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How to Recover Euclidean Invariance from an Ondelette Cluster Expansion (*)

by

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ABSTRACT. — The d-dimensional Lemarié basis is not the tensor product of the j-dimensional basis with the $(d\text{-}j)$-dimensional basis, and the linear relation between these bases is complicated. Nevertheless, we find a j-dimensional regularization of the d-dimensional Lemarié basis defined by an arbitrary cutoff in scale Λ. This orthonormal set has the important property that the subspace spanned by it is precisely $L^2(\mathbf{R}^{d-j}) \otimes L^j_\Lambda$, where L^j_Λ is the span of the Λ-scale cutoff of the j-dimensional Lemarié basis. We use this tensor product theorem to recover Euclidean invariance from the ondelette cluster expansion. We will also apply it to Ward identities in a future paper.

RÉSUMÉ. — La base de Lemarié en dimension d n'est pas le produit tensoriel de la base de dimension j et de la base de dimension $d\text{-}j$. La relation linéaire entre ces deux bases est compliquée. Néanmoins, nous exhibons une régularisation j-dimensionnelle de la base de Lemarié en dimension d définie par un cut-off arbitraire à l'échelle Λ. Cet ensemble orthonormal a la propriété importante que le sous-espace qu'il engendre est précisément $L^2(R^{d-j}) \otimes L^j_\Lambda$ où L^j_Λ est l'ensemble engendré par la partie de la base de Lemarié avec cutoff d'échelle Λ. Nous utilisons ce théorème de produit tensoriel pour retrouver l'invariance Euclidienne dans la « cluster-expansion » en ondelettes. Nous appliquerons ce résultat aux identités de Ward dans un article suivant.

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1. INTRODUCTION

The problem of proving Euclidean invariance of a very singular field interaction controlled by a phase cell cluster expansion that breaks Euclidean invariance has been a recurring headache in constructive quantum field theory. This is particularly true in the case where a multi-scale lattice structure is built into the phase space localization. In this case a noteworthy success in proving symmetry can be found in [*1*], where Euclidean invariance is recovered in the continuum limit of the lattice abelian $\text{Higgs}_{2,3}$ models. The aim of our paper is to show how Euclidean invariance can be proven for any model controlled by a *specific kind* of phase cell expansion—namely, the ondelette cluster expansion [*2*]. We concentrate on the ϕ_3^4 quantum field theory, but only in the interests of concreteness.

Thus our purpose here is not to « sell » the ondelette cluster expansion, but to show that it passes a test that every phase cell expansion *must* pass. The claim has often been made—even by this author—that with this kind of cluster expansion « there are no cutoffs. » Instead of getting involved in an esthetic dispute over what this really means, we simply meet the legitimate objection that this phase space localization « shares the problem of cutoffs. » Actually it should be emphasized that, *a priori*, ondelettes destroy physical positivity as well as Euclidean invariance, but we will show how both are restored.

Our approach combines a standard idea with a non-trivial tensor product theorem for ondelettes. The standard idea is to super-impose a spherically symmetric cutoff in both position space and momentum space. If the input estimates for the given phase cell expansion are uniform with respect to this cutoff, then a double-limit argument allows one to recover rotational invariance, and the rest of the argument is easy. The sticking point is the uniformity of the input estimates. For the ondelette decomposition, and indeed for any phase space localization with a multi-scale lattice structure, the more familiar ultraviolet cutoffs—e. g., sharp or exponential—will not do, because they destroy the scaling properties of the phase cell estimates. One must use the inverse power cutoff given by the factor $(p^2 + m^2)^{-1/\kappa}$ in momentum space, because it is the only candidate for reasonable scale homogeneity at small length scales. On the other hand, this ultraviolet cutoff suffers from a loss in the *degree* of regularity as $\kappa \to \infty$. Now for models whose ultraviolet divergences are marginal—e. g., ϕ_2^{2n} interactions—this poses no problem. But for the ϕ_3^4 theory this defect is serious: although the phase cell estimates are uniform in κ, it is not clear that there is a continuation of the rotational symmetry into the region $\kappa \geq 2$ (where ultraviolet singularities develop) because we do not have analyticity in κ. Actually we believe that some kind of continuation argument is possible, but we propose a solution to the rotational invariance problem which demands far less information about the cluster expansion and can be adapted to recover physical positivity as well.

