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The large-scale limit of Dyson's hierarchical vector valued model at low temperatures. The non-Gaussian case

PART I: LIMIT THEOREM FOR THE AVERAGE SPIN

by

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RÉSUMÉ. — Nous étudions la limite thermodynamique du modèle hiérarchique de Dyson vectoriel invariant par rotation à basse température. Ce modèle dépend d'un paramètre c qui joue un rôle analogue à la dimension. Le cas $\sqrt{2} < c < 2$ a été étudié dans [5] et nous considérons ici le cas $1 < c < \sqrt{2}$ qui donne des limites thermodynamiques non gaussiennes.

Dans la première partie nous étudions l'action de la renormalisation sur ce modèle, et nous établissons la convergence pour des normalisations non triviales. Dans la seconde partie nous étudions la limite thermodynamique de l'état de Gibbs en imposant un petit champ magnétique que nous faisons tendre vers zéro avec le volume.

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1. INTRODUCTION

First we formulate the problem we are investigating. We consider Dyson's hierarchical vector valued model which is defined in the following way: Put $\mathbb{Z} = \{ 1, 2, \dots \}$ and define the hierarchical distance $d(\cdot, \cdot)$ on $\mathbb{Z}$ by the formula

$$d(i,j) = \begin{cases} 0 & \text{if} \quad i = j \\ 2^{n(i,j)-1} & \text{if} \quad i \neq j \end{cases} \tag{1.1}$$

with $n(i,j) = \{\min n$, there is an integer k such that $(k-1)2^n < i, j \leq k2^n\}$. We call a sequence $\sigma = \{\sigma(i), i \in \mathbb{Z}\}$ a configuration, and assume in this work that $\sigma(i) \in \mathrm{R}^p$ with some $p \geq 2$ for all $i \in \mathbb{Z}$. In order to define our model we introduce the following Hamiltonian function $\mathcal{H}(\sigma)$ in the space of configurations

$$\mathcal{H}(\sigma) = -\sum_{i \in \mathbb{Z}} \sum_{\substack{j \in \mathbb{Z} \\ j \geq i}} \mathrm{U}(i,j)\sigma(i)\sigma(j) \tag{1.2}$$

$$\mathrm{U}(i,j) = d(i,j)^{-a} \quad \text{if} \quad i \neq j, \tag{1.2$'$}$$

where a, $1 < a < 2$, is a parameter of the model, and $\sigma(i)\sigma(j)$ denotes scalar product. (In the sequel we shall use $c = 2^{2-a}$ instead of the parameter a.) We also introduce the function

$$\mathcal{H}_n(\sigma) = -\sum_{i=1}^{2^n} \sum_{j=i+1}^{2^n} \mathrm{U}(i,j)\sigma(i)\sigma(j), \quad \sigma = \{\sigma(1), \ldots, \sigma(2^n)\} \tag{1.2$''$}$$

and a free measure ν on R^p defined by the formula

$$\frac{d\nu}{dx} = p(x) = p(x,\ t) = \mathrm{C}(t) \exp\left\{ -\frac{x^2}{2} - \frac{t}{4} | x |^4 \right\} \tag{1.3}$$

where $t > 0$ is a sufficiently small fixed constant. It is another parameter of the model. The constant $\mathrm{C}(t)$ is chosen in such a way that $p(x)$ be a density function.

Dyson has introduced such a model in [*12*] and [*13*]. It is a simplified version of one-dimensional Ising type models with long range interaction. Many physical phenomenas can be studied more simply in this model. Dyson introduced it to study phase transitions and Thouless effect. Later it became clear that this model is also very appropriate to study renormalization type problems (see [*6*], [*8*], [*9*]). The aim of the present work is also to study a renormalization type problem. We want to understand the consequences of continuous symmetry in renormalization type problems.

There is a standard way to define equilibrium states at temperature T for a model with Hamiltonian $\mathcal{H}$ and free measure ν (see e. g. [*19*], [*20*]). For the sake of completeness we recall it in Appendix D. In Appendix E we also prove that the measure constructed in Part II is an equilibrium state. In this work we are mainly interested in the large-scale limit of the equilibrium state of Dyson's model. The notion of large-scale limit is defined in many places, e. g. in [*20*] or at the beginning of Part II of this work. (In Part I this notion does not appear yet.) In Section 7 of [*6*] it is pointed out that the large-scale limits in Dyson's model are essentially different in the case $\sigma(i) \in \mathrm{R}^1$ and in the vector-valued case $\sigma(i) \in \mathrm{R}^p$, $p \geq 2$.

Moreover, in the vector valued case the situations $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$ also differ. The large-scale limit in the case $\sqrt{2} < c < 2$ is described in [5]. The corresponding result for the case $1 < c < \sqrt{2}$ is formulated in [6] with a sketch of proof. (The case $c = \sqrt{2}$ deserves special attention, but we are going to investigate it elsewhere.) The aim of the present work is to give a rigorous proof of the result about the model with $1 < c < \sqrt{2}$ on the basis of the ideas of [6]. During the proof we had to overcome several technical difficulties which we found interesting in themselves. The proof can be split up into the solution of two analytical problems which are fairly independent. In the first part of this work we study the first of them which deals with the behaviour of Gibbs states without boundary conditions. (See Appendix D for explanation of this terminology.) The most essential differences between cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$ appear at this point.

The first problem can be formulated directly, and it has a special interest. For all n, $n = 0, 1, 2, \ldots$ and $T > 0$ consider the probability measure $\mu_n = \mu_{n,T}$ on $(R^p)^{2^n}$ with the density function

$$p_n(x_1, \ldots, x_{2^n}) = p_n(x_1, \ldots, x_{2^n}, T, t), \quad x_j \in R^p$$

defined by the formula

$$p_n(x_1, \ldots, x_{2^n}) = \frac{1}{Z_n(T, t)} \exp\left\{ -\frac{1}{T} \sum_{i=1}^{2^n} \sum_{j=i+1}^{2^n} U(i,j) x_i x_j \right\} \prod_{i=n}^{2^n} p(x_i, t) \quad (1.4)$$

where $U(i,j)$ is defined by formulas (1.1) and (1.2), $p(x, t)$ by (1.3) and $Z_n(T, t)$ is an appropriate norming constant with which $p_n(x_1, \ldots, x_{2^n})$ is a probability density function. Let $(\sigma(1), \ldots, \sigma(2^n))$ be a $\mu_{n,T}$ distributed random vector, and let $p_n(x, T) = p_n(x, T, t)$ denote the density function of the average $\frac{1}{2^n} \sum_{j=1}^{2^n} \sigma(j)$. We are interested in the asymptotic behaviour of $p_n(x, T)$ as $n \to \infty$. Let us remark that $p_n(x, T)$ depends on x only through $|x|$, i. e. if we define the function $\bar{p}_n(y, t)$, $y \in R^1$ as $\bar{p}_n(y, t) = p_n((y, 0), t)$, $0 = (0, \ldots, 0) \in R^{p-1}$ then $p_n(x, t) = \bar{p}_n(|x|, T)$. Now we formulate Theorem 1, the main result of Part I. Its main content is that for $1 < c < \sqrt{2}$ the density function $p_n(x, T)$ satisfies a limit theorem with an unusual normalization. The limit distribution is not normal, its density is defined by an integral equation. Theorem 2 also contains some information about the smoothness of the functions $p_n(x, T)$ and their decrease at infinity.

THEOREM 1. — *For* $1 < c < \sqrt{2}$ *there exist some* $T_0 > 0$ *and* $t_0 > 0$ *such* *that for all* $0 < T < T_0$ *and* $0 < t < t_0$ *there are some* $\overline{M} = \overline{M}(c, t, T) > 0$ *and* $n_0 = n_0(c, t, T)$ *such that for* $n > n_0$

$$c^{-n}p_n(x, T) = c^{-n}\overline{p}_n(|x|, T) =$$
$$= B(\overline{M}, T)\exp\left(-\frac{a_0c^n}{T}\overline{M}(|x| - \overline{M})\right)\cdot g\left(\frac{a_1c^n}{T}\overline{M}(|x| - \overline{M})\right) + r_n(x) \quad (1.5)$$

with $a_0 = \dfrac{2}{2-c}$ *and* $a_1 = a_0 + 1$, *where* $B(\overline{M}, T) > 0$ *is an appropriate norming constant, the function* $g(x)$ *is the solution of the integral equation*

$$g(x) = \left(\frac{2}{c\sqrt{\pi}}\right)^{p-1}\int_{R^p}\exp(-v^2)g\left(\frac{x}{c} + u + \frac{v^2}{2}\right)g\left(\frac{x}{c} - u + \frac{v^2}{2}\right)dudv \quad (1.6)$$

where $u \in R^1$, $v \in R^{p-1}$, v^2 *denotes scalar product, and the error term* $r_n(x)$ *satisfies the inequality*

$$|r_n(x)| \leqq K(\overline{M}, T)q^n \quad \text{for all} \quad x \geq 0$$

with some $K(\overline{M}, T) > 0$ *and* $0 < q < 1$, *where* q *depends only on the parameter* c, $1 < c < \sqrt{2}$.

The equation (1.6) has a unique solution in the class of functions $\mathscr{A} = \left\{ g(x), \int e^{tx}|g(x)|dx < \infty \text{ if } |t| < t_0(g), t_0(g) > 0 \right\}$ *beside the trivial one* $g(x) \equiv 0$, *and this function appears in formula (1.6). It also satisfies the relation* $g(x) > 0$ *for all* x, *and* $\exp(-a_0x)g(a_1x) < K(\alpha)\exp(-\alpha|x|)$ *for all* $x \in R^1$ *if* $\alpha < \dfrac{6c-4}{c(2-c)}$.

The functions $\overline{p}_n(x, T)$ *and* $r_n(x)$ *also satisfy the inequalities*

$$\left|\frac{d^j}{dx^j}c^{-n}\overline{p}_n(x, T)\right| \leqq c^{nj}K(\overline{M}, T)\exp\left(\frac{-\mu c^n\overline{M}}{T}|x - \overline{M}|\right) \quad \text{for all} \quad x > 0$$
$$j = 0, 1, 2, \quad n > n_0 \quad (1.7)$$

$$\left|\frac{d^j}{dx^j}r_n(x)\right| \leqq c^{nj}K(M, T)q^n, \quad j = 0, 1, 2 \quad \text{for all} \quad x > 0, \quad n > n_0 \quad (1.8)$$

with some $K(\overline{M}, T) > 0$, $\mu > 0$ *and* $0 < q < 1$, *where* μ *and* q *depend only on the parameter* c.

The number $\overline{M}$ *satisfies the relation*

$$\overline{M}^2 = \frac{a_0 - T}{tT} + R(T, t) \quad (1.9)$$

with some $|R(T, t)| \leqq$ const. *such that* $R(T, t) \to 0$ *if* $T \to 0$ *and* $t \to 0$.

We shall prove (Lemma 12) that the function $g(x)$ is the density function of an appropriately defined quadratic form of independent normal variables.

Theorem 1 means in particular that in the case $1 < c < \sqrt{2}$ if $(\sigma(1), \ldots, \sigma(2^n))$ is a $\mu_{n,\mathrm{T}}$ distributed random vector then the density functions of the random variables $\frac{c^n}{2^n}\left|\sum_{j=1}^{2^n} \sigma(j)\right| - c^n\overline{\mathrm{M}}$ tend to the density function

$$\text{const.}\ \exp\left(-\frac{a_0}{\mathrm{T}}\mathrm{M}x\right)g\left(\frac{a_1}{\mathrm{T}}\mathrm{M}x\right) \quad \text{as} \quad n \to \infty .$$

The behaviour of Dyson's model in the case $\sqrt{2} < c < 2$ or with scalar valued spins is essentially different. In these cases the random vectors $2^{-\frac{n}{2}}\left|\sum_{j=1}^{2^n} \sigma(j)\right| - 2^{\frac{n}{2}}\mathrm{M}_n$ tend in distribution to a normal law with zero expectation, $\mathrm{M}_n \to \overline{\mathrm{M}}$ with some $\overline{\mathrm{M}}$ if $\mathrm{M}_n = \mathrm{E}\,|\,\sigma(j)\,|$ (expectation is taken with respect to the measure $\mu_{n,\mathrm{T}}$). (See [*5*] Appendix, [*6*] and [*9*].) This means that in these two cases we have to normalize differently.

In Part II of this work we show that this difference is also inherited in the behaviour of equilibrium states. Moreover, we describe the large-scale limit of the equilibrium state, and show that its component in the direction of the magnetization is a quadratic functional of a Gaussian field.

Let us remark that in our model both the Hamiltonian function $\mathcal{H}(\sigma)$ and the free measure $\nu(\sigma)$ defined in (1.2) and (1.3) remain invariant if all spins $\sigma(j)$, $j \in \mathbb{Z}$, are rotated in the same way. Such an invariance is called an $\mathrm{O}(p)$ continuous symmetry in the physics literature. Actually this continuous symmetry is the cause of the results in our model. The real problem we are going to study is the consequences of continuous symmetries. We expect that results analogous to those of this paper also hold for translation invariant models with a continuous symmetry on the three dimensional lattice. We formulate this conjecture in a more explicit form in the second part of this work.

In that part we need some more information about the behaviour of the function $p_n(x, \mathrm{T})$ than that given in Theorem 1. Hence we prove the following

THEOREM 2. — *Under the conditions of Theorem 1 there exist some integer n_0 and positive real numbers ε, q, B, K, L and δ depending on the parameters c, T and t in such a way that for $n > n_0$*

$$c^{-n}\overline{p}_n(x, \mathrm{T}) = \mathrm{B}\exp\left(-\frac{a_0 c^n}{\mathrm{T}}\overline{\mathrm{M}}(x - \overline{\mathrm{M}})\right)g\left(\frac{a_1 c^n}{\mathrm{T}}\overline{\mathrm{M}}(x - \overline{\mathrm{M}})\right)(1 + r_n(x)) \quad (1.10)$$

with some $|r_n(x)| < \mathrm{K}q^n$, $0<q<1$, *in the interval* $-\varepsilon n c^{-n} < x - \overline{\mathrm{M}} < \varepsilon n^{1/\alpha} c^{-n}$, *where* $\alpha = \dfrac{\log 2}{\log c}$, *and*

$$c^{-n}\overline{p}_n(x, \mathrm{T}) < \mathrm{K}q^n \exp(-\mathrm{L}(c^n|x - \overline{\mathrm{M}}|)^{2+\delta}) \tag{1.11}$$

if $x > \overline{\mathrm{M}} + \varepsilon n^{1/\alpha} c^{-n}$.

Relation (1.10) can be considered as the multiplicative version of relation (1.5). In order to deduce it from (1.5) we have to give a good lower bound on $c^{-n}\overline{p}_n(x, \mathrm{T})$ in the interval $-\varepsilon n c^{-n} < x - \overline{\mathrm{M}} < \varepsilon n^{1/\alpha} c^{-n}$. The main content of formula (1.11) is that for $x \gg \overline{\mathrm{M}}$ a much sharper upper bound can be given than that in (1.7). On the other hand for $0 < x < \overline{\mathrm{M}}$ the bound in (1.7) cannot be improved considerably, at most a better constant μ can be written in the exponent.

The appearance of the number $\alpha = \dfrac{\log 2}{\log c}$ has a deeper reason. The decrease of the limit function $g(x)$ at plus infinity is of order $\exp(-\text{const}\,|x|^\alpha)$. The size of the typical region where the good asymptotic formula (1.10) is proved is also connected with the tail behaviour of the limit function. This typical region is chosen in such a way that the density function $\overline{p}_n(x, \mathrm{T})$, after an appropriate scaling, is exponentially small outside of this region. (Observe the term q^n, $q < 1$, in formula (1.11).) The typical region is not symmetric with respect to the origin, because the decrease of the function $\overline{p}_n(x, \mathrm{T})$ is different for positive and negative arguments. For negative x formula (1.7) gives a good bound on $\overline{p}_n(x, \mathrm{T})$ outside of the typical region.

In Theorems 1 and 2 we have assumed that the free measure ν is defined by (1.3). We could have considered a more general class of free measures. Theorems 1 and 2 can be proved without any essential change if

$$\frac{d\nu}{dx} = p(x, t) = \mathrm{C}(t)\exp\left(-\frac{x^2}{2} - \frac{t}{4}|x|^4 + \mathrm{R}(x, t)\right)$$

with some function R such that $\left|\dfrac{\partial^j \mathrm{R}}{\partial x^j}\right| < \mathrm{C}t^{1+\varepsilon}|x|^{4-j}$ with some $\mathrm{C} > 0$, $j = 0, 1, 2, 3, 4$.

2. ON THE CONTENT OF THEOREM 1. CONVERGENCE TO THE SOLUTION OF THE FIXED POINT EQUATION

It is proved e. g. in Section 4 of [*6*] that the function $p_n(x, \mathrm{T})$ defined in Section 1 satisfies the recursive relations

$$p_n(x, \mathrm{T}) = \mathrm{C}_n(\mathrm{T})\int \exp\left(\frac{c^{n-1}}{\mathrm{T}}(x^2 - u^2)\right)p_{n-1}(x-u, \mathrm{T})p_{n-1}(x+u, \mathrm{T})du \tag{2.1}$$

$$p_0(x, \mathrm{T}) = p_0(x) = \mathrm{C}(t)\exp\left\{-\frac{x^2}{2} - \frac{t}{4}x^4\right\} \tag{2.1'}$$

where $C(t)$ and $C_n(T)$ are appropriate norming constants which turn p_n into a density function. (These are formulas (4.2) and (4.2)′ in [2]. For the sake of completeness we also present their proof in Appendix A.) Thus Theorem 1 actually formulates the properties of the function defined by relations (2.1) and (2.1)′, and we have to study these formulas. We can simplify them a little by introducing the functions

$$q_n(x,T)=B_n \exp\left(\frac{a_0}{2a_1}c^n x^2\right)p_n\left(\sqrt{\frac{T}{a_1}}x, T\right),\ a_0=\frac{2}{2-c},\ a_1=a_0+1, \quad (2.2)$$

with some constant B_n to be defined later.

