ANNALES DE L'I. H. P., SECTION A

MAURIZIO SERVA

# Relativistic stochastic processes associated to Klein-Gordon equation

*Annales de l'I. H. P., section A*, tome 49, nº 4 (1988), p. 415-432
<http://www.numdam.org/item?id=AIHPA_1988__49_4_415_0>

*Ann. Inst. Henri Poincaré,*
Vol. 49, n° 4, 1988, p. 415-432. *Physique théorique*

# Relativistic stochastic processes associated to Klein-Gordon equation

by

**Maurizio SERVA**
Dipartimento di Fisica, Universita' « La Sapienza »,
00185 Roma, Italy and INFN, Sezione di Roma, Italy

ABSTRACT. — A stochastic interpretation of Klein-Gordon equation is proposed. The starting point is the construction of relativistically covariant diffusions satisfying a property similar but not identical to the Markov property. It is then shown that the continuity equation associated with these processes together with the relativistic version of mean Newton equation are equivalent to the Klein-Gordon equation.

RÉSUMÉ. — Nous proposons une interprétation stochastique de l'équation de Klein-Gordon. Le point de départ est la construction d'une diffusion relativiste covariante qui satisfait une propriété similaire (mais pas identique) à la propriété de Markov. Nous montrons ensuite que l'équation de continuité associée à ce processus plus la version relativiste de l'équation de Newton en moyenne sont équivalentes à l'équation de Klein-Gordon.

## 1. INTRODUCTION

In the last years considerable efforts have been made to provide a stochastic mathematical formulation of equations describing quantum mechanical systems. The reason for this can be partially found in the technical potentialities of the probabilistic approach and partially in the atmosphere of new interest inside the scientific for theories in which probability and stochastic processes play an important role [4], [5].

In this paper we give a probabilistic interpretation of Klein-Gordon equation by using relativistically covariant Bernstein diffusions. The approach we make has many common points with the method used by Nelson [*6*], [*7*] to obtain Schrödinger equation and, from a mathematical point of view, it represents a relativistic version of it.

The theory of Nelson, known as Stochastic Mechanics has received some attention from physicists during the last twenty years and the original idea has been enriched with important contributions; we refer for example, to the incorporation of spin [*8-11*], to the variational approach [*12-14*], to the operative definition of momentum [*15-17*], to the reformulation of the theory in every representation [*8*], [*18-20*], to the extensions to non-flat manifolds [*21*], [*22*], and so on. Also this theory suggested the way to face and solve for the first time some problems of tunnelling in multiwell potentials [*1-3*].

In spite of this progress, and in spite of various efforts, a complete relativistic generalization has not been achieved up to now.

The reason for this is not directly related to stochastic mechanics but it originates from the well known difficulties connected with relativistically covariant Markov diffusions. Hakim [*23*], in fact, has shown that the expectation of the square of position increment during an infinitesimal time interval cannot be left proportional to the time interval itself in a relativistically covariant frame.

To make clear the content of this paper it is useful to recall briefly the main aspects of Nelson's theory.

Stochastic mechanics hypothizes that the motion of a point-like particle is determined by the joint action of classical forces and of a random disturbance of non specified origin producing continuous but non differentiable trajectories. This assumption has both a kinematical content and a dynamical one; the first is carried out by the fact that the increment of spatial position during an infinitesimal interval of time is the sum of a « classical » term (the product of a drift by the time interval) and a brownian increment, the second is realized assuming that Newton equation still holds in average. In this frame the possible trajectories of a particle are the realizations of a Wiener process whose drift and initial density are the solution of continuity and Newton equations.

Since the dynamics of systems described by the Schrödinger equation are time reversible, it is not astonishing that stochastic mechanics also is reversible [*24*]. Past and future play, in fact, the same role in the theory so that the evolution of the system will be described as a Markov process if a prevision about the future, knowing the past, is wanted and as an « anti-Markov » process (a Markov process with inverted time) if a description of the past, knowing the future, is needed. The two processes are equivalent and also enter symmetrically into the definition of the mean acceleration that is fundamental for the derivation of the dynamical part of the theory.

The situation is completely analogous to the classical mechanics where the particle trajectory into a certain time interval, once given a velocity field, is reversible and completely determined fixing the initial or otherwise the final position.

In the stochastic frame, the time reversal invariance says that the probability associate to a single path is the same if we consider it as the realization of the Markov process or of the anti-Markov one. This implies that the continuous, non-differentiable trajectories (that are symmetric with respect to time inversion) are generate by a mechanism which remains partially arbitrary.

The above considerations are, in our opinion, of great importance for the physical understanding of the theory and they play an important role in the present work.

The efforts of physicists working on the relativistic generalization of the theory have been mostly concentrated on the attempt to circunvent the problem pointed out by Hakim. Since covariant diffusion in physical time were not available it was clear from the beginning that some other kind of processes had to be used to construct relativistic stochastic mechanics. Attempts have been made in various directions; some authors, for example, have introduced additive hypothesis like discrete time [*25*] or extended fluid elements [*26*]. In this paper we are concerned with the research of a description not containing additive assumptions; for this reason we review similar attempts part of which related to the Dirac equation and part to Klein-Gordon equation.

One approach (not hystorically the first) has been inspired by recent papers that provide a path integral description of $1 + 1$ dimensional Dirac equation [*28*], [*29*]. This works on path integral exploit the formal analogy with a « heat » equation associated to a process with the speed of light which inverts the direction of motion at Poisson distributed random times. It should be kept in mind that they do not provide a truly stochastic description of a Dirac particle as well as Feynman path integral does not provide a truly stochastic description of a non relativistic system. In order to obtain stochastic mechanics it is necessary to interprete the continuity equation satisfied by the quantum mechanical probability density as a forward Kolmogorow equation [*30*]. This procedure for the $1 + 1$ dimensional Dirac particle case leads us to consider processes which are a generalization of the ones associated to the path integral formulation. For these generalized processes the probability of an inversion of velocity during an infinitesimal time is not anymore a constant but depends on the position and on the direction of motion.

Unfortunately this approach, due to G. F. De Angelis, G. Jona-Lasinio, N. Zanghi and the present author, has been only partially extended to the $3 + 1$ dimensional case and it does not provide a derivation of the dynamical part of the theory.

A different direction of research has been indicated by F. Guerra and P. Ruggiero [*31*], [*32*]. Taking into account that time and space formally play an identical role in relativistic theories they propose to consider four-dimensional diffusions with an invariant time as parameter while the physical time is a random variable.

