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Phase diagram of the Potts model in an external magnetic field

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ABSTRACT. — We study the effect of an external field, h, on the d-dimensional ($d \geq 2$) q-state Potts models. Whenever q and the inverse temperature, β, are large we prove that there exist two unique open trajectories of phase coexistence, in the (h, β)-plane, starting at the zero field transition point $\beta_t(d, q)$. More precisely,

— For $0 < h < h(\beta, d, q)$ the « 0 »-ordered phase (in the direction of h) coexists with the disordered one.

— For $h < 0$ the others $(q - 1)$ ordered phases coexist with the disordered one.

Moreover the surface tensions between coexisting phases are strictly positive and satisfy the prewetting inequalities.

RÉSUMÉ. — Sur un réseau cubique de dimension $d \geq 2$ nous étudions le modèle de Potts à q états de « spin » soumis à un champ magnétique h.

Pour q et β (l'inverse de la température) assez grands ; nous montrons que dans le plan (β, h) il existe deux trajectoires uniques de coexistence de phases pour h différent de zéro. A savoir,

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— Pour $0 < h < h(\beta, q, d)$ la phase ordonnée « 0 » favorisée par le champ coexiste avec la phase désordonnée.
— Pour $h < 0$ les $(q - 1)$ autres phases ordonnées coexistent avec la phase désordonnée.

En plus les tensions superficielles, entre les phases qui coexistent, sont strictement positives et satisfont aux inégalités de mouillage.

1. INTRODUCTION

The q-state Potts model [1] is a generalization of the Ising [2] model to more than two components, and has been a subject of increasingly intense research in recent years, both in two and higher dimensions, because of its richness and its connexions with other systems with physical interest. A summary of results and references can be found in a review article by Wu [13].

An interesting property of the model is that the nature of the phase transition depends on the number of states q. However there exists a critical value of q, $q_c(d)$ depending on the space dimension d, where the phase transition is of first order, if $q > q_c(d)$, otherwise it is continuous.

In the absence of an external magnetic field, it was established by Baxter [4] that $q_c(2) = 4$. This result is also firmly supported by series expansions [5], [6]. In dimension $d > 2$ it was proved [7] by using the standard Pirogov-Sinai (P.S.) theory [8], [9] with the help of the duality transformation, that there exists a unique point of first order phase transition $\beta_t(q)$ where q ordered phases and the disordered one coexist, whenever q is large enough.

In this paper we consider the effect of an external field on the d-dimensional ($d \geq 2$) Potts model; however for the two components ($q = 2$) Potts model that is the Ising model, it is well known from rigorous theorems [10], [11] that an applied magnetic field destroys the transition and the partition function has no singularities for $h \neq 0$. For $q > 2$ the situation is less clear but some numerical calculations has been performed. Goldschmit [12] use the $1/q$ expansion for the partition function and found that for $q \leq q_c(d)$ an external field applied in the direction of a state has the same effects as for the Ising model, i. e. rubs out the transition, while for $q > q_c(d)$ there is a first order phase transition line starting at the zero field transition point and terminating in a critical point in the interior of the h – T plane, with h positive. Although mean field theory clearly fails in general, Mittag *et al.* [13] point out that the theory provides an accurate description of the Potts model transition when the number of components is large. Indeed they have conjectured that it becomes exact in the limit $q \uparrow \infty$. However it is this conjecture that it is proved and applied to the Potts model in an external magnetic field by Griffiths *et al.* [*14*]. On the other hand this model was also solved analytically on Bethe lattices by Peruggi *et al.* [*15*] for the ferromagnetic interactions they found a phase diagram which is similar to that suggested by [*12*] and [*14*] for $h > 0$. A rigorous result about the phase diagram of the model in an external positive magnetic field is derived by Bricmont *et al.* [*24*] using the reflexion positivity.

In the present article we shall use as in [*7*], the standard P.S. theory combined with the duality transformation which will serve to transform a disordered boundary condition (b.c.) into ordered (b.c.) in the dual model. This approach enables us to study the Potts model in an external field in the both positive and negative field regions. We prove the existence of two unique open trajectories, of phase coexistence, starting at the zero field transition point, $\beta_t(q, d)$, where the disordered phase coexists with the 0-ordered one in the positive field region and with the other $(q - 1)$ ordered phases in the negative field region.

The paper is organized as follows:

In Section 2 we introduce the model with the help of the cell complex formalism and we formulate our main results. In Section 3 we give a formulation of the model in terms of the P.S. theory. Section 4 contains the proof of the results.

2. DEFINITIONS AND RESULTS

2.1. Definitions.

2.1.1. Cell Complex formalism

Here we will introduce some notions on the cell complex formalism which is very convinient when dealing with the duality transformation which is crutial in the approach proposed in [*16*], [*7*].

We will denote $\mathbf{L}$ the cell complex associated to the lattice $\mathbf{Z}^d$ $(d \geq 2)$ and L^p, $p = 0, 1, \ldots, d$, the set of p-cells, s^p in L, to each p-cell (i. e. a site $(p = 0)$, a link $(p = 1)$, a plaquette $(p = 2)$, ...) is assigned a non-negative integer, p, called its dimension. Each p-cell is in correspondence with another p-cell $(- s^p)$ i. e. a cell with opposite orientation.

A p-chain c^p over the coefficient domain, G, (G is a ring with unity) is an odd function of p-cells over G and it may be written as a sum of monomial chains,

$$c^p = \sum_{n=1}^{N^p(\mathrm{L})} m_n^p s^p, \qquad m_n^p \in \mathrm{G}$$

here the group law is denoted additively and $N^p(\mathbf{L})$ is the rank of the group, $C^p(\mathbf{L}, \mathbf{Z})$ of integral p-chains. Taking into account that a p-cell is a particular integral p-chain one defines as usual, the boundary, $\partial : C^p(\mathbf{L}, \mathbf{Z}) \to C^{p-1}(\mathbf{L}, \mathbf{Z})$ and the coboundary, $\partial^* : C^p(\mathbf{L}, \mathbf{Z}) \to C^{p+1}(\mathbf{L}, \mathbf{Z})$, operators.

A finite cell complex K of $\mathbf{L}$, $K \subset \mathbf{L}$ is called to be closed (resp. open) if it contains with every cell also the cells on its boundary (resp. coboundary), we will denote by $\overline{K}$ the closure of the complex K, i. e. the minimal closed complex containing K.

A homomorphism σ^p from $C^p(\mathbf{L}, \mathbf{Z})$ into an abelian group G is called a G-valued p-chain. The set of G-valued p-chains of K forms an abelian group denoted $C^p(\mathbf{L}, G)$. Any $\sigma^p \in C^p(\mathbf{L}, G)$ is determined by its value on the p-chains s^p. One may define the differential $d : C^p(\mathbf{L}, G) \to C^{p+1}(\mathbf{L}, Z)$ and the codifferential, $d^* : C^p(\mathbf{L}, G) \to C^{p-1}(\mathbf{L}, G)$, operators such that

$$d\sigma^p(c^{p+1}) = \sigma^p(\partial c^{p+1}), \qquad \text{and} \qquad d^*\sigma^p(c^{p-1}) = \sigma^p(\partial^* c^{p-1}).$$