Let $\phi(x)$ be the free Euclidean scalar field in three dimensions with mass m. The ondelette cluster expansion for ϕ_3^4 [3], [4], [5] is based on the decomposition

$$\phi(x) = \sum_k \alpha_k u_k(x), \tag{1.1}$$

where

$$u_k = (-\Delta + m^2)^{-1/2}\Phi_k \tag{1.2}$$

and Φ_k runs over the modified Lemarié basis [5]. Actually we are defining the random variables α_k by

$$\alpha_k = \phi((-\Delta + m^2)^{1/2}\Phi_k) \tag{1.3}$$

and associating the regularization

$$\phi_A x) = \sum_{k\in A} \alpha_k u_k(x) \tag{1.4}$$

to every finite set A. The corresponding regularization of the ϕ_3^4 interaction will be denoted by $\lambda I_A(\phi)$, where λ is the coupling constant. Let $\langle \cdot \rangle_0$ denote the free expectation functional and set $Z_A = \langle e^{-\lambda I_A(\phi)} \rangle_0$. The ondelette cluster expansion proves:

THEOREM [3]. — For sufficiently small λ and for arbitrary Schwartz functions $f_1, \ldots, f_n$, the limit $\langle \Pi_{i=1}^n \phi(f_i) \rangle_\lambda$ of

$$Z_A^{-1} \left\langle e^{-\lambda I_A(\phi)} \prod_{i=1}^n \phi(f_i) \right\rangle_0 \tag{1.5}$$

exists as A approaches the set of *all* modes in the sense of set inclusion.

The problem is to show that this limit is Euclidean invariant.

THEOREM 1.1. — $\langle \Pi_{i=1}^n \phi(f_i) \rangle_\lambda$ is invariant with respect to unit scale translations.

The proof of this theorem is a standard type of argument [7] which involves grubbing around with the convergence tails of the given cluster expansion. The key input, of course, is the invariance of the modified Lemarié basis with respect to unit scale transitions.

LEMMA 1.2. — Rotations and unit-scale translations generate the group of Euclidean motions (with respect to invariance of $\langle \Pi_{i=1}^n \phi(f_i) \rangle_\lambda$).

Proof. — It suffices to show that *all* translations along *one* axis are generated. Now suppose a translation by x is generated. By the Pythagorean Theorem it follows that translation by $x\sqrt{2}$ is also induced. But differences between induced translations are also induced, so translation by $x(\sqrt{2} - 1)$ is generated. It follows from this inductive step and the unit translation that $(\sqrt{2} - 1)^N$-translation is induced for an arbitrary integer N. But integer linear combinations of these numbers are dense in $\mathbf{R}$, so the continuity property of $\langle \Pi_{i=1}^{n} \phi(f_i) \rangle_\lambda$ with respect to translations generates arbitrary real transitions. ■

Thus the Euclidean invariance problem reduces to rotational invariance, which involves the difficulty we have already mentioned.

Our first step in getting around the problem is to note that rotational invariance with respect to every coordinate plane is enough. On the other hand, the modified Lemarié basis has seven-dimensional subspaces invariant with respect to 90° rotations. By taking a sequence of sets A having this symmetry when one removes the phase cell cutoff, we see that $\langle \Pi_{i=1}^{n} \phi(f_i) \rangle_\lambda$ has this symmetry, and so the goal of this paper is to prove:

THEOREM 1.3. — $\langle \Pi_{i=1}^{n} \phi(f_i) \rangle_\lambda$ is rotationally invariant in the $x_1 x_2$ coordinate plane.