A process of this type has very particular features, in fact, since time has a diffusive nature, the particle can change direction in his temporal motion. In this frame, a particle moving backward in time can be interpreted as an antiparticle while the points of inversion of temporal motion can be interpreted as points of creation or destruction of pairs of particles. The reason for this is of simple understanding; let us take the example of a particle moving forward in time till the instant $t_0$ where it starts to move backward, an observer sees a pair of particles at any time before $t_0$ and no particles afterwards, therefore he decides that at time $t_0$ a particle and an antiparticle have been annihilated.

Generally, a trajectory can intersect much more than once a hypersurface of constant $t$ so that a theory of this kind can be thought to describe a many particle system with destruction and creation of pairs. One possible interpretation of this fact is that the four-dimensional one-particle process is only a mathematical « trick » to describe a many particles reality; from this point of view the invariant time has no physical meaning and it is only an instrument utilized to associate a probability measure to any possible realization of the « true » many particle process.

Another possible interpretation, mathematically equivalent to the first, is that the one-particle trajectories have physical reality and that a microscopic particle can really invert the temporal motion. From this point of view the invariant time is again an instrument to associate a probability to any realization but also it appears as a generalization of proper time. For example, in the present model, it turns out that the equation $dx^\mu dx_\mu = mc^2(d\tau)^2$ holds in average providing a probabilistic definition of the invariant parameter $\tau$.

In spite of that, $\tau$ is not proportional to the time of the system rest frame (in our model such a frame does not exist) and it has not analogous « configurational » meaning. In other words an orbserver, looking at a single realization, is not able to associate an invariant time to each point of the trajectory, nevertheless he can give to $\tau$ an averaged operative definition.

A partial result (that also has inspired the present work) in the direction of construction of a relativistic process of this kind has been obtained by F. Guerra. He considers a four-dimensional markovian diffusion satisfying stochastic differential equations of the type

$$dq^\mu(\tau) = v^\mu(q(\tau), \tau)d\tau + dw^\mu \qquad \text{with} \qquad \mathrm{E}[dw^\nu dw^\mu] = 2\nu\eta^{\nu\mu}d\tau .$$

The problem arises from the fact that the metric $\eta^{\nu\mu}$ cannot be taken equal to $g^{\nu\mu} = \mathrm{diag}(1, -1, -1, -1)$ since it must be positive definite. The author makes a choice that remains partially arbitrary and the origin of which is difficult to motivate, furthermore additional problems arise in relation to the definition of a stochastic version of second principle.

In this paper we adopt a different approach. We propose to abandon the Markov property in order to define diffusions satisfying a slightly different property that is, in this context, as « natural » as the Markov one.

We have previously remarked that the time reversibility of classical trajectories is maintained in Nelson theory and it leads to a description which considers together a Markov process and an anti-Markov process. In the relativistic formulation of stochastic mechanics we also expect invariance with respect to inversion of invariant time (charge conjugation). This symmetry suggests that this theory too can be constructed utilizing two processes which together take into account of the reversible nature of diffusive trajectories. Nevertheless, we are not obliged to consider a Markov process and then a complementary anti-Markov one; on the contrary we can make the choice in the « natural » larger class of Bernstein processes [*33-35*] that also contains combinations of Markov processes and anti-Markov processes. Relativistic covariance restricts the choice of such combinations to two of them: the first with the physical time diffusing forward with respect to the invariant parameter and the space position diffusing backward; the second (the complementary one) with time diffusing backward and position diffusing forward. We must also consider the pairs of processes that are obtained from the first via Lorentz boost. The continuity equation which is associated with these diffusions have a manifestly covariant form (the laplacian of brownian motions is substituted by a dalambertian). We emphasize that it is completely natural to have select Bernstein diffusions since we want to describe a reversible dynamics and we also emphasize that the asymmetry between space and time (that disappears if we consider the pair of processes together) is the direct consequence of a request of covariance.

We have seen that a process of this type crosses many times a given hypersurface $t = t_0$ appearing to an observer as a cloud of particles with total charge $\pm e$ where pairs are continuously created and annihilated. It is shown in the present article that the cloud spreads on a sphere of radius $\langle \Delta x \rangle \cong h/mc$ in agreement with relativistic quantum position indeterminacy.

The work is organized as follows: In section 2 we describe the main characteristics of the processes that we introduce and in particular we explain the nature of Bernstein property which replaces the Markov property. In section 3 the continuity equation, associated to these diffusions, is found and it is shown to be relativistically covariant. In section 4 the natural generalization of forward and backward mean derivatives are given in order to state, in section 5, the relativistic analog of mean Newton equation.

In section 5 it is also shown that the continuity equation together with the relativistic Newton mean equation are equivalent to the Klein-Gordon equation. The last one can be considered, in this context, as a linearization of the first two.

In the appendix it is shown that the classical equation $dx^\nu dx_\nu = mc^2(d\tau)^2$ confining the 4-momentum of a relativistic system into an hyperboloid, still holds in average providing the physical meaning of invariant time.

## 2. BERNSTEIN PROCESSES

The symmetry of physical trajectories with respect to the inversion of invariant time allows us to utilize Bernstein processes to obtain the relativistic stochastic mechanics.

In this section we describe this class of processes without any reference to physics and we also select and describe a sub-class directly related to the present work.

The fundamental property of Bernstein processes is the following:

$$\mathbb{E}[f(x(\tau))/\mathrm{P}(\leq a), \mathrm{F}(\geq b)] = \mathbb{E}[f(x(\tau))/x(a); x(b)] \tag{2.1}$$

for each function $f$ and for $a \leq \tau \leq b$.

The first expectation is conditioned by fixing the past of the process up to time $a$ and the future after time $b$, while in the second only $x(a)$ and $x(b)$ are fixed. In other words; the probabilistic knowledge of the process at time $\tau$, if the past is completely known until time $a$ and the future is completely known after time $b$, is the same if only $x(a)$ and $x(b)$ are known.

This time symmetric property is satisfied both by Markov processes and by anti-Markov processes and it is also satisfied by combinations of the two.

Let us make a first simple example; let us consider the two dimensional process $(x(\tau), y(\tau))$ defined by the equations

$$\begin{aligned} x(\tau) &= x_i + w_1(\tau - \mathrm{S}) \\ y(\tau) &= t_f + w_2(\mathrm{T} - \tau) \end{aligned} \tag{2.2}$$

where $w_1$ and $w_2$ are standard brownian noises. The process (2.2) is defined only for $\mathrm{S} \leq \tau \leq \mathrm{T}$.