A straightforward calculation shows that (2.1) and (2.1)′ imply that

$$q_n(x,T)=\overline{C}_n(T)\int_{R^p}\exp(-c^{n-1}u^2)q_{n-1}(x-u,T)q_{n-1}(x+u,T)du \quad (2.3)$$

$$q_0(x,T) = \overline{C}_0(T)\exp\left(\frac{a_0-T}{2a_1}x^2 - \frac{t}{4}\frac{T^2}{a_1^2}x^4\right) \quad (2.3)'$$

with some $\overline{C}_n(T) > 0$. Observe that $q_n(x,T)$ is also rotation invariant, i. e. $q_n(x,T) = \overline{q}_n(|x|,T)$ for the function $\overline{q}_n(z,T) = q_n((z,0),T)$, $z \in R^1$, $0 = (0,\dots,0) \in R^{p-1}$. Also the relation $\overline{q}_n(x,T) = \overline{q}_n(-x,T)$ holds. Choose the constant B_n in (2.2) in such a way that

$$\int_0^\infty \overline{q}_n(x,T)dx = 1. \quad (2.4)$$

Clearly

$$p_n(x,T) = \frac{1}{B_n}\exp\left(-\frac{a_0}{2T}c^n x^2\right)q_n\left(\sqrt{\frac{a_1}{T}}x, T\right), \quad (2.5)$$

and (2.3) implies that

$$\overline{q}_{n+1}(x,T)=$$
$$=K_n\int\exp(-c^n(u^2+v^2))\overline{q}_n(\sqrt{(x+u)^2+v^2},T)\overline{q}_n(\sqrt{(x-u)^2+v^2},T)dudv \quad (2.6)$$

Given the function $\overline{q}_n(x,T)$ we define a number M_n and a function $f_n(x) = f_n(x,T)$ by the formulas

$$M_n = M_n(T) = \int_0^\infty x\overline{q}_n(x,T)dx \quad (2.7)$$

$$f_n(x) = f_n(x,T) = c^{-n}\overline{q}_n\left(M_n + \frac{x}{c^n}, T\right). \quad (2.7)'$$

Clearly, $f_n(x) \geq 0$, $f_n(x - c^n M_n) = f_n(-x - c^n M_n)$, $\int_{-c^n M_n}^{\infty} f_n(x)dx = 1$, $\int_{-c^n M_n}^{\infty} x f_n(x)dx = 0$ and $\bar{q}_n(x, T) = c^n f_n(c^n(x - M_n), T)$. The reason for introducing the function f_n and number M_n is that we expect that this is the right rescaling of the function q_n for which $M_n \to M$ and $f_n(x) \to g_M(x)$ with some appropriate $M > 0$ and g_M as $n \to \infty$. (See Section 7 of [*6*] for a heuristic argument.) We shall prove the following

THEOREM 1'. — *Under the conditions of Theorem 1 the limit* $\lim_{n\to\infty} M_n = M$ *exists, and*

$$M^2 = \frac{a_1(a_0 - T)}{tT^2} + R(t, T) \tag{2.8}$$

with some $|R(t, T)| \leq$ const. *and* $R(t, T) \to 0$ *as* $T \to 0$ *and* $t \to 0$. *Moreover, there is some* $n_0 = n_0(c, t, T)$ *such that for* $n > n_0$

$$M_n = M + \frac{c}{4(c-1)M} c^{-n} + \delta(n) \cdot c^{-n}, \quad |\delta(n)| \leqq K \cdot c^{-n} \tag{2.8'}$$

with some $K = K(c)$. *For* $n > n_0$ *there is some* $0 < q < 1$ *and* $K > 0$ *depending on the parameter c such that*

$$\left| \frac{d^j}{dx^j}\left[f_n(x, T) - Mg\left(M\left(x + \frac{c}{4(c-1)M}\right)\right)\right]\right| \leqq Kq^n, \; j = 0, 1, 2, \; x > -c^n M_n \tag{2.9}$$

where g(x) is the same function as in Theorem 1. We also have

$$\left| \frac{d^j}{dx^j} f_n(x, T)\right| < KM^{j+1} \exp(-\mu M|x|), \quad j = 0, 1, 2, \quad x > -c^n M_n \tag{2.9'}$$

for $n > n_0$, *with some* $\mu > 0$ *and* $K > 0$ *depending only on c.*

We shall deduce Theorem 1 from Theorem 1'. In order to study f_n and M_n we introduce the integral operators $\bar{Q}_{n,M}$, $n = 1, 2, \ldots, M > 0$ defined for functions $f \in \mathscr{A}_{n,M}$,

$$\begin{aligned} \mathscr{A}_{n,M} = \{ f : R^1 \to R^1, \; & f(x) \text{ is continuous}, \; 0 \leq f(x) \leq K \text{ for all } x \in R^1, \\ & f(x) < K \exp(-\alpha x) \text{ with some } K = K(f) > 0 \text{ and } \alpha = \alpha(f) > 0 \\ & \text{for } x > 0, \text{ there is some } x > -c^n M \text{ where } f(x) > 0 \}, \end{aligned}$$

by the formula

$$\begin{aligned} \bar{Q}_{n,M} f(x) = \int \exp\left(-\frac{u^2}{c^n} - v^2\right) & f\left(c^n\left(\sqrt{\left(M + \frac{x}{c^{n+1}} + \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - M\right)\right) \\ & f\left(c^n\left(\sqrt{\left(M + \frac{x}{c^{n+1}} - \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - M\right)\right) du dv . \end{aligned} \tag{2.10}$$

For the sake of simpler notations we shall restrict ourselves from now on to two-dimensional models, i. e. $R^p = R^2$. In this case $u \in R^1$ and $v \in R^1$ in (2.10). Observe that $\overline{Q}_{n,M}f(x)$ depends on the values of $f(x)$ only for $x > -c^n M$, and $\overline{Q}_{n,M}f(x) = \overline{Q}_{n,M}f(-x - 2c^{n+1}M)$ what can be seen by applying the substitution $(u, v) \to (-u, -v)$ in the integral defining $\overline{Q}_{n,M}f$. Moreover, $\overline{Q}_{n,M}f \in \mathscr{A}_{n+1,M}$ for $f \in \mathscr{A}_{n,M}$, since $\overline{Q}_{n,M}f(cx) > 0$ if $f(x) > 0$, $0 \leq \overline{Q}_{n,M}f(x) \leqq c^n \pi \sup |f(x)|^2$, and we get, by splitting up the domain of integration in (2.10) to $\left\{(u,v), |u| < \frac{x}{2}\right\}$ and $\left\{(u,v), |u| > \frac{x}{2}\right\}$, that for $x > 0$

$$|\overline{Q}_{n,M}f(x)| \leqq K^2 c^n \pi \left(\exp\left(-\frac{\alpha}{2c}x\right) + \exp\left(-\frac{x^2}{4c^n}\right)\right) \leqq \overline{K}\exp(-\overline{\alpha}x).$$

Put

$$m_n = \frac{\int_{-c^{n+1}M}^{\infty} x\overline{Q}_{n,M}f(x)dx}{\int_{-c^{n+1}M}^{\infty} \overline{Q}_{n,M}f(x)dx}, \tag{2.11}$$

and define the normalization of the operator $\overline{Q}_{n,M}$

$$Q_{n,M}f(x) = \frac{\overline{Q}_{n,M}f(x + m_n)}{\int_{-c^{n+1}M}^{\infty} \overline{Q}_{n,M}f(x)dx} \tag{2.12}$$

for $f \in \mathscr{A}_{n,M}$. (The above formulas are meaningful since $\overline{Q}_{n,M}f \in \mathscr{A}_{n+1,M}$.)

Let us define

$$Q_n(f(x), M) = \left(Q_{n,M}f(x), M + \frac{m_n}{c^{n+1}}\right) \tag{2.13}$$

for $f \in \mathscr{A}_{n,M}$, $M > 0$. We claim that the relation

$$Q_n(f_n(x), M_n) = (f_{n+1}(x), M_{n+1}) \tag{2.13)'}$$

holds for the functions f_n and numbers M_n defined in (2.7) and (2.7)', and $f_n \in \mathscr{A}_{n,M}$. This can be seen by observing that by (2.6) and the definition of f_n

$$\overline{q}_{n+1}\left(M_n + \frac{x}{c^{n+1}}\right) = \\ = K_n \int \exp\left(-\frac{u^2}{c^n} - v^2\right) f_n\left(c^n\left(\sqrt{\left(M_n + \frac{x}{c^{n+1}} + \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - M_n\right)\right) \\ f\left(c^n\left(\sqrt{\left(M_n + \frac{x}{c^{n+1}} - \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - M_n\right)\right) dudv = K_n \overline{Q}_{n,M_n} f_n(x),$$

hence

$$f_{n+1}(x) = Q_{n,M_n} f_n(x) \tag{2.14}$$

and

$$M_{n+1} = M_n + \frac{m_n}{c^{n+1}} \tag{2.14'}$$

(The constants K, C, K_n, C_n, etc. will denote appropriate multiplying factors in the sequel. The same letter may denote different numbers in different formulas.)

If $\frac{x}{c^{n+1}}$, $\frac{u}{c^n}$ and $\frac{v^2}{c^n}$ are much smaller than M then a simple Taylor expansion yields that

$$c^n\left(\sqrt{\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - M\right) =$$

$$= c^n M\left(\sqrt{\left(1 + \frac{x}{c^{n+1}M} \pm \frac{u}{c^n M}\right)^2 + \frac{v^2}{c^n M^2}} - 1\right) \sim \frac{x}{c} \pm u + \frac{v^2}{2M}. \tag{2.15}$$

This relation suggests to approximate the operators $\overline{Q}_{n,M}$ and $Q_{n,M}$ by $\overline{T}_M$ and T_M defined by the formulas

$$\overline{T}_M f(x) = \int \exp(-v^2) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) du dv \tag{2.16}$$

and

$$T_M f(x) = \frac{2}{c\sqrt{\pi}} \overline{T}_M f\left(x - \frac{c}{4M}\right). \tag{2.17}$$

As we shall see later, T_M maps a density function with zero expectation to such a function again, hence T_M is the natural approximation of the operator $Q_{n,M}$. By relation (2.14) we can write

$$f_{n+1}(x) = T_{M_n} f_n(x) + \varepsilon_n(x) \tag{2.18}$$

with $\varepsilon_n(x) = Q_{n,M_n} f_n(x) - T_{M_n} f_n(x)$, and because of (2.15) we expect that $\varepsilon_n(x)$ is a small error term. We also expect that the limit $M = \lim_{n\to\infty} M_n > 0$ exists.

Given a number $M > 0$ we look for the solution of the fixed point equation $f = T_M f$ and investigate the speed of convergence of the sequence $T_M^n f$, $n = 1, 2, \ldots$ to this fixed point for a general function f as $n \to \infty$. If this convergence turns out to be sufficiently fast then it is natural to expect that our sequence f_n tends to the fixed point as $n \to \infty$.

For our purposes it will be sufficient to investigate the operators $\overline{T}_M$ and T_M in the spaces $\mathscr{A}$ and $\mathscr{A}_0 \subset \mathscr{A}$

$$\mathscr{A} = \left\{ f, \int e^{sx} |f(x)| dx < \infty \text{ if } |s| < s(f) \text{ with some } s(f) > 0 \right\} \tag{2.19}$$

and

$$\mathscr{A}_0 = \left\{ f,\ f \in \mathscr{A},\ \int f(x)dx = 1,\ \int xf(x)dx = 0 \right\}.$$

We can work better with the Fourier transforms $\tilde{\tilde{T}}_M\tilde{f}$ and $\tilde{T}_M\tilde{f}$. (We define the operator $\tilde{\tilde{T}}_M$ and $\tilde{T}_M$ by the identities $\tilde{\tilde{T}}_M\tilde{f} = (\bar{T}_M f)^{\sim}$ and $\tilde{T}_M\tilde{f} = (T_M f)^{\sim}$.) We get, by applying the change of variables $z = \frac{x}{c} + u + \frac{v^2}{2M}$ and $y = \frac{x}{c} - u + \frac{v^2}{2M}$ instead of x and u, that

$$\begin{aligned}\tilde{\tilde{T}}_M\tilde{f}(\xi) &= \int \exp(i\xi x - v^2) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) dxdudv = \\ &= \frac{c}{2}\int f(y)f(z)\exp\left(i\xi\frac{c}{2}\left(z + y - \frac{v^2}{M}\right) - v^2\right)dydzdv = \\ &= \frac{c}{2}\tilde{f}\left(\frac{c}{2}\xi\right)^2 \int \exp\left(-v^2\left(1 + \frac{ic\xi}{2M}\right)\right)dv = \\ &= \frac{c}{2}\sqrt{\pi}\frac{\tilde{f}\left(\frac{c}{2}\xi\right)^2}{\sqrt{1 + i\frac{c\xi}{2M}}} \quad \text{for} \quad f \in \mathscr{A}\end{aligned} \tag{2.20}$$

and

$$\tilde{T}_M\tilde{f}(\xi) = \frac{\exp\left(i\frac{c}{4M}\xi\right)}{\sqrt{1 + i\frac{c\xi}{2M}}}\tilde{f}\left(\frac{c}{2}\xi\right)^2, \quad f \in \mathscr{A}. \tag{2.20$'$}$$

Since $\tilde{f}(0) = \int f(x)dx$ and $\tilde{f}'(0) = i\int xf(x)dx$ relation (2.20)′ implies that

$$\int T_M f(x)dx = \left[\int f(x)dx\right]^2 \tag{2.21}$$

and

$$\int xT_M f(x)dx = c\int f(x)dx \cdot \int xf(x)dx. \tag{2.21$'$}$$

As a consequence, $\int T_M f(x)dx = 1$ and $\int xT_M f(x)dx = 0$ if $f \in \mathscr{A}_0$, and this relations explain our scaling in the definition of the operator T_M.

By (2.20)′ the fixed point equation $f = T_M f$ can be rewritten in the space of Fourier transforms as

$$\tilde{f}(\xi) = \frac{\exp\left(i\frac{c}{4M}\xi\right)}{\sqrt{1 + i\frac{c\xi}{2M}}}\tilde{f}^2\left(\frac{c}{2}\xi\right)$$

or, by taking logarithm,

$$\log \tilde{f}(\xi) = 2\log \tilde{f}\left(\frac{c}{2}\xi\right) + i\frac{c}{4M}\xi - \frac{1}{2}\log\left(1 + i\frac{c}{2M}\xi\right). \quad (2.22)$$

We are looking for the solution of the equation (2.22) in the space $f \in \mathscr{A}_0$ in the form $\log \tilde{f}(\xi) = \sum_{k=2}^{\infty} \alpha_k \xi^k$. (Observe that for $f \in \mathscr{A}_0$

$$\log \tilde{f}(0) = \frac{d}{dx}\log \tilde{f}(0) = 0,$$

therefore $\alpha_0 = \alpha_1 = 0$ in the above expansion.) Then by (2.22)

$$\sum_{k=2}^{\infty}\left(1 - 2\left(\frac{c}{2}\right)^k\right)\alpha_k\xi^k = \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\left(-\frac{ic}{2M}\right)^k\xi^k$$

and

$$\alpha_k = \frac{(-ic)^k}{2k\left(1 - \left(\frac{c}{2}\right)^k\right)(2M)^k}.$$

In such a way we have defined $\log \tilde{f}(\xi)$ in a small neighbourhood of zero, and it is analytic there. Then by (2.22) it can be continued analytically to the whole real line, and this analytic continuation gives the solution of the fixed point equation.

If $\log \tilde{g}(\xi)$, $g \in \mathscr{A}_0$, is analytic in a small neighbourhood of zero then it can be written in the form $\log \tilde{g}(\xi) = \sum_{k=2}^{\infty} d_k \xi^k$ with some coefficients d_k, and the same calculation as before supplies that $\log \widetilde{T_M g}(\xi) = \sum_{k=2}^{\infty} T_{M,k}(d_k)\xi^k$ with

$$T_{M,k}(d) = 2\left(\frac{c}{2}\right)^k d + \frac{1}{2k}\left(-\frac{ic}{2M}\right)^k.$$

Since $2\left(\frac{c}{2}\right)^k < 1$ for $k \geq 2$ if $1 < c < \sqrt{2}$ the coefficients of the Taylor series of the function $\log \tilde{T}^n_M \tilde{g}(\xi)\ T^{(n)}_{M,k}(d_k)$, $k = 2, 3, \ldots$, tend to α_k exponentially fast as $n \to \infty$. This means that the convergence to the solution of the fixed point equation is sufficiently fast.

One has to overcome several technical difficulties when trying to turn the above heuristic argument into a rigorous proof. In the next section we explain which are the main difficulties during the proof of Theorem 1′ and how we want to overcome them.

3. ON THE STRATEGY OF THE PROOF: THE INDUCTIVE PROCEDURE

The main difficulty in the proof of Theorem 1′ consists in the justification of formula (2.18) together with a good bound on $\varepsilon_n(x)$ in it. Let us remark that such a relation can be expected only for large n. Indeed, when the operator $Q_{n,M}$ is approximated by T_M then the kernel $\exp\left(-\frac{u^2}{c^n} - v^2\right)$ in the integral defining $Q_{n,M}f$ is changed to $\exp(-v^2)$, and this change causes a negligible error only if n is large. For small n we need a different method to control the behaviour of the function $f_n(x)$.

Let us first consider the starting function $q_0(x, T)$ defined in (2.3)′. Simple calculation shows that if $T < a_0$ then the function $\bar{q}_0(x, T)$ has two maxima (in the variable x) in the points $\pm \hat{M}_0$

$$\hat{M}_0^2 = \frac{a_1(a_0 - T)}{T^2 t} \tag{3.1}$$

and

$$\bar{q}_0(x, T) = \bar{C}_0(T) \exp\left\{ -\frac{a_0 - T}{a_1}(x - \hat{M}_0)^2\left(1 + \frac{(x - \hat{M}_0)}{2\hat{M}_0}\right)^2 \right\}. \tag{3.1′}$$

We shall show that if $\hat{M}_0$ is sufficiently large then

$$f_0(x) = f_0(x, T) = n\left(x, \sqrt{\frac{a_1}{2(a_0 - T)}}\right) + R_1$$

if $x > -M_0$, $M_0 = \hat{M}_0 + \bar{R}_1$ with some negligible error terms R_1 and $\bar{R}_1$, where $n(x, \sigma)$ denotes the normal density function $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ with expectation zero and variance σ^2. Moreover, we shall see that for small n (depending on $\hat{M}_0$) $M_n \sim \hat{M}_0$, and the operator $\bar{Q}_{n,M} f_n$ can be well approximated by

$$\hat{T}_n f_n(x) = \int \exp\left(-\frac{u^2}{c^n} - v^2\right) f_n\left(\frac{x}{c} - u\right) f_n\left(\frac{x}{c} + u\right) du dv =$$
$$= \sqrt{\pi} \int \exp\left(-\frac{u^2}{c^n}\right) f_n\left(\frac{x}{c} - u\right) f_n\left(\frac{x}{c} + u\right) du,$$

i. e. a small error is committed even if the argument of f_n in the integral $Q_{n,M}f_n$ is approximated not by (2.15), but the term $\frac{v^2}{2M^2}$ is dropped in its right-hand side. (But this is not true for large n.) Observe that $\hat{T}_n$ turns an almost Gaussian density function with variance σ^2 to an almost Gaussian density function with variance $\frac{c^2}{2}\cdot\sigma^2$. By refining the above argument we shall be able to prove the following

PROPOSITION 1. — *For all integers* $N \geq 1$ *there is some* $K = K(N) > 0$ *such that if* $\hat{M}_0^2 = \frac{a_1(a_0 - T)}{T^2 t} > K$, $0 < T < \frac{1}{10}$ *then for* $n \leq N$

$$\left|\frac{d^j}{dx^j}\left[f_n(x) - n\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)\right]\right| \leqq \frac{B(n)}{\sqrt{M_n}} \exp\left(-2\left(\frac{2}{c}\right)^n |x|\right),$$

$$\textit{if} \quad j = 0, 1, 2, \quad |x| < \log M_n \quad (3.2)$$

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq B(n) \exp\left(-\left(\frac{2}{c}\right)^n \left|2x + \frac{x^2}{M_n c^n}\right|\right),$$

$$\textit{if} \quad x > -c^n M_n, \quad j = 0, 1, 2 \quad (3.3)$$

and

$$|M_n - M_0| \leqq \frac{B(n)}{\sqrt{\hat{M}_0}}, \quad (3.4)$$

where $\sigma_0^2 = \frac{a_1}{2(a_0 - T)}$, *and* $B(n)$ *is some appropriate multiplying factor depending on* n *but not on* $\hat{M}_0$ *and* c.

Let us now turn to the investigation of $Q_{n,M}f_n$ in the case of large n. A calculation of the error in the approximation (2.15) suggests that

$$|\overline{Q}_{n,M}f_n(x) - \overline{T}_M f_n(x)| \leqq \text{const} \cdot q^n \cdot \sup |f_n'(y)|^2 \quad (3.5)$$

with some $0 < q < 1$. It is relatively simple to demonstrate formula (3.5), but it is useful only if we have an additional estimate on $\sup |f_n'(y)|$ which shows that the dominating term on the right hand side of (3.5) is q^n and not $\sup |f_n'(y)|^2$. To prove this additional estimate we have to carry out a much more refined analysis where the function f_n is bounded simultaneously with its Fourier transform.

More precisely, since $f_n(x - c^n M_n) = f_n(x + c^n M_n)$, the function f_n has a peak not only at zero but also at $-2c^n M_n$. As a consequence, the Fourier transform of f_n does not behave nicely, and it is useful to make a regularization of $f_n(x)$ and to work with its Fourier transform.

DEFINITION. — *Let us choose some fixed function* $\phi \in C_0^\infty(\mathbb{R})$ *such that* $1 \geq \phi(x) \geq 0$ *for all* $x \in \mathbb{R}^1$, $\phi(x) = 1$ *for* $|x| \leq 1$ *and* $\phi(x) = 0$ *for* $|x| \geq 2$. *Put* $\phi_n(x) = \phi\left(\frac{1}{100} c^{-n/2} x\right)$. *Given some function f,* $f(x) \geq 0$, $\int |x| f(x) dx < \infty$, *we define its n-th regularization* $\phi_n(f)$ *as* $\phi_n(f)(x) = \frac{1}{A_n} \phi_n(x + B_n) f(x + B_n)$ *with* $A_n = \int \phi_n(x) f(x) dx$, *and* $B_n = \frac{1}{A_n} \int x \phi_n(x) f(x) dx$, *provided that the above formula is meaningful, i. e.* $A_n > 0$.