What have we gained from this reduction? The idea is that if we impose a *phase cell* cutoff in the *3rd* coordinate direction then all ultraviolet divergences are *marginal*. With this regularization imposed, we can therefore use the $(p^2 + m^2)^{-1/k}$ ultraviolet cutoff to establish rotational invariance in the perpendicular plane.

This strategy is not as easy as it sounds, because it is not all clear what we mean by a « phase cell cutoff in the 3rd coordinate direction. » The difficulty is that the *three dimensional modified Lemarié basis is not the tensor product of the two-dimensional modified Lemarié basis with the one-dimensional one*. Indeed, the linear relation between these bases is complicated, so it is not immediately clear how to truncate a basis of ondelettes « in the 3rd coordinate direction » without affecting the other coordinate directions as well. One could probably get around this problem by basing the expansion formalism on the tensor product of ondelette bases, but it would vastly complicate the case structure of the expansion. Each phase cell could be large scale in one direction and small scale in another.

Note. — Elements of the modified Lemarié basis will always be referred to as Lemarié functions, even if they are the special unit-scale functions that modify the Lemarié basis.

In Section 2 we find a one-dimensional regularization of the modified Lemarié basis satisfying the following conditions:

a) If $\Lambda = 2^{-\rho}$ is the scale of the cutoff, then all Lemarié functions *down to that scale* are included.

b) At scales $< \Lambda$ the orthonormal functions are two-dimensional Lemarié functions tensored with (one-dimensional) Λ-*scale functions only.*

c) If $\mathcal{H}_\Lambda$ is the subspace of $L^2(\mathbf{R}^3)$ spanned by our orthonormal set, then $\mathcal{H}_\Lambda = L^2(\mathbf{R}^2) \otimes \mathcal{L}_\Lambda$, where $\mathcal{L}_\Lambda$ is the subspace spanned by the Λ-scale cutoff of the one-dimensional Lemarié basis.

This last condition is a non-trivial theorem (Theorem 2.2) which depends on the block spin construction [*8*] of Lemarié functions in a significant way. Theorems of this nature will also be applied to Ward identities in a future paper.

Having outlined the strategy, we must now be more precise about how the reasoning goes. For $R > 0$ let ζ_R be a spherically symmetric C^∞ function such that $\zeta_R(x) = 1$ for $|x| \le R$, $\zeta_R(x) = 0$ for $|x| \ge R + 1$, and the partial derivatives of ζ_R are bounded uniformly in R. Let

$$\phi_\Lambda^{(\kappa,R)} = \zeta_R(-\Delta + m^2)^{-\frac{1}{\kappa}}\phi_\Lambda, \tag{1.6}$$

$$\phi_\Lambda = \sum_{k \in \mathbf{B}_\Lambda} \alpha_k(-\Delta + m^2)^{-\frac{1}{2}}\varphi_k, \tag{1.7}$$

where $\{ \varphi_k \mid k \in \mathbf{B}_\Lambda \}$ is our new orthonormal set associated with the scale Λ. Let $\lambda I_\Lambda^{(\kappa,R)}(\phi)$, $\lambda\, I_\Lambda(\phi)$ be the corresponding regularizations of the ϕ_3^4 interaction. Now in spite of our unusual kind of ultraviolet cutoff,

$$Z_\Lambda^{(\kappa,R)^{-1}} \left\langle e^{-\lambda I_\Lambda^{(\kappa,R)}(\phi)} \prod_{i=1}^{n} \phi(f_i) \right\rangle_0 \tag{1.8}$$

is well-defined for $\kappa < \infty$ by very basic estimation of convergent integrals and this is the purpose of the Λ-regularization; but the main point is that this regularization is rotationally invariant in the x_1x_2 coordinate plane. *This is a consequence of the tensor product theorem.* Since

$$\sum_{k \in \mathbf{B}_\Lambda} \hat{\varphi}_k(p)\overline{\hat{\varphi}_k(p')} = \delta(p_1 - p'_1)\delta(p_2 - p'_2)K_\Lambda(p_3, p'_3) \tag{1.9}$$