We see that $x(\tau)$ is markovian with initial value $x(\mathrm{S}) = x_i$ and that $y(t)$ is anti-markovian with final value $y_f$. It is easy to check that the combination of the two satisfies (2.1) and therefore is a Bernstein process.

The transition probability density of finding $x(\tau)$ in $x$ and $y(\tau)$ in $y$, once given the « initial » values $x_i$ and $y_f$, is:

$$p(x, y; \tau/x_i, \mathrm{S}; y_f, \mathrm{T}) = \frac{1}{2\pi} \frac{1}{(\tau-\mathrm{S})^{1/2}(\mathrm{T}-\tau)^{1/2}} \exp\left\{ -\frac{(x-x_i)^2}{2(\tau-\mathrm{S})} - \frac{(y-y_f)^2}{2(\mathrm{T}-\tau)} \right\}. \tag{2.3}$$

It is remarkable that this probability satisfies the equation

$$\partial_\tau p = 1/2 \cdot [\partial_x \partial_x p - \partial_y \partial_y p]. \tag{2.4}$$

In analogy with the markovian diffusions it is possible to define:

$$x(\tau) = x_i + \int_S^\tau b_1(x(u), y(u))du + w_1(\tau - S)$$

$$y(\tau) = y_f - \int_\tau^T b_2(x(u), y(u))du + w_2(T - \tau) \tag{2.5}$$

where $b_1$ and $b_2$ are two assigned drifts and again $S \leq \tau \leq T$.

Also this process satisfies (2.1).

The construction of the relativistic stochastic mechanics will be made by utilizing diffusions of the type (2.5) so let us look at this more carefully. We can easily convince ourselves that (2.5) satisfies a stronger property than (2.1), in fact the probabilistic knowledge that we have of the system at time $\tau$ if we know all the past of $x(u)$ before time $a$ and all the future of $y(u)$ after time $b$ is the same we have if we only know $x(a)$ and $y(b)$.

In other words:

$$\mathbb{E}[g(x(\tau), y(\tau))/P_x(\leq a), F_y(\geq b)] = \mathbb{E}[g(x(\tau), y(\tau))/x(a), y(b)] \tag{2.6}$$

for each function $g$ and for $S \leq a \leq \tau \leq b \leq T$.

A symbolic way of writing the differential equation satisfied by process (2.5) could be:

$$dx = b_1(x, y)d\tau + dw_1$$

$$dy = b_2(x, y)d\tau + \widetilde{dw_2} \tag{2.7}$$

where the anti-markovian nature of the Wiener increment in the second equation is marked by a tilde.

In the following sections we will consider processes of type (2.5) with an invariant time as parameter while the physical time itself will be a stochastic process.

We will also require from these diffusions to be relativistic covariant.

## 3. CONTINUITY EQUATION

We will start the construction of relativistic stochastic mechanics introducing the process with invariant parameter $\tau$ defined by

$$\boldsymbol{x}(\tau) = \boldsymbol{x}_i + \int_S^\tau \boldsymbol{b}_+(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\,\boldsymbol{w}(\tau - S)$$

$$ct(\tau) = ct_f - \int_\tau^T b_+^0(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\,w_0(T - \tau) \tag{3.1}$$

where $c$ is the speed of light, $\hbar$ is the Planck constant and $S \leq \tau \leq T$.

The realizations of (3.1) are trajectories in the 4-dimensional space-time; $x(\tau)$ is the spatial position of the system while $t(\tau)$ is the physical time. We suppose the drift does not depend explicitly on the invariant parameter (having the dimensions of a time over a mass). We observe that the three-dimensional brownian noise appearing in the first equation is markovian while the unidimensional one that appears in the second is anti-markovian.

Together with the class of processes (3.1) we must consider the complementary class obtained by reversing the invariant time. Furthermore, for reasons of covariance, we must also consider the more general class which originates from the first two by a Lorentz transformation. This more general class of processes provides the ingredients of relativistic stochastic mechanics.

Let us, for the moment, consider only processes of type (3.1).

The associate probability of transition (the probability of finding the system in $x$ and in $t$ at time $\tau$ if it was in $x_i$ at time S and it will be in $t_f$ at time T) is implicitly given by

$$p(\boldsymbol{x}, t; \tau \mid \boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T}) = \mathbb{E}[\delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\tau))] \tag{3.2}$$

where $x(\tau)$ and $t(\tau)$ are the processes (3.1).

In order to obtain the equation satisfied by (3.2) we observe at first that the following equalities hold:

$$\lim_{\sigma\downarrow\tau} \mathbb{E}\left[\frac{\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau)}{\sigma - \tau} \mid \boldsymbol{x}(\tau) = \boldsymbol{x},\ t(\sigma) = t\right] = \boldsymbol{b}_+(\boldsymbol{x}, t)$$

$$\lim_{\sigma\downarrow\tau} \mathbb{E}\left[\frac{|\,\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau)\,|^2}{\sigma - \tau} \mid \boldsymbol{x}(\tau) = \boldsymbol{x},\ t(\sigma) = t\right] = 3\hbar$$

$$\lim_{\sigma\downarrow\tau} \mathbb{E}\left[\frac{c\,[t(\sigma) - t(\tau)]}{\sigma - \tau} \mid \boldsymbol{x}(\tau) = \boldsymbol{x},\ t(\sigma) = t\right] = b_+^0(\boldsymbol{x}, t)$$

$$\lim_{\sigma\downarrow\tau} \mathbb{E}\left[\frac{c^2\,[t(\sigma) - t(\tau)]^2}{\sigma - \tau} \mid \boldsymbol{x}(\tau) = \boldsymbol{x},\ t(\sigma) = t\right] = \hbar\,. \tag{3.3}$$

It should be remarked that the conditions in the above expectations fix $x(\tau)$ at the beginning of invariant time interval and $t(\tau)$ at the end.