(Let us remark that although the Fourier transforms $\tilde{f}_n(\zeta)$ and $\tilde{\phi}_n(f_n)(\zeta)$ are not similar, nevertheless the functions $Q_{n,M} f_n(x)$ and $Q_{n,M} \phi_n(f_n)(x)$ are for typical x (x not very far from the origin) close to each other since the main contribution to the integrals defining them are in a small neighbourhood of zero, where f_n and $\phi_n(f_n)$ are close to each other. This is the reason why we can use our information about $\tilde{\phi}_n(f_n)(\zeta)$ in the investigation of $f_n(x)$.) First we shall prove the following

COROLLARY OF PROPOSITION 1. — *Under the conditions of Proposition 1 we have for* $n \leq N$

$$|\tilde{\phi}_n(f_n)(t + is)| \leqq \frac{\exp(\beta_n s^2)}{1 + \alpha_n t^2} \quad |s| < 2 \cdot (\sqrt{2}/c)^n, \quad t \in \mathbb{R}^1, \tag{3.6}$$

and

$$\left| \frac{d^j}{dx^j} f_n(x) \right| \leqq \frac{10^5}{\beta_n^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta_n}} \left| 2x + \frac{x^2}{c^n M_n} \right| \right) \quad \text{if} \quad x > -c^n M_n \tag{3.7}$$

with $\alpha_n = \frac{1}{200} \left(\frac{c^2}{2} \right)^n$ *and* $\beta_n = \left(\frac{c^2}{2} \right)^n$.

We shall formulate an inductive assumption about the functions $f_n(x)$ and $\tilde{\phi}_n(f_n(t + is))$ for all n. But first we have to understand their behaviour better. Formula (3.2) shows that $\sup f_0(x) < C$ with some bound C independent of $\hat{M}_0$, and for small n $\sup_x f_{n+1}(x) \sim f_{n+1}(0) \sim \frac{\sqrt{2}}{c} \sup_x f_n(x)$. On the other hand it is natural to expect that for large n, $f_n(x) \sim g_M(x)$, where g_M is the solution of the fixed point equation $g_M = T_M g_M$. Since $g_M(x) = M g_1\left(\frac{x}{M}\right)$, hence $\sup_x g_M(x) \sim \text{const.}\, M$. The above considerations suggest the following picture: for small n the value of $\sup_x f_n(x)$ is growing exponentially fast, first at rate $\sqrt{2}/c$ then slower and slower, and finally for large n it gets stabilized at const. $\hat{M}_0$. If $\sup_x f_n(x) = K_n$ then $f_n(x)$ is negligible small outside a region of size $\frac{1}{K_n}$, i. e. as the function $f_n(x)$ is growing it gets more localized. This behaviour of the function f_n is reflected in a slightly hidden way in the properties I(n) and J(n) defined below.

Let us fix some positive integer N, and introduce the sequences α_n and β_n (with starting index N) as

$$\alpha_N = \frac{1}{200}\left(\frac{c^2}{2}\right)^N \tag{3.8}$$

$$\alpha_{n+1} = \frac{c^2}{2}(1 - c^{-\frac{n}{2}})\alpha_n + \frac{10^{-12}}{M_n^2} \quad \text{for} \quad n \geq N, \tag{3.8$'$}$$

and

$$\beta_N = \left(\frac{c^2}{2}\right)^N \tag{3.9}$$

$$\beta_{n+1} = \frac{c^2}{2}(1 + c^{-\frac{n}{2}})\beta_n + \frac{10}{M_n^2} \quad \text{for} \quad n \geq N, \tag{3.9$'$}$$

where M_n is defined in formula (2.7).

Now we define

Property I(n)

Let $n \geq N$. *The function* f_n *satisfies Property* I(n) *(with starting index* N *and multiplying factor* C*) if*

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^n M_n}\right|\right)$$

$$\text{for} \quad j = 0, 1, 2, \quad x > -c^n M_n \tag{3.10}$$

with the above defined β_n, *and the number* M_n *defined in (2.7).*

and

Property J(n)

Let $n \geq N$. *The function* $f_n(x)$ *satisfies Property* J(n) *(with starting index* N*) if*

$$|\tilde{\phi}_n(f_n)(t + is)| \leqq \frac{\exp \beta_n s^2}{1 + \alpha_n t^2} \quad \text{for} \quad |s| < \frac{2}{\sqrt{\beta_n}}. \tag{3.11}$$

The content of Properties I(n) and J(n) is very similar. As we shall see later, $C_1 < \frac{\beta_n}{\alpha_n} < C_2$ with some appropriate $C_1 > 0$ and $C_2 > 0$ for all n. It is natural that the bound on the right-hand of (3.11) depends on s^2 and t^2, since in the Taylor expansion of $\tilde{\phi}_n(f_n)(z)$ the first term disappears because of the relation $\frac{d}{dx}\tilde{\phi}_n(f_n(\zeta))\Big|_{\zeta=0} = \int x\phi_n(f_n)(x)dx = 0$. Formula (3.11) in a small neighbourhood of zero tells us that the variance of a random variable with density function $\phi_n(f_n)(x)$ is between $2\alpha_n$ and $2\beta_n$, hence $\phi_n(f_n)(x)$ is essentially concentrated in a domain of size const. $\frac{1}{\sqrt{\beta_n}}$. The bound on $\tilde{\phi}_n(f_n)(t+is)$ in the complex domain tells, roughly speaking, that $\phi_n(f_n)(x)$ tends to zero with the rate $\exp\left(-\frac{2|x|}{\sqrt{\beta_n}}\right)$ as $|x| \to \infty$.

Finally we remark that the smoothness of the function $\phi_n(f_n)$ is connected with the decrease of its Fourier transform at infinity. Property J(n) states a decrease of order $O(t^{-2})$. This is a weaker property than the second order differentiability of the function f_n imposed in Property I(n), but it is enough for our purposes.

In the exponent at the right-hand side of (3.10) the term $2x$ is essential, and the term $\frac{x^2}{c^n M_n}$ could be omitted. In that case the proof could be carried out with some small changes, only it would become considerably longer. The same remark applies for the formulas in Propositions 2 and 3.

The main step of the proof of Theorem 1′ is the following

PROPOSITION 3. — *The multiplying factor* C *in Property* I(n) *can be chosen in such a way (e.g. any* $C \geq e^{1000}$ *is an appropriate choice) that if* $N \geq N_0(c, C)$ *with some appropriate threshold* N_0 *depending only on* c *and* C, $n \geq N$, $|M_n - M_{n-1}| < 1$, $M_n > K(c)$ *with some* $K(c) > 0$ *independent of* n, $f_n(x)$ *satisfies Properties* I(n) *and* J(n) *(with the above defined multiplying factor* C *and starting index* N*)* $100 > \beta_n > \max\left(\frac{9}{M_n^2}, 4^{-n}\right)$, *then* $f_{n+1}(x)$ *satisfies Properties* I($n+1$) *and* J($n+1$) *(with the same parameters* C *and* N*), and*

$$M_{n+1} = M_n - \frac{1}{4c^n M_n} + \frac{\gamma(n)}{c^n}, \quad |\gamma(n)| < C_1 c^{-n}\sqrt{\beta_n} \tag{3.12}$$

with some absolute constant C_1. *Moreover,*

$$\left|\frac{d^j}{dx^j}[f_{n+1}(x) - T_{M_n}\phi_n(f_n)(x)]\right| \leqq$$

$$\leqq \frac{C_1 C^4}{\beta_n^{(j+1)/2}} c^{-n}\left[\exp\left(-\frac{1}{\sqrt{\beta_{n+1}}}\left|2x + \frac{x^2}{c^{n+1}M_{n+1}}\right|\right) + \exp\left(-\frac{2|x|}{\sqrt{\beta_{n+1}}}\right)\right] \tag{3.13}$$

for $x > -c^{n+1}M_{n+1}$, $j = 0, 1, 2$, *and*

$$\left|\frac{d^j}{dx^j}T_{M_n}\phi_n(f_n)(x)\right| \leqq \frac{C_1C^2}{\beta_n^{(j+1)/2}}\exp\left(-\frac{2|x|}{\sqrt{\beta_{n+1}}}\right), \quad x\in R, \quad j=0,1,2,3,4 \tag{3.14}$$

with some absolute constant C_1.

As a consequence, if $\hat{M}_0^2 = \dfrac{a_1(a_0 - T)}{T^2 u}$ *is sufficiently large and* $0 < T < \dfrac{1}{10}$ *then Properties* I(n) *and* J(n) *hold for all* $f_n(x)$, $n \geq N$, *if the parameters* C *and* N *are appropriately chosen, and in this case relations (3.12), (3.13) and (3.14) also hold.*

We shall prove Proposition 3 with the help of the following Proposition 2 which can be considered as a more refined and elaborated version of formula (3.5). We recall that the operators $Q_{n,M}$ and $\overline{Q}_{n,M}$ were defined in Section 2 for $f \in \mathscr{A}_{n,M}$.

PROPOSITION 2. — *Given some positive integer n and real numbers* $M > 100$, $\varepsilon > 0$ *let us consider some* $f \in \mathscr{A}_{n,M}$ *such that* $\int_{-c^nM}^{\infty} f(x)dx = 1$, $\int_{-c^nM}^{\infty} xf(x)dx = 0$, *and*

$$\left|\frac{d^jf(x)}{dx^j}\right| \leqq C\beta^{-\frac{j+1}{2}}\exp\left(-\frac{1}{\sqrt{\beta}}\left|2x + \frac{x^2}{c^nM}\right|\right), \quad j=0,1,2, \quad x > -c^nM, \tag{3.15}$$

with some β *and* C *such that* $100 > \beta > \max\left(\dfrac{4}{(1-\varepsilon)M^2}, 4^{-n}\right)$, *and let* $n > n_0(c, C)$, *where the threshold* $n_0(c, C)$ *depends only on c and* C. *Let* $\dfrac{1}{2} > \varepsilon > 10c^{-\frac{n}{4}}$. *Then there exists some* $C(\varepsilon) > 0$ *depending only on* ε *such that*

$$\left|\frac{d^j}{dx^j}Q_{n,M}f(x)\right| \leqq C(\varepsilon)C^2\beta^{-\frac{j+1}{2}}\exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x + \frac{x^2}{c^{n+1}\overline{M}}\right|\right) \tag{3.16}$$

for $x > -c^{n+1}\overline{M}$, $j = 0, 1, 2$

$$\left|\frac{d^j}{dx^j}T_M\phi_n(f)(x)\right| \leqq C(\varepsilon)C^2\beta^{-\frac{j+1}{2}}\exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right) \tag{3.17}$$

for $j = 0, 1, 2, 3, 4$, $x \in R^1$, *and*

$$\left|\frac{d^j}{dx^j}[Q_{n,M}f(x) - T_M\phi_n(f)(x)]\right| \leqq$$
$$\leqq \frac{C(\varepsilon)C^4}{\beta^{(j+1)/2}}c^{-n}\left[\exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x + \frac{x^2}{c^{n+1}\overline{M}}\right|\right) + \exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right)\right] \tag{3.18}$$

for $x > -c^{n+1}\overline{\mathrm{M}}$, $j = 0, 1, 2$, *where*

$$\overline{\mathrm{M}} = \mathrm{M} + \frac{m_n}{c^n} \tag{3.19}$$

and m_n *is defined by formula (2.11). We also have*

$$m_n = -\frac{c}{4\mathrm{M}} + \gamma_n, \quad |\gamma_n| < \mathrm{C}_1\mathrm{C}^2 c^{-n}\sqrt{\beta} \tag{3.19)'$$

with some absolute constant $\mathrm{C}_1 > 0$.

In formulas (2.10)-(2.13) we have defined $\mathrm{Q}_n(f(x), \mathrm{M}) = (\mathrm{Q}_{n,\mathrm{M}}f(x), \overline{\mathrm{M}})$, and in Proposition 2 we have estimated $\mathrm{Q}_n(f(x), \mathrm{M})$ with the help of $f(x)$ and M. We would try to deduce Proposition 3 from Proposition 2 with the choice $f = f_n$, $\beta = \beta_n$ and $\mathrm{M} = \mathrm{M}_n$. (The number $\varepsilon > 0$ appears in Proposition 2 for some technical reasons. At a later step of the proof we need an almost optimal multiplying factor inside the exponent of formulas (3.16)-(3.18).) The main difficulty which does not allow to deduce Proposition 3 directly from Proposition 2 is that the multiplying factor in (3.16) is $\mathrm{C}(\varepsilon)\mathrm{C}^2$ which is too large to deduce Property $\mathrm{I}(n+1)$ with the same multiplying factor C which appears in Property $\mathrm{I}(n)$. We can overcome this difficulty by investigating the functions f simultaneously with their Fourier transform. This is the reason why we have formulated both Properties $\mathrm{I}(n)$ and $\mathrm{J}(n)$. Our induction procedure works only for large n, hence we have proved the Corollary of Proposition 1 which allows us to start the induction from a large starting index. Then the following observations help us to carry out our induction procedure.

We have a better multiplying constant in the exponent of formula (3.16) than we need in Property $\mathrm{I}(n+1)$. Hence we can prove the inequality appearing in Property $\mathrm{I}(n+1)$ for large x by slightly decreasing the multiplying factor in the exponent of (3.16). On the other hand formula (3.18) (observe that there is a factor c^{-n} on the right hand side) guarantees that a negligible error arises if $f_{n+1} = \mathrm{Q}_{n,\mathrm{M}_n}f_n$ is changed to $\mathrm{T}_{\mathrm{M}_n}\phi_n(f_n)$. Then, since the operator T_M can be naturally investigated in the space of the Fourier transforms, we can complete the proof of Property $\mathrm{I}(n+1)$ under the conditions of Proposition 3 with the help of Property $\mathrm{J}(n)$. The important point is that the bound we can give on $\mathrm{T}_\mathrm{M}\phi_n(f_n)$ with the help of Property $\mathrm{J}(n)$ does not depend on the constant C appearing in Property $\mathrm{I}(n)$.

The proof of Property $\mathrm{J}(n+1)$ is similar. With the help of Proposition 2 the problem can be reduced to the bounding of $\tilde{\mathrm{T}}_{\mathrm{M}_n}\tilde{\phi}_n(f_n)(t+is)$ which can be done with the help of Property $\mathrm{J}(n)$. The remaining statements of Proposition 3 can be deduced from Proposition 2 with some work.

Let us remark that we have bounded $f_n(x)$ together with its first two derivatives although only the bound given for $f_n(x)$ is interesting for us. But, since a Taylor expansion is applied in the inductive proof we need some information about $f_n'(x)$ in order to bound $f_{n+1}(x)$. On the other hand, the operators Q_n and T, similarly to the convolution operator, have some smoothing properties, and it helps us to carry out the inductive procedure without weakening the smoothness conditions during the subsequent steps of induction. In particular, let us remark that in formulas (3.14) and (3.17) four derivatives of $T_M\phi_n(f)$ could be bounded with the help of only two derivatives of the function f.

Formula (3.13) in Proposition 3 can be considered as a more exact version of formula (2.18). It enables us to carry out the heuristic argument at the end of Section 2 in a precise form. In such a way we can prove that $\tilde{\phi}_n(f_n)(t)$ tends to $\tilde{g}_M(t)$ exponentially fast if $t \in D$, where D is a small but fixed neighbourhood of zero, and g_M is the solution of the fixed point equation $g_M = T_M g_M$. However, this knowledge is not sufficient to prove Theorem 1'. But by exploiting that $\tilde{\phi}_{n+1}(f_{n+1})(t)$ can be well approximated by $\tilde{T}_M\tilde{\phi}_n(f_n)(t)$ and that formula (2.20) gives us the estimate

$$|\tilde{T}_M\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| = |\tilde{T}_M\tilde{\phi}_n(f_n)(t) - \tilde{T}_M\tilde{g}_M(t)| = \left(1 + \frac{c^2}{4M^2}t^2\right)^{-1/2} \cdot$$

$$\cdot \left| \tilde{\phi}_n(f_n)\left(\frac{c}{2}t\right)^2 - \tilde{g}_M\left(\frac{c}{2}t\right)^2 \right| \leqq 2\left(1 + \frac{c^2}{4M^2}\right)^{-1/2} \left| \tilde{\phi}_n(f_n)\left(\frac{c}{2}t\right) - \tilde{g}_M\left(\frac{c}{2}t\right) \right|$$

we can give a good bound on $\sup_{t \in D_n} |\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)|$ in an exponentially increasing domain D_n. Then by bounding $\tilde{\phi}_n(f_n)(t)$ and $\tilde{g}_M(t)$ for large t and applying inverse Fourier transformation we are able to prove Theorem 1'.

Theorem 1 can be deduced from Theorem 1' by expressing $p_n(x, T)$ with the help of $f_n(x)$ and M_n. The main difficulty of this deduction is connected with the following problem: it follows from Theorem 1' that for large n, $q_n\left(\sqrt{\frac{a_1}{T}}x, T\right)$ is essentially concentrated in the domain $\left| |x| - \sqrt{\frac{T}{a_1}}M \right| < \text{const.}\, nc^{-n}$, and we want to prove the same for $p_n(x, T)$ which can be expressed by $q_n(x, T)$ with the help of relation (2.5). To show this property we have to prove that for $|x| < \sqrt{\frac{T}{a_1}}M$ the decrease of $q_n\left(\sqrt{\frac{a_1}{T}}x, T\right)\Big/ q_n(M, T)$ (in the variable x) is faster than the increase of $\exp\left(-\frac{a_0}{2T}c^n x^2 + \frac{a_0}{2T}c^n \frac{T}{a_1}M^2\right)$. We can prove this statement by determining which number μ, $\mu > 0$ can be written in formula (2.9)', i. e. we need a better understanding about the decrease of the function f_n outside the typical region. The essential technical difficulty after this step is to give a good asymptotic value for the norming constant B_n in formula (2.5).

The proof of Theorem 2 requires an even better understanding of the behaviour of the functions $f_n(x, T)$ and the function $g(x)$ defined by formula (1.6) for $|x| \to \infty$. We shall discuss the content of Theorem 2 and the difficulties arising during its proof in Section 9.

4. THE PROOF OF PROPOSITION 1 AND ITS COROLLARY. THE FIRST STEP OF THE INDUCTIVE PROCEDURE

First we prove formulas (3.2), (3.3) and (3.4) for $n = 0$. The function $\bar{q}_0(x, T)$ can be written in the form given by formulas (3.1) and (3.1)', where the norming constant $\bar{C}_0(T)$ is given by (2.4). We have

$$\frac{1}{\bar{C}_0(T)} = \int_0^\infty \exp\left[- \frac{a_0 - T}{a_1}(x - \hat{M}_0)^2\left(1 + \frac{x - \hat{M}_0}{2\hat{M}_0}\right)^2\right] dx =$$

$$= \int_{|x-\hat{M}_0|<\hat{M}_0/2} + \int_{\substack{|x-\hat{M}_0|>\hat{M}^0/2 \\ x>0}} = I_1 + I_2$$

and

$$I_1 = \int_{-\hat{M}_0/2}^{\hat{M}_0/2} \exp\left(-\frac{a_0-T}{a_1}x^2\right)\left(1-\frac{a_0-T}{a_1}\left(\frac{x^3}{\hat{M}_0}+\frac{x^4}{4\hat{M}_0^2}+O\left(\frac{x^4}{\hat{M}_0^2}\right)\right)\right)dx =$$

$$= \sqrt{\frac{a_1\pi}{a_0-T}} + O\left(\frac{1}{\hat{M}_0^2}\right)$$

$$I_2 = O\left(\frac{1}{\hat{M}_0^2}\right).$$

Hence

$$\bar{C}_0(T) = \sqrt{\frac{a_0 - T}{a_1\pi}} + O\left(\frac{1}{\hat{M}_0^2}\right). \tag{4.1}$$

Similarly,

$$M_0 - \hat{M}_0 = \bar{C}(T)\int_0^\infty (x-\hat{M}_0)\exp\left[-\frac{a_0-T}{a_1}(x-\hat{M}_0)^2\left(1+\frac{x-\hat{M}_0}{2\hat{M}_0}\right)^2\right]dx =$$

$$= O\left(\frac{1}{\hat{M}_0}\right). \tag{4.2}$$

Put $\sigma_0^2 = \dfrac{a_1}{2(a_0 - T)}$. By relations (4.1) and (4.2)

$$n(x,\sigma_0) - f_0(x) = n(x,\sigma_0)\left[1-\left(1+O\left(\frac{1}{M_0^2}\right)\right)\right.$$

$$\left.\exp\left\{-\frac{\left(x+O\left(\frac{1}{M_0}\right)\right)^2}{2\sigma_0^2}\left(1+\frac{x+O\left(\frac{1}{M_0}\right)}{2M_0}\right)^2+\frac{x^2}{2\sigma_0^2}\right\}\right],$$

hence

$$| n(x,\sigma_0)-f_0(x)| \leqq n(x,\sigma_0)\cdot\frac{O(|x|^3+1)}{M_0} \leqq \frac{\exp\left(-\frac{x^2}{2\sigma_0^2}-2\sigma_0^2\right)}{\sqrt{M_0}} \leqq$$

$$\leqq \frac{\exp(-2|x|)}{\sqrt{M_0}} \quad \text{for} \quad |x| < \log M_0 \,,$$

and similarly

$$\left|\frac{d^j}{dx^j}(n(x,\sigma_0)-f_0(x))\right| \leqq \frac{\exp(-2|x|)}{\sqrt{M_0}} \quad \text{for} \quad |x| < \log M_0, \ j=0,1,2\,.$$

We claim that

$$f_0(x) \leqq C\exp\left(-\left|2x+\frac{x^2}{M_0}\right|+16\sigma_0^2\right) \quad \text{for} \quad x > -M_0\,.$$

Indeed, if $\hat{M}_0$ is sufficiently large $\left(\hat{M}_0 > \frac{80\sigma^2}{3} \text{ is e. g. enough}\right)$ then for $x > M_0$

$$f_0(x)=C\exp\left(-\frac{\left(x+O\left(\frac{1}{M_0}\right)\right)^2}{2\sigma_0^2}\left(1+\frac{x+O\left(\frac{1}{M_0}\right)}{2M_0}\right)^2\right) \leqq C\exp\left(-\frac{x^2}{10\sigma_0^2}\right) =$$

$$=C\exp\left(-\frac{3x^2}{80\sigma_0^2}\right)\exp\left(-\frac{x^2}{16\sigma_0^2}\right) \leqq C\exp\left(-\frac{3x^2}{80\sigma_0^2}\right)\exp(-2|x|+16\sigma_0^2) \leqq$$

$$\leqq C\exp\left(-\left|2x+\frac{x^2}{M_0}\right|+16\sigma_0^2\right).$$

Similarly

$$\left|\frac{d^j}{dx^j}f_0(x)\right| \leqq C\exp\left(-\left|2x+\frac{x^2}{M_0}\right|\right) \quad \text{if} \quad x > -M_0, \ j=0,1,2\,.$$

The above relations imply formulas (3.2)-(3.4) for $n = 0$. For $0 < n < N$ we prove them by induction with the help of the following two lemmas.

