cap \mathrm{H}^{2,2}$ and $r^2 w \in \mathrm{L}^2$. Then we have

$$\| \nabla w \|_b \leqq \begin{cases} \mathrm{C}t^{-4/(4-\beta)} \| \mathrm{J}^2 w \|^{2/(4-\beta)} \| w \|^{(2-\beta)/(4-\beta)} + \mathrm{C}t^{-2} \| \mathrm{J}^2 w \|, & t > 0, \quad (2.18) \\ \mathrm{C} \cdot (\| \Delta w \| + \| \nabla w \|), & (2.19) \end{cases}$$

where $0 < \beta < 4/3$ if $n = 3$, $0 < \beta < 2$ if $n = 4$, $0 < \beta \leqq 2$ if $n \geqq 5$.

Proof. — We put $\zeta_1 = (1 + r)^{-(2k+1)}$, $0 \leqq 2k < n - 2$. Since $w \in \mathrm{H}_0^{1,2}$ we have with $v = \mathrm{S}(-t)w$

$$\begin{aligned} t^2 \| \nabla w \|_b^2 &= -t^2 \int \nabla(x \,|\, \partial_j w \,|^2) dx = -\int \nabla(x \,|\, it\partial_j w \,|^2) dx \\ &= -\int \nabla(x \,|\, (x_j + it\partial_j) w \,|^2) dx = -\int \nabla(x \,|\, \mathrm{J}_j w \,|^2) dx \\ &= -t^2 \int \nabla(x \,|\, \nabla \mathrm{S}(-t) w \,|^2) dx = -t^2 \int \nabla(x \,|\, \nabla v \,|^2) dx \\ &= -t^2 \int_{\partial \mathrm{D}} |\, \nabla v \,|^2 (x \cdot n) d\sigma \leqq \max_{x \in \partial \mathrm{D}} \zeta_1^{-1} \left(\int_{\partial \mathrm{D}} -t^2 \zeta_1 \,|\, \nabla v \,|^2 (x \cdot n) d\sigma \right), \\ &\leqq \mathrm{C} \cdot \left(-\int \nabla(\zeta_1 x t^2 \,|\, \nabla v \,|^2) dx \right), \end{aligned} \tag{2.20}$$

where we have used the boundedness of $\partial \mathrm{D}$. Since

$$\nabla(\zeta_1 x) = n\zeta_1 + r\partial_r \zeta_1 = (n - (2k+1)) r (1+r)^{-1} \zeta_1 \geqq (n - 2k - 2)\zeta_1 \geqq 0,$$

we obtain by (2.20)

$$\| \nabla w \|_b^2 \leqq \mathrm{C} \sum_{|\alpha| = 2} \| \zeta \partial^\alpha v \| \, \| \zeta \nabla v \|, \tag{2.21}$$

where $\zeta=(1+r)^{-k}$. On the other hand, integration by parts and the Schwarz inequality give

$$\begin{aligned}\sum_{|\alpha|=2} \| \zeta\partial^\alpha v \|^2 &= \sum_{j,l=1}^{n}\Big(\int \partial_j(\zeta^2(\partial_l v \cdot \partial_j \partial_j \partial_l \bar{v} - \partial_j v \cdot \partial_l^2 \bar{v})dx \\ &\qquad - 2\int \zeta\partial_j\zeta(\partial_l v \cdot \partial_j \partial_l \bar{v} - \partial_j v \cdot \partial_l^2 \bar{v})dx\Big) + \| \zeta\Delta v \|^2 \\ &\leqq \Big| \sum_{j,l=1}^{n} \int \partial_j(\zeta^2(\partial_l v \partial_j \partial_l v - \partial_j v \partial^2 v))dx \Big| + \frac{1}{2}\sum_{|\alpha|=2} \| \zeta\partial^\alpha v \|^2 + \\ &\quad + C\sum_{j,l=1}^{n} \| \partial_j\zeta \cdot \partial_l v \|^2 + \| \zeta\Delta v \|^2 . \end{aligned} \tag{2.22}$$

In the same way as in the proof of (16) (Chapter 1 in [3]), the first term of the R. H. S. of (2.22) is dominated by

$$\frac{1}{4}\sum_{|\alpha|=2} \| \zeta\partial^\alpha v \|^2 + C \| \zeta\nabla v \|^2 . \tag{2.23}$$

Therefore by virtue of (2.22) and (2.23)

$$\sum_{|\alpha|=2} \| \zeta\partial^\alpha v \|^2 \leqq C \cdot (\| \zeta\Delta v \|^2 + \| \zeta\nabla v \|^2). \tag{2.24}$$

A direct calculation shows

$$\| \zeta\nabla v \|^2 = \frac{1}{2}((\Delta\zeta^2)v, v) - (\zeta^2\Delta v, v). \tag{2.25}$$

Since $\Delta\zeta^2 \leqq 2k(2k+2-n)(1+r)^{-2k-2}$, we get by (2.25)

$$\| \zeta\nabla v \|^2 \leqq k(2k+2-n) \| (1+r)^{-1-k} v \|^2 + \| \Delta v \| \, \| (1+r)^{-2k} v \| . \tag{2.26}$$