The derivative of $p$ with respect to the invariant time is found by calculating the limit:

$$\begin{aligned}
&\lim_{\sigma\downarrow\tau} (\sigma - \tau)^{-1}\mathbb{E}[\delta(\boldsymbol{x} - \boldsymbol{x}(\sigma))\delta(t - t(\sigma)) - \delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\tau))] = \\
&\quad = \lim_{\sigma\downarrow\tau} (\sigma - \tau)^{-1}\mathbb{E}[-(\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau))\boldsymbol{\nabla}[\delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\sigma))] + \\
&\quad + \frac{1}{2}\,[(\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau))\boldsymbol{\nabla}]^2\,[\delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\sigma))] + \\
&\quad - (t(\sigma) - t(\tau))\partial_t\,[\delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\sigma))] + \\
&\qquad - \frac{1}{2}\,[(t(\sigma) - t(\tau))\partial_t]^2\,[\delta(\boldsymbol{x} - \boldsymbol{x}(\tau))\delta(t - t(\sigma))] + o(\sigma - \tau)]
\end{aligned} \tag{3.4}$$

taking into account the equalities (3.3) and the independence of $\boldsymbol{x}(\tau)$, $t(\sigma)$ from the increments $\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau)$, $t(\sigma) - t(\tau)$ we see that the following equation holds

$$\partial_\tau p = - \partial_\mu(b^\mu_+ p) - (\hbar/2)\partial^\mu\partial_\mu p \tag{3.5}$$

where $\partial_\mu\partial^\mu \equiv c^{-2}\partial_t^2 - \Delta$ and $\partial_\mu(b^\mu_+ p) \equiv c^{-1}\partial_t(b^0_+ p) + \nabla(\boldsymbol{b}_+ p)$. The relativistic covariance of the equation (3.5) is explicit.

The « initial » conditions assigned to the solution $p$ are

$$\int d\boldsymbol{x}_i p(\boldsymbol{x}, t; \mathrm{S} \mid \boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T}) = \delta(t - t_f)$$

$$\int dt_f p(\boldsymbol{x}, t; \mathrm{T} \mid \boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T}) = \delta(\boldsymbol{x} - \boldsymbol{x}_i) \tag{3.6}$$

they are a consequence of definitions (3.1), (3.2) and they correspond to the stochastic « initial » values $\boldsymbol{x}(\mathrm{S}) = \boldsymbol{x}_i$, $t(\mathrm{T}) = t_f$.

The transition probability of the diffusion $(\boldsymbol{x}'(\tau), t'(\tau))$ obtained from (3.1) by a Lorentz boost of velocity $\boldsymbol{v}$ also satisfies the covariant equation (3.5). The « initial » conditions will be given fixing $\boldsymbol{v}\boldsymbol{x}'(\mathrm{S}) + |\,\boldsymbol{v}\,|^2 t'(\mathrm{S})$, $\boldsymbol{x}'(\mathrm{S}) - (\boldsymbol{v}/|\,\boldsymbol{v}\,|^2)[\boldsymbol{v}\boldsymbol{x}'(\mathrm{S})]$ and $t'(\mathrm{T}) + (\boldsymbol{v}/c)\boldsymbol{x}'(\mathrm{T})$.

Therefore, any process belonging to the general class is associated to the covariant equation (3.5) and to one of the above « initial » conditions; furthermore, there is a reference frame in which its stochastic equations take the form (3.1).

Let us consider again only diffusions (3.1). The (scalar) probability density of finding the system in the point $\boldsymbol{x}, t$ of space-time at the instant $\tau$ is:

$$\rho(\boldsymbol{x}, t; \tau) = \int d\boldsymbol{x}_i dt_f p(\boldsymbol{x}, t; \tau \mid \boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T})\Pi(\boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T}) \tag{3.7}$$

where $\Pi(\boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T})$ is the probability density that the system is in $\boldsymbol{x}_i$ at time S and in $t_f$ at time T for which the following identity holds

$$\Pi(\boldsymbol{x}_i, \tau; t_f, \tau) \equiv \rho(\boldsymbol{x}, t; \tau)\,. \tag{3.8}$$

The scalar probability density (3.7) also satisfies an equation (3.5) as well as the probability densities associated with processes of the general class.

It should be remarked that only the probability densities which are stationary solutions of an equation (3.5) have physical relevance since $\tau$ is not directly observable.

The situation is identical in classical relativistic mechanics. Let us consider, in fact, the classical relativistic equations of a particle in a velocity field (they can be obtained from (3.1) with the assumption $\hbar = 0$); they lead to the continuity equation

$$\partial_\tau \rho = - \partial_\mu(b^\mu \rho) \equiv c^{-1}\partial_t(b_0 \rho) + \nabla(\boldsymbol{b}\rho) \tag{3.9}$$

where $\tau$ is now the proper time. The above equation becomes the « true » continuity equation only after we put $\partial_\tau \rho = 0$.

In this classical frame the four-component density $\rho b_0$ (which in a four-dimensional picture is the mean flux crossing in the point $\boldsymbol{x}$ a hypersurface of constant $t$) has a direct physical meaning since it is the probability of finding the system in $\boldsymbol{x}$ at the (physical) time $t$. Furthermore $\boldsymbol{u} \equiv \boldsymbol{b}/b_0$ is the velocity of the particle and $\boldsymbol{b}\rho = (\rho b_0)\boldsymbol{u}$ is the three-dimensional flux density.

In the stochastic version, as it be will shown in section 5, it is possible to define a four-component density and flux density with an analogous meaning.

## 4. MEAN DERIVATIVES

In the previous section we have shown the main characteristics of the diffusions we need in order to construct relativistic stochastic mechanics. Now we are ready to extend the definition of mean derivatives previously introduced by Nelson for Markov processes.

Let us consider again a process of type (3.1). Because of the anti-markovian nature of the Wiener noise appearing in the second of the (3.1) it is natural to generalize Nelson's forward derivative in the following way.

$$\begin{aligned}
D^+F(\boldsymbol{x}, t; \tau) \equiv & \lim_{\sigma \downarrow \tau} \mathbb{E}[F(\boldsymbol{x}(\sigma), t(\sigma), \sigma) - F(\boldsymbol{x}(\tau), t(\tau), \tau) \mid \boldsymbol{x}(\tau) = \boldsymbol{x}, t(\sigma) = t](\sigma - \tau)^{-1} \\
= & \lim_{\sigma \downarrow \tau} \mathbb{E}[(\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau))\nabla F(\boldsymbol{x}, t; \tau) + \frac{1}{2}[(\boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau))\nabla]^2 F(\boldsymbol{x}, t; \tau) + \\
& + (t(\sigma) - t(\tau))\partial_t F(\boldsymbol{x}, t; \tau) - \frac{1}{2}[(t(\sigma) - t(\tau))\partial_t]^2 F(\boldsymbol{x}, t; \tau) + \\
& + (\sigma - \tau)\partial_\tau F(\boldsymbol{x}, t; \tau) + o(\sigma - \tau) \mid \boldsymbol{x}(\tau) = \boldsymbol{x}, t(\sigma) = t](\sigma - \tau)^{-1} = \\
= & \, b_+^\mu(\boldsymbol{x}, t)\partial_\mu F(\boldsymbol{x}, t; \tau) - \frac{1}{2}\hbar\partial_\mu\partial^\mu F(\boldsymbol{x}, t; \tau) + \partial_\tau F(\boldsymbol{x}, t; \tau) \qquad (4.1)
\end{aligned}$$

where the position has been fixed at the beginning of the invariant parameter interval and the time at the end. This derivative has a manifestly covariant form and it becomes a standard one in the classical limit $\hbar \to 0$.