LEMMA 1. — *Given some integer n, $0 \leq n < N$, and $M > 0$ such that $M > 4\cdot\left(\frac{2}{c}\right)^N$ let us consider some $f \in \mathcal{A}_{n,M}$ which satisfies the inequality*

$$\left|\frac{d^j}{dx^j}f(x)\right| \leqq B(n)\exp\left(-\left(\frac{2}{c}\right)^n\left|2x+\frac{x^2}{c^n M}\right|\right) \quad \text{for} \quad j=0,1, x>-c^n M \quad (4.3)$$

with some $\mathrm{B}(n) > 0$. *Then*

$$\left| \frac{d^j}{dx^j} \overline{\mathrm{Q}}_{n,\mathrm{M}} f(x) \right| \leqq \mathrm{B}(n+1) \exp\left(-\left(\frac{2}{c}\right)^{n+1} \left| 2x + \frac{x^2}{c^{n+1}\mathrm{M}} \right| \right) \quad \text{for} \quad j=0,1,2 \tag{4.4}$$

with some appropriate $\mathrm{B}(n+1)$ *which depends only on n and* $\mathrm{B}(n)$.

LEMMA 2. — *Let n,* M *and* $f(x)$ *satisfy the conditions of Lemma 1. Moreover, let* $f(x)$ *be such that* $\int_{-c^n\mathrm{M}}^{\infty} f(x)dx = 1$, $\int_{-c^n\mathrm{M}}^{\infty} xf(x)dx = 0$ *and*

$$\left| \frac{d^j}{dx^j} \left[f(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right) \right] \right| \leqq \frac{\mathrm{B}(n)}{\sqrt{\mathrm{M}}} \exp\left(-2\left(\frac{2}{c}\right)^n |x| \right) \tag{4.5}$$

for $j = 0, 1, 2$ *and* $|x| < \log \mathrm{M}$, *and let* $\mathrm{M} > \mathrm{K}(\mathrm{N})$ *with some sufficiently large* $\mathrm{K}(\mathrm{N}) > 0$. *Then*

$$\left| \frac{d^j}{dx^j} \left[\overline{\mathrm{Q}}_{n,\mathrm{M}} f(x) - \lambda_n \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^{n+1} \sigma_0\right) \right] \right| \leqq \frac{\mathrm{B}(n+1)}{\sqrt{\mathrm{M}}} \exp\left(-2\left(\frac{2}{c}\right)^{n+1} |x| \right) \tag{4.6}$$

for $j = 0, 1, 2$ *and* $|x| < \left(1 + \frac{c-1}{4}\right) \log \mathrm{M}$, *where*

$$\lambda_n = \frac{c}{2\sigma_0} \left(\frac{\sqrt{2}}{c}\right)^n \left(\frac{1}{c^n} + \frac{2^n}{c^{2n}\sigma_0^2}\right)^{-1/2},$$

and $\mathrm{B}(n+1)$ *depends only on* $\mathrm{B}(n)$ *and n.*

Proof of Lemma 1. — Let us introduce the notation

$$l_{n,\mathrm{M}}^{\pm}(x, u, v) = c^n \left(\sqrt{\left(\mathrm{M} + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - \mathrm{M} \right). \tag{4.7}$$

A simple calculation shows that

$$2l_{n,\mathrm{M}}^{\pm}(x, u, v) + \frac{l_{n,\mathrm{M}}^{\pm}(x, u, v)^2}{\mathrm{M}c^n} = \frac{v^2}{\mathrm{M}} + 2\left(\frac{x}{c} \pm u\right) + \frac{1}{c^n\mathrm{M}}\left(\frac{x}{c} \pm u\right)^2. \tag{4.8}$$

We claim that

$$\left| 2l_{n,\mathrm{M}}^{+}(x, u, v) + \frac{l_{n,\mathrm{M}}^{+}(x, u, v)^2}{\mathrm{M}c^n} \right| + \left| 2l_{n,\mathrm{M}}^{-}(x, u, v) + \frac{l_{n,\mathrm{M}}^{-}(x, u, v)^2}{\mathrm{M}c^n} \right| \geqq$$

$$\geqq \frac{2}{c} \left| 2x + \frac{x^2}{c^{n+1}\mathrm{M}} \right| - \frac{2}{\mathrm{M}} \left(v^2 + \frac{u^2}{c^n} \right). \tag{4.8$'$}$$

Indeed, we get from (4.8) with the help of the inequalities $|\mathrm{A}| + |\mathrm{B}| \geq \mathrm{A} + \mathrm{B}$

and $|A| + |B| \geq -A - B$ that the left-hand side of (4.8)′ can be estimated from below both by

$$\frac{2}{c}\left(2x+\frac{x^2}{c^n M}\right)+\frac{2}{M}\left(v^2+\frac{u^2}{c^n}\right) \quad \text{and} \quad -\frac{2}{c}\left(2x+\frac{x^2}{c^n M}\right)-\frac{2}{M}\left(v^2+\frac{u^2}{c^n}\right).$$

These estimates imply (4.8)′.

Relations (4.8)′, (4.3) and the inequality $M > 4\cdot\left(\frac{2}{c}\right)^N > 4\cdot\left(\frac{2}{c}\right)^n$ imply that

$$\exp\left(-\frac{u^2}{c^n}-v^2\right)f(l^+_{n,M}(x,u,v))f(l^-_{n,M}(x,u,v))\leqq$$

$$\leqq B(n)^2\exp\left\{-\left(\frac{2}{c}\right)^{n+1}\left|2x+\frac{x^2}{c^{n+1}M}\right|-\left(\frac{u^2}{c^n}+v^2\right)\left(1-\frac{2}{M}\left(\frac{2}{c}\right)^n\right)\right\}\leqq$$

$$\leqq B(n)^2\exp\left(-\left(\frac{2}{c}\right)^{n+1}\left|2x+\frac{x^2}{c^{n+1}M}\right|\right)\cdot\exp\left(-\frac{1}{2}\left(\frac{u^2}{c^n}+v^2\right)\right). \tag{4.9}$$

We get formula (4.4) for $j = 0$ by integrating inequality (4.9) with respect to the variables u and v. The case $j = 1, 2$ can be investigated similarly. The quantities $\frac{d}{dx}\overline{Q}_{n,M}f(x)$ and $\frac{d^2}{dx^2}\overline{Q}_{n,M}f(x)$ can be expressed as

$$\frac{d}{dx}\overline{Q}_{n,M}f(x)=$$

$$=\frac{2}{c}\int\exp\left(-\frac{u^2}{c^n}-v^2\right)\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)\left[\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)^2+\frac{v^2}{c^n}\right]^{-1/2}\cdot$$

$$\cdot f'(l^+_{n,M}(x,u,v))f(l^-_{n,M}(x,u,v))dudv \tag{4.10}$$

and

$$\frac{d^2}{dx^2}\overline{Q}_{n,M}f(x)=\frac{4}{c^2}\int\exp\left(-\frac{u^2}{c^n}-v^2\right)\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)\left(M+\frac{x}{c^{n+1}}-\frac{u}{c^n}\right)\cdot$$

$$\cdot\left[\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)^2+\frac{v^2}{c^n}\right]^{-1/2}\left[\left(M+\frac{x}{c^{n+1}}-\frac{u}{c^n}\right)^2+\frac{v^2}{c^n}\right]^{-1/2}\cdot$$

$$\cdot f'(l^+_{n,M}(x,u,v))f'(l^-_{n,M}(x,u,v))dudv-$$

$$-\frac{4}{c^{n+2}}\int u\exp\left(-\frac{u^2}{c^n}-v^2\right)\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)\left[\left(M+\frac{x}{c^{n+1}}+\frac{u}{c^n}\right)^2+\frac{v^2}{c^n}\right]^{-1/2}$$

$$f'(l^+_{n,M}(x,u,v))f(l^-_{n,M}(x,u,v))dudv. \tag{4.10$'$}$$

Relations (4.10) and (4.10)′ can be obtained by differentiating formula (2.10) after the change of variable $u' = M + \frac{x}{c^{n+1}} - \frac{u}{c^n}$ and then (when calculating the second derivative) $u'' = M + \frac{x}{c^{n+1}} + \frac{u}{c^n}$. Observe that the second derivative of $Q_{n,M}f$ is expressed in (4.10)′ with the help of the first derivative of f. This means in particular that for the existence of the second derivative of $\overline{Q}_{n,M}f$ it is enough that f is once differentiable. Since

$$\left|\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)\left[\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}\right]^{-1/2}\right| \leqq 1$$

relations (4.10) and (4.10)′ imply that

$$\left|\frac{d}{dx}\overline{Q}_{n,M}f(x)\right| \leqq \frac{2}{c}\int \exp\left(-\frac{u^2}{c^n} - v^2\right)\left|f'(l_{n,M}^+(x,u,v))\right|\left|f(l_{n,M}^-(x,u,v))\right|dudv \tag{4.11}$$

and

$$\left|\frac{d^2}{dx^2}\overline{Q}_{n,M}f(x)\right| \leqq \frac{4}{c^2}\int \exp\left(-\frac{u^2}{c^n} - v^2\right)\left|f'(l_{n,M}^+(x,u,v))\right|\left|f'(l_{n,M}^-(x,u,v))\right|dudv +$$
$$+ \frac{4}{c^{n+2}}\int |u|\exp\left(-\frac{u^2}{c^n} - v^2\right)\left|f'(l_{n,M}^+(x,u,v))\right|\left|f(l_{n,M}^-(x,u,v))\right|dudv \qquad (4.11)'$$

Formula (4.4) for $j = 1, 2$ follows from (4.3), (4.8), (4.11) and (4.11)′ in the same way as it was proved for $j = 0$.

Proof of Lemma 2. — First we consider the case $j = 0$. We show that

$$\left|l_{n,M}^{\pm}(x,u,v) - \left(\frac{x}{c} \pm u + \frac{v^2}{2M}\right)\right| \leqq 5\left(\frac{v^4}{c^n M^3} + \frac{x^2 + u^2}{c^n M}\right) \tag{4.12}$$

if $|x| < \frac{1}{4}c^{n+1}M$, $|u| < \frac{1}{4}c^n M$, $v^2 < c^n M$. Indeed,

$$\left|l_{n,M}^{\pm}(x,u,v) - \left(\frac{x}{c} \pm u + \frac{v^2}{2M}\right)\right| =$$
$$= \left|c^n\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)\left(\sqrt{1 + \frac{v^2}{c^n\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^2}} - 1\right) - \frac{v^2}{2M}\right| \leqq$$
$$\leqq \frac{v^2}{2M}\left|\frac{1}{1 + \frac{x}{c^{n+1}M} \pm \frac{u}{c^n M}} - 1\right| + \frac{v^4}{c^n\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^3} \leqq$$
$$\leqq \frac{v^2}{M}\left|\frac{x}{c^{n+1}M} \pm \frac{u}{c^n M}\right| + \frac{4v^2}{c^n M^3} \leqq 5\left(\frac{v^4}{c^n M^3} + \frac{x^2 + u^2}{c^n M}\right).$$

Define the set $A = A(M) = \left\{ (u, v), |u| < \frac{c-1}{2c} \log M, |v| < M^{1/4} \right\}$. First we prove that

$$\left| \int_A \exp\left(-\frac{u^2}{c^n} - v^2 \right) f(l_{n,M}^+(x, u, v)) f(l_{n,M}^-(x, u, v) du dv - \lambda_n \varkappa\left(x, \left(\frac{c}{\sqrt{2}}\right)^{n+1} \sigma_0 \right) \right| \leqq$$
$$\leqq \frac{\overline{B}(n)}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^{n+1} |x| \right). \quad (4.13)$$

For this aim we show that for $r_n^\pm(x, u, v) = f(l_{n,M}^\pm(x, u, v)) - \varkappa\left(\frac{x}{c} \pm u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0 \right)$

$$|r_n^\pm(x, u, v)| \leqq \frac{\tilde{B}(n)}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^n \left| \frac{x}{c} \pm u \right| \right) \quad \text{if} \quad (u, v) \in A,$$
$$|x| < \left(1 + \frac{c-1}{4} \right) \log M. \quad (4.14)$$

Indeed, it follows from (4.12) that for $|x| < \left(1 + \frac{c-1}{4}\right) \log M$, $(u, v) \in A$, $\left| l_{n,M}^\pm(x, u, v) - \left(\frac{x}{c} \pm u\right) \right| \leq \frac{1}{\sqrt{M}}$. Hence

$$\left| \frac{x}{c} \pm u \right| < \left(\frac{c+3}{4c} + \frac{c-1}{2c} \right) \log M < \log M, \ |l_{n,M}^\pm(x, u, v)| < \log M,$$

and by (4.5)

$$\left| f(l_{n,M}^\pm(x, u, v)) - \varkappa\left(\frac{x}{c} \pm u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0 \right) \right| \leqq \frac{3B(n)}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^n \left| \frac{x}{c} \pm u \right| \right) +$$
$$+ \left| \varkappa\left(l_{n,M}^\pm(x, u, v), \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0 \right) - \varkappa\left(\frac{x}{c} \pm u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0 \right) \right| \leqq$$
$$\leqq \frac{3B'}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^n \left| \frac{x}{c} \pm u \right| \right) + \frac{C}{\sqrt{M}} \exp\left(-\frac{1}{4}\left(\frac{2}{c^2}\right)^n \frac{1}{\sigma_0^2} \left(\frac{x}{c} \pm u\right)^2 \right).$$

Because of the inequality $-\frac{1}{4}\left(\frac{2}{c^2}\right)^n \frac{1}{\sigma_0^2}\left(\frac{x}{c} \pm u\right)^2 + 2\left(\frac{2}{c}\right)^n \left| \frac{x}{c} \pm u \right| - 4 \cdot 2^n \sigma_0^2 \leq 0$

$$\exp\left(-\frac{1}{4}\left(\frac{2}{c^2}\right)^n \frac{1}{\sigma_0^2} \left(\frac{x}{c} \pm u\right)^2 \right) \leqq \exp\left(-2\left(\frac{2}{c}\right)^n \left| \frac{x}{c} \pm u \right| + 4 \cdot 2n\sigma_0^2 \right),$$

and the above relations imply (4.14).

We have for $|x| < \left(1 + \frac{c-1}{4}\right)\log M$

$$\int_A \exp\left(-\frac{u^2}{c^n} - v^2\right)\varkappa\left(\frac{x}{c} + u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)\varkappa\left(\frac{x}{c} - u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)dudv =$$

$$= \int \frac{2^n}{2\pi c^{2n}\sigma_0^2}\exp\left(-\frac{u^2}{c^n} - v^2 - \frac{2^n}{c^{2n}\sigma_0^2}\left(\frac{x^2}{c^2} + u^2\right)\right)dudv + R_n(x) =$$

$$= \lambda_n \varkappa\left(x, \left(\frac{c}{\sqrt{2}}\right)^{n+1}\sigma_0\right) + R_n(x), \tag{4.15}$$

with

$$|R_n(x)| = \left|\int_{R^2 - A}\right| \leq C\exp\left(-\frac{1}{\sigma_0^2}\left(\frac{c-1}{2c}\log M\right)^2\right) \leqq$$

$$\leqq \frac{B}{\sqrt{M}}\exp\left(-2\left(\frac{2}{c}\right)^{n+1}|x|\right), \tag{4.15)'$$

$\Bigg($Here we applied that for large M and $\varepsilon = \frac{1}{2\sigma_0^2}\left(\frac{c-1}{2c}\right)^2$ $\varepsilon \log^2 M > \left(\frac{2}{c}\right)^{n+1}|x|$ and $\exp\left(-\varepsilon(\log^2 M) < \frac{1}{\sqrt{M}}\right).\Bigg)$ Hence to prove (4.13) it is enough to have a good bound on

$$I_1 = \int_A r_n^{\pm}(x, u, v)\varkappa\left(\frac{x}{c} \mp u, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)dudv$$

and

$$I_2 = \int_A r_n^{+}(x, u, v) r_n^{-}(x, u, v) dudv .$$

We get from (4.14), integrating first by the variable v that

$$I_1 \leqq \frac{B(n)'}{\sqrt{M}}\int \exp\left[-2\left(\frac{2}{c}\right)^n\left|\frac{x}{c} + u\right| - \frac{1}{2}\left(\frac{2}{c^2}\right)^n\frac{1}{\sigma_0}\left(\frac{x}{c} - u\right)^2\right]du =$$

$$= \frac{B(n)'}{\sqrt{M}}\int \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{2}{c^2}\right)^n u^2 - 2\left(\frac{2}{c}\right)^n\left|\frac{2x}{c} - u\right|\right]du .$$

Because of the evenness of the left hand side (4.13) in the variable x we may assume that $x \geq 0$. We get that

$$I_1 \leqq \frac{B(n)'}{\sqrt{M}}\left\{\int_{-\infty}^{2x/c}\exp\left[-\frac{1}{2}\left(\frac{2}{c^2}\right)^n\frac{u^2}{\sigma_0^2} + 2\left(\frac{2}{c}\right)^n u\right]du \cdot \exp\left(-2\cdot\left(\frac{2}{c}\right)^{n+1}x\right) + \right.$$

$$\left. + \int_{2x/c}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{2}{c^2}\right)^n\frac{u^2}{\sigma_0^2} - 2\left(\frac{2}{c}\right)^n u + 2\cdot\left(\frac{2}{c}\right)^{n+1}x\right]du\right\} \leqq$$

$$\leqq \frac{B(n)''}{\sqrt{M}}\left[\exp\left(-2\left(\frac{2}{c}\right)^{n+1}x\right) + \exp\left(-\frac{1}{2}\left(\frac{2}{c^2}\right)^n\frac{4x^2}{c^2\sigma_0^2}\right)\right].$$

(The last integral can be bounded by constant times the value of the integrand in $2x/c$.)