for some regular kernel $K_\Lambda(t, t')$, we have

$$\langle \hat{\phi}_\Lambda(p)\hat{\phi}_\Lambda(p') \rangle_0 =$$

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' K_\Lambda(t, p_3)K_\Lambda(t', p'_3) \frac{1}{(p_1 - p'_1)^2 + (p_2 - p'_2)^2 + (t - t')^2 + m^2}, \tag{1.10}$$

which proves the desired symmetry for (1.8). One can now preserve this symmetry by first taking the $\kappa = \infty$ & $R = \infty$ limits and then taking the $\Lambda = 0$ limit.

THEOREM 1.4. — For sufficiently small λ

$$\left\langle \prod_{i=1}^{n} \phi(f_i) \right\rangle_{\lambda} = \lim_{\Lambda \to 0} \lim_{\kappa, \mathrm{R} \to \infty} \ (1.8). \tag{1.11}$$

The proof of this theorem actually depends on the estimates for the φ_k functions that we prove in Section 2, but given these estimates, it is clear that the expansion formalism is unchanged and that convergence is uniform with respect to Λ, κ, &R. The identification of limits is due to the equation

$$(1.5) = \lim_{\Lambda \to 0} \lim_{\kappa, \mathrm{R} \to \infty} \mathrm{Z}_{\Lambda,\mathrm{A}}^{(\kappa,\mathrm{R})-1} \left\langle e^{-\lambda \mathrm{I}_{\Lambda}^{(\kappa,\mathrm{R})}}(\phi) \prod_{i=1}^{n} \phi(f_i) \right\rangle_{0}, \tag{1.12}$$

where $\lambda \mathrm{I}_{\Lambda,\mathrm{A}}(\phi)$ is the ϕ_3^4 interaction corresponding to the regularization

$$\phi_{\Lambda,\mathrm{A}}^{(\kappa,\mathrm{R})} = \zeta_{\mathrm{R}}(-\Delta + m^2)^{-1/\kappa} \phi_{\Lambda,\mathrm{A}}, \tag{1.13}$$

$$\phi_{\Lambda,\mathrm{A}} = \sum_{k \in \mathrm{A} \cap \mathbf{B}_{\Lambda}} \alpha_k (-\Delta + m^2)^{-1/2} \varphi_k. \tag{1.14}$$

The equation holds because $\varphi_k = \Phi_k$ down to scale Λ. Indeed the Λ-dependence disappears when $\Lambda \leq \min \{ \mathrm{L}_k \mid k \in \mathrm{A} \}$, where $\mathrm{L}_k = 2^{-r}$ is the scale of φ_k.

We close the introduction with a remark on how to prove physical positivity. One dimensionally regularizes the modified Lemarié basis in the above manner (yet to be described) but in every direction except the direction of Osterwalder-Schrader reflection. One can prove—in the same way we prove Theorem 2.2 below—that the subspace spanned by the resulting orthonormal set is a tensor product with $\mathrm{L}^2(\mathbf{R})$ as a factor, and so the regularized free covariance has the reflection positive structure in that direction [7]. The limit argument is straightforward because no ultraviolet cutoffs are super-imposed.

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APPENDIX

In this section we define our one-dimensional regularization, establish the fundamental estimates on the new functions, and prove the tensor product theorem.