For processes that take the form (3.1) only after a Lorentz transformation of velocity $\boldsymbol{v}$ the derivative must be defined fixing the conditions

$$\begin{aligned}
\boldsymbol{v}\boldsymbol{x}(\tau) + |\boldsymbol{v}|^2 t(\tau) &= \boldsymbol{v}\boldsymbol{x} + |\boldsymbol{v}|^2 t \qquad (4.2) \\
\boldsymbol{x}(\tau) - (\boldsymbol{v}/|\boldsymbol{v}|^2)(\boldsymbol{v}\boldsymbol{x}(\tau)) &= \boldsymbol{x} - (\boldsymbol{v}/|\boldsymbol{v}|^2)(\boldsymbol{v}\boldsymbol{x}) \\
t(\sigma) + \boldsymbol{v}\boldsymbol{x}(\sigma)/c^2 &= t + \boldsymbol{v}\boldsymbol{x}/c^2 .
\end{aligned}$$

Furthermore, the stochastic equations satisfied by these processes can be written in the differential form (with the convention (2.7))

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{b}_+(\boldsymbol{x}, t)dt + \sqrt{\hbar}\,[d\boldsymbol{w} + (\gamma - 1)\boldsymbol{n}(\boldsymbol{n}d\boldsymbol{w}) + \gamma\beta\boldsymbol{n}\widetilde{dw_0}\,] \\ cdt &= b^0_+(\boldsymbol{x}, t)dt + \sqrt{\hbar}\,[\gamma\widetilde{dw_0} + \gamma\beta(\boldsymbol{n}d\boldsymbol{w})] \end{aligned} \tag{4.3}$$

where $\boldsymbol{n} = \boldsymbol{v}/|\boldsymbol{v}|$, $\beta = |\boldsymbol{v}|/c$, $\gamma = (1 - \beta^2)^{-1/2}$ and $\boldsymbol{w}$ and $w_0$ are as usual Wiener noises. It is easy to check that relations (4.3) together with conditions (4.2) define a mean forward derivative identical to (4.1).

Since both continuity equation and mean derivative are independent of the $\boldsymbol{v}$ appearing in the stochastic equation (4.3), something like a gauge invariance comes out. In other words, since the continuity equation and the mean derivative are determined only by the drift they can be associated with any possible cocktail of processes which have the same drift but different « privileged reference frame velocities $\boldsymbol{v}$ ».

Let us now come back to the processes (3.1). We can define a second mean derivative, that corresponds to Nelson's backward mean derivative fixing symmetric conditions with respect to (4.1)

$$D^-F(\boldsymbol{x}, t\,;\tau) \equiv \lim_{\sigma\downarrow\tau} \mathbb{E}\,[F(\boldsymbol{x}(\sigma), t(\sigma)\,;\sigma) - F(\boldsymbol{x}(\tau), t(\tau)\,;\tau)\,|\,\boldsymbol{x}(\sigma) = \boldsymbol{x}, t(\tau) = t\,](\sigma - \tau)^{-1} \tag{4.4}$$

This expectation can be calculated following a standard procedure (see for example reference [*24*]).

We have

$$D^-F = \partial_\tau F + b^\mu_- \partial_\mu F + \frac{1}{2}\hbar\partial_\mu\partial^\mu F \tag{4.5}$$

where we use:

$$b^\mu_- \equiv b^\mu_+ + \hbar\frac{\partial^\mu\rho}{\rho}. \tag{4.6}$$

This further mean derivative also has a manifestly covariant form and it can be associated with any cocktail of processes (4.3).

The system whose probability density is a solution of (3.5) and which generates the derivatives (4.1), (4.5) is (invariant) time symmetric, for example, it is possible to rewrite its continuity equation in the form:

$$\partial_\tau\rho = -\,\partial_\mu(b^\mu_-\rho) + \frac{1}{2}\hbar\partial_\mu\partial^\mu\rho \tag{4.7}$$

in which the sign in front of the Dalembertian has changed and which can be derived from the invariant time reversed stochastic equations

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{b}_-(\boldsymbol{x}, t)dt + \sqrt{\hbar}\,[\widetilde{d\boldsymbol{w}} + (\gamma - 1)\boldsymbol{n}(\boldsymbol{n}\widetilde{d\boldsymbol{w}}) + \gamma\beta\boldsymbol{n}dw_0\,] \\ cdt &= b^0_-(\boldsymbol{x}, t)dt + \sqrt{\hbar}\,[\gamma dw_0 + \gamma\beta(\boldsymbol{n}\widetilde{d\boldsymbol{w}})] \end{aligned} \tag{4.8}$$

where we use the usual convention (2.7) and notations (4.3). Furthermore, starting from (4.8) and (4.4), it is possible to obtain the derivative (4.5) directly while the derivative (4.1) is obtained by following the procedure shown in reference [*24*].

It is easy to be convinced that equations (4.3) (together with the initial density $\Pi(\boldsymbol{x}_i, \mathrm{S}; t_f, \mathrm{T})$) and equations (4.8) (together with the initial density $\Pi(\boldsymbol{x}_i, \mathrm{T}; t_f, \mathrm{S})$) associate the same probability to a given trajectory of the stochastic system.

Therefore, we have arrived at the conclusion that such a system can be described by one of the processes (4.3) as well as by one of the processes (4.8) independently of $\boldsymbol{v}$ and also it can be described by any possible cocktails of these.

This freedom of choice comes out from relativistic covariance and invariant time reversibility. About this last symmetry we remark that the process described by equations (4.3) and the process described by equations (4.8) belong to two complementary class linked by invariant time reversal.

We conclude this section with two definitions:

$$b^\mu \equiv \frac{1}{2}(b^\mu_+ + b^\mu_-)$$

$$\delta b^\mu \equiv \frac{1}{2}(b^\mu_+ - b^\mu_-) = -\frac{1}{2}\hbar\frac{\partial^\mu \rho}{\rho}. \tag{4.9}$$

## 5. DYNAMICAL EQUATIONS

In the previous section we have introduced all the kinematical ingredients necessary to the construction of a relativistic stochastic mechanics. In particular we have defined two mean derivatives in analogy with the non-relativistic theory.