Since $-\frac{1}{2}\left(\frac{2}{c^2}\right)^n \frac{4x^2}{c^2\sigma_0^2} \leqq -2\left(\frac{2}{c}\right)^{n+1} x + 2^n\sigma_0^2$

$$I_1 \leqq \frac{\overline{B}(n)}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^{n+1}|x|\right) \quad \text{if} \quad |x| < \left(1+\frac{c-1}{4}\right)\log M. \tag{4.16}$$

On the other hand by (4.14)

$$I_2 \leqq \frac{\tilde{B}^2(n)\sqrt{\pi}}{M}\int \exp\left[-2\left(\frac{2}{c}\right)^n\left(\left|\frac{x}{c}+u\right|+\left|\frac{x}{c}-u\right|\right)\right]du =$$

$$= \frac{\tilde{B}^2(n)\sqrt{\pi}}{M}\left(\frac{2}{c}|x| + \frac{1}{2}\left(\frac{c}{2}\right)^n\right)\exp\left(-2\left(\frac{2}{c}\right)^{n+1}|x|\right),$$

hence

$$I_2 \leqq \frac{B(n)'}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^{n+1}|x|\right) \quad \text{if} \quad |x| < \left(1+\frac{c-1}{4}\right)\log M. \tag{4.17}$$

Relations (4.15), (4.15)', (4.16) and (4.17) imply (4.13). On the other hand we get by integrating (4.9) that

$$\left|\int_{R^2-A} \exp\left(-\frac{u^2}{c^n}-v^2\right) f(l_{n,M}^+(x,u,v)) f(l_{n,M}^-(x,u,v))\,du\,dv\right| \leqq$$

$$\leqq \frac{B(n)}{\sqrt{M}} \exp\left(-2\left(\frac{2}{c}\right)^{n+1}|x|\right)$$

for $|x| < \left(1+\frac{c-1}{4}\log M\right)$. (Observe that in this case we make a negligible error by omitting the term $\frac{x^2}{c^{n+1}M}$ from the exponent.) This relation together with (4.13) imply Lemma 2 in the case $j = 0$.

To investigate the cases $j = 1, 2$ we need the following identities

$$\frac{d}{dx}\mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^{n+1}\sigma_0\right) =$$

$$= \lambda_n^{-1}\cdot\frac{2}{c}\int \exp\left(-\frac{u^2}{c^n}\right)\mathscr{n}'\left(\frac{x}{c}+u, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\mathscr{n}\left(\frac{x}{c}-u, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)du \tag{4.18}$$

$$\frac{d^2}{dx^2}\mathscr{n}\left(x,\left(\frac{c}{\sqrt{2}}\right)^{n+1}\sigma_0\right)=$$

$$=\lambda_n^{-1}\cdot\frac{4}{c^2}\int\exp\left(-\frac{u^2}{c^n}\right)\mathscr{n}'\left(\frac{x}{c}+u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\mathscr{n}'\left(\frac{x}{c}-u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)du-$$

$$-\frac{4}{c^{n+2}}\int u\exp\left(-\frac{u^2}{c^n}\right)\mathscr{n}'\left(\frac{x}{c}+u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\mathscr{n}\left(\frac{x}{c}-u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)du, \tag{4.18$'$}$$

where $\mathscr{n}'(x,\sigma)$ denotes the first derivative of the normal density $\mathscr{n}(x,\sigma)$. They can be obtained by differentiating the identity

$$\mathscr{n}\left(x,\left(\frac{c}{\sqrt{2}}\right)^{n+1}\sigma_0\right)=$$

$$=\lambda_n^{-1}\int\exp\left(-\frac{u^2}{c^n}\right)\mathscr{n}\left(\frac{x}{c}+u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\mathscr{n}\left(\frac{x}{c}-u,\left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)du.$$

Lemma 2 in the case $j = 1, 2$ can be obtained similarly to the case $j = 0$, only in this case formulas (4.18) and (4.18)′ have to be compared with formulas (4.10) and (4.10)′ in the domain $(u, v) \in A$. In this comparison the inequality

$$\left(M+\frac{x}{c^{n+1}}\pm\frac{u}{c^n}\right)\left[\left(M+\frac{x}{c^{n+1}}\pm\frac{u}{c^n}\right)^2+\frac{v^2}{c^n}\right]^{-1/2}-1\leq\frac{10c^{-n}}{M^{3/2}}$$

$$\text{if}\quad |x|<\left(1+\frac{c-1}{4}\right)\log M,\quad (u,v)\in A$$

can be applied.

Proof of Proposition 1. — Let us assume that Proposition 1 holds for n, $n < N$. We shall prove it for $n + 1$. We have

$$f_{n+1}(x) = Q_{n,M_n}f_n(x) = \frac{1}{A_n}\overline{Q}_{n,M_n}f_n(x+m_n)$$

with $A_n=\int_{-c^{n+1}M_n}^{\infty}\overline{Q}_{n,M_n}f_n(x)dx$, and $m_n=\frac{1}{A_n}\int_{-c^{n+1}M_n}^{\infty}f_n(x)dx$. We apply Lemmas 1 and 2 with the choice $f = f_n$ and $M = M_n$. Then we get

$$A_n=\int_{|x|<\log M_n}+\int_{\substack{|x|>\log M_n\\ x>-c^{n+1}M_n}}=\lambda_n+O\left(\frac{B(n+1)}{\sqrt{M_n}}\right)$$

where λ_n is defined in Lemma 2, and similarly

$$m_n = \frac{1}{A_n}\int x\left[\overline{Q}_{n,M_n}f_n(x) - \lambda_n \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^{n+1}\sigma_0\right)\right]dx + O\left(\frac{1}{M_n}\right) = \\ = O\left(\frac{B(n+1)}{\sqrt{M_n}}\right).$$

Then, since $M_{n+1} = M_n + m_n c^{-(n+1)}$, Lemmas 1 and 2 clearly imply Proposition 1 for $n + 1$.

Proof of the Corollary.

a) Proof of formula (3.6).

Let us first consider the case $|s| < 2\left(\frac{\sqrt{2}}{c}\right)^n$, $|t| < 10\left(\frac{\sqrt{2}}{c}\right)^n$. If $\hat{M}_0$ is sufficiently large then so are M_n, $n \leq N$. In this case relations (3.2) and (3.3) imply that the functions $\phi_n(x)f_n(x)$ and $\phi_n(f_n)(x)$ are close to each other, and the relations

$$\left|\frac{d^j}{dx^j}\left[\phi_n(f_n)(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right]\right| \leqq 10^{-50}2^{-n}\exp\left(-2\left(\frac{2}{c}\right)^n|x|\right) \quad (4.19)$$

for $|x| < 50c^{n/2}$, $j = 0, 1, 2$, $n \leq N$ and

$$\left|\frac{d^j}{dx^j}\phi_n(f_n)(x)\right| \leqq \left|\frac{d^j}{dx^j}\mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right| + 10^{-50}\cdot 2^{-n}\exp\left(-2\left(\frac{2}{c}\right)^n|x|\right) \quad (4.20)$$

for $x \in R^1$, $j = 0, 1, 2$, $n \leq N$ hold true. Since

$$\int\left[\phi_n(f_n)(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right]dx = \\ = \int x\left[\phi_n(f_n)(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right]dx = 0$$

hence

$$\tilde{\phi}_n(f_n)(t + is) = \int \exp(itx - sx)\phi_n(f_n)(x)dx = \\ = \int \exp(itx - sx)\mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)dx + \\ + \int \exp(itx - sx)\left[\phi_n(f_n)(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right]dx = \\ = \exp\left(\frac{(s - it)^2}{2}\left(\frac{c^2}{2}\right)^n\sigma_0^2\right) + \int\left[\exp(itx - sx) - 1 - (it - s)x\right] \\ \left[\phi_n(f_n)(x) - \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n\sigma_0\right)\right]dx = I_1 + I_2.$$

Since

$$|\exp(itx - sx) - 1 - (it - s)x| \leqq (s^2 + t^2)x^2 \exp(|sx|)$$

we get from (4.19) and (4.20) (assuming that M_n is sufficiently large) that

$$I_2 \leqq (s^2 + t^2)\left[10^{-50} \cdot 2^{-n} \int x^2 \exp\left(|sx| - 2\left(\frac{2}{c}\right)^n |x|\right)dx + \right.$$
$$\left. + 2\int_{|x|>50c^{n/2}} x^2 \exp(|sx|)\mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)dx \leqq 10^{-40}\left(\frac{c^2}{2}\right)^n (s^2 + t^2).\right.$$

Thus we get that

$$|\tilde{\phi}_n(f_n)(t + is)| \leqq \exp\left(\left(\frac{c^2}{2}\right)^n \frac{s^2 - t^2}{2}\sigma_0^2\right) + 10^{-40}\left(\frac{c^2}{2}\right)^n (s^2 + t^2).$$

If $0 < T < \frac{1}{10}$ then $\frac{5}{8} < \sigma_0^2 < \frac{7}{8}$, and we can write

$$|\tilde{\phi}_n(f_n)(t + is)| \leqq \exp\left(\frac{c^2}{2}\right)^n \frac{s^2 - t^2}{2}\sigma_0^2\left(1 + \frac{1}{100}\left(\frac{c^2}{2}\right)^n (s^2 + t^2)\right) \leqq$$
$$\leqq \exp\left[\left(\frac{c^2}{2}\right)^n s^2 - \frac{1}{3}\left(\frac{c^2}{2}\right)^n t^2\right] \leqq \frac{\exp \beta_n s^2}{1 + \alpha_n t^2}$$

if $|s| < 2\left(\frac{\sqrt{2}}{c}\right)^n$, $|t| < 10\left(\frac{\sqrt{2}}{c}\right)^n$.

In the case $|s| < 2(\sqrt{2}/c)^n$, $|t| > 10\left(\frac{\sqrt{2}}{c}\right)^n$ we get, integrating by parts twice, that

$$\tilde{\phi}_n(f_n)(t + is) = \frac{1}{(t + is)^2} \int \frac{d^2}{dx^2}(\phi_n(f_n)(x)) \exp(itx - sx)dx,$$

and by formula (4.20)

$$\left|\int \exp(itx - sx)\frac{d^2}{dx^2}\phi_n(f_n)(x)dx\right| \leqq \int \exp(|sx|)\left[\left|\frac{d^2}{dx^2}\mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)\right| + \right.$$
$$\left. + 10^{-50} \cdot 2^{-n} \exp\left(-2\left(\frac{2}{c}\right)^n |x|\right)\right]dx \leqq 10\left(\frac{c^2}{2}\right)^n.$$

Hence

$$|\tilde{\phi}_n(f_n)(t + is)| \leqq \frac{10}{t^2}\left(\frac{c^2}{2}\right)^n \leqq \frac{1}{1 + \alpha_n t^2} \leqq \frac{\exp \beta_n s^2}{1 + \alpha_n t^2}$$

also in the case $|s| < 2\left(\frac{\sqrt{2}}{c}\right)^n$, $|t| > 10\left(\frac{\sqrt{2}}{c}\right)^n$.

b) Proof of formula (3.7).

By formula (3.2) we have for $|x| < \log M_n$

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq \frac{B(n)}{\sqrt{M_n}} \exp\left(-\left(\frac{2}{c}\right)^n |x|\right) + \left|\frac{d^j}{dx^j} \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)\right|.$$

On the other hand

$$\frac{B(n)}{\sqrt{M_n}} \exp\left(-\left(\frac{2}{c}\right)^n |x|\right) \leqq \frac{1}{\beta_n^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^n M_n}\right|\right),$$

and

$$\left|\frac{d^j}{dx^j} \mathscr{n}\left(x, \left(\frac{c}{\sqrt{2}}\right)^n \sigma_0\right)\right| \leqq \left(\frac{\sqrt{2}}{c}\right)^{n(j+1)} \left(1+\left(\left(\frac{\sqrt{2}}{c}\right)^n |x|\right)^j\right) \exp\left(-\frac{1}{2}\left(\frac{2}{c^2}\right)^n \frac{x^2}{\sigma_0^2}\right) =$$

$$= \frac{1}{\beta_n^{(j+1)/2}} \left(1+\left(\frac{|x|}{\sqrt{\beta_n}}\right)^j\right) \exp\left(-\frac{1}{2}\left(\frac{|x|}{\sqrt{\beta_n}\sigma_0}\right)^2 - 8\sigma_0^2 + 8\sigma_0^2\right) \leqq$$

$$\leqq \frac{1}{\beta_n^{(j+1)/2}} \left(1+\frac{|x|^j}{\beta_n^{(j+1)/2}}\right) \exp\left(-\frac{4|x|}{\sqrt{\beta_n}} + 8\sigma_0^2\right) \leqq$$

$$\leqq \frac{9999}{\beta_n^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^n M_n}\right|\right)$$

for $j = 0, 1, 2$. If $|x| > \log M_n$, $x > -c^n M_n$ we get from (3.3) that

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq$$

$$\leqq B(n) \exp\left(-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^n M}\right|\right) \cdot \exp\left[\left(\left(\frac{\sqrt{2}}{c}\right)^n - \left(\frac{2}{c^2}\right)^n\right)\left|2x + \frac{x^2}{c^n M_n}\right|\right] \leqq$$

$$\leqq \frac{1}{\beta_n^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^n M}\right|\right).$$

5. THE PROOF OF PROPOSITION 2

We prove Proposition 2 with the help of a series of Lemmas. Lemmas 3, 4, 5 and 6 enable us to estimate the functions $\overline{Q}_{n,M} f(x)$, $\overline{T}_M \phi_n(f)(x)$ together with their difference. We can consider their normalization $Q_{n,M} f(x)$ and $T_M \phi_n(f)(x)$ with the help of Lemma 7.

LEMMA 3. — *Given some positive integer n and real numbers $\beta > 0$, $M > 10^6$ let $f \in \mathscr{A}_{n,M}$ be such that*

$$\left| \frac{d^j}{dx^j} f(x) \right| \leqq \frac{C}{\beta^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta}} \left| 2x + \frac{x^2}{c^n M} \right| \right) \text{ for } x > -c^n M, j = 0, 1, 2. \tag{5.1}$$

There exists some threshold $n_0(c, C)$ such that if $n > n_0(c, C)$, and

$$\beta > \frac{4}{(1-\varepsilon)M^2} \quad \text{with some } \frac{1}{2} > \varepsilon > 10c^{-n/4}, \qquad \beta < 100 \tag{5.2}$$

then

$$\left| \frac{dj}{dx^j} \overline{Q}_{n,M} f(x) \right| < C_1(\varepsilon) \frac{C^2}{\beta^{(j+1)/2}} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right), \ j = 0, 1, 2$$

for $x > -c^{n+1}M$ with some $C_1(\varepsilon) > 0$ depending only on ε.

Proof of Lemma 3. — Let us first consider the case $j = 0$. It follows from (4.8)′ and (5.1) that

$$\exp\left(-\frac{u^2}{c^n} - v^2 \right) f(l^+_{n,M}(x, u, v)) f(l^-_{n,M}(x, u, v)) \leqq$$
$$\leqq \frac{C^2}{\beta} \exp\left(-\frac{2}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right) \exp\left(-\left(\frac{u^2}{c^n} + v^2 \right)\left(1 - \frac{2}{\sqrt{\beta}M} \right) \right). \tag{5.3}$$

Since $1 - \dfrac{2}{\sqrt{\beta}M} > 1 - \sqrt{1-\varepsilon} > 0$ by (5.2) we get by integrating (5.3) that

$$|\overline{Q}_{n,M} f(x)| \leqq \frac{C^2 c^{n/2} \pi}{\beta} \left(1 - \frac{2}{\sqrt{\beta}M} \right)^{-1} \exp\left(-\frac{2}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right). \tag{5.4}$$

For $\{ x \mid x > -c^{n+1}M, |x| > c^{n/2}\sqrt{M} \}$ relation (5.4) implies Lemma 3. since in this case $\exp\left(-\dfrac{\varepsilon}{\sqrt{\beta}} \left| 2x + \dfrac{x^2}{c^{n+1}M} \right| \right) \leqq C(\varepsilon)\sqrt{\beta} c^{-n/2}$. In order to prove Lemma 3 for small $|x|$ we need a different bound on $\overline{Q}_{n,M} f(x)$. We claim that for $|x| < c^{n/2}\sqrt{M}$

$$\exp\left(-\left(\frac{u^2}{c^n} + v^2 \right) \right) f(l^+_{n,M}(x, u, v)) f((^-_{n,M}(x, u, v)) \leqq$$
$$\leqq \frac{C^2}{\beta} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right) \cdot$$
$$\exp\left[-\left(1 - \frac{2(1-\varepsilon)}{\sqrt{\beta}M} \right)\left(\frac{u^2}{c^n} + v^2 \right) \right] \exp\left(-\frac{3|u|\varepsilon}{\sqrt{\beta}} \right). \tag{5.3)'$$

Indeed, by relation (4.8)

$$\left[2l^{+}_{n,\mathrm{M}}(x,u,v)+\frac{l^{+}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right]-\left[2l^{-}_{n,\mathrm{M}}(x,u,v)+\frac{l^{-}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right]= \\ =\left|4u+\frac{4ux}{c^{n+1}\mathrm{M}}\right|>3|u|$$

for $|x|<c^{n/2}\sqrt{\mathrm{M}}$, and in this case this relation, (4.8)′ and the inequality

$$\exp\left(-\frac{u^2}{c^n}-v^2\right)f(l^{+}_{n,\mathrm{M}}(x,u,v))f(l^{-}_{n,\mathrm{M}}(x,u,v))\leqq \\ \leqq\frac{\mathrm{C}^2}{\beta}\exp\left\{-\frac{1-\varepsilon}{\sqrt{\beta}}\left(\left|2l^{+}_{n,\mathrm{M}}(x,u,v)+\frac{l^{+}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right|\right.\right. \\ \left.+\left|2l^{-}_{n,\mathrm{M}}(x,u,v)+\frac{l^{-}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right|\right)-\frac{\varepsilon}{\sqrt{\beta}}\left|\left[2l^{+}_{n,\mathrm{M}}(x,u,v)+\frac{l^{+}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right]\right. \\ \left.\left.-\left[2l^{-}_{n,\mathrm{M}}(x,u,v)+\frac{l^{-}_{n,\mathrm{M}}(x,u,v)^2}{c^n\mathrm{M}}\right]\right|\right\}$$

imply (5.3)′. For $|x|<c^{n/2}\sqrt{\mathrm{M}}$ we get Lemma 3 with $j=0$ by integrating relation (5.3)′. (In this case the multiplying factor before the exponent will be appropriate for our purposes because of the term $\exp\left(-\frac{3\varepsilon|u|}{\sqrt{\beta}}\right)$.)

The cases $j=1,2$ are similar, only in these cases formulas (4.11) and (4.11)′ have to be applied.

LEMMA 4. — *Let f, β, M and n satisfy the conditions of Lemma 3, and let $\beta>4^{-n}$, $\int_{-c^n\mathrm{M}}^{\infty}f(x)dx=1$, $\int_{-c^n\mathrm{M}}^{\infty}xf(x)dx=0$. Then*

$$\left|\frac{d}{dx^j}\left[\overline{\mathrm{Q}}_{n,\mathrm{M}}f(x)-\overline{\mathrm{Q}}_{n,\mathrm{M}}\phi_n(f)(x)\right]\right| \\ \leqq 2^{-n}\frac{\mathrm{C}^2\mathrm{C}_1(\varepsilon)}{\beta^{(j+1)/2}}\exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x+\frac{x^2}{c^{n+1}\mathrm{M}}\right|\right)\qquad j=0,1,2,\ x>-c^{n+1}\mathrm{M}$$

where the regularization $\phi_n(f)(x)$ is defined in the Definition given in Section 3.