2.1.2. Potts model in an external field

Hereafter we will consider $G = Z_q = \{0, \dots, q\}$ the group of integers with the addition modulo q as a group law. Then it is clear that $C^0(\mathbf{L}, Z_q) \equiv C^0(\mathbf{L})$ is the configuration space. Namely, to each site $s^0 \in \mathbf{L}$ we associate a « spin » variable $\sigma(s^0)$ with values in Z_q. Here σ is a configuration in $C^0(\mathbf{L})$, $\sigma \in C^0(\mathbf{L})$. We consider V an open cell complex of $\mathbf{L}$, $V \subset \mathbf{L}$ and $\overline{V}$ its closure such that $V = \overline{V} \backslash B(\overline{V})$ ($B(\overline{V})$ is the boundary of $\overline{V}$ defined equals $\overline{\mathbf{L} \backslash \overline{V}} \cap \overline{V}$). Defining, $C^0(\overline{V})$, the restriction of the configuration space $C^0(\mathbf{L})$ to $\overline{V}$ and restrinting the operators ∂, ∂^* and d, d^* to $\overline{V}$ and $C^0(\overline{V})$ respectively. We define the hamiltonian

$$-\beta H_{\overline{V}}(\sigma) = \beta \sum_{s^1 \in \overline{V}} \delta(d\sigma(s^1)) + \beta h \sum_{s^0 \in \overline{V}} \delta(\sigma(s^0)) \tag{2.2.1}$$

here β is the inverse temperature and h is a real parameter. δ is the Kronecker symbol satisfying: $\delta(\alpha) = 1$ if $\alpha = 0 \bmod (q)$ and $\delta(\alpha) = 0$ otherwise.

Remark. — It is clear from (2.2.1) that the hamiltonian $H_{\overline{V}}(\sigma)$ is with free « f » boundary conditions (b.c.) since $\overline{V}$ is closed.

To consider different kind of boundary conditions we shall introduce a characteristic function, χ, specifying configurations on the boundary $B(\overline{V})$ of $\overline{V}$. In a such way we introduce the equilibrium conditional Gibbs measure:

$$\mu^{\text{b.c.}}_{\overline{V},\beta,h}(\sigma) = (Z(\overline{V}, H_{\overline{V}} \mid \text{b.c.}))^{-1} e^{-\beta H_{\overline{V}}(\sigma)} \chi^{\text{b.c.}}_{\overline{V}}(\sigma) \tag{2.2.2}$$

here the partition function, $Z(\overline{V}, H_{\overline{V}} \mid \text{b.c.})$, is defined such that

$$\sum_{\sigma \in C^0(\overline{V})} \mu^{\text{b.c.}}_{\overline{V},\beta,h}(\sigma) = 1\,. \tag{2.2.3}$$

For a function, g, we denote

$$\langle g \rangle_{\overline{\mathrm{V}}}^{\mathrm{b.c.}} = \sum_{\sigma \in \mathrm{C}^0(\overline{\mathrm{V}})} \mu_{\overline{\mathrm{V}},\beta,h}^{\mathrm{b.c.}}(\sigma)$$

its expectation value in $\overline{\mathrm{V}}$ and $\langle g \rangle^{\mathrm{b.c.}}(\beta, h)$ the corresponding infinite volume limit, $\overline{\mathrm{V}} \uparrow \mathrm{L}$, with respect to boundary conditions. In this article we will be interested in the following (b.c.)

— Free « f » (b.c.) described by $\chi_{\overline{\mathrm{V}}}^{f}(\sigma) \equiv 1$

— « α » $\in \mathrm{Z}_q$ (b.c.) defined by

$$\chi_{\overline{\mathrm{V}}}^{\alpha}(\sigma) = \prod_{s^0 \in \mathrm{C}^0(\overline{\mathrm{V}})} \delta(\sigma(s^0) - \alpha)\,. \tag{2.2.4}$$

2.2. **Results.**

The phase diagram of the q-state Potts model without an external field was studied by different authors [*17*], [*18*], [*19*], [*16*], [*7*]. It exhibits a first order phase transition at some temperature $\beta_t^{-1}(d, q)$ (for q large enough) where q ordered phases and the disordered one coexist. Here we prove the following theorem for the Potts model submitted to an external field.

To state the theorem we will denote $\mathrm{L}_{0,\mathrm{dis}}$ and $\mathrm{L}_{\alpha \neq 0,\mathrm{dis}}$ the trajectories contained, respectively, in a tiny strip of width of order $\mathrm{O}(2e^{-\beta/4})$ (β large) around the two following trajectories in the (β, h) plane:

$$d(\beta - \ln(1 + (e^{\beta} - 1)/q)) + \beta h - \ln(1 + (e^{\beta h} - 1)/q) - \ln(q) = 0$$
$$d(\beta - \ln(1 + (e^{\beta} - 1)/q)) - \ln(1 + (e^{\beta h} - 1)/q) - \ln(q) = 0\,.$$

THEOREM 2.2.1. — Whenever q is large than $3\,d(1 + \ln(\mu(d))+(d-1)\ln(2)$ and β large than $3\,(1 + \ln(2\mu(d))$, here $\mu(d)$ is a geometric parameter depending on the dimension d of the lattice, there exist two unique open trajectories $\mathrm{L}_{0,\mathrm{dis}}$ and $\mathrm{L}_{\alpha \neq 0,\mathrm{dis}}$ of phases coexistence. More precisely,

ci) For $h \in (0, (\ln(q) - 3d(1 + \ln(\mu(d))) + (d-2)\ln(2))/\beta)$ at least two phases, the 0-ordered phase and the disordered, one coexist on $\mathrm{L}_{0,\mathrm{dis}}$.

cii) For $h < 0$ at least q phases, $(q-1)$ ordered phases and the disordered one coexist on $\mathrm{L}_{\alpha \neq 0,\mathrm{dis}}$.

Comments.

1) To prove the theorem it is sufficient to prove the following two statements

— on the line $\mathrm{L}_{0,\mathrm{dis}}$ it is

$$\langle \delta(\sigma(s^0) >^{0} > 1/2\,, \qquad < \delta(\sigma(s^0) >^{f} < 1/2$$
$$\langle \delta(d\sigma(s^1) >^{0} > 1/2, \qquad < \delta(d\sigma(s^1) >^{f} < 1/2$$

— on the line $L_{\alpha\neq 0,\mathrm{dis}}$ it is

$$< \delta(\sigma(s^0) - \alpha) >^{\alpha\neq 0} > 1/2, \quad < \delta(\sigma(s^0) - \alpha) >^f < 1/2$$
$$< \delta(d\sigma(s^1)) >^{\alpha\neq 0} > 1/2 \quad , \quad < \delta(d\sigma(s^1)) >^f < 1/2 \quad .$$

2) On can use the results of Section 3 to prove that

— On the line $L_{0,\mathrm{dis}}$ the surface tension, $\tau^{0,f}$, between the 0-ordered phase and the disordered one is strictly positive.

— On the line $L_{\alpha\neq 0,\mathrm{dis}}$ the surface tensions τ^{α_1,α_2} and $\tau^{\alpha,f}$, between ordered phases and between an ordered phase and the disordered one are strictly positives.

Since the correlations inequalities suggested in [*20*] and [*25*] for the Potts model in an external positive magnetic field can be extended to the situation where the magnetic field is negative [*26*] therefore the surface tensions τ^{α_1,α_2}, $\tau^{\alpha_1,f}$ and $\tau^{\alpha_2,f}$ satisfy the following prewetting inequality:

$$\tau^{\alpha_1,\alpha_2} \geq \tau^{\alpha_1,f} + \tau^{\alpha_2,f}$$

here α_1 and α_2 belongs to Z_q.

Comments on the inequality ([1]).