Hölder's inequality gives

$$\| (1+r)^{-2k} v \| \leqq C \| (1+r)^{-1-k} v \|^{\beta/2} \| v \|^{1-(\beta/2)}. \tag{2.27}$$

Thus by (2.26), (2.27) and Hölder's inequality, we see that

$$\| \zeta\nabla v \|^2 \leqq C \| \Delta v \|^{4/(4-\beta)} \| v \|^{2(2-\beta)/(4-\beta)}. \tag{2.28}$$

From (2.21), (2.24) and (2.28) we have

$$\begin{aligned}\| \nabla w \|_b^2 &\leqq C \cdot (\| \Delta v \| + \| \nabla v \|^{2/(4-\beta)} \| v \|^{(2-\beta)/(4-\beta)}) \\ &\qquad \times \| \Delta v \|^{2/(4-\beta)} \| v \|^{(2-\beta)/(4-\beta)} \\ &\leqq C \cdot (\| \Delta v \|^2 + \| \Delta v \|^{4/(4-\beta)} \| v \|^{2(2-\beta)/(4-\beta)}). \end{aligned} \tag{2.29}$$

Since $J^2 w = S(t)(-t^2 \Delta v)$, (2.29) implies (2.18). In the same way as in the proofs of (2.21) and (2.24), we have

$$\| \nabla w \|_b^2 \leqq C \sum_{|\alpha|=2} \| \zeta \partial^\alpha w \| \ \| \zeta \nabla w \| , \tag{2.30}$$

$$\sum_{|\alpha|=2} \| \zeta \partial^\alpha w \| \leqq C \| \zeta \Delta w \|^2 + \| \zeta \nabla w \|^2) . \tag{2.31}$$

(2.19) follows from (2.30) and (2.31). Q. E. D.

LEMMA 2.5. — We assume that the assumptions of lemma 2.4 are satisfied. Then we have

$$\sum_{|\alpha|=2} \| J^\alpha w \| \leqq C \cdot (\| J^2 w \| + t^{2(2-\beta)/(4-\beta)} \| J^2 w \|^{2/(4-\beta)} \| w \|^{(2-\beta)/(4-\beta)}) .$$

Proof. — We have for $v = S(-t) w$

$$\sum_{|\alpha|=2} \| \partial^\alpha v \|^2 \leqq \left| \sum_{j,l=1}^{n} \int \partial_j (\partial_l v \cdot \partial_j \partial_l \overline{v} - \partial_j v \cdot \partial_l^2 \overline{v}) dx \right| + \| \Delta v \|^2 . \tag{2.32}$$

In the same way as in the proof of (16) (Chapter 1 in [3]), The first term of the R. H. S. of (2.32) is dominated by

$$\frac{1}{2} \sum_{|\alpha|=2} \| \partial^\alpha v \|^2 + C \| \zeta \nabla v \|^2 , \tag{2.33}$$

Since ∂D is bounded. Thus we have by (2.32) and (2.33)

$$\sum_{|\alpha|=2} \| \partial^\alpha v \|^2 \leqq C(\| \Delta v \|^2 + \| \zeta \nabla v \|^2) . \tag{2.34}$$

In the same way as in the proof of (2.30), we get the desired estimate. Q. E. D.

LEMMA 2.6. — Let u be the solution of (1.1)-(1.3). Then we have

$$\| Ku(t) \|^2 \leqq CI(\phi)(1+t)^{2(2-\beta)/(3-\beta)}(1 + \log (1 + t))^{(4-\beta)/(3-\beta)} ,$$

where β is the same one as that of lemma 2.4.

Proof. — We multiply (2.14) by $\overline{Ku}$ and take the imaginary part to obtain

$$\frac{d}{dt} \| Ku \|^2 + \operatorname{Im} \int \nabla(\nabla Ku \cdot (-2itr\partial_r \overline{u})) dx = 0 . \tag{2.35}$$

We apply (2.5) and the Schwarz inequality to (2.35). Then we have

$$
\begin{aligned}
&\frac{d}{dt}\| \mathrm{K}u \|^2 \\
&= -\mathrm{Im}\int \partial_j(x_j(r^2\nabla u + (n-2)it\nabla u + 2itr\partial_r \nabla u + 2it^2\nabla\partial_t u) \times (2it\nabla \bar{u}))dx \\
&\qquad\qquad \leqq \mathrm{C}t^2(\| \partial_r \nabla u \|_b + t \| \nabla \partial_t u \|_b) \| \nabla u \|_b . \qquad (2.36)
\end{aligned}
$$