Proof of Lemma 4. — We have

$$|\mathrm{A}_n-1|=\left|\int_{-c^n\mathrm{M}}^{\infty}\left[\phi\left(\frac{1}{100}c^{-n/2}x\right)-1\right]f(x)dx\right|\leqq\int_{\substack{x>-c^n\mathrm{M}\\|x|>100c^{n/2}}}f(x)dx\leqq \\ \leqq\int_{\substack{x>-c^n\mathrm{M}\\|x|>100c^{n/2}}}\frac{\mathrm{C}}{\sqrt{\beta}}\exp\left(-\frac{|x|}{\sqrt{\beta}}\right)dx\leqq 2^{-n}$$

if $n > n_0(c, C)$ is sufficiently large. Similarly,

$$|A_nB_n| = \left| \int_{-c^nM}^{\infty} xf(x)[\phi_n(x) - 1]dx \right| \leqq$$

$$\leqq \int_{\substack{|x|>100c^{n/2} \\ x>-c^nM}} |x| \exp\left(-\frac{|x|}{\sqrt{\beta}}\right) dx < \sqrt{\beta}\cdot 2^{-n}.$$

It follows from the above estimates and relation (5.1) that

$$\left| \frac{d^j}{dx^j} [f(x) - \phi_n(f)(x)] \right| \leqq \frac{2^{-n}C}{\beta^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta}} \left| 2x + \frac{x^2}{c^nM} \right| \right),$$

$$j = 0, 1, 2, \qquad x > -c^nM \quad (5.5)$$

and

$$\left| \frac{dj}{dx^j} \phi_n(f)(x) \right| \leqq \frac{C}{\beta^{(j+1)/2}} \exp\left(-\frac{1}{\sqrt{\beta}} \left| 2x + \frac{x^2}{c^nM} \right| \right),$$

$$j = 0, 1, 2, \quad x > -c^nM. \quad (5.6)$$

The proof of Lemma 3 with some slight changes yields that relations (5.5) and (5.6) imply the inequalities

$$\left| \frac{dj}{dx^j} \left[\int \exp\left(-\frac{u^2}{c^n} - v^2 \right) f_i(l^{\mp}_{n,M}(x,u,v)) \right] \left[f(l^{\mp}_{n,M}(x,u,v)) - \phi_n(f)(l^{\mp}_{n,M}(x,u,v)) \right] dudv \right|$$

$$\leqq 2^{-n} \frac{C(\varepsilon)C^2}{\beta^{(j+1)/2}} \exp\left(\frac{2(1-\varepsilon)}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right), \qquad x > -c^{n+1}M$$

for $j = 0, 1, 2$, $i = 1, 2$ with $f_1(x) = f(x)$, $f_2(x) = \phi(f)(x)$. Because of the special quadratic form of the operator $\overline{Q}_{n,M}$ these estimates imply Lemma 4.

LEMMA 5. — *Under the conditions of Lemma 4*

$$\left| \frac{dj}{dx^j} (\overline{Q}_{n,M}\phi_n(f)(x) - \overline{T}_M\phi_n(f)(x)) \right|$$

$$\leqq \frac{C(\varepsilon)C^2}{\beta^{(j+1)/2}} c^{-n} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right) \quad (5.7)$$

for $x > -c^{n+1}M$, $\quad j = 0, 1, 2$.

Proof of Lemma 5. — Let us first consider the case $j = 0$. The main step of the proof is to check (5.7) for small x. The main contribution to the integrals $\overline{Q}_{n,M}\phi_n(f)(x)$ and $\overline{T}_M\phi_n(f)(x)$ is given by small u and v, and for such values we need a good asymptotics of the integrands. Let us consider the case $|x|<c^{n/2}\sqrt{M}$, define the set $D(n)=\left\{(u,v),|u|<10c^{n/2}\sqrt{M},\ \frac{v^2}{M}<10c^{n/2}\sqrt{M}\right\}$, and give a good bound on

$$J(x,u,v)=\exp\left(-\frac{u^2}{c^n}-v^2\right)\phi_n(f)(l^+_{n,M}(x,u,v))\phi_n(f)(l^-_{n,M}(x,u,v))-$$
$$-\exp(-v^2)\phi_n(f)\left(\frac{x}{c}+u+\frac{v^2}{2M}\right)\phi_n(f)\left(\frac{x}{c}-u+\frac{v^2}{2M}\right).$$

We claim that

$$|J(x,u,v)|\leqq\frac{C_1C^2}{\beta}\left(\frac{u^2+x^2}{c^n}+\frac{v^4}{c^nM}+\frac{u^6+x^6}{c^{3n}}+\left(\frac{v^4}{Mc^n}\right)^3\right)\cdot$$
$$\cdot\exp\left(-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}+u+\frac{v^2}{2M}\right|-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}-u+\frac{v^2}{2M}\right|-v^2\right)\quad(5.8)$$

if $|x|<c^{n/2}\sqrt{M}$ and $(u,v)\in D(n)$.

Indeed, by (5.6) $\left(\text{the term } \frac{x^2}{c^nM} \text{ can be dropped from the exponent in (5.6)}\right)$ and (4.12)

$$\left|\phi_n(f)(l^{\pm}_{n,M}(x,u,v))-\phi_n(f)\left(\frac{x}{c}\pm u+\frac{v^2}{2M}\right)\right|$$
$$\leqq 5\left(\frac{v^4}{c^nM^3}+\frac{x^2+u^2}{c^nM}\right)\cdot\sup_{\frac{x}{c}\pm u+\frac{v^2}{2M}<\xi<l^{\pm}_{n,M}(x,u,v)}\left|\frac{d}{d\xi}\phi_n(f)(\xi)\right|$$
$$\leqq\frac{C'C}{M\beta}c^{-n}\left(\frac{v^2}{M}+x^2+u^2\right)\exp\left(-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}\pm u+\frac{v^2}{2M}\right|\right)\quad(5.9)$$

and

$$|\exp(-u^2/c^n)-1|\leqq u^2/c^n.\quad(5.10)$$

First we show that

$$|J(x,u,v)|\leqq\left\{\left(1+\frac{u^2}{c^n}\right)\left[\phi_n(f)\left(\frac{x}{c}+u+\frac{v^2}{2M}\right)+\frac{C'C}{\sqrt{\beta}}c^{-n}\left(\frac{v^4}{M}+x^2+u^2\right)\right.\right.$$
$$\left.\left.\exp\left(-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}+u+\frac{v^2}{2M}\right|\right)\right]\cdot\left[\phi_n(f)\left(\frac{x}{c}-u+\frac{v^2}{2M}\right)+\frac{C'C}{\sqrt{\beta}}c^{-n}\left(\frac{v^4}{M}+x^2+u^2\right)\right.\right.$$
$$\left.\left.\exp\left(-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}-u+\frac{v^2}{2M}\right|\right)\right]-\phi_n(f)\left(\frac{x}{c}+u+\frac{v^2}{2M}\right)\phi_n(f)\left(\frac{x}{c}-u+\frac{v^2}{2M}\right)\right\}$$
$$\exp(-v^2).\quad(5.11)$$

Indeed, write

$$\phi_n(f)(l^+_{n,M}(x, u, v)) = \phi_n(f)\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) + \varepsilon_1 ,$$

$$\phi_n(f)(l^-(x, u, v)) = \phi_n(f)\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) + \varepsilon_2 , \quad \exp(-u^2/c^n) = 1 + \varepsilon_3$$

in the definition of $J(x, u, v)$, and carry out all multiplication both in the expression $J(x, u, v)$ and the right hand side of (5.11). Then the relations $\frac{1}{M\beta} < \frac{1}{\sqrt{\beta}}$, (5.9) and (5.10) imply that each term in the expression $J(x, u, v)$ is majorized by the corresponding term at the right hand side of (5.11). Similar argument shows that the right hand side of (5.11) increases if $\phi_n(f)\left(\frac{x}{c} \pm u + \frac{v^2}{2M}\right)$ is replaced by its upper bound

$$C\beta^{-1/2}\exp\left(-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c} \pm u + \frac{v^2}{2M}\right|\right).$$

(This upper bound follows from (5.6).) Hence we get that

$$| J(x, u, v) | \leqq \exp\left\{-\frac{2}{\sqrt{\beta}}\left|\frac{x}{c}+u+\frac{v^2}{2M}\right| - \frac{2}{\sqrt{\beta}}\left|\frac{x}{c}-u+\frac{v^2}{2M}\right| - v^2\right\}$$
$$\frac{\bar{C}C^2}{\beta}\left\{\left(\left(1 + \frac{u^2}{c^n}\right)\left(1 + c^{-n}\left(\frac{v^4}{M} + x^2 + u^2\right)\right)\right)^2 - 1\right\}$$

and this estimate implies (5.8).

Since $\frac{2}{\sqrt{\beta}}\frac{v^2}{M} - v^2 < -(1 - \sqrt{1 - \varepsilon})v^2$ by (5.2) relation (5.8) yields that

$$| J(x, u, v) | \leqq \frac{C_1C^2}{\beta}c^{-n}\left(1 + u^2 + x^2 + \frac{u^6 + x^6}{c^{2n}}\right)\left(1 + \frac{v^4}{M} + \frac{v^{12}}{c^{2n}M^3}\right)$$
$$\exp\left\{-\frac{2}{\sqrt{\beta}}\left(\left|\frac{x}{c} + u\right| + \left|\frac{x}{c} - u\right|\right) - (1 - \sqrt{1 - \varepsilon})v^2\right\} \quad (5.8)'$$

if $| x | < c^{n/2}\sqrt{M}$ and $(u, v) \in D(n)$.

Integrating this inequality and using the change of variables $\bar{u} = \frac{u}{\sqrt{\beta}}$ we get that

$$\int_{D[n)} | J(x, u, v) | \, dudv \leqq \frac{C_1C^2}{\sqrt{\beta}}c^{-n}\int\left(1 + x^2 + \frac{x^6}{c^{2n}} + \beta u^2 + \beta^3\frac{u^6}{c^{2n}}\right)$$
$$\exp\left(-2\left|\frac{x}{c\sqrt{\beta}} + u\right| - 2\left|\frac{x}{c\sqrt{\beta}} - u\right|\right)du\int\left(1 + \frac{v^4}{M} + \frac{v^{12}}{c^{2n}M^3}\right)$$
$$\exp(-(1 - \sqrt{1 - \varepsilon})v^2)dv \leqq \quad (5.12)$$

$$\leqq \frac{C(\varepsilon)C^2}{\sqrt{\beta}} c^{-n} \exp\left(-\frac{4(1-\varepsilon/3)}{c\sqrt{\beta}}|x|\right)$$
$$\leqq \frac{C(\varepsilon)C^2}{\sqrt{\beta}} c^{-n} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x+\frac{x^2}{c^{n+1}M}\right|\right) \text{ if } |x| < c^{n/2}M.$$

Observe that if $|x| < c^{n/2}\sqrt{M}$ and $(u, v) \notin D(n)$ then either

$$\left|\frac{x}{c}+u+\frac{v^2}{2M}\right| > 200c^{n/2} \quad \text{or} \quad \left|\frac{x}{c}-u+\frac{v^2}{2M}\right| > 200c^{n/2}$$

therefore

$$\int_{R^2-D[n)} \exp(-v^2)\phi_n(f)\left(\frac{x}{c}+u+\frac{v^2}{2M}\right)\phi_n(f)\left(\frac{x}{c}-u+\frac{v^2}{2M}\right)dudv = 0$$
$$\text{if } |x| < c^{n/2}\sqrt{M}. \quad (5.13)$$

Indeed, if $\frac{v^2}{M} > 10c^{n/2}\sqrt{M}$ then

$$\left|\frac{x}{c}+u \text{ sign } u+\frac{v^2}{2M}\right| > \frac{v^2}{2M} - \frac{|x|}{c} > 4c^{n/2}\sqrt{M} > 200c^{n/2},$$

and if $|u| > 10c^{n/2}\sqrt{M}$,

$$\frac{v^2}{M} < 10c^{n/2}\sqrt{M} \quad \text{then} \quad \left|\frac{x}{c}+u+\frac{v^2}{2M}\right| > |u| - |x| - \frac{v^2}{2M} > 200c^{n/2}.$$

On the other hand relation (5.3)′, for the function $\phi_n(f)$, yields that

$$\int_{R^2-D(n)} \exp\left(-\left(\frac{u^2}{c^n}+v^2\right)\right)\phi_n(f)(l^+_{n,M}(x,u,v))\phi_n(f)(l^-_{n,M}(x,u,v))dudv \leqq$$
$$\leqq \frac{C^2C(\varepsilon)}{\sqrt{\beta}} c^{-n} \exp\left(-\frac{2(1-\varepsilon)}{\sqrt{\beta}}\left|2x+\frac{x^2}{c^{n+1}M}\right|\right) \text{ if } |x| < c^{n/2}\sqrt{M}. \quad (5.13)'$$

Relations (5.12), (5.13) and (5.13)′ imply Lemma 5 for $|x| < c^{n/2}\sqrt{M}$, $j = 0$.

If $|x| > c^{n/2}\sqrt{M}$ then relation (5.4) for the function $\phi_n(f)$ gives that

$$|\overline{Q}_{n,M}\phi_n(f)(x)| \leqq \frac{C(\varepsilon)C^2}{\sqrt{\beta}} c^{-n} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x+\frac{x^2}{c^{n+1}M}\right|\right) \quad (5.14)$$

for $|x| > c^{n/2}\sqrt{M}$, $x > -c^{n+1}M$.

If $x > c^{n/2}\sqrt{M}$ then $\frac{x}{c} + u \text{ sign } u + \frac{v^2}{2M} > 200c^{n/2}$ $(M > 10^6)$, hence $\phi_n(f)\left(\frac{x}{c} + u \text{ sign } u + \frac{v^2}{2M}\right) = 0$, and

$$\overline{T}_M \phi_n(f)(x) = 0 \quad \text{if} \quad x > c^{n/2}\sqrt{M}. \tag{5.15}$$

If $x < -c^{n/2}\sqrt{M}$ then relation (5.6) $\left(\text{dropping the term } \frac{x^2}{c^n M} \text{ from the exponent, what can be done, since } \phi_n(f)(x) = 0 \text{ for } |x| > 200c^{n/2}\right)$ and (5.2) imply that

$$|\overline{T}_M \phi_n(f)(x)| \leqq \frac{C^2}{\beta} \int \exp\left\{ -v^2 - \frac{2}{\sqrt{\beta}}\left(\left|\frac{x}{c} + u + \frac{v^2}{2M}\right| + \left|\frac{x}{c} - u + \frac{v^2}{2M}\right|\right)\right\}$$

$$dudv \leqq \frac{C^2}{\beta} \int \exp\left(-\frac{2}{\sqrt{\beta}}\left(1 - \frac{\varepsilon}{3}\right)\left(\left|\frac{x}{c} + u\right| + \left|\frac{x}{c} - u\right|\right)\right) du \int \exp(-\sqrt{1-\varepsilon}\,v^2)$$

$$dv \leqq \frac{C(\varepsilon)C^2}{\sqrt{\beta}}\left(1 + \frac{|x|}{\sqrt{\beta}}\right) \exp\left(-\frac{4(1-\varepsilon/3)}{c\sqrt{\beta}} x\right)$$

$$\leqq \frac{C(\varepsilon)C^2}{\sqrt{\beta}} \exp\left(-\frac{4(1-\varepsilon/2)}{c\sqrt{\beta}}|x|\right).$$

Hence

$$|\overline{T}_M \phi_n(f)(x)| \leqq \frac{C^2 C(\varepsilon)}{\sqrt{\beta}} c^{-n} \exp\left(-\frac{2(1-\varepsilon)}{c\sqrt{\beta}}\left|2x + \frac{x^2}{c^{n+1}M}\right|\right) \tag{5.15$'$}$$

if $-c^{n+1}M < x < -c^{n/2}M$.

For $|x| > c^{n/2}\sqrt{M}$, $x > -c^{n+1}M$, $j = 0$ Lemma 5 follows from (5.14), (5.15), (5.15)′.

In the case $j = 1,2$ we compare formulas (4.10) and (4.10)′ with the identities

$$\frac{d}{dx}\overline{T}_M f(x) = \frac{2}{c}\int \exp(-v^2) f'\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) dudv,$$

$$\frac{d^2}{dx^2}\overline{T}_M f(x) = \frac{4}{c^2}\int \exp(-v^2) f'\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f'\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) dudv,$$

and apply the inequalities

$$\left| \frac{M + \dfrac{x}{c^{n+1}} \pm \dfrac{u}{c^n}}{\sqrt{\left(M + \dfrac{x}{c^{n+1}} \pm \dfrac{u}{c^n}\right)^2 + \dfrac{v^2}{c^n}}} - 1 \right| \leqq 5 \frac{v^2}{c^n M}, \quad \text{if } |x| < c^{n/2}M, \ (u, v) \in D(n),$$

$$\left| \left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)\left(\left(M + \frac{x}{c^{n+1}} \pm \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}\right)^{-1/2} \right| \leqq 1 .$$

For $j = 1$ almost the same proof works as for $j = 0$ with some slight modifications. An additional multiplying term $\frac{1}{\sqrt{\beta}}$ appears in the estimates when we have to bound f' instead of f. In the case $j = 2$ we estimate similarly, but we have to show that the second term in (4.10)′ is negligible small, it can be bounded by $\frac{C(\varepsilon)C^2}{\beta^{3/2}} c^{-n} \exp\left(- \frac{2(1-\varepsilon)}{\sqrt{\beta c}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right)$.

This follows from the estimate

$$\int \frac{|u|}{\sqrt{\beta}} \exp\left(- \frac{u^2}{c^n} - v^2 \right) | f'(l^+_{n,M}(x, u, v)) | \, | f(l^-_{n,M}(x, u, v)) | \, du dv \leqq$$

$$\leqq \frac{C^2 C(\varepsilon)}{\beta^{3/2}} \exp\left(- \frac{2-\varepsilon}{c\sqrt{\beta}} \left| 2x + \frac{x^2}{c^{n+1}M} \right| \right), \qquad (5.16)$$

which can be proved similarly to Lemma 3, by observing that the integrand on the left hand side of (5.16) can be bounded similarly to (5.3) and (5.3)′, only $\frac{1}{\beta}$ must be replaced by $\frac{1}{\beta^{1/2}}$ and $\exp\left(- \frac{3\varepsilon |u|}{\sqrt{\beta}} \right)$ by $\frac{1}{\varepsilon} \exp\left(- \frac{2\varepsilon |u|}{\sqrt{\beta}} \right)$ in (5.3)′ and the right hand side of (5.3) must be multiplied by $\beta^{-1} |u|$. (This multiplying term in (5.3) causes no problem, because we need (5.3) only to deduce (5.4) for $|x| > c^{n/2}\sqrt{M}$, and in this case the pre-exponential term in (5.4) need not be bounded sharply).

LEMMA 6. — *If the functions $f_1(x)$ and $f_2(x)$ satisfy the inequalities*

$$\left| \frac{d^j}{dx^j} f_i(x) \right| \leqq \frac{C}{\beta^{(j+1)/2}} \exp\left(- \frac{2|x|}{\sqrt{\beta}} \right), \qquad j = 0, 1, 2 ; \qquad i = 1, 2$$

$$\left| \frac{d^j}{dx^j} [f_1(x) - f_2(x)] \right| \leqq \frac{C\delta}{\beta^{(j+1)/2}} \exp\left(- \frac{2|x|}{\sqrt{\beta}} \right), \qquad j = 0, 1, 2$$

for all $x \in \mathrm{R}^1$ *with some* $\mathrm{C} > 0$, $\delta > 0$, $\beta > \frac{4}{\mathrm{M}^2(1-\varepsilon)}$ *with some* $\varepsilon > 0$ *then*

$$\left| \frac{d^j}{dx^j} [\overline{\mathrm{T}}_{\mathrm{M}} f_1(x) - \overline{\mathrm{T}}_{\mathrm{M}} f_2(x)] \right| \leqq \mathrm{C}_1(\varepsilon) \frac{\mathrm{C}^2\delta}{\beta^{(j+1)/2}} \exp\left(- \frac{4(1-\varepsilon)}{c\sqrt{\beta}} |x| \right)$$

for all $x \in \mathrm{R}^1$ *and* $j = 0, 1, 2, 3, 4$. *In particular, by choosing* $f_2(x) = 0$ *we get that*

$$\left| \frac{d^j}{dx^j} \overline{\mathrm{T}}_{\mathrm{M}} f_1(x) \right| \leqq \mathrm{C}_1(\varepsilon) \frac{\mathrm{C}^2}{\beta^{(j+1)/2}} \exp\left(- \frac{4(1-\varepsilon)}{c\sqrt{\beta}} |x| \right), \qquad j = 0, 1, 2, 3, 4.$$

(Observe that we get a bound on the first four derivatives of $\overline{\mathrm{T}}_{\mathrm{M}} f_1(x)$ and $\overline{\mathrm{T}}_{\mathrm{M}} f_2(x)$ with the help of the first two derivatives of $f_1(x)$ and $f_2(x)$.)

Proof of Lemma 6.

We have

$$\frac{d^j}{dx^j} [\overline{\mathrm{T}}_{\mathrm{M}} f_1(x) - \overline{\mathrm{T}}_{\mathrm{M}} f_2(x)]$$

$$= \left(\frac{2}{c}\right)^j \int e^{-v^2} \left\{ f_1^{(l)}\left(\frac{x}{c} + u + \frac{v^2}{2\mathrm{M}}\right) \left[f_1^{(j-l)}\left(\frac{x}{c} - u + \frac{v^2}{2\mathrm{M}}\right) - f_2^{(j-l)}\left(\frac{x}{c} - u + \frac{v^2}{2\mathrm{M}}\right) \right] \right.$$

$$\left. + f_2^{(j-l)}\left(\frac{x}{c} - u + \frac{v^2}{2\mathrm{M}}\right) \left[f_1^{(l)}\left(\frac{x}{c} + u + \frac{v^2}{2\mathrm{M}}\right) - f_2^{(l)}\left(\frac{x}{c} + u + \frac{v^2}{2\mathrm{M}}\right) \right] \right\} du dv$$

with $l = \left[\frac{j}{2}\right]$, where [ ] denotes integer part.