Let β large and consider the Ising model with a strictly positive magnetic field. It is well known that there exists a unique Gibbs state corresponding to the state $(+)$ and in this case the surface tension, $\tau^{+,-}(\beta, h)$ vanishes. The same situation may occur in some models considered by the P.S. theory.

Now we consider the Potts model with a magnetic field in the direction of the « 0 » state. Here the situation is very different on the two lines of phase coexistence. Namely,

a) For $0 < h < h(\beta, d, q)$ if one consider the instabe « $\alpha \neq 0$ » (b.c.) our system will prefer the stable « f » (b.c.) instead of the « 0 » (b.c.) which is also a stable boundary condition.

To see that we will consider the inequality $\tau^{0,\alpha} \geq \tau^{0,f} + \tau^{\alpha,f}$ and suppose that the « 0 » stable (b.c.) is prefered to the instable « α » then like for the Ising model the surface tension $\tau^{0,\alpha}$ vanishes but this contradicts the fact that the surface tension $\tau^{0,f}$ is strictly positive on the line $L_{\alpha\neq 0,\mathrm{dis}}$.

Therefore to satisfy the inequality our system will prefer the « f » (b.c.) instead of « α » (b.c.), then one obtain rigorousely that

$$\tau^{0,\alpha} \geqslant \tau^{0,f} > 0 .$$

b) For $0 < h$ the same arguments hold for the « 0 » instable phase in this case.

([1]) L. L. is gratefull to Alain Messager about discussions on this comments.

3. FORMULATION IN TERMS OF PIROGOV-SINAI THEORY

3.1. Motivations.

In this article we will consider the coexistence of ordered phases and of the disordered one. One believes that the ($\alpha \neq 0$)-ordered phase and the 0-ordered one coexist only at $h = 0$, hence two situations, corresponding respectively, to h positive and h negative will be analyzed.

To expand partition functions in terms of contours separating these expected stable « pure » phases we will describe how our procedure is achieved. We consider a volume $\overline{\mathrm{V}}$ (a closed finite subcomplex of **L**) with all spin variables on the sites of the boundary of $\overline{\mathrm{V}}$ fixed equals to $\alpha \in \mathrm{Z}_q$. It then follows that for every configuration $\sigma \in \mathrm{C}^0(\overline{\mathrm{V}})$ there exists a component of ordered links $\delta(d\sigma(s^1)=1$, and $\sigma(s^0)=\alpha)$ connected to the boundary of $\overline{\mathrm{V}}$ and isolated from other components by a region of disordered links ($\delta(d\sigma(s^1) = 0$). The latter region will be decomposed into disjoint connectivity components called *external contours.* The connected components in the complement of the union of external contours and of the component connected to the boundary of $\overline{\mathrm{V}}$ are with boundary conditions different from α. On partition functions of these components we perform the duality transformation to obtain partition functions with « 0 » (b.c.) in the dual lattice.

At the end we get a model with contours separating regions with ordered configurations of the original model and of its dual transform. We apply the P.S. theory as in [7] to obtain an expression of partition functions in terms of two standard contour models with parameters. q of these parameters specify ordered phases of the original model and one parameter specify the ordered phase of the dual model which yields a relevant information about the « disordered » phase of the original model.

3.2. Duality transformation.

The cell complex **L*** is said to be the dual of the cell complex **L** if there exists a one to one correspondence $s^p \leftrightarrow s^{d-p}$, between p-cells of **L** and $(d-p)$-cells of **L***. The lattice $(\mathrm{Z}^d)^* = \{ x^1 + 1/2, \ldots, x^d + 1/2,\ i=1, \ldots, d \}$ is the dual lattice of Z^d.

To any cell complex $\mathrm{K} \subset \mathrm{L}$ there exists a dual complex K^*, $\mathrm{K}^* \subset \mathbf{L}^*$, such that if K is closed (resp. open) then K^* is open (resp. closed). Hereafter we will denote K_* a finite subcomplex of **L*** while K^* is the dual complex of K.

To introduce the dual transform of partition functions and expectations, we define the hamiltonian of the dual model,

$$- \mathrm{H}_{\mathrm{K}^*}(\sigma) = \beta^* \sum_{s^{d-1} \in \mathrm{K}^*} \delta(d\sigma(s^{d-1})) + (\beta h)^* \sum_{s^d \in \mathrm{K}^*} \delta(\sigma(s^d)) \qquad (3.2.1)$$

here the operator d is restricted to K^*.

We notice that if K is open (resp. closed) then the hamiltonian in (3.2.1) is with « 0 » (b.c.) (resp. « f » b.c.)
$(\beta h)^*$ and β^* are respectively defined by:

$$(e^{\beta h} - 1)(e^{(\beta h)^*} - 1) = q = (e^{\beta} - 1)(e^{\beta^*} - 1). \qquad (3.2.2)$$

To introduce the duality relation for partition functions and expectations we will consider $\mathrm{K} \subset \mathbf{L}$ as closed subcomplex of $\mathbf{L}$ and $\mathrm{Z}(\mathrm{K}, \mathrm{H}_{\mathrm{K}})$ the partition function with free boundary condition on K. By mean of the conditional Gibbs measure in (2.2.2) we compute the expectations $< \delta(d\sigma(\sigma^1)) >^f(\beta, h)$ and $< \delta(\sigma(\sigma^0)) >^f(\beta, h)$.

PROPOSITION 3.2.1.

$$a) \quad \mathrm{Z}(\mathrm{K}, \mathrm{H}_{\mathrm{K}}) = \left(\frac{e^{\beta} - 1}{q}\right)^{\mathrm{N}^1(\mathrm{K})} (e^{\beta h} - 1)^{\mathrm{N}^0(\mathrm{K})} \mathrm{Z}(\mathrm{K}^*, \mathrm{H}_{\mathrm{K}^*}). \qquad (3.2.3)$$

$$b) \quad \langle \delta(d\sigma(s^1)) - \lambda(\beta)/2 \rangle_{\mathrm{K}}^{f} = \frac{\lambda(\beta)}{2} - \lambda'(\beta, q)\lambda(\beta) \langle \delta(\sigma(s^{d-1})) \rangle_{\mathrm{K}^*}^{0}. \qquad (3.2.4)$$

$$c) \quad \left\langle \delta(\sigma(s^0)) - \frac{\lambda(\beta h)}{2} \right\rangle_{\mathrm{K}}^{f} = \frac{\lambda(\beta h)}{2} - \lambda(\beta h)\lambda'(\beta h) \langle \delta(d\sigma(s^d)) \rangle_{\mathrm{K}^*}^{0} \qquad (3.2.5)$$

here $\lambda(x) = e^x/(e^x - 1)$ *and* $\lambda'(x, q) = q/(q + e^x - 1)$.

Proof. — The statements above are a slight modification of the results from Proposition 3.1 in [*16*].

3.3. **Definition of contours.**

To introduce contours we shall define the notion of the envelope of a set Q^p of lattice p-cells contained in L^p, $\mathrm{Q}^p \subset \mathrm{L}^p$. We will denote $\overline{\mathrm{Q}}^p$ the closure of Q^p and define the *envelope* $\mathrm{E}(\mathrm{Q}^p)$ of Q^p as the maximal closed complex of $\mathbf{L}$ whose sets of lattice r-cells $r \leq p$ coincide with $\overline{\mathrm{Q}}^p$, $\mathrm{E}(\mathrm{Q}^p) \cap \mathrm{L}^p = \mathrm{Q}^p$. An explicit expression is $\mathrm{E}(\mathrm{Q}^p) = \cup_{q=p+1}^{d} \mathrm{E}^q(\mathrm{Q}^p) \cup \overline{\mathrm{Q}}^p$ with $\mathrm{E}^p(\mathrm{Q}^p) = \mathrm{Q}^p$ and $\mathrm{E}^q(\mathrm{Q}^p) = \{ s^q \in \mathrm{L}^q \mid \text{all } s^{q-1} \text{ of } \partial s^q \text{ belongs to } \mathrm{E}^{q-1}(\mathrm{Q}^p)$

whenever $q \geq p + 1$. We define the *fringe*, $F(Q^p)$ of Q^p as $F(Q^p) = \mathbf{L} \setminus \{ E(L^0 \setminus L^0 \cap E(Q^p)) \cup E(Q^p) \}$.