This section, on the contrary, is devoted to the dynamics of the theory. The main assumption we make is that classical equations still hold in average. In other words we state

$$\frac{1}{2}(\mathrm{D}^+\mathrm{D}^- + \mathrm{D}^-\mathrm{D}^+)x^\nu = \frac{e}{c}\mathrm{F}^{\nu\mu}\frac{(\mathrm{D}^+ + \mathrm{D}^-)}{2}x_\mu \tag{5.1}$$

where the mean 4-acceleration is defined in analogy to Nelson's acceleration and where $e$ is the elettrical charge and $\mathrm{F}^{\nu\mu}$ is the elettromagnetical tensor

$$\mathrm{F}^{\mu\nu} = \partial^\mu \mathrm{A}^\nu - \partial^\nu \mathrm{A}^\mu. \tag{5.2}$$

We have previously assumed that the drift does not depend explicitely on $\tau$, now we also assume

$$b^\mu(\boldsymbol{x}, t) = - \partial^\mu S(\boldsymbol{x}, t) - (e/c)A^\mu(\boldsymbol{x}, t). \tag{5.3}$$

With this assumption, once the derivatives are explicitely calculated, equation (5.1) may be rewritten:

$$\partial^\nu [(\partial^\mu S + (e/c)A^\mu)(\partial_\mu S + (e/c)A_\mu) - (\hbar^2 \partial^\mu \partial_\mu \rho^{1/2})/\rho^{1/2}] = 0 \tag{5.4}$$

and so

$$(\partial^\mu S + (e/c)A^\mu)(\partial_\mu S + (e/c)A_\mu) - (\hbar^2 \partial^\mu \partial_\mu \rho^{1/2})/\rho^{1/2} = m^2c^2 \tag{5.5}$$

where $m$ is a constant that we identify with the mass of the particle.

Since we assume, in analogy with the classical case, that $b^+$ and $b^-$ are not explicit functions of $\tau$ and since $b^+ - b^- = \hbar\partial \log \rho$, we also implicitely assume that the density $\rho$ does not depends directly on the invariant time. In other words, the physical solutions of equation (4.7) are the stationary solutions for which:

$$\partial_\tau \rho = -\partial_\mu(b^\mu_+ \rho) - (\hbar/2)\partial^\mu\partial_\mu \rho = -\partial_\mu(b^\mu \rho) = \partial^\mu [(\partial_\mu S + (e/c)A_\mu)\rho] = 0. \tag{5.6}$$

It is easy to show that the complex function

$$\Psi(\boldsymbol{x}, t) \equiv \rho^{1/2}(\boldsymbol{x}, t) \exp \{ iS(\boldsymbol{x}, t)/\hbar \} \tag{5.7}$$

satisfies the Klein-Gordon equation

$$(i\hbar\partial^\mu - (e/c)A^\mu)(i\hbar\partial_\mu - (e/c)A_\mu)\Psi = m^2c^2\Psi \tag{5.8}$$

and so we arrive at the conclusion that the quantum behaviour of a system described by a function $\Psi$ satisfying the Klein-Gordon equation has an explanation in an underlying stochastic process.

A problem could arise at this point; the scalar density $\rho$ is positive and the same could be expected for the fourth component density $\rho b_0$ but, on the contrary, it is sometimes negative at least in some regions of space-time. This fact has a simple explanation which leads us to interpret $\rho b_0$ as the charge density.

We first of all observe that $\rho b_0$, in the 4-dimensional picture, represents the mean flux crossing a hypersurface of constant $t$ in a point $\boldsymbol{x}$. In the classical case this flux is always positive because the fourth component of the trajectory of the system is a monotone growing function of proper time, while in the quantum case it is not monotone and it can cross the hypersurface in both directions.

The crossings from region $t \leq t_0$ to region $t \geq t_0$ give a positive contribution to the fourth component density while the crossings from region $t \geq t_0$ to region $t \leq t_0$ give negative contribution.

An observer, looking at time $t_0$, will see a particle with charge $+ e$ if crossing is from region $t \leq t_0$ to region $t \geq t_0$ and a particle with charge $- e$ if the motion is opposite. Furthermore, when the same particle crosses twice the hypersurface, he will see a pair of particles with opposite charge and, when the particle crosses it $n$ times, he will see $n$ particles with total charge zero if $n$ is even and with total charge $e$ if $n$ is odd.

It is easy to be convinced that, from the observer point of view, the stochastic system will appear as a kind of dressed particle i. e. a cloud of particles with total charge $e$ where pairs are continuously created and annihilated.

Some easy calculations show that the cloud spreads on a sphere of radius $\langle \Delta x \rangle \cong \hbar/mc$ according to relativistic quantum position indeterminacy.

We make these calculations for a free particle « at rest » for which $\boldsymbol{b}^+ = \boldsymbol{b}^- = 0$ and $b_0^+ = b_0^- = mc$. This simple system can be described by the equations:

$$\begin{aligned} \boldsymbol{x}(\tau) - \boldsymbol{x}(\mathrm{T}) &= (\hbar)^{1/2}\boldsymbol{w}(\mathrm{T} - \tau) \\ ct(\tau) - ct(\mathrm{S}) &= mc(\tau - \mathrm{S}) + (\hbar)^{1/2}w_0(\tau - \mathrm{S}) \end{aligned} \tag{5.9}$$

the first process is anti-markovian while the second one is markovian.

We put $\boldsymbol{x}(\mathrm{T}) = \boldsymbol{x}_0$ and $t(\mathrm{S}) = t_0$ so that the particle is in $t_0$ at the (invariant) time S. In the following (invariant) instants the particle can cross the hypersurface $t = t_0$ many more times; but after the time $\mathrm{S} + \Delta\tau$ for which:

$$(mc\Delta\tau)^2 \cong |\hbar^{1/2}(w_0(\mathrm{S} + \Delta\tau) - w_0(\mathrm{S}))|^2 \tag{5.10}$$

the positive drift contribution $mc\Delta\tau$ is greater than the brownian contribution and it brings the system far from $t_0$. Therefore we can assume that after the (invariant) time $\mathrm{S} + \Delta\tau$ the system does not actually cross again the hypersurface.