For the modified Lemarié basis we let Φ denote the special unit scale function whose $\mathbf{Z}^3$-translates complete the basis. Recall [5] that Φ is given by

$$\hat{\Phi}(p) = \prod_{\mu=1}^{3} \frac{(-1)^{\frac{1}{2}M} e^{i\frac{1}{2}Mp_\mu} \hat{\chi}(p_\mu)^{M+1}}{\left[\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu+2\pi n)|^{2M+2}\right]^{1/2}} \equiv \prod_{\mu=1}^{3} v(p_\mu), \tag{2.1}$$

where χ is the characteristic function of the unit interval [0, 1]. For any lower scale $L=2^{-r}$ we let Φ_s denote the function for which $L^{-3/2}\Phi_s(L^{-1}x)$ generates the sub-level associated with the sth coordinate direction. Recall [5] that Φ_s is given by

$$\hat{\Phi}_1(p) = \frac{(1-e^{ip_1})^{M+1}\hat{\chi}(p_1)^{M+1}}{w(p_1)^{1/2} \sum_{n=-\infty}^{\infty} |\hat{\chi}(p_1+2\pi n)|^{2M+2}} v(p_2)v(p_3),$$

$$\hat{\Phi}_2(p) = v(2p_1) \frac{(1-e^{ip_2})^{M+1}\hat{\chi}(p_2)^{M+1}}{w(p_2)^{1/2} \sum_{n=-\infty}^{\infty} |\hat{\chi}(p_2+2\pi n)|^{2M+2}} v(p_3), \tag{2.2.2}$$

$$\hat{\Phi}_3(p) = v(2p_1)v(2p_2) \frac{(1-e^{ip_3})^{M+1}\hat{\chi}(p_3)^{M+1}}{w(p_3)^{1/2} \sum_{n=-\infty}^{\infty} |\hat{\chi}(p_3+2\pi n)|^{2M+2}}, \tag{2.2.3}$$

where

$$w(t) = \frac{|1-e^{it}|^{2M+2}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(t+2\pi n)|^{2M+2}} + \frac{|1+e^{it}|^{2M+2}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(t+2\pi n+\pi)|^{2M+2}}. \tag{2.3}$$

The translates $\Phi_{s,m}$ are given by

$$\hat{\Phi}_{1,m}(p) = \exp\left(i2m_1p_1 + i\sum_{\mu=2}^{3} m_\mu p_\mu\right)\hat{\Phi}_1(p), \tag{2.4.1}$$

$$\hat{\Phi}_{2,m}(p) = \exp\left(i2\sum_{\mu=1}^{2} m_\mu p_\mu + im_3p_3\right)\hat{\Phi}_2(p), \tag{2.4.2}$$

$$\hat{\Phi}_{3,m}(p) = \exp(i2m\cdot p)\hat{\Phi}_3(p). \tag{2.4.3}$$

Taking the translation rules into account one can see that 7 ondelettes are naturally associated with the L-scale cube; there are 8 associated with the unit-scale cube.

We introduce the *one-dimensional regularization* $\mathbf{B}_\Lambda$ of this basis for a fixed but arbitrary scale $\Lambda = 2^{-\rho}$: *down to that scale we take the same functions as before.* For $L < \Lambda$ we throw away the 3rd sub-level and replace $L^{-3/2}\Phi_{1,m}(L^{-1}x)$ & $L^{-3/2}\Phi_{2,m}(L^{-1}x)$ with $\varphi^{(L)}_{1,m}$ & $\varphi^{(L)}_{2,m}$, where

$$\hat{\varphi}^{(L)}_1(p) = L\Lambda^{\frac{1}{2}} \frac{(1 - e^{iLp_1})^{M+1}\hat{\chi}(Lp_1)^{M+1}}{w(Lp_1)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} | \hat{\chi}(Lp_1 + 2\pi n)|^{2M+2}} v(Lp_2)v(\Lambda p_3), \tag{2.5.1}$$

$$\hat{\varphi}^{(L)}_2(p) = L\Lambda^{\frac{1}{2}} v(2Lp_1) \frac{(1 - e^{iLp_2})^{M+1}\hat{\chi}(Lp_2)^{M+1}}{w(Lp_2)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} | \hat{\chi}(Lp_2 + 2\pi n)|^{2M+2}} v(\Lambda p_3), \tag{2.5.2}$$

$$\hat{\varphi}^{(L)}_{1,m}(p) = \exp\,(i2Lm_1p_1 + iLm_2p_2 + i\Lambda m_3p_3)\hat{\varphi}^{(L)}_1(p), \tag{2.6.1}$$
$$\hat{\varphi}^{(L)}_{2,m}(p) = \exp\,(i2Lm_1p_1 + i2Lm_2p_2 + i\Lambda m_3p_3)\hat{\varphi}^{(L)}_2(p). \tag{2.6.2}$$

Given the calculation in the introduction of [*8*], one can easily convince oneself that these functions form an orthonormal set. Orthogonality in at least one variable is guaranteed for any two such functions—whether they live on the same scale or not.