Consider thus a configuration $\sigma \in C^0(\mathbf{L})$ such that

$$M^1(\sigma) = \{ s^1 \in L^1 \mid d\sigma(s^1) \neq 0 \bmod (q) \} \quad \text{is finite},$$
$$| M^1(\sigma) | < \infty \qquad \text{and} \qquad Q^1(\sigma) = \{ s^1 \in L^1 \mid d\sigma(s^1) = 0 \bmod (q) \},$$

the unique infinite component of $\mathbf{L} \setminus M^1(\sigma)$. We shall denote $F(\sigma)$ the fringe of $Q^1(\sigma)$, $F(\sigma) = F(Q^1(\sigma))$.

A pair $\tilde{\gamma} = (\gamma, \sigma_\gamma)$ where γ is a connectivity component of $F(\sigma)$ and σ_γ the restriction of the configuration σ to the complex γ, will be called an external contour of σ. A pair $\tilde{\gamma} = (\gamma, \sigma_\gamma)$ with γ a subcomplex of $\mathbf{L}$ and σ_γ a configuration on it will be called a contour if there exists a configuration $\sigma \in C^0(\mathbf{L})$ such that $| M^1(\sigma) | < \infty$ and $\tilde{\gamma}$ is its external contour. Whenever $\tilde{\gamma}$ is a contour we call the complex γ its support, $\gamma = \operatorname{supp} \tilde{\gamma}$, and introduce the complexes: Ext γ as the envelope of the set of lattice sites of the unique infinite component of $\mathbf{L} \setminus \gamma$, $V(\gamma) = \mathbf{L} \setminus \operatorname{Ext} \gamma$ and $\operatorname{Int} \gamma = V(\gamma) \setminus \gamma$.

Two contours $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ with disjoint supports $\gamma_i \cap \gamma_j = \phi$ are called mutually compatible. $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ are mutually compatible external contours if $V(\gamma_i) \subset \operatorname{Ext} \gamma_j$ and $V(\gamma_j) \subset \operatorname{Ext} \gamma_i$. We define $\tilde{\theta} = \{ \tilde{\gamma}_1 \ldots, \tilde{\gamma}_i, \ldots \tilde{\gamma}_n \}$ a family of mutually compatible external contours and define $\theta = \operatorname{supp} \tilde{\theta} = \cup_i \gamma_i V(\theta) = \mathbf{L} \setminus \operatorname{Ext} \theta$, $\operatorname{Int} \theta = V(\theta) \setminus \theta$. Let $K \subset \mathbf{L}$ we will denote $\operatorname{Ext}_K \theta = K \cap \operatorname{Ext} \theta$.

For contours on the dual lattice $\mathbf{L}^*$ (defined in the same way as above with $\mathbf{L}$ replaced by $\mathbf{L}^*$) we shall denote $\tilde{\gamma}_* = (\gamma_*, \sigma_{\gamma_*})$. Notice that γ_* as a support of a contour is an open cell complex of $\mathbf{L}^*$ while γ^* is the dual of the complex γ and it is thus a closed complex of $\mathbf{L}^*$ for every $\tilde{\gamma} = (\gamma, \sigma_\gamma)$.

3.4. **Inductive expression of partition functions.**

To expand partition functions in terms of contours, we introduce the notion of disordered « dis » boundary conditions. In accordance with definitions of Section 3.3, the partition functions with « dis » (b.c.) are defined on $V(\theta)$, with θ a family of external contours on $\mathbf{L}$

$$Z(V(\theta), H_{V(\theta)}) = \sum_{\sigma \in C^0(V(\theta))} e^{-H_{V(\theta)}(\sigma)} \chi^{\text{dis}}_{V(\theta)}(\sigma). \tag{3.4.1}$$

$$\text{with } \chi^{\text{dis}}_{V(\theta)}(\sigma) = \prod_{\gamma \in \theta} \chi^{\text{dis}}_{V(\gamma)}(\sigma) = \prod_{\gamma \in \theta} \prod_{s^1 \in \gamma} (1 - \delta(d\sigma(s^1) - \alpha)) \tag{3.4.2}$$

here the differential operator d is restricted to $V(\theta)$ and $\alpha \in Z_q$

$$Z(V(\theta_*), H_{V(\theta^*)} \mid \text{dis}) = \sum_{\sigma \in C^{d-1}(V(\theta_*))} e^{-H_{V(\theta_*)}(\sigma)} \chi^{\text{dis}}_{V(\theta_*)}(\sigma). \tag{3.4.3}$$

$$\chi^{\text{dis}}_{V(\theta_*)}(\sigma) = \prod_{\gamma_* \in \theta_*} \chi^{\text{dis}}_{V(\gamma_*)}(\sigma) = \prod_{\gamma_* \in \theta_*} \prod_{s^{d-1} \in \gamma_*} (1 - \delta(\sigma(s^{d-1}))). \tag{3.4.4}$$

Now we state a lemma (closer to Lemma 2.4.1 from [7]) which serves as a starting for expanding partition functions in terms of contour models.

LEMMA 3.4.1. — *Let* V *and* V_* *be two closed cell complexes of* **L** *and* $\mathbf{L}^*$ *and* θ, θ_* *families of external contours in* **L** *and* $\mathbf{L}^*$, *respectively, then*

a) $$Z(\overline{V}, H_{\overline{V}} \mid \alpha) = \sum_{\theta \subset V} e^{\beta N^1(\text{Ext}_{\overline{V}}(\theta)) + \beta h \delta(\alpha) N^0(\text{Ext}_{\overline{V}}(\theta))} Z(V(\theta), H_{V(\theta)} \mid \text{dis}).$$

b) $$Z(\overline{V}_*, H_{\overline{V}^*} \mid 0) = \sum_{\theta_* \subset V_*} e^{\beta^* N^{d-1}(\text{Ext}_{\overline{V}_*}(\theta_*)) + (\beta h)^* N^d(\text{Ext}_{\overline{V}_*}(\theta_*))} Z(V(\theta_*), H_{V(\theta_*)} \mid \text{dis}).$$

c) $$Z(V(\theta), H_{V(\theta)} \mid \text{dis}) = \prod_{\gamma \in \theta} Z(V(\gamma), H_{V(\gamma)} \mid \text{dis}).$$

d) $$Z(V(\theta_*), H_{V(\theta_*)} \mid \text{dis}) = \prod_{\gamma_* \in \theta_*} Z(V(\gamma_*), H_{V(\gamma_*)} \mid \text{dis}).$$

e) $$Z(V(\gamma), H_{V(\gamma)} \mid \text{dis}) = \\ = g(\gamma, \beta, h) \left(\frac{e^{\beta} - 1}{q} \right)^{N^1(\text{Int}\,\gamma)} (e^{\beta h} - 1)^{N^0(\text{Int}\,\gamma)} Z((\text{Int}\,\gamma)^*, H_{(\text{Int}\,\gamma)^*} \mid 0).$$

f) $$Z(V(\gamma_*), H_{V(\gamma_*)} \mid \text{dis}) = g(\gamma_*, \beta^*, (\beta h)^*)$$

$$\left(\frac{e^{(\beta h)^*} - 1}{q} \right)^{N^d(\text{Int}\,\gamma_*)} (e^{\beta h} - 1)^{N^{d-1}(\text{Int}\,\gamma_*)} (1 - e^{-\beta h})^{-N^d(\gamma_*)} q^{N^{d-1}(\gamma_*)}$$

$$Z((\text{Int}\,\gamma)^*, H_{(\text{Int}\,\gamma_*)^*} \mid 0).$$

here $$V(\theta) \cap L^1 \subset V$$

and $$V(\theta_*) \cap L^{d-1} \subset V_*$$

Remark. — It is easy to prove that $|g(\gamma, \beta, h)|$ and $|g(\gamma_*, \beta^*, (\beta h)^*)|$ are less than one.