The relation (5.10) gives

$$\Delta\tau \cong \hbar\backslash m^2c^2 . \tag{5.11}$$

On the other side we see that during this interval of (invariant) time $\Delta\tau$, in which the system can still be found with reasonable probability in $t_0$, the position spreads on a sphere of radius

$$\langle \Delta x \rangle \cong |(\hbar)^{1/2}(w(\mathrm{T}) - w(\mathrm{T} - \Delta\tau))| \cong \hbar/mc \tag{5.12}$$

that is the radius of the region of the space in which the particle can cross the hypersurface. An observer, looking at (physical) time $t_0$ sees a cloud of particles with linear dimension $\hbar/mc$.

## 6. CONCLUSIONS

The model presented here gives a reasonable solution to the problem of constructing relativistic diffusion associated to Klein-Gordon equation.

It also turns out that it shows some physical features of relativistic quantum mechanics like pairs production and position indeterminacy.

We think that this model also has an interest in view of possible extensions and generalizations. A description of the Dirac particle, for example, seems possible. First calculations show that, to reach this goal, some constraints on the trajectories should be imposed. Once this is done, it turns out that at least in the $1 + 1$ dimensional case, the trajectories have the speed of light in agreement with the assumption made in reference [*30*].

## ACKNOWLEDGMENTS

It is a pleasure to thank M. Cini for his continuous interest and for illuminating discussion. The author is also grateful to G. Jona-Lasinio for stimulating comments and to F. Guerra for many discussions.

# APPENDIX

In classical mechanics the motion of a particle in an electromagnetical field is confined into the hyperboloid of equation

$$m^2c^2 = (dx_\mu/d\tau)(dx^\mu/d\tau) \tag{A.1}$$

that defines the invariant mass.

In the present stochastic model, the particle is not anymore confined into a hyperboloid but equation (A.1) still holds in average. In other words the limit for $\Delta\tau \to 0$ of a certain mean of:

$$(\Delta x_\mu \Delta x^\mu)/(\Delta\tau)^2 \equiv c^2 [(t(\sigma)-t(\tau))(t(\tau)-t(\eta))]/(\Delta\tau)^2 \times - [(\boldsymbol{x}(\sigma)-\boldsymbol{x}(\tau))(\boldsymbol{x}(\tau)-\boldsymbol{x}(\eta))]/(\Delta\tau)^2 \tag{A.2}$$

with $\sigma - \tau = \tau - \eta = \Delta\tau > 0$, is equal to $m^2c^2$.

In (A.2) we only consider products of increments in neighbouring intervals of invariant time because an average made on the same intervals diverges when $\Delta\tau \to 0$ and therefore it has no physical meaning.

In order to calculate (A.2) we consider at first a system that is in $\boldsymbol{x}(\eta)$ at time $\eta$ and in $t(\sigma)$ at time $\sigma$. According to equations (3.1) we have:

$$\begin{aligned} \boldsymbol{x}(\tau) - \boldsymbol{x}(\eta) &= \int_\eta^\tau \boldsymbol{b}_+(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\boldsymbol{\omega}(\tau - \eta) \\ c(t(\sigma) - t(\tau)) &= \int_\tau^\sigma b_+^0(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\tilde{\omega}_0(\sigma - \tau) \end{aligned} \tag{A.3}$$

with $\boldsymbol{\omega}(\tau - \eta) = \boldsymbol{w}(\tau - \mathrm{S}) - \boldsymbol{w}(\eta - \mathrm{S})$ and $\tilde{\omega}_0(\sigma - \tau) = w_0(\mathrm{T} - \sigma) - w_0(\mathrm{T} - \tau)$.

Then we consider the system that is in $\boldsymbol{x}(\sigma)$ at time $\sigma$ and in $t(\tau)$ at time $\tau$. According to equations (4.8); we have,

$$\begin{aligned} \boldsymbol{x}(\sigma) - \boldsymbol{x}(\tau) &= \int_\tau^\sigma \boldsymbol{b}_-(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\tilde{\boldsymbol{\omega}}(\sigma - \tau) \\ c(t(\tau) - t(\eta)) &= \int_\eta^\tau b_-^0(\boldsymbol{x}(u), t(u))du + \sqrt{\hbar}\omega_0(\tau - \eta)\,. \end{aligned} \tag{A.4}$$

All the brownian increments in (A.3) and in (A.4) are independent. On the contrary, the integrals in equation (A.3) depend on the brownian increments in (A.4) and also the integrals in (A.4) depend on brownian increments in (A.3).

The more reasonable mean of (A.2) is

$$\{ \mathbb{E}[(\Delta x_\mu \Delta x^\mu)/(\Delta\tau)^2 \,|\, \boldsymbol{x}(\eta)=\boldsymbol{x},\, t(\sigma)=t] + \mathbb{E}[(\Delta x_\mu \Delta x^\mu)/(\Delta\tau)^2 \,|\, \boldsymbol{x}(\sigma)=\boldsymbol{x},\, t(\eta)=t] \}/2 \tag{A.5}$$

where $(\Delta x^\mu \Delta x_\mu)/(\Delta\tau)^2$ is given by equation (A.2) together with equations (A.3) and (A.4).

In order to calculate exactly the limit of (A.5) it is sufficient to take (A.3) and (A.4) till the order $(\Delta\tau)^{3/2}$. For example with respect to conditions $\boldsymbol{x}(\eta) = \boldsymbol{x}$ and $t(\sigma) = t$ we have:

$$\begin{aligned} \boldsymbol{x}(\tau) - \boldsymbol{x}(\eta) = \boldsymbol{b}_+(\boldsymbol{x}, t)\Delta\tau &+ \int_\eta^\tau [(\boldsymbol{x}(u) - \boldsymbol{x}(\eta))\nabla]\boldsymbol{b}_+(\boldsymbol{x}, t)du + \\ &+ \int_\eta^\tau [(t(u) - t(\sigma))\partial_t]\boldsymbol{b}_+(\boldsymbol{x}, t)du + \sqrt{\hbar}\boldsymbol{\omega}(\tau - \eta) + o(\Delta\tau^{3/2}) \end{aligned} \tag{A.6}$$

we also have:

$$x(\sigma) - x(\tau) = b\xi(x, t)\Delta\tau + \int_\tau^\sigma [(x(u) - x(\eta))\nabla]b_-(x, t)du + $$
$$+ \int_\tau^\sigma [(t(u) - t(\sigma))\partial_t]b_-(x, t)du + \sqrt{\hbar}\tilde{\omega}(\sigma - \tau) + o(\Delta\tau^{3/2}) \quad (A.7)$$