Lemmas 2.1-2.2 and (2.36) yield

$$\| \mathrm{K}u \|^2 \leqq \mathrm{CI}(\phi)(1+t)^2. \qquad (2.37)$$

By lemma 2.2, lemma 2.4 (2.18), (2.36) and (2.37) we see that

$$
\begin{aligned}
\frac{d}{dt}\| \mathrm{K}u \|^2 &\leqq \mathrm{C}t^2(t^{-1/2}(1+t)^{-1/2}\mathrm{I}^{1/2} + t \| \nabla\partial_t u \|_b) \\
&\quad \times (t^{-4/(4-\beta)} \| \mathrm{K}u \|^{2/(4-\beta)}\mathrm{I}^{(2-\beta)/2(4-\beta)} + t^{-2} \| \mathrm{K}u \|) \\
&\leqq \mathrm{CI}^{(2-\beta)/2(4-\beta)}(1+t)^{2(2-\beta)/(4-\beta)} \\
&\quad \times (t^{-1/2}(1+t)^{-1/2}\mathrm{I}^{1/2} + t \| \nabla\partial_t u \|_b) \| \mathrm{K}u \|^{2/(4-\beta)} .
\end{aligned}
$$

From this, (2.37), lemma 2.1 and the Schwarz inequality it follows that

$$
\begin{aligned}
\| \mathrm{K}u \|^b &\leqq \| \mathrm{K}u(1) \|^{b_1} + \mathrm{CI}^{b_2/2}\int_1^t (1+s)^{2b_2}(s^{-1/2}(1+s)^{-1/2}\mathrm{I}^{1/2} \\
&\quad + s \| \nabla\partial_s u \|_b)ds \leqq \mathrm{CI}^{b_1/2}\cdot(1+t)^{2b_2}(1+\log(1+t) \\
&\quad + \mathrm{CI}^{b_2/2}\cdot(1+t)^{2b_2}\left(\int_1^t s \| \nabla\partial_s u \|_b^2 ds\right)^{1/2}\left(\int_1^t s^{-1}ds\right)^{1/2} \\
&\leqq \mathrm{CI}^{b_1/2}\cdot(1+t)^{2b_2}(1+\log(1+t)), \qquad (2.38)
\end{aligned}
$$

where $b_1 = 2(3-\beta)/(4-\beta)$, $b_2 = (2-\beta)/(4-\beta)$. Lemma 2.6 follows from (2.38) immediately. Q. E. D.

Proof of Theorem 1. — By Sobolev's inequality (see [*1*], [*3*], [*4*]) we have

$$
\| \psi \|_p \leqq \begin{cases} \mathrm{C} \| \nabla\psi \|^{1-\gamma}\cdot \sum_{|\alpha|=2} \| \partial^\alpha \psi \|^\gamma, & (2.39) \\ \mathrm{C}t^{-1-\gamma} \| \mathrm{J}\psi \|^{1-\gamma}\cdot \sum_{|\alpha|=2} \| \mathrm{J}^\alpha \psi \|, \quad t > 0, & (2.40) \end{cases}
$$

where $p = 2n/(n-2-2\gamma) \geqq 2$, $0 \leqq \gamma \leqq 1/2$ if $n = 3$, $0 \leqq \gamma < 1$ if $n = 4$, $0 \leqq \gamma \leqq 1$ if $n \geqq 5$. We have by lemma 2.1, lemma 2.5, (2.37) and (2.40)

$$
\begin{aligned}
\| u(t) \|_p &\leqq \mathrm{CI}^{(1-\gamma)/2}t^{-1-\gamma}(\| \mathrm{J}^2 u \| + t^{2b_2} \| \mathrm{J}^2 u \|^{2/(4-\beta)}\mathrm{I}^{b_2/2})^\gamma \\
&\leqq \mathrm{CI}^{b_2/2}t^{-1-\gamma+2b_2} \| \mathrm{J}^2 u \|^{(1-b_2)/2} \\
&\leqq \mathrm{CI}^{1/2}t^{-1-\gamma}(1+t)^{2b_2\gamma(4-\beta)/(3-\beta)}(1+\log(1+t))^{\gamma/(3-\beta)} \\
&\leqq \mathrm{CI}^{1/2}t^{-1-\gamma}(1+t)^{2(2-\beta)\gamma/(3-\beta)}(1+\log(1+t))^{\gamma/(3-\beta)}, \quad t>0. \qquad (2.41)
\end{aligned}
$$

From (2.39) it is clear that

$$\| u(t) \|_p \leqq C I^{1/2}. \tag{2.42}$$

Theorem 1 *follows from* (2.41) *and* (2.42). *Q. E. D.*

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REFERENCES

[*1*] A. FRIEDMAN, *Partial Differential Equations:* Holt, Rinehart and Winston, New York, 1969.

[*2*] Y. JING-QI, Comportement à l'infini des solutions d'une équation de Schrödinger non linéaire dans un domaine extérieur, *C. R. Acad. Sc. Paris*, t. **294**, 1982, p. 163-166.

[*3*] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd English ed., Gordon and Breach, New York, 1969.

[*4*] O. A. LADYZHENSKAYA and N. N. URAL'CEVA, *Linear and Quasilinear Equations of Elliptic Type*, Academic Press, New York, 1968.

[*5*] Y. TSUTSUMI, Global solution of the nonlinear Schrödinger equation in exterior domains, *Comm. PDE*, t. **8**, 1983, p. 1337-1374.

[*6*] K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[*7*] Y. CHEN, Global existence of the solution of nonlinear Schrödinger equations in exterior domains, *Acta Mathematicae Applicatae Sinica*, t. **2**, (3), 1985, p. 191-212.

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