Proof of the lemma. — To prove the statement *a*) we use the definition of contours introduced in Sect. 3.3 to write

$$Z(\overline{V}, H_{\overline{V}} \mid \alpha) = \sum_{\theta \subset V} \Bigg\{ \sum_{\sigma \in C^0(\mathrm{Ext}_{\overline{V}}(\theta))} \prod_{s^1 \in \mathrm{Ext}_{\overline{V}}(\theta))} e^{\beta\delta(d_e\sigma(s'))} \prod_{s^0 \in \mathrm{Ext}_{\overline{V}}\,\theta} e^{\beta h \delta(\sigma(s^0))} \delta(\sigma(s^0) - \alpha)$$

$$\sum_{\mu \in C^0(V(\theta))} \prod_{s^1 \in \theta} (1 - \delta(d_e\sigma(s^1) + d_V \mu(s^1)) e^{-H_{V(\theta)}(\mu, \sigma)} \Bigg\}. \qquad (3.4.5)$$

here d_e and d_V denote the restriction of the differential operator d to $\mathrm{Ext}_{\overline{V}}\,\theta$ and $V(\theta)$ respectively.

Taking into account that

$$e^{\beta h \delta(s^0)} \delta(\sigma(s^0) - \alpha) = e^{\beta\delta(\alpha)} \delta(\sigma(s^0) - \alpha).$$

and that the characteristic function

$$\prod_{s^0 \in \mathrm{Ext}_{\overline{V}}(\theta)} \delta(\sigma(s^0) - \alpha)$$

allows us to restrict all configurations on $\mathrm{Ext}_{\overline{V}}\,\theta$ to the value α, thus the expression (3.4.5) equals,

$$\sum_{\theta \subset V} e^{\beta N^1(\mathrm{Ext}_{\overline{V}}\theta) + \beta h \delta(\alpha) N^0(\mathrm{Ext}_{\overline{V}}\theta)} \sum_{\sigma \in C^0(V(\theta))} e^{-H_{V(\theta)}(\sigma)} \chi^{\alpha}_{V(\theta)}(\sigma).$$

Referring to the relations (3.4.1) and (3.4.3) we derive the statements *a*) and *b*). The statements *c*) and *d*) are a consequence of the definitions (3.4.2) and (3.4.4). To prove the statements *e*) and *f*) we apply the statement *a*) of Proposition 3.2.1.

Now we define the « diluted » and « crystal » partition functions

$$Z^{\mathrm{diL.}}_{\alpha}(\overline{V}, H_{\overline{V}}) = e^{-\beta N^1(B(\overline{V})) - \beta h N^0(B(\overline{V}))\delta(\alpha)} Z(\overline{V}, H_{\overline{V}} \mid \alpha). \qquad (3.4.6)$$

$$Z^{\mathrm{crys.}}(\gamma, H_{V(\gamma)}) = Z(V(\gamma), H_{V(\gamma)} \mid \mathrm{dis}).$$

$$Z^{\mathrm{diL.}}(\overline{V}_*, H_{\overline{V}_*}) = e^{-\beta^* N^{d-1}(B(\overline{V}_*))} \left(\frac{e^{\beta} - 1}{q}\right)^{N^{d-1}(V_*)} (e^{\beta h} - 1)^{N^d(V_*)} Z(\overline{V}_*, H_{\overline{V}_*} \mid 0).$$

$$(3.4.7)$$

$$Z^{\mathrm{crys.}}(\gamma_*, H_{V(\gamma_*)}) = \left(\frac{e^{\beta} - 1}{q}\right)^{N^{d-1}(V(\gamma_*))} (e^{\beta h} - 1)^{N^d(V(\gamma_*))} Z(V(\gamma_*), H_{V(\gamma_*)} \mid \mathrm{dis}).$$

Under the above definitions Lemma 3.4.1 reads,

LEMMA 3.4.2.

a) $$Z^{\mathrm{diL.}}_{\alpha}(\overline{V}, H_{\overline{V}}) = \sum_{\theta \subset V} e^{\beta N^1(\mathrm{Ext}_V\,\theta) + \beta h \delta(\alpha) N^0(\mathrm{Ext}_V\,\theta)} Z^{\mathrm{crys.}}(\theta, H_{V(\theta)}).$$

$b)$ $$Z^{\text{diL.}}(\overline{V}_*, H_{\overline{V}_*}) = \sum_{\theta_* \subset V^*} e^{\ln(1+(e^\beta-1)/q)N^{d-1}(\text{Ext}_{V_*}\theta_*)} e^{\ln(q+e^{\beta h}-1)N^d(\text{Ext}_{V_*}\theta_*)} Z^{\text{crys.}}(\theta_*, H_{V(\theta_*)}).$$

$c)$ $$Z^{\text{crys.}}(\gamma, H_{V(\gamma)}) = g(\gamma, \beta, h) Z^{\text{diL.}}(V(\gamma)^*, H_{V(\gamma)^*}).$$

$d)$ $$Z^{\text{crys.}}(\gamma_*, H_{V(\gamma_*)}) = g(\gamma_*, \beta^*, (\beta h)^*)(e^\beta - 1)^{N^{d-1}(\gamma_*)} e^{\beta h N^d(\gamma_*)} Z^{\text{diL.}}(V(\gamma_*)^*, H_{V(\gamma_*)^*}).$$

The prefactor in front of the expression (3.4.7) assures that both diluted partition functions defined in (3.4.6) and (3.4.7) yield the free energy,

$$F(\beta, h) = \lim_{V \nearrow L} \left(\frac{1}{N^0(\overline{V})}\right) \ln Z^{\text{diL.}}(\overline{V}, H_{\overline{V}}) = \lim_{V^* \nearrow L^*} \left(\frac{1}{N^d(\overline{V}^*)}\right) \ln Z^{\text{diL.}}(\overline{V}^*, H_{\overline{V}^*}). \tag{3.4.8}$$

here the limit is over complexes approaching **L** and **L*** in the Van Hove sense. Namely

$$\lim_{\overline{V} \nearrow L} N^0(B(\overline{V}))/N^0(\overline{V}) = \lim_{\overline{V}^* \nearrow L^*} N^d(B(\overline{V}^*))/N^d(V^*) = 0.$$

Proof. — It is well known that the free energy, $F(\beta, h)$, as defined above is independent of the boundary conditions. In accordance with the statement $a)$ of Proposition 3.2.1 which relate the partition function with free « f » (b.c.) in the original model to the partition function with « 0 » (b.c.) of the dual model and taking into account that

$$\left(\frac{e^\beta - 1}{q}\right)^{N^1(V)} (e^{\beta h} - 1)^{N^0(V)} = \left(\frac{e^\beta - 1}{q}\right)^{dN^0(V)} (e^{\beta h} - 1)^{N^0(V)} \left(\frac{e^\beta - 1}{q}\right)^{N^1(V) - dN^0(V)}.$$

with

$$\lim_{V \nearrow L} \frac{(N^1(V) - dN^0(V))}{N^0(V)} = 0.$$

thus we recover the equality stated in (3.4.8).