In the $\rho = \infty$ $(\Lambda = 0)$ limit we obviously recover our modified Lemarié basis. Moreover, for finite ρ the ondelette cluster expansion for the ρ-cutoff induced on the ϕ^4_3 theory converges uniformly in ρ. Indeed for scales $L < \Lambda$ the convergence problem becomes effectively two-dimensional *without complicating anything in the expansion formalism.* The larger scale in the 3rd coordinate direction is fixed as far as case structure is concerned. From the standpoint of basic ondelette estimates the scheme is that for modes k such that $L_k \geq \Lambda$, we have by [*5*]—since $\varphi_k = \Phi_k$ in this case—

$$|(D^\beta\varphi_k)(x)| \leq cL_k^{-\frac{3}{2}-\beta} \prod_{\mu=1}^{3} (1 + L_k^{-1}|x_\mu - x^{(k)}_\mu|)^{-N} \tag{2.7}$$

for all real $\beta < M$ and all positive integers $N \leq c(M, \beta)$, where $x^{(k)}$ denotes the center of the cube associated with k and

$$D \equiv (-\Delta + m^2)^{1/2}. \tag{2.8}$$

On the other hand if $L_k < \Lambda$, the basic ondelette estimate is two-dimensional.

THEOREM 2.1. — Let k be a mode for which $L_k < \Lambda$, where « mode » is now defined by the one-dimensional regularization of the modified Lemarié basis. Then for all real $\beta < M$ and all positive integers $N \leq c(M, \beta)$,

$$|(D^\beta\varphi_k)(x)| \leq c\Lambda^{-\frac{1}{2}}L_k^{-1-\beta} \prod_{\mu=1}^{2} (1 + L_k^{-1}|x_\mu - x^{(k)}_\mu|)^{-N}(1 + \Lambda^{-1}|x_3 - x^{(k)}_3|)^{-N}, \tag{2.9}$$

where $x^{(k)}$ denotes the center of the $L_k \times L_k \times \Lambda$ rectangular solid associated with k.

Proof. — The problem easily reduces to showing that the L^1-norm of

$$\prod_{\mu=1}^{3} \frac{\partial^N}{\partial p^N_\mu}[D(p)^\beta\hat{\varphi}^{(L)}_\iota(p)] \tag{2.10}$$

is bounded by

$$c\Lambda^{N-\frac{1}{2}}L^{2N-1-\beta} \tag{2.11}$$

for the given ranges of parameter values. The input estimates are

$$\left|\frac{\partial^{|m|}}{\partial p^m}\hat{\varphi}_l^{(L)}(p)\right| \leq cL^{m_1+m_2+1}\Lambda^{m_3+\frac{1}{2}}|Lp_\iota|^{M+1-m_\iota}(1+\Lambda|p_3|)^{-M-1}, \tag{2.12}$$

$$\left|\frac{\partial^{|n|}}{\partial p^n}D(p)^\beta\right| \leq c(p^2+m^2)^{\frac{1}{2}\beta-\frac{1}{2}|n|}, \tag{2.13}$$

for $|p_\mu| \leq L^{-1}$, $\mu = 1, 2$, and $-\infty < p_3 < \infty$, where m & n are multi-indices and $m_\mu + n_\mu = N$. In the region $|p_\mu| \geq L^{-1}$, $\mu = 1, 2$, we use (2.13) and

$$\left|\frac{\partial^{|m|}}{\partial p^m}\hat{\varphi}_l^{(L)}(p)\right| \leq c\prod_{\mu=1}^{2}(1+L|p_\mu|)^{-M-1}(1+\Lambda|p_3|)^{-M-1}L^{m_1+m_2+1}\Lambda^{m_3+\frac{1}{2}}. \tag{2.14}$$