Hence

$$\left| \frac{d^j}{dx^j} [\overline{\mathrm{T}}_{\mathrm{M}} f_1(x) - \overline{\mathrm{T}}_{\mathrm{M}} f_2(x)] \right| \leqq$$

$$\leqq 2\left(\frac{2}{c}\right)^j \mathrm{C}^2 \delta \beta^{-(j+2)/2} \int \exp\left\{ -v^2 - \frac{2}{\sqrt{\beta}} \left(\left| \frac{x}{c} + u + \frac{v^2}{2\mathrm{M}} \right| + \left| \frac{x}{c} - u + \frac{v^2}{2\mathrm{M}} \right| \right) \right\}$$

$$du dv \leqq 2\left(\frac{2}{c}\right)^j \mathrm{C}^2 \delta \beta^{-(j+2)/2} \int \exp\left(- \frac{2}{\sqrt{\beta}} \left(\left| \frac{x}{c} + u \right| + \left| \frac{x}{c} - u \right| \right) \right) du \,.$$

$$\cdot \int \exp\left(- \frac{2}{\sqrt{\beta \mathrm{M}}} \right) \Big) dv \,.$$

By calculating the above integrals we get that

$$\left|\frac{d^j}{dx^j}[\overline{T}_M f_1(x) - \overline{T}_M f_2(x)]\right| \leqq$$

$$\leqq 2\left(\frac{2}{c}\right)^j \frac{C^2\delta\sqrt{\pi}}{\beta^{(j+1)/2}}\left(1 - \frac{2}{\sqrt{\beta}M}\right)^{-1/2}\left(1 + \frac{4|x|}{c\sqrt{\beta}}\right)\exp\left(-\frac{4|x|}{c\sqrt{\beta}}\right) \leqq$$

$$\leqq \frac{C_1(\varepsilon)C^2}{\beta^{(j+1)/2}}\exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right).$$

Lemma 6 is proved.

LEMMA 7. — *Let f, n and β satisfy the conditions of Lemma 4. Then*

$$R_0 = \int_{-c^{n+1}M}^{\infty} \overline{Q}_{n,M} f(x) dx = \frac{c\sqrt{\pi}}{2} + \gamma_n,$$

$$R_1 = \int_{-c^{n+1}M}^{\infty} x\overline{Q}_{n,M} f(x) dx = -\frac{c^2\sqrt{\pi}}{8M} + \delta_n$$

with $|\gamma_n| < C_1 C^2 c^{-n}$, $|\delta_n| < C_1 C^2 c^{-n}\sqrt{\beta}$.

Proof of Lemma 7. — We can write both for $j = 0$ and $j = 1$

$$R_j = \int_{-c^{n+1}M}^{\infty} x^j[\overline{Q}_{n,M} f(x) - \overline{Q}_{n,M}\phi_n(f)(x)]dx +$$

$$+ \int_{-c^{n+1}M}^{\infty} x^j[\overline{Q}_{n,M}\phi_n(f)(x) - \overline{T}_M\phi_n(f)(x)]dx + \int_{-\infty}^{-c^{n+1}M} -x^j\overline{T}_M\phi_n(f)(x)dx +$$

$$+ \int x^j\overline{T}_M\phi_n(f)(x)dx = I_1^{(j)} + I_2^{(j)} + I_3^{(j)} + I_4^{(j)}. \tag{5.17}$$

It follows from (2.20) that

$$I_4^{(0)} = \int \overline{T}_M\phi_n(f)(x)dx = \frac{c\sqrt{\pi}}{2}, \quad I_4^{(1)} = \int x\overline{T}_M\phi_n(f)(x)dx = -\frac{c^2\sqrt{\pi}}{8M}. \tag{5.18}$$

By applying Lemma 4 and Lemma 5 with $\varepsilon = \frac{1}{10}$ we get that

$$|I_1^{(j)}| \leqq \frac{2^{-n}C_1C^2}{\sqrt{\beta}}\int_{-c^{n+1}M}^{\infty}|x|^j\exp\left(-\frac{|x|}{\sqrt{\beta}}\right)dx \leqq 2^{-n}C_1C^2\beta^{j/2} \tag{5.19}$$

$$|I_2^{(j)}| \leqq c^{-n}C_1C^2\beta^{j/2} \tag{5.19'}$$

and by Lemma 6

$$|I_3^{(j)}| \leqq C_1C^2\frac{1}{\sqrt{\beta}}\int_{-\infty}^{-c^{n+1}M}|x|^j\exp\left(-\frac{|x|}{\sqrt{\beta}}\right)dx \leqq c^{-n}\overline{C}_1C^2\beta^{j/2}. \tag{5.19''}$$

Relations (5.17)-(5.19)″ imply Lemma 7.

Proof of Proposition 2. — The relation

$$Q_{n,M}f(x) = \frac{1}{R_0}\,\overline{Q}_{n,M}f(x + m_n) \tag{5.20}$$

holds with

$$m_n = \frac{R_1}{R_0} = -\frac{c}{4M} + \gamma_n \qquad \text{with} \qquad |\gamma_n| < C_1C^2c^{-n}\sqrt{\beta} \quad (5.20)'$$

$\left(\text{observe that } \frac{1}{\sqrt{\beta}} < M\right)$, and this implies (3.19) and (3.19)'. Relation (3.16) follows from (5.20), (5.20)' and Lemma 3, relation (3.17) from Lemma 6 and (2.17).

We claim that

$$\left|\frac{d^j}{dx^j}\left[\frac{1}{R_0}\overline{T}_M\phi_n(f)(x + m_n) - T_M\phi_n(f)(x)\right]\right| \leqq$$

$$\leqq \frac{C(\varepsilon)C^4}{\beta^{(j+1)/2}}c^{-n}\exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right) \tag{5.21}$$

for $j = 0, 1, 2$. Indeed,

$$T_M\phi_n(f)(x) - \frac{1}{R_0}\overline{T}_M\phi_n(f)(x + m_n) = \left(\frac{2}{c\sqrt{\pi}} - \frac{1}{R_0}\right)\overline{T}_M\phi_n(f)(x + m_n) +$$

$$+ \frac{2}{c\sqrt{\pi}}\left(\overline{T}_M\phi_n(f)\left(x - \frac{c}{4M}\right) - \overline{T}_M\phi_n(f)(x + m_n\right),$$

and by Lemma 6 and (5.20)'

$$\left|\frac{d^j}{dx^j}\left[\overline{T}_M\phi_n(f)\left(x - \frac{c}{4M}\right) - \overline{T}_M\phi_n(f)(x + m_n)\right]\right| \leqq$$

$$\leqq \left|m_n + \frac{c}{4M}\right| \left|\frac{d^{j+1}}{dx^{j+1}}\overline{T}_M\phi_n(f)(x)\Big|_{x=\xi}\right| \leqq$$

$$\leqq \frac{C(\varepsilon)C^4}{\beta^{(j+1)/2}}c^{-n}\exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right) \quad \text{with some } \xi \in \left[x - \frac{c}{4M}, x + m_n\right].$$

Similarly

$$\left|\left(\frac{2}{c\sqrt{\pi}} - \frac{1}{R_0}\right)\overline{T}_M\phi_n(f)(x+m_n)\right| \leqq \frac{C(\varepsilon)C^4}{\beta^{(j+1)/2}}\exp\left(-\frac{4(1-\varepsilon)}{c\sqrt{\beta}}|x|\right)$$

and these relations imply (5.21). Relation (3.18) follows from Lemmas 4, 5 and relations (5.20), (5.21).

6. THE PROOF OF PROPOSITION 3 AND SOME OF ITS CONSEQUENCES. THE SECOND STEP OF THE INDUCTIVE PROCEDURE

We prove Proposition 3 with the help of the following two lemmas.

LEMMA 8. — *Under the conditions of Proposition 2 for* $|s| < \frac{\sqrt[6]{2}}{c}\frac{2}{\sqrt{\beta}}\sqrt{2}$.

a)
$$|\tilde{\phi}_{n+1}(Q_{n,M}f)(t+is) - \tilde{T}_M\tilde{\phi}_n(f)(t+is)| \begin{cases} \leqq C_1(C^4+1)c^{-n}\beta(s^2+t^2) & (6.1) \\ \leqq C_1(C^4+1)c^{-n}\dfrac{1}{\beta t^2} & (6.2) \end{cases}$$

b)
$$|\tilde{T}_M\tilde{\phi}_n(f)(t+is)| \leqq \frac{C_1C^2}{\beta^2 t^4}.$$

LEMMA 9. — *If the Fourier transform* $\tilde{f}$ *of the function* $f(x)$ *satisfies the inequality*

$$|\tilde{f}(t+is)| \leqq \frac{\exp \beta s^2}{1+\alpha t^2} \quad \textit{for} \quad |s| < \frac{2}{\sqrt{\beta}}, \quad t \in R^1$$

and $\beta > \frac{9}{M^2}$, $\alpha > \frac{10^{-12}}{M^2}$ *with some* $M > 1$, $\beta > \alpha$ *then*

a) $\left|\frac{d^j}{dx^j}T_M f(x)\right| \leqq \frac{C_1}{\alpha^{(j+1)/2}}\exp\left(-\frac{4|x|}{c\sqrt{\beta}}\right)$, $j=0,1,2$ *with some absolute constant* $C_1 > 0$.

b)
$$|\tilde{T}_M\tilde{f}(t+is)| \leqq \frac{\exp\left\{\left(\frac{c^2}{2}\beta + \frac{10}{M^2}\right)s^2\right\}}{1+\left(\frac{c^2}{2}\alpha + \frac{10^{-11}}{9M^2}\right)t^2} \quad \textit{for} \quad |s| < \frac{4}{c\sqrt{\beta}}.$$

Proof of Lemma 8. — First we show that it follows from Proposition 2 that

$$\left|\frac{d^j}{dx^j}[\phi_{n+1}(Q_{n,M}f)(x) - T_M\phi_n(f)(x)]\right| \leqq$$
$$\leqq \frac{C_1(C^4+1)c^{-n}}{\beta^{(j+1)/2}}\exp\left(-\frac{2^{3/4}}{c\sqrt{\beta}}|2x|\right) \qquad j=0,1,2. \quad (6.3)$$

If we replace $\phi_{n+1}(Q_{n,M}f)(x)$ by $\phi_{n+1}(x)Q_{n,M}f(x)$ in (6.3) then this modified version of relation (6.3) holds. Indeed, this follows from (3.18) for $|x| < 100c^{n/2}(\phi_{n+1}(x) = 1$ for $|x| < 100c^{n/2})$ and from the bounds given on $Q_{n,M}(f)(x)$ and $T_M\phi_n(f)(x)$ in (3.16) and (3.17) for $|x| > 100c^{n/2}$. (Observe that the support of $\phi_{n+1}(x)Q_{n,M}f(x)$ is in the set $|x| < 200c^{n/2}$.) Since

$$\left|\int \phi_{n+1}(x)Q_{n,M}f(x)dx - 1\right| = \int [1 - \phi_{n+1}(x)]Q_{n,M}f(x)dx \leqq$$
$$\leqq \frac{C^2}{\sqrt{\beta}}\sqrt{\beta}\exp\left(-\frac{c^{n/2}}{\sqrt{\beta}}\right) \leqq C^2c^{-n}$$

and

$$\left|\int x\phi_{n+1}(x)Q_{n,M}f(x)dx\right| = \left|\int x(\phi_{n+1}(x) - 1)Q_{n,M}f(x)dx\right| \leqq C^2c^{-n}\sqrt{\beta},$$

simple calculation shows that (6.3) also holds in his original form. Since

$$\tilde{\phi}_{n+1}(Q_{n,M}f)(0) - \tilde{T}_M\tilde{\phi}_n(f)(0) = \frac{d}{dt}[\tilde{\phi}_{n+1}(Q_{n,M}f)(t) - \tilde{T}_M\phi_n(f)(t)]\,|_{t=0} = 0\,,$$

we have for $|s| \leqq \frac{\sqrt{2}}{c}\frac{2}{\sqrt{\beta}}\sqrt[6]{2}$

$$|\tilde{\phi}_{n+1}(Q_{n,M}f)(t+is) - \tilde{T}_M\tilde{\phi}_n(f)(t+is)| \leqq$$
$$\leqq 2(t^2+s^2) \sup_{|\mathrm{Im}\,\xi| < \frac{\sqrt{2}}{c}\frac{2}{\sqrt{\beta}}\sqrt[6]{2}} \left|\frac{d^2}{d\xi^2}[\tilde{\phi}_{n+1}(Q_{n,M}f)(\xi) - \tilde{T}_M\tilde{\phi}_n(f)(\xi)]\right| \leqq$$
$$\leqq 2(t^2+s^2)\int \exp\left(\frac{\sqrt{2}}{c}\frac{2}{\sqrt{\beta}}\sqrt[6]{2}|x|\right)x^2|\phi_{n+1}(Q_{n,M}f)(x) - T_M\phi_n(f)(x)|\,dx. \tag{6.4}$$

Relation (6.1) follows from (6.3) (with $j = 0$) and (6.4). To prove (6.2) we integrate by parts the Fourier transform formula twice. We get that $\tilde{\phi}_{n+1}(Q_{n,M}f)(t+is) - \tilde{T}_M\tilde{\phi}_n(f)(t+is) =$

$$= -\frac{1}{\sqrt{2\pi}(t+is)^2}\int \exp\left[(it-s)x\right]\frac{d^2}{dx^2}(\phi_{n+1}(Q_{n,M}f)(x) - T_M\phi_n(f)(x))dx\,.$$

Hence relation (6.3) (with $j = 2$) yields that

$$|\tilde{\phi}_{n+1}(Q_{n,M}f)(t+is) - \tilde{T}_M\tilde{\phi}_n(f)(t+is)| \leqq$$
$$\leqq \frac{C_1(C^4+1)}{t^2\beta^{3/2}}c^{-n}\int \exp\left(|sx| - \frac{2^{3/4}}{c\sqrt{\beta}}|2x|\right)dx \leqq$$
$$\leqq \frac{C_1'(C^4+1)}{\beta t^2}c^{-n} \quad \text{for} \quad |s| < \frac{\sqrt{2}}{c}\frac{2}{\beta}\sqrt[6]{2}.$$

Similarly, integration by parts four times and relation (3.17) yield that

$$|\tilde{T}_M\tilde{\phi}_n(f)(t+is)| \leqq \frac{1}{t^4}\int \exp(|sx|)\left|\frac{d^4}{dx^4}T_M\phi_n(f)(x)\right| dx \leqq$$
$$\leqq \frac{C_1C^2}{\beta^2 t^4} \quad \text{for} \quad |x| < \frac{2\sqrt{2}}{c\sqrt{\beta}}\sqrt[6]{2}.$$

Lemma 8 is proved.

Proof of Lemma 9. — First we estimate $\tilde{T}_M\tilde{f}$. For $|s| < \dfrac{1}{c\sqrt{\beta}}$ we have by (2.20)′

$$|\tilde{T}_M\tilde{f}(t+is)| = \left|\tilde{f}\left(\frac{c}{2}(t+is)\right)\right|^2 \frac{\exp\left(-\dfrac{c}{4M}s\right)}{\sqrt{\left|1+i\dfrac{2}{2M}(t+is)\right|}} \leqq$$

$$\leqq \frac{\exp\left(\dfrac{c^2}{2}\beta s^2 - \dfrac{c}{4M}s\right)}{\left(1+\dfrac{c^2}{4}\alpha t^2\right)^2 \sqrt[4]{\left(1-\dfrac{c}{2M}s\right)^2 + \dfrac{c^2}{4M^2}t^2}} = \tag{6.5}$$

$$= \frac{\exp\left(\dfrac{c^2}{2}\beta s^2\right)}{\left(1+\dfrac{c^2}{4}\alpha t^2\right)^2} \frac{\exp\left(-\dfrac{c}{4M}s\right)}{\sqrt{1-\dfrac{c}{2M}s}} \frac{1}{\sqrt[4]{1+\dfrac{c^2t^2}{4M^2}\left(1-\dfrac{c}{2M}s\right)^{-2}}}.$$

Since for $|s| < \dfrac{4}{c\sqrt{\beta}}$ and $\beta > \dfrac{9}{M^2}$, $1 - \dfrac{c}{2M}s < \dfrac{5}{3}$ and

$$\frac{\exp\left(-\dfrac{c}{4M}s\right)}{\sqrt{1-\dfrac{c}{2M}s}} = \exp\left\{-\frac{1}{2}\left[\frac{c}{2M}s - \log\left(1-\frac{c}{2M}s\right)\right]\right\} \leqq \exp\left(\frac{10}{M^2}s^2\right)$$

we get from (6.5) that under the conditions of Lemma 9

$$|\tilde{T}_M\tilde{f}(t+is)| \leqq \frac{\exp\left\{\left(\dfrac{c^2}{2}\beta + \dfrac{10}{M^2}\right)s^2\right\}}{\left(1+\dfrac{c^2}{4}\alpha t^2\right)^2 \sqrt[4]{1+\dfrac{9c^2}{100}\dfrac{t^2}{M^2}}} \quad \text{for} \quad |s| < \frac{4}{c\sqrt{\beta}}. \tag{6.6}$$

We estimate $\dfrac{d^j}{dx^j}T_Mf(x)$ by applying inverse Fourier transformation and integrating on the line $-\frac{4 \operatorname{sign} x}{c\sqrt{\beta}} i + t$. We get from (6.6) for $j=0, 1, 2$ that

$$\frac{d^j}{dx^j} \mathrm{T_M} f(x) = \frac{1}{2\pi} \int \left(t + \frac{4 \operatorname{sign} x}{c\sqrt{\beta}} i \right)^j \cdot \exp\left[- ix\left(t - \frac{4 \operatorname{sign} x}{c\sqrt{\beta}} i \right)\right] \widetilde{\mathrm{T}}_{\mathrm{M}} \tilde{f}\left(t + \frac{4 \operatorname{sign} x}{c\sqrt{\beta}} i \right) dt$$

and

$$\left| \frac{d^j}{dx^j} \mathrm{T_M} f(x) \right| \leqq$$

$$\leqq \frac{1}{2\pi} \exp\left\{ - \frac{4 \mid x \mid}{c\sqrt{\beta}} + \left(8 + \frac{160}{c^2 \beta \mathrm{M}^2} \right) \right\} \int 2\left[\mid t \mid^j + \left(\frac{4}{c\sqrt{\beta}} \right)^j \right] \frac{dt}{\left(1 + \frac{c^2}{4} \alpha t^2 \right)^2} \leqq$$

$$\leqq \mathrm{C}_1 \exp\left(- \frac{4 \mid x \mid}{c\sqrt{\beta}} \right) \alpha^{-(j+1)/2} \int \left(\mid t \mid^2 + \left(\frac{\alpha}{\beta} \right)^{j/2} \right) \frac{1}{(1+t^2)^2} dt \leqq$$

$$\leqq \pi \mathrm{C}_1 \alpha^{-(j+1)/2} \exp\left(- \frac{4 \mid x \mid}{c\sqrt{\beta}} \right),$$

since $\alpha/\beta \leqq 1$. Part *a*) is proved.

We claim that

$$\left(1 + \frac{c^2}{2} \alpha t^2 \right) \sqrt[4]{1 + \frac{9c^2}{100} \frac{t^2}{\mathrm{M}^2}} \geqq 1 + \frac{c^2}{2} \alpha t^2 + \frac{10^{-11}}{9} \frac{t^2}{\mathrm{M}^2} \quad \text{if} \quad \alpha > \frac{10^{-12}}{\mathrm{M}^2}. \tag{6.7}$$

Relations (6.6) and (6.7) imply part *b*) of Lemma 9.