3.5. **Partition functions in terms of contour models.**

In the following we will express the partition function $Z(\overline{V}, H_{\overline{V}} \mid \alpha)$ and $Z(\overline{V}_*, H_{\overline{V}^*} \mid 0)$ and corresponding probabilities of external contours in terms of $(q+1)$ contour models $\Phi_\alpha, \alpha = 0 \ldots, q-1$ and Φ_*, living on contours of **L** and **L*** respectively. The former describe the q ordered phases of the model and the latter (taking into account the duality transformation) will yield some information about the disordered phase of the same model.

To describe the phase diagram of our model we introduce as in the P.S. theory, contour models with parameters b_α, b_*. The transition lines $L_{0,\mathrm{dis}}$ and $L_{\alpha\neq 0,\mathrm{dis}}$ will be identified as the unique lines for which $b_0 = b_* = 0$ (corresponding to h positive in the sense that will be clarified later) and $b_{\alpha\neq 0} = b_* = 0$ (corresponding to a negative magnetic field).

These lines turn out to be near the lines determined respectively by the equations

$$d\left(\beta - \operatorname{Ln}\left(1 + \frac{e^\beta - 1}{q}\right)\right) + \beta h - \operatorname{Ln}(q + e^{\beta h} - 1) = 0\,.$$

$$d\left(\beta - \operatorname{Ln}\left(1 + \frac{e^\beta - 1}{q}\right)\right) - \operatorname{Ln}(q + e^{\beta h} - 1) = 0\,.$$

Considering $D(\overline{V})$ the set of families, ∂, of mutually compatible contours in $\overline{V}$ satisfying $V(\gamma) \cap L^1 \subset V$ we will write

$$Z(\overline{V} \mid \phi_\alpha) = \sum_{\partial \subset D(\overline{V})} \phi_\alpha(\partial)\,.$$

with

$$\phi_\alpha(\partial) = \prod_{\gamma \in \partial} \phi_\alpha(\gamma)\,.$$

Referring the reader to [*21*] for the theory of contour models we just recall that introducing

$$Z(\gamma \mid \phi_\alpha) = \phi_\alpha(\gamma) Z(\operatorname{Int} \gamma \mid \phi_\alpha)\,.$$

one has

$$Z(\overline{V} \mid \phi_\alpha) = \sum_{\partial \subset D(\overline{V})} \prod_{\gamma \in \theta} Z(\gamma \mid \phi_\alpha) = \sum_{\theta \subset V} Z(\theta \mid \phi_\alpha)\,.$$

with θ a family of external contours in V.

The partition function of a contour model Φ_α on contours of $\mathbf{L}$ with a parameter b_α are defined

$$Z(\overline{V} \mid \phi_\alpha, b_\alpha) = \sum_{\theta \subset V} e^{b_\alpha N^0(V(\theta))} Z(\theta \mid \phi_\alpha)\,.$$

and those of a contour models Φ_* on contours of $\mathbf{L}^*$ with parameter b_* by

$$Z(\overline{V}_* \mid \phi_*, b_*) = \sum_{\theta^* \subset V^*} e^{b_* N^d(V(\theta^*))} Z(\theta_* \mid \phi_*)\,.$$

DÉFINITION. — *Φ_α (resp. Φ_*) is a τ-functional if for every γ (resp. γ_*) it satisfies*

$$|\Phi_\alpha(\gamma)| < e^{-\tau N^1(\gamma)} \qquad (\text{resp. } |\Phi_*(\gamma)| < e^{-\tau N^d}(\gamma_*))$$

here τ is a fixed parameter depending on the dimension d of the lattice. More precisely $\tau > 1 + \ln(2\mu(d))$ here $\mu(d)$ is a geometric parameter.

In particular if Φ_α and Φ_* are τ-functionals the limits

$$f(\phi_\alpha) = \lim_{\overline{V}\nearrow L} \frac{1}{N^0(\overline{V})} \operatorname{Ln} Z(\overline{V} \mid \phi_\alpha).$$

$$f(\phi_*) = \lim_{\overline{V}_*\nearrow L^*} \frac{1}{N^d(\overline{V}_*)} \operatorname{Ln} Z(\overline{V}_* \mid \phi_*).$$

exist and satisfy

$$|f(\phi_\alpha)| \leqslant e^{-\tau}, \qquad |f(\phi_*)| \leqslant e^{-\tau}.$$

The boundary terms

$$\Delta(\overline{V} \mid \phi_\alpha) = \operatorname{Ln} Z(\overline{V} \mid \phi_\alpha) - N^0(\overline{V}) f(\phi_\alpha).$$

$$\Delta(\overline{V}_* \mid \phi_*) = \operatorname{Ln} Z(\overline{V}_* \mid \phi_*) - N^d(\overline{V}_*) f(\phi_*).$$

may be evaluated,

$$|\Delta(\overline{V} \mid \phi_\alpha)| \leqslant N^0(B(\overline{V}))e^{-\tau}, \qquad |\Delta(\overline{V}_* \mid \phi_*)| \leqslant N^{d-1}(B(\overline{V}_*))e^{-\tau}.$$

Now we will formulate a proposition about the equivalence with the contour models which will serve to prove our main theorem

PROPOSITION 3.5.1. — *Suppose that $\beta > 3\tau$ and q large enough then there exist τ-functionals Φ_α and Φ_* and parameters b_α, b_* such that*

a) $\beta d + \beta h \delta(\alpha) + b\alpha + f(\phi_\alpha) = F(\beta, h) =$

$$= \operatorname{Ln}\left(1 + \frac{e^\beta - 1}{q}\right) + \operatorname{Ln}(q + e^{\beta h} - 1) + b_* + f(\phi_*). \quad (3.5.1)$$

and

b) $$Z(\gamma \mid \phi_\alpha) = e^{-b\alpha N^0(V(\gamma)) - \beta N^1(V(\gamma)) - \beta h \delta(\alpha) N^0(V(\gamma))} Z_\alpha^{\text{crys.}}(\gamma, H_{V(\gamma)}). \quad (3.5.2)$$

for every contour γ on $\mathbf{L}$

c) $$Z(\gamma_* \mid \phi_*) = e^{-b_* N^d(V(\gamma_*)) - \ln\left(1 + \frac{e^\beta - 1}{q}\right) N^{d-1}(V(\gamma_*))} e^{-\operatorname{Ln}(q + e^{\beta h} - 1) N^d(V(\gamma_*))} Z^{\text{crys.}}(\gamma_*, H_{V(\gamma_*)}). \quad (3.5.3)$$

for every contour γ_* on $\mathbf{L}^*$.

d) *Moreover* $\min((b_\alpha)_\alpha, b_*) = 0$ *and there exist unique transition lines* $L_{0,\text{dis}}$, $L_{\alpha\neq 0,\text{dis}}$ *such that on the line* $L_{0,\text{dis}}$ *it is* $b_0 = b_* = 0$ *and on the line* $L_{\alpha\neq 0,\text{dis}}$ *it is* $b_{\alpha\neq 0} = b_* = 0$.