Now if we make the change of variables

$$p'_\mu = Lp_\mu, \qquad \mu = 1, 2.$$
$$p'_3 = \Lambda p_3,$$

and use $L^{n_3} \leq \Lambda^{n_3}$, we obtain the bound (2.11) provided that our momentum bounds yield integrability in each region. This can be arranged choosing N sufficiently small relative to M & β, provided $\beta < M$.

REMARK. — Estimate (2.9) is « better » than (2.7) because it is applied to only a two-dimensional counting problem in the cluster expansion. For example, it is easy to see that the usual cancellation of the ϕ_3^4 mass bubble is unnecessary below scale Λ because divergences are two-dimensional there.

Having established that the one-dimensionally regularized set B_Λ does no harm to the cluster expansion, we now prove the motivating result.

THEOREM 2.2. — Let $\mathscr{H}_\Lambda$ be the subspace of $L^2(\mathbb{R}^3)$ spanned by the orthonormal set B_Λ defined above. Then

$$\mathscr{H}_\Lambda = L^2(\mathbb{R}^2) \otimes \mathscr{L}_\Lambda, \tag{2.15}$$

where $\mathscr{L}_\Lambda$ is the subspace of $L^2(\mathbb{R})$ spanned by the Λ-scale cutoff of the *one*-dimensional modified Lemarié basis.

Proof. — First we show that $\mathscr{H}_\Lambda \subset L^2(\mathbb{R}^2) \otimes \mathscr{L}_\Lambda$. It suffices to express an arbitrary φ_k in the one-dimensionally regularized set B_Λ as a linear combination of functions of the form $f \otimes q_l$, where $f \in L^2(\mathbb{R}^2)$ and q_l is a one-dimensional Lemarié function with scale $\geq \Lambda$. Now if $L_k \geq \Lambda$, then φ_k is a three-dimensional Lemarié function (by definition of our orthonormal set). In this case there are two possibilities (see (2.1) & (2.2.*i*)). Either φ_k already has the form $f \otimes q_l$ or

$$\hat{\varphi}_k(p) = \hat{\psi}_\lambda(p_1, p_2)\exp(iL_k m_3 p_3)v(L_k p_3), \tag{2.16}$$

where ψ_λ is a two-dimensional Lemarié function with scale $L_\lambda = L_k$. The problem, then, is to show that $e^{iL_k m_3 t}v(L_k t)$ lies in $\mathscr{L}_\Lambda$. But this is equivalent to showing that it is orthogonal to $\mathscr{L}_\Lambda^\perp$, and this orthogonality is a natural consequence of the block spin construction [8] of the Lemarié basis. Indeed, for any Lemarié function $q_l \in \mathscr{L}_\Lambda^\perp$, we have

$$\int \hat{q}_l(t)\overline{\hat{\chi}(L_k t)^{M+1}\exp(iL_k nt)}dt = 0 \tag{2.17}$$

for all n because q_l ultimately arises from ± 1 block spin assignments at scale $L_l \leq L_k$ (recall that (2.17) is only one form of the consequence). On the other hand, it follows from Poisson summation that

$$e^{im_3t}v(t) = \frac{(-1)^{\frac{1}{2}M} e^{i\frac{1}{2}Mt}\hat{\chi}(t)^{M+1}}{\left[\prod_{l=-\infty}^{\infty} |\hat{\chi}(t+2\pi l)|^{2M+2}\right]^{\frac{1}{2}}} e^{im_3t}$$
$$= (-1)^{\frac{1}{2}M} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} d\omega \frac{e^{i\left(\frac{1}{2}M+m_3-n\right)\omega}}{\left[\sum_{l=-\infty}^{\infty} |\hat{\chi}(\omega+2\pi l)|^{2M+2}\right]^{\frac{1}{2}}} e^{int}\hat{\chi}(t)^{M+1}, \tag{2.18}$$