from which we obtain:

$$\mathbb{E}[-(x(\sigma) - x(\tau))(x(\tau) - x(\eta))/(\Delta\tau)^2 \mid x(\eta) = x, t(\sigma) = t] = b_-(x, t)b_+(x, t) + \hbar\nabla b_-(x, t) + \dots \quad (A.8)$$

where we have taken into account that:

$$\mathbb{E}[(x(u) - x(\eta))\omega(\tau - \eta)] = \mathbb{E}[(x(\tau) - x(\eta))\omega(\tau - \eta)] + o(\Delta\tau) =$$
$$= \hbar^{1/2}\Delta\tau + o(\Delta\tau) \quad (A.9)$$

for $u \geq \tau$ and

$$\mathbb{E}[(x(u) - x(\eta))\tilde{\omega}(\sigma - \tau)] = o(\Delta\tau) \quad (A.10)$$

for $u \leq \tau$. In an identical way it is easily found:

$$\mathbb{E}[c^2(t(\sigma) - t(\tau))(t(\tau) - t(\eta))/(\Delta\tau)^2 \mid x(\eta) = x, t(\sigma) = t] = b^0_-(x, t)b^0_+(x, t) + \hbar\partial_t b^0_-(x, t) + \dots \quad (A.11)$$

and also

$$\mathbb{E}[(\Delta x_\mu \Delta x^\mu)/(\Delta\tau)^2 \mid x(\sigma) = x, t(\eta) = t] = b_{+\mu}b^\mu_- + \hbar\partial_\mu b^\mu_+ + \dots \quad (A.12)$$

finally collecting (A.10), (A.11) and (A.12) we obtain the limit for $\Delta\tau \mapsto 0$ of (A.5) to be equal to:

$$b_{+\mu}b^\mu_- + \hbar\partial_\mu(\delta b^\mu) = m^2c^2 \quad (A.13)$$

where the last equality is simply given by (5.5). Equation (A.13), that reduces, to (A.1) in the classical limit $\hbar \to 0$, defines an invariant mass and can be thought as a constraint on the drift.

## BIBLIOGRAPHY


[1] G. Jona-Lasinio, F. Martinelli, E. Scoppola, *Comm. Math. Phys.*, t. **80**, 1981, p. 223.
[2] G. Jona-Lasinio, F. Martinelli, E. Scoppola, *Ann. Inst. Henri Poincaré*, t. **42**, no. 1985, p. 73.
[3] G. Jona-Lasinio, F. Martinelli, E. Scoppola, *Mathematics + Physics, Lectures on Recent Results*, Vol. II, L. Streit (ed.), World Scientific Publ., Singapore.
[4] M. Cini, *Italian Studies in the fundations of Quantum Physics*, Reidel Publishing Company, 1986.
[5] M. Cini, *La Nuova Critica*, t. **75-76**, 1985, p. 70.
[6] E. Nelson, *Phys. Rev.*, t. **150**, 1966, p. 1079.
[7] E. Nelson, *Dynamical Theories of Brownian motion* (Princeton University, Princeton, N. Y., 1967).
[8] T. G. Dankel, Jr,. *Archiv. Rational Mech. Anal.*, t. **37**, 1971, p. 192.
[9] T. G. Dankel, Jr., *J. Math. Phys.*, t. **18**, 1977, p. 253.
[10] W. G. Faris, *Found. Phys.*, t. **13**, 1982, p. 1.
[11] G. F. De Angelis and G. Jona-Lasinio, *J. Phys. A: Math. Gen.*, t. **15**, 1982, p. 2053.
[12] K. Yasue, *J. Math. Phys.*, t. **22**, 1981, p. 1010.
[13] K. Yasue, *J. Funct. Anal.*, t. **41**, 1981, p. 327.
[14] F. Guerra and L. Morato, *Phys. Rev.*, t. **D 27**, 1983, p. 1774.
[15] D. Shucker, *J. Funct. Anal.*, t. **38**, 1980, p. 146.

[16] M. SERVA, *Lett. Nuovo Cimento.*, t. **41**, 1984, p. 198.
[17] E. A. CARLEN, *Ann. Inst. Henri Poincaré*, t. **A 42**, 1985, p. 407.
[18] F. GUERRA and R. MARRA, *Phys. Rev.*, t. **D 29**, 1984, p. 1647.
[19] M. CINI and M. SERVA, *J. Phys. A: Math. Gen.*, t. **19**, 1986, p. 1163.
[20] G. F. DE ANGELIS, *Fundamental aspect of quantum theory*, Plenum Press, New York, 1986.
[21] D. DOHRN and F. GUERRA, *Lett. Nuovo Cimento.*, t. **22**, 1978, p. 121.
[22] L. M. MORATO, *J. Math. Phys.*, t. **23**, 1982, p. 1020.
[23] R. HAKIM, *Math. Phys.*, t. **9**, 1968, p. 1805.
[24] F. GUERRA, *Phys. Rep.*, t. **77**, 1981, p. 263.
[25] W. J. LEHR and H. Y. PARK, *J. Math. Phys.*, t. **18**, 1977, p. 1235.
[26] J.-P. VIGIER, *Lett. Nuovo Cimento.*, t. **24**, 1979, p. 265.
[27] L. DE LA PENA-AUERBACH, *J. Math. Phys.*, t. **12**, 1971, p. 453.
[28] GAVEAU, JACOBSON, KAC and SHULMAN, *Phys. Rev. Lett.*, t. **53**, 1984, p. 419.
[29] Ph. BLANCHARD, Ph. COMBE, M. SIRUGUE and M. SIRUGUE-COLLIN, Preprint BiBoS, Bielefeld, 1985.
[30] G. F. DE ANGELIS, G. JONA-LASINIO, M. SERVA and N. ZANGHI, *J. Phys. A: Math. Gen.*, t. **19**, 1986, p. 865.
[31] F. GUERRA and P. RUGGIERO, *Lett. Nuovo Cimento*, t. **23**, 1978, p. 529.
[32] F. GUERRA, Bi.Bo.S., Preprint, t. **167**, 1986.
[33] S. BERNSTEIN, In *Verh. des Intern. Mathematikerkongr.*, Zurich (1932), Band 1.
[34] J. C. ZAMBRINI, *Phys. Rev.*, t. **A 33**, 1986, p. 1532.
[35] J. C. ZAMBRINI, *J. Math. Phys.* (to be published).

*(Manuscrit reçu le 25 mai 1988)*