Remark. — *We notice that the relation (3.5.2) leads to*

$$Z_\alpha^{\mathrm{diL.}}(\overline{V}, H_{\overline{V}}) = e^{\beta N^1(V) + \beta h\delta(\alpha)N^0(V)} \sum_{\theta\subset V} e^{b_\alpha N^0(V(\theta))} Z(\theta \mid \phi_\alpha)\,.$$

By defining

$$Z(\overline{V} \mid \phi_\alpha, b_\alpha) = \sum_{\theta\subset V} e^{b_\alpha N^0(V(\theta))} Z(\theta \mid \phi_\alpha)\,.$$

one obtains

$$Z(\overline{V} \mid \phi_\alpha, b_\alpha) = e^{-\beta N^1(V) - \beta h\delta(\alpha)N^0(V)} Z_\alpha^{\mathrm{diL.}}(\overline{V}, H_{\overline{V}})\,. \tag{3.5.4}$$

We use the relation (3.5.3) and proceed as above to get,

$$Z(\overline{V}_* \mid \phi_*, b_*) = Z^{\mathrm{diL.}}(\overline{V}_*, H_{\overline{V}_*})\, e^{-\mathrm{Ln}\left(1+\frac{e^\beta - 1}{q}\right)N^{d-1}(V_*) - \mathrm{Ln}\,(q+e^{\beta h}-1)N^d(V_*)}\,. \tag{3.5.5}$$

These relations imply that the contour models Φ_α and Φ_* reproduce the probability of external contours governed by the hamiltonian $H_{\overline{V}}$ and its dual $H_{\overline{V}_*}$ under « α » (b.c.) and « f » (b.c.) respectively. This allows to distinguish the ordered phases and the disordered one by evaluating the expectations, $\langle\, \delta(\sigma(s^p))\,\rangle$ for some fixed p-cell s^p.

Proof of Proposition 3.5.1. — Following the inductive procedure of the P.S. theory, initiated in [*23*] and extended to the Potts model in [*7*] we shall construct contour functionals Φ_α and Φ_*. We observe that for every positive parameters a_α and a_* one may define by induction in $N^1(V(\gamma))$ and in $N^{d-1}(V(\gamma_*))$ contour functionals $\Phi_\alpha^{a_\alpha}$ and $\Phi_*^{a_*}$ satisfying

$$Z(\gamma \mid \phi_\alpha^{a_\alpha}) = e^{-a_\alpha N^0(V(\gamma)) - \beta N^1(V(\gamma)) - \beta h\delta(\alpha)N^0(V(\gamma))} Z^{\mathrm{crys.}}(\gamma, H_{V(\gamma)})\,. \tag{3.5.6}$$

$$Z(\gamma_* \mid \phi_*^{a_*}) = e^{-a_* N^d(V(\gamma_*)) - \mathrm{Ln}\left(1+\frac{e^\beta-1}{q}\right)N^{d-1}(V(\gamma_*))}$$
$$e^{-\mathrm{Ln}\,(q+e^{\beta h}-1)N^d(V(\gamma_*))} Z^{\mathrm{crys.}}(\gamma_*, H_{V(\gamma_*)})\,. \tag{3.5.7}$$

and define

$$\overline{\phi}_\alpha^{a_\alpha}(\gamma) = \min\left(\phi_\alpha^{a_\alpha}(\gamma), e^{-\tau N^1(\gamma)}\right).$$
$$\overline{\phi}_\alpha^{a_*}(\gamma_*) = \min\left(\phi_*^{a_*}(\gamma_*), e^{-\tau N^d(\gamma_*)}\right).$$

and

$$b_\alpha = \mathrm{Sup}\,\{\, a_\alpha \mid a_\alpha + d\beta + \beta h\delta(\alpha) + f(\overline{\phi}_\alpha^{a_\alpha}) \leqslant F(\beta, h)\,\}\,.$$

$$b_* = \mathrm{Sup}\left\{ a_* \mid a_* + \mathrm{Ln}\left(1+\frac{e^\beta-1}{q}\right) + \mathrm{Ln}\,(q+e^{\beta h}-1) \leqslant F(\beta, h) - f(\overline{\phi}_*^{a_*}) \right\}.$$

In fact

$$a_\alpha + \beta h\delta(\alpha) + d\beta + f(\overline{\phi}_\alpha^{a_\alpha}) = F(\beta, h) = \mathrm{Ln}\left(1+\frac{e^\beta-1}{q}\right) + \mathrm{Ln}\,(q+e^{\beta h}-1) + f(\overline{\phi}_*^{a_*})\,. \tag{3.5.9}$$

because one prove that $f(\overline{\Phi}_\alpha^{a_\alpha})$ and $f(\overline{\Phi}_*^{a_*})$ are continuous in a_α and a_* respectively (see [27] and [7]).

By induction in $N^1(V(\gamma))$ and in $N^{d-1}(V(\gamma_*))$ one proves that $\Phi_\alpha^{b_\alpha}$ and $\Phi_*^{b_*}$ are τ-functionals. Namely

$$\overline{\phi}_\alpha^{b_\alpha} = \phi_\alpha^{b_\alpha}, \qquad \overline{\phi}_*^{b_*} = \phi_*^{b_*}.$$

Proceeding, as usual, by supposing that for all $\hat{\gamma}$ with $N^1(V(\gamma)) \leq k$ and all $\hat{\gamma}_*$ with $N^{d-1}(V(\gamma_*)) \leq k$ it is $\Phi_\alpha^{b_\alpha} = \overline{\Phi}_\alpha^{b_\alpha}$ and $\Phi_*^{b_*} = \overline{\Phi}_*^{b_*}$ and consider a contour γ with $N^1(V(\gamma)) \leq k + 1$, hence it follows from the induction hypothesis that

$$Z\left(\text{Int } \gamma \mid \phi_\alpha^{b_\alpha}\right) = Z\left(\text{Int } \gamma \mid \overline{\phi}_\alpha^{b_\alpha}\right) = e^{f(\overline{\phi}_\alpha^{b_\alpha}) N^0(\text{Int } \gamma) + \Delta(\text{Int } \gamma \mid \overline{\phi}_\alpha^{b_\alpha})}. \quad (3.5.10)$$

Since N^1 $(\text{Int } \gamma) = N^{d-1}$ $((\text{Int } \gamma)^*) \leq k$ then for any $\hat{\gamma}_* \subset (\text{Int } \gamma)^*$ we get

$$Z(V(\gamma)^*, \phi_*^{b_*}) = Z(V(\gamma)^* \mid \overline{\phi}_*^{b_*}) = e^{f(\overline{\phi}_*^{b_*}) N^d(V(\gamma)^*) + \Delta(V(\gamma)^* \mid \overline{\phi}_*^{b_*})}. \quad (3.5.11)$$

Now we refer to the relation (3.5.10) to infer that,

$$\phi_\alpha^{b_\alpha}(\gamma) = e^{-b_\alpha N^0(V(\gamma)) - \beta N^1(V(\gamma)) - \beta h \delta(\alpha) N^0(V(\gamma))} Z^{\text{crys.}}(\gamma, H_{V(\gamma)}) / Z\left(\text{Int } \gamma \mid \overline{\phi}_\alpha^{b_\alpha}\right).$$

From the statement $c)$ of the lemma (3.4.2) it follows

$$Z^{\text{crys.}}(\gamma, H_{V(\gamma)}) = g(\gamma, \beta, h) Z^{\text{diL.}}(V(\gamma)^*, H_{V(\gamma)^*}). \quad (3.5.12)$$