where the convergence of such a sum has already been established in [8]. Thus

$$\int \hat{q}_l(t)e^{iL_km_3t}v(L_kt)dt = 0, \tag{2.19}$$

and so the orthogonality of $e^{iL_km_3t}v(L_kt)$ so $\mathscr{L}_\Lambda^\perp$ is verified. This concludes the case $L_k \geq \Lambda$. If $L_k < \Lambda$, then φ_k has the form

$$\hat{\varphi}_k(p) = \hat{\psi}_\lambda(p_1, p_2)e^{i\Lambda m_3p_3}v(\Lambda p_3). \tag{2.20}$$

But the same reasoning that we have just given shows that $e^{i\Lambda m_3t}v(\Lambda t)$ is also orthogonal to $\mathscr{L}_\Lambda^\perp$. This completes the argument that $\mathscr{H}_\Lambda \subset L^2(\mathbb{R}^2) \otimes \mathscr{L}_\Lambda$.

To prove that $L^2(\mathbb{R}^2) \otimes \mathscr{L}_\Lambda \subset \mathscr{H}_\Lambda$ we pick an arbitrary $\psi_\lambda \otimes q_l$ with $L_l \geq \Lambda$ and try to express it as a linear combination of elements from our orthonormal set B_Λ. There are 3 cases to consider:

a) $L_\lambda > L_l$,

b) $L_l \geq L_\lambda \geq \Lambda$,

c) $\Lambda > L_\lambda$.

In case *a)* we simply express $\hat{\psi}_\lambda(p_1, p_2)$ as a linear combination of the functions

$$\prod_{\mu=1}^{2} (e^{i2L_lm_\mu p_\mu}v(2L_lp_\mu)). \tag{2.21}$$

This is possible because

$$\hat{\chi}(2^rt)^{M+1} = 2^{-rM-r}(1+e^{i2^{r-1}t})^{M+1}\dots(1+e^{i2t})^{M+1}(1+e^{it})^{M+1}\hat{\chi}(t)^{M+1}, \tag{2.22}$$

and we can appeal to (2.18) and

$$e^{im_\mu s}\frac{(1-e^{is})^{M+1}\hat{\chi}(s)^{M+1}e^{i\frac{1}{2}Ms}}{\left[\sum_{l=-\infty}^{\infty} |\hat{\chi}(s+2\pi l)|^{2M+2}\right]^{\frac{1}{2}}} = \sum_{k=0}^{M+1}(-1)^k\binom{M+1}{k}\sum_{n=-\infty}^{\infty} e^{ins}\hat{\chi}(s)^{M+1}.$$
$$\int_0^{2\pi} d\omega \frac{e^{i\left(\frac{1}{2}M+k+m_\mu-n\right)\omega}}{\left[\sum_{l=-\infty}^{\infty} |\hat{\chi}(\omega+2\pi l)|^{2M+2}\right]^{\frac{1}{2}}}. \tag{2.23}$$

But a function whose Fourier transform has the form

$$\prod_{\mu=1}^{2} (e^{i2L_l m_\mu p_\mu} v(2L_l p_\mu)) \hat{q}_l(p_3) \tag{2.24}$$

is precisely an L_l-scale Lemarié function and therefore a member of B_Λ in this case.

In case *b*) we play exactly the same game with the roles of ψ_λ and q_l reversed (except the factor 2 in (2.21) is missing). The result is a linear combination of L_λ-scale Lemarié functions, which belong to B_Λ in this case as well.

In case *c*) the problem is to express q_l as a linear combination of the functions $e^{i\Lambda m p} v(\Lambda p_3)$. But this possibility follows from (2.18), (2.22) & (2.23) again. ■

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