The relations (3.5.5) and (3.5.11) lead to the equality

$$Z^{\text{diL.}}(V(\gamma)^*, H_{V(\gamma)^*}) = e^{-\text{Ln}\left(1 + \frac{e^\beta - 1}{q}\right) N^{d-1}((\text{Int } \gamma)^*)}$$
$$e^{-\text{Ln}(q + e^{\beta h} - 1) N^d((\text{Int } \gamma)^*)} Z(V(\gamma)^*, \phi_*^{b_*}, b_*).$$

and to the inequality

$$Z^{\text{diL.}}(V(\gamma)^*, H_{V(\gamma)^*}) \leqslant e^{-\text{Ln}\left(1 + \frac{e^\beta - 1}{q}\right) N^{d-1}((\text{Int } \gamma)^*)}$$
$$e^{-\text{Ln}(q + e^{\beta h} - 1) N^d((\text{Int } \gamma)^*) + b_* N^d((\text{Int } \gamma)^*)} Z(V(\gamma)^* \mid \overline{\phi}_\alpha^{b_\alpha}). \quad (3.5.13)$$

Combining the equalities (3.5.10), (3.5.11), (3.5.12) and the inequality (3.5.13) and using the identity (3.5.1) and taking into account that $N^0(V(\gamma)) = N^0$ $(\text{Int } \gamma)$ and $|g(\gamma, \beta, h)| \leq 1$ we get

$$\phi_\alpha^{b_\alpha}(\gamma) \leqslant e^{-\beta(N^1(V(\gamma)) - d N^0(\text{Int } \gamma))} e^{2e^{-\tau N^1(\gamma)}}$$
$$e^{-\text{Ln}\left(1 + \frac{e^\beta - 1}{q}\right)(N^1(\text{Int } \gamma) - d N^0(\text{Int } \gamma))}$$
$$e^{-\Delta(\text{Int } \gamma \mid \overline{\phi}_\alpha^{b_\alpha}) + \Delta(V(\gamma)^*, \overline{\phi}_*^{b_*})}. \quad (3.5.14)$$

Since

$$\left|\Delta\left(\text{Int } \gamma \mid \overline{\phi}_\alpha^{b_\alpha}\right)\right| \leqslant e^{-\tau} N^0(B(\text{Int } \gamma)), \ \left|\Delta(V(\gamma)^* \mid \overline{\phi}_*^{b_*}\right| \leqslant e^{-\tau} N^{d-1} B(V(\gamma)^*).$$

by noticing that

$$N^1(\text{Int } \gamma) - d N^0(\text{Int } \gamma) = -N^1(\gamma)/2.$$

Therefore the expression (3.5.14) is bounded such that

$$\phi_\alpha^{b_\alpha}(\gamma) \leqslant e^{-\frac{\beta}{2}N^1(\gamma)} e^{-\mathrm{Ln}\left(1+\frac{e^\beta-1}{q}\right)\frac{N^1(\gamma)}{2}} e^{4e^{-\tau}N^1(\gamma)}.$$

Choosing β satisfying the inequality

$$\beta = \ln(1 + (e^\beta - 1)/q) > 2(\tau + 4e^{-\tau})$$

we obtain the desired result. Namely

$$\phi_\alpha^{b_\alpha}(\gamma) \leqslant e^{-\tau N^1(\gamma)}.$$

To prove that $\Phi_*^{b_*}$ is a τ-functional one proceeds similarly for γ_* with $N^{d-1}(V(\gamma_*)) \leq k$ to get

$$\phi_*^{b_*}(\gamma_*) \leqslant e^{-\tau N^d(\gamma_*)}.$$

The proof of the statement *d*) of the proposition is similar to that in [*7*] and [*23*].

4. PROOF OF THEOREM 2.2.1

4.1. Proof of the statement *ci*).

First step :

Taking into account that $\langle \delta(\sigma(s^p)) \rangle^0 = 1 - \mathrm{Prob}(\sigma(s^p) \neq 0 \mid «0»)$, here $\mathrm{Prob}(\sigma(s^p) \neq 0 \mid «0»)$ is the probability that $\sigma(s^p) \neq 0$ given «0» (b.c.). Therefore for every configuration $\sigma \in C^0(L)$ with $\sigma(s^p) = 0$ for $s^0 \in L\backslash V$ there exists a contour γ belonging to the family, θ, of external contours, $\theta \subset V$, such that

$$\mathrm{Prob}(\sigma(s^0) \neq 0 \mid «0») \leqslant \sum_{\gamma: s^0 \in V(\gamma)} \sum_{\theta \subset V: \gamma \in \theta} e^{b_0 N^0(V(\theta))} Z(\theta \mid \phi_0^{b_0}) / Z(\overline{V} \mid \phi_0^{b_0}, b_0).$$

$$\mathrm{Prob}(d\sigma(s^1) \neq 0 \mid «0») \leqslant \sum_{\gamma: s^1 \in V(\gamma)} \sum_{\theta \subset V: \gamma \in \theta} e^{b_0 N^0(V(\theta))} Z(\theta \mid \phi_0^{b_0}) / Z(\overline{V} \mid \phi_0^{b_0}, b_0).$$

Remembering that on the line $L_{0,\mathrm{dis.}}$ it is $b_0 = 0$ and referring to the proposition 3.5.1 to get that Φ_0 is a τ-functional and using standard arguments to obtain,

$$\langle \delta(\sigma(s^0)) \rangle^0 \geq 1 - \varepsilon(q) \quad \text{and} \quad \langle \delta(d\sigma(s^1)) \rangle^0 \geq 1 - \varepsilon'(q) \qquad (4.1.1)$$

here $\varepsilon(q)$ and $\varepsilon'(q)$ go to zero as q goes to infinity.

Second step:

Using the same procedure as in the first step one obtains for the dual model

$$\langle \delta(\sigma(s^d)) \rangle^0 \geq 1 - \varepsilon''(q) \quad \text{and} \quad \langle \delta(d\sigma(s^{d-1})) \rangle^0 \geq 1 - \varepsilon'''(q). \qquad (4.1.2)$$

According to the duality relations from statements *b*) and *c*) from Proposition 3.2.1 the relation (4.1.2) lead to

$$\langle \delta(\sigma(s^0)) \rangle^f < 1/2 \quad \text{and} \quad \langle \delta(d\sigma(s^1)) \rangle^f < 1/2. \tag{4.1.3}$$

Combining the relations (4.1.1) and (4.1.3) we get the statement *ci*) of the theorem.

4.2. Proof of the statement *cii*).

First step:

To prove that $\langle \delta(\sigma(s^0) - \alpha) \rangle^{\alpha \neq 0} > 1/2$, $\langle \delta(d\sigma(s^1)) \rangle^{\alpha \neq 0} > 1/2$ and $\langle \delta(\delta\sigma(s^1)) \rangle^f < 1/2$ on the $L_{\alpha \neq 0, \text{dis.}}$ we proceed as in section 4.1.

Second step:

To prove that $\langle \delta(\sigma(s^0) - \alpha) \rangle^f < 1/2$ we first notice that $\langle \delta(\sigma(s^0) - v) \rangle^f = \langle \delta(\sigma(s^0) - \rho) \rangle^f$ for each v and each ρ different from 0. Taking into account that $\sum_{\omega=0}^{q-1} \langle \delta(\sigma(s^0) - \omega \rangle^f = 1$. Thus

$$\langle \delta(\sigma(s^0) - \alpha) \rangle^f = 1/(q - 1 - \langle \delta(\sigma(s^0)) \rangle^f/(q - 1) < 1/(q - 1).$$

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