Relation (6.7) is equivalent to

$$\sqrt[4]{1 + \frac{9c^2}{100} \frac{t^2}{\mathrm{M}^2}} - 1 \geqq \frac{10^{-11}}{9} \frac{t^2}{\mathrm{M}^2} \frac{1}{1 + \frac{c^2}{2} \alpha t^2} \quad \text{if} \quad \alpha > \frac{10^{-12}}{\mathrm{M}^2}. \tag{6.7'}$$

Since $\sqrt[4]{x+1} - 1 \geqq \frac{x}{4}(1+x)^{-3/4}$ for $x > 0$ the left hand side of (6.7)′ can be estimated as

$$\sqrt[4]{1 + \frac{9c^2}{100} \frac{t^2}{\mathrm{M}^2}} - 1 \geqq \frac{9c^2}{400} \frac{t^2}{\mathrm{M}^2} \left(1 + \frac{9c^2}{100} \frac{t^2}{\mathrm{M}^2} \right)^{-3/4}. \tag{6.8}$$

Observe that $(1+x)^{3/4} < 1 + x^{3/4}$ for $x > 0$, since for $f(x) = (1+x)^{3/4} - x^{3/4} - 1$

$f(0)=0$, and $f'(x) \leq 0$ for $x > 0$. This estimate together with (6.8) imply that the left hand side of (6.7)' can be estimated as

$$\sqrt[4]{1+\frac{9c^2}{100}\frac{t^2}{\mathrm{M}^2}}-1 \geqq \frac{9c^2}{400}\frac{t^2}{\mathrm{M}^2}\,\frac{1}{1+\left(\frac{9c^2}{100}\frac{t^2}{\mathrm{M}^2}\right)^{3/4}}$$

and since $\alpha > 10^{-12}\frac{1}{\mathrm{M}^2}$ relation (6.7)' follows from the inequality

$$1+\left(\frac{9c^2}{100}\frac{t^2}{\mathrm{M}^2}\right)^{3/4} \leqq \frac{81.10^{11}}{400}c^2\left(1+\frac{c^2}{2}10^{-12}\frac{t^2}{\mathrm{M}^2}\right),$$

or equivalently

$$\left(\frac{9c^2}{100}\frac{t^2}{\mathrm{M}^2}\right)^{3/4} \leqq \left(\frac{81}{400}10^{12}c^2-1\right)+\frac{81c^4}{8\,000}\frac{t^2}{\mathrm{M}^2}.$$

The last relation follows from the inequality $a+b \geqq 4.3^{-1/4}a^{3/4}b^{1/4}$ with the choice $a=\frac{81}{8\,000}\frac{t^2}{\mathrm{M}^2}$ and $b=\frac{81}{4\,000}10^{12}c^2-1$. Lemma 9 is proved.

Proof of Proposition 3. — First we show some properties of the numbers α_n, β_n defined in (3.8), (3.8)', (3.9), (3.9)'. We claim that

$$\beta_n > \alpha_n > \frac{1}{2}10^{-13}\beta_n \tag{6.9}$$

if the starting index N in the definition of α_n and β_n is larger than some $\mathrm{N}_0(c)$. Indeed, let N be so large that $\prod_{j=\mathrm{N}}^{\infty}\frac{1+c^{j/2}}{1-c^{j/2}} < 2$.

Since $\min\left(\frac{\beta_n}{\alpha_n},1\right) \leqq \frac{\beta_{n+1}}{\alpha_{n+1}} \leqq \max\left(\frac{1+c^{n/2}}{1-c^{n/2}}\cdot\frac{\beta_n}{\alpha_n},10^{13}\right)$ simple induction yields that $\beta_n > \alpha_n > \prod_{j=\mathrm{N}}^{n-1}\left(\frac{1-c^{j/2}}{1+c^{j/2}}\right)\cdot 10^{-13}\beta_n$ for all $n \geq \mathrm{N}$. This implies (6.9). Under the conditions of Proposition 3

$$\beta_n > \frac{5}{\mathrm{M}_n^2},\quad \beta_n < 1,\quad \frac{1}{\sqrt{\beta_{n+1}}} < \frac{\sqrt{2}}{c}\frac{1}{\sqrt{\beta_n}},\quad \text{and}\quad \beta_{n+1} < 3\beta_n. \tag{6.10}$$

The last relation follows from the following estimates: $\beta_n > \frac{10}{\mathrm{M}_{n-1}^2} > \frac{5}{\mathrm{M}_n^2}$ and $\beta_{n+1} \leq \beta_n + \frac{10}{\mathrm{M}_n^2}$, hence $\beta_{n+1} \leq 3\beta_n$. Similarly

$$\alpha_{n+1} < 3\alpha_n. \tag{6.10'}$$

Now we prove that Properties I(n), J(n) and the additional conditions on M_n, β_n and n in Proposition 3 imply Properties I($n+1$) and J($n+1$). In the proof we shall apply Proposition 2 and Lemma 8 with $f(x) = f_n(x)$, $M = M_n$, $\beta = \beta_n$ and $\varepsilon = 1 - 2^{-1/4}$ and Lemma 9 with $f(x) = \phi_n(f_n)(x)$, $\beta = \beta_n$, $\alpha = \alpha_n$ and $M = M_n$. Observe that $\overline{M} = M_{n+1}$ in Proposition 2 with this choice. First we prove I($n+1$).

If $|x| > C_2\sqrt{\beta_n}\log C$, $x > -c^{n+1}M_{n+1}$, where $C_2 > 0$ will be appropriately chosen then by (3.16)

$$\left|\frac{d^j}{dx^j} f_{n+1}(x)\right| \leqq C_1C^2\beta_n^{-(j+1)/2} \exp\left(-\frac{2^{3/4}}{2\sqrt{\beta_n}}\left|\frac{x^2}{c^{n+1}M_{n+1}} + 2x\right|\right) \leqq$$
$$\leqq C\beta_n^{-(j+1)/2} \exp\left(-\frac{1}{\sqrt{\beta_{n+1}}}\left|\frac{x^2}{c^{n+1}M_{n+1}} + 2x\right|\right)$$
$$\text{for } j = 0, 1, 2, \quad x > -c^{n+1}M_{n+1},$$

since

$$\exp\left[\left(-\frac{2^{3/4}}{c\sqrt{\beta_n}} + \frac{1}{\sqrt{\beta_{n+1}}}\right)\left|2x + \frac{x^2}{c^{n+1}M_{n+1}}\right|\right] \leqq$$
$$\leqq \exp\left(-\frac{1}{10\sqrt{\beta_n}}\left|2x + \frac{x^2}{c^{n+1}M_{n+1}}\right|\right) < \frac{1}{CC_1}$$

if $x > -c^{n+1}M_{n+1}$ and $|x| > C_2\sqrt{\beta_n}\log C$ with a sufficiently large absolute constant C_2.

If $|x| < C_2\sqrt{\beta_n}\log C$ and $n > n(C, c)$ part *a*) of Lemma 9 and (3.18) imply that

$$\left|\frac{d^j}{dx^j} f_{n+1}(x)\right| \leqq \left|\frac{d^j}{dx^j} T_{M_n}\phi_n(f_n)(x)\right| + \left|\frac{d^j}{dx^j}\left[Q_{n,M_n}f_n(x) - T_{M_n}\phi_n(f_n)(x)\right]\right| \leqq$$
$$\leqq C_1\alpha_n^{-(j+1)/2}\exp\left(-\frac{4|x|}{c\sqrt{\beta_n}}\right) +$$
$$+ c^{-n}C_1C^4\beta_n^{-(j+1)/2}\left[\exp\left(-\frac{2^{3/4}}{\sqrt{\beta_n}c}\left|2x + \frac{x^2}{c^{n+1}M_{n+1}}\right|\right) + \right.$$
$$\left. + \exp\left(-\frac{2.2^{3/4}}{\sqrt{\beta_n}c}|x|\right)\right]. \tag{6.11}$$

If n is so large that $c^{-n}C_1C^4 < \frac{C}{8}$ and for

$$|x| < C_2\sqrt{\beta_n}\log C \quad \frac{1}{\sqrt{\beta_n}}\frac{x^2}{c^{n+1}M_{n+1}} < \frac{1}{c^{n+1}}C_2^2(\log C)^2 < 1/2$$

(this allows to replace $-\dfrac{4|x|}{c\sqrt{\beta_n}}$ and $-\dfrac{2.2^{3/4}}{c\sqrt{\beta_n}}|x|$ by

$$-\frac{1}{\sqrt{\beta_{n+1}}}\left|2x+\frac{x^2}{c^{n+1}M_{n+1}}\right|+1$$

at the right hand side of (6.11)) then (6.11) implies that

$$\left|\frac{d^j}{dx^j}f_{n+1}(x)\right| \leqq \left(3C_1\alpha_n^{-(j+1)/2}+\frac{C}{2}\beta_n^{-(j+1)2}\right)\exp\left(-\frac{1}{\sqrt{\beta_{n+1}}}\left|2x+\frac{x^2}{c^{n+1}M_{n+1}}\right|\right) \quad (6.12)$$

for $j=0,1,2,|x|<C_2\sqrt{\beta_n}\log C$. If C is so large that $3C_1\alpha_n^{-(j+1)/2}<\frac{1}{2}C\beta_n^{-(j+1)/2}$, $j=0, 1, 2$, then relation (6.12) implies $I(n+1)$ for $|x|<C_2\sqrt{\beta_n}\log C$. Such a choice of C is possible since C_1 is an absolute constant, and relation (6.9) holds.

Now we prove $J(n+1)$. First we consider the case $|t|>\dfrac{C_3C}{\sqrt{\beta_n}}$ with a sufficiently large absolute constant C_3. Since $\dfrac{2}{\sqrt{\beta_{n+1}}}<\dfrac{\sqrt{2}}{c}\dfrac{2}{\sqrt{\beta_n}}\sqrt[6]{2}<\dfrac{4}{c\sqrt{\beta_n}}$ we can apply Lemmas 8 and 9 for $|s|<\dfrac{2}{\sqrt{\beta_{n+1}}}$. Part *b*) and formula (6.2) from Part *a*) of Lemma 8 imply that

$$|\tilde{\varphi}_{n+1}(f_{n+1})(t+is)|\leqq\frac{C_1C^2}{\beta_n^2t^4}+\frac{C_1C^2}{\beta_nt^2}c^{-n}\leqq\frac{10^{-14}}{\beta_nt^2}$$

if n is so large that $C_1C^4c^{-n}<\frac{1}{2}10^{-14}$, and C_3 is so large that

$$\frac{C_1C^2}{\beta_nt^2}<\frac{C_1}{C_3^2}<\frac{1}{2}10^{-14}.$$

Let C_3 be so large that also the relation $\alpha_nt^2>\dfrac{1}{2.10^{13}}C_3^2C^2>\dfrac{1}{2.10^{13}}C_3^2>1$ holds for $|t|>C_3C\beta_n^{-1/2}$. (We may assume that $C>1$.) Then

$$|\tilde{\varphi}_n(f_n)(t+is)|\leqq\frac{1}{5\alpha_n^2t^2}\leqq\frac{1}{1+4\alpha_nt^2}\leqq\frac{\exp\beta_{n+1}s^2}{1+\alpha_{n+1}t^2}.$$

For $|t| < \dfrac{C_3 C}{\sqrt{\beta_n}}$, $|s| < \dfrac{2}{\sqrt{\beta_{n+1}}}$, $n > n(c, C)$ Part *b*) of Lemma 9 and (6.1) imply that

$$|\tilde{\phi}_{n+1}(f_{n+1})(t+is)| \leqq \frac{\exp\left[\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)s^2\right]}{1+\left(\frac{c^2}{2}\alpha_n + \frac{1}{9}\frac{10^{-11}}{M_n^2}\right)t^2} + C_1 C^4 c^{-n}(s^2+t^2)\beta_n. \quad (6.12)'$$

For $|t| < \dfrac{CC_3}{\sqrt{\beta_n}}$, $n > n(c, C)$ $1+\left(\dfrac{c^2}{2}\alpha_n + \dfrac{1}{9M_n^2}10^{-11}\right)t^2 \leqq 2C_3^2C^2$,

$$1+\left(\frac{c^2}{2}\alpha_n + \frac{1}{9M_n^2}10^{-11}\right)t^2 - \frac{c^2}{2}\alpha_n c^{-n/2}t^2 > 0,$$

hence

$$\frac{\exp\left\{\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)s^2\right\}}{1+\left(\frac{c^2}{2}\alpha_n + \frac{1}{9M_n^2}\cdot 10^{-11}\right)t^2 - \frac{c^2}{2}\alpha_n c^{-n/2}t^2} - \frac{\exp\left\{\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)s^2\right\}}{1+\left(\frac{c^2}{2}\alpha_n + \frac{1}{9M_n^2}\right)t^2 10^{-11}} \geqq \frac{\alpha_n c^{-n/2}t^2}{10C_3^4C^4} \quad (6.13)$$

and

$$\frac{\exp\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)s^2 + \frac{c^2}{2}\beta_n c^{-n/2}s^2}{1+\alpha_{n+1}t^2} - \frac{\exp\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)^2 s^2}{1+\alpha_{n+1}t^2} \geqq \frac{\beta_n c^{-n/2}s^2}{10C_3^4C^4}. \quad (6.13)'$$

Relations (6.13) and (6.13)′ imply that

$$\frac{\exp \beta_{n+1}s^2}{1+\alpha_{n+1}t^2} - \frac{\exp\left[\left(\frac{c^2}{2}\beta_n + \frac{10}{M_n^2}\right)s^2\right]}{1+\left(\frac{c^2}{2}\alpha_n + \frac{1}{9M_n^2}10^{-11}\right)t^2} \geqq \frac{1}{10C_3^4C^4}c^{-n/2}\beta_n(s^2+t^2) \quad (6.14)$$

Relations (6.12)′ and (6.14) imply $J(n+1)$ if $n > n(c, C)$. Relations (3.12), (3.13) and (3.14) are straightforward consequences of Proposition 2. Let us choose some $C > 0$ and $N_0 = N_0(c, C)$ in such a way that for $n \geq N_0$

and this C all conditions imposed on them during the proof of Properties I$(n+1)$ and J$(n+1)$ be satisfied. Let us choose some $N \geq N_0$ and $\overline{K}(N)$ in such a way that Proposition 1 holds for $n \leq N$ if $\hat{M}_0^2 > \overline{K}(N)$ and $0 < T < \frac{1}{10}$. Moreover, let $\hat{M}_0$ be so large that $|M_N - M_{N-1}| < 1, M_N > K(c) + \sum_{j=N}^{\infty} c^{-j}$, where $K(c)$ is the same number which appears in the formulation of Proposition 3, and $\beta_N = \left(\frac{c^2}{2}\right)^N > \max\left(\frac{9}{M_N^2}, 4^{-N}\right)$. Such a choice of $\hat{M}_0$ is possible because of relation (3.4). Properties I(N) and J(N) (with the above constants C and N) hold if we choose $\hat{M}_0$ in the above way, and in this case $M_N > K(c)$, $|M_N - M_{N-1}| < 1$, $100 > \beta_N > \max\left(\frac{9}{M_n^2}, 4^{-N}\right)$. Then, because of the above proof of I$(n+1)$ and J$(n+1)$ from I(n) and J(n) and relation (3.12), a simple induction yields that Properties I(n) and J(n) (with the parameters C and N) hold for all $n \geq N$, and also the relations $|M_{n+1} - M_n| < c^{-n} \leq 1 \quad M_{n+1} > \tilde{K}(n+1) > K(c)$ with $\tilde{K}(n) = \sum_{j=n}^{\infty} c^{-j} + K(c)$, $\beta_{n+1} > \max\left(\frac{9}{M_{n+1}^2}, 4^{-(n+1)}\right)$ are valid. Proposition 3 is proved.

The following two lemmas are consequences of Proposition 3.

LEMMA 10. — *If* $\hat{M}_0^2 = \frac{a_1(a_0 - T)}{Tu^2} > K, 0 < T < \frac{1}{10}$, *with some* K *then the limit* $\lim_{n\to\infty} M_n = M$ *exists and* $|\hat{M}_0 - M| < 1$. *For all* $\varepsilon > 0$ *there is some* $K(\varepsilon) > 0$ *such that if* $\hat{M}_0 > K(\varepsilon)$ *then* $|M - \hat{M}_0| < \varepsilon$. *Also the relations*

$$M_n = M + \frac{c}{4(c-1)M} c^{-n} + \delta(n) \cdot c^{-n}, \qquad |\delta(n)| \leqq \overline{K}(c) c^{-n} \quad (6.15)$$

and

$$\beta_n < \frac{1}{M^2 \mu^2} \qquad \textit{with} \qquad \mu = \frac{1}{6}\left(1 - \frac{c^2}{2}\right)^{1/2} \quad \textit{if} \quad n > n_0(\hat{M}_0, c) \quad (6.16)$$

hold with some appropriate $\overline{K}(c)$ *and* $n_0(\hat{M}_0, c)$.

(Here in the definition of β_n we fix some starting index N for which $\beta_N = \left(\frac{c^2}{2}\right)^N$.)

LEMMA 11. — *Under the conditions of Theorem 1 there is some* $n_0 = n_0(\hat{M}_0, c)$ *such that for* $n > n_0$

$$\left| \frac{d^j}{dx^j} f_n(x) \right| < CM^{j+1} \exp\left(-\frac{\mu M}{2} |x| \right) \quad \textit{if} \quad x > -c^n M_n, \qquad j = 0, 1, 2 \tag{6.17}$$

$$| \phi_{n+1}(f_{n+1})(x) - T_M \phi_n(f_n)(x) | \leqq c^{-n} CM \exp\left(-\frac{\mu M}{2} |x| \right) \quad \textit{for all} \quad x \in \mathbf{R}^1 \tag{6.18}$$

where C *is some absolute constant, and* μ *is the same as in Lemma 10.*

Proof of Lemma 10. — It follows from (3.12) that $\lim M_n = M$ exists, and if we choose properly the starting index N in Proposition 3 and also $\hat{M}_0$ is sufficiently large then $| M_N - \hat{M}_0 | < \frac{\varepsilon}{2}$ and $| M_N - \overline{M} | < \frac{\varepsilon}{2}$. Since $M_n > K$ for all n $|M_n - M| \leqq \sum_{j=n}^{\infty} | M_n - M_{n-1} | < Kc^{-n}$. Substituting this relation into (3.12) we get that

$$M_{n+1} - M_n = -\frac{1}{4c^n M} + r(n) \quad \text{with} \quad |r(n)| < Kc^{-n}.$$

Summing up this relation for all $j > n$ we get (6.15). To prove (6.16) let us introduce the auxiliary sequence $\overline{\beta}_n$, $\overline{\beta}_N = 100$, $\overline{\beta}_{n+1} = \frac{1}{2}\left(\frac{c^2}{2} + 1\right)\overline{\beta} + \frac{15}{M^2}$. Then $\overline{\beta}_n > \beta_n$ for all $n \geq N$. On the other hand if $\overline{\beta}$ is the solution of the fixed point equation $\overline{\beta} = \frac{1}{2}\left(\frac{c^2}{2} + 1\right)\overline{\beta} + \frac{15}{M^2}$ then $\beta_n \to \overline{\beta} = \frac{30}{M^2}\left(1 - \frac{c^2}{2}\right)^{-1/2}$. Hence for $n > n_0(\hat{M}_0, c)$ $\beta_n < \overline{\beta}_n < \frac{6}{5}\overline{\beta}$, i. e. relation (6.16) holds.

Proof of Lemma 11. — Relation (6.17) immediately follows from (3.10), (6.15) and the inequality $\beta_n > \frac{10}{M_n^2} > \frac{9}{M^2}$.

We claim that

$$| T_M \phi_n(f_n)(x) - T_{M_n} \phi_n(f_n)(x) | \leqq c^{-n} CM \exp\left(-\frac{\mu M}{2} |x| \right) \tag{6.19}$$

and

$$| \phi_{n+1}(x) f_{n+1}(x) - \phi_{n+1}(f_{n+1})(x) | \leqq c^{-n} CM \exp\left(-\frac{\mu M}{2} |x| \right), \tag{6.20}$$

for all x.

Relation (6.18) follows from (6.19), (6.20), (6.16), the inequality $\beta > \frac{9}{M^2}$ and relation (3.18) for $|x| < 100c^{n/2}$ and relations (3.16), (3.17) for $|x| > 100c^{n/2}$.

For $|x| > 50c^{n/2}$ relations (6.19) and (6.20) follow from the bounds given on the terms $T_M\phi_n(f_n)(x)$, $T_{M_n}\phi_n(f_n)(x)$, $\phi_{n+1}(f_{n+1})(x)$ and $\phi_{n+1}(x)\cdot f_{n+1}(x)$ in Proposition 3, Lemmas 6 and 10.

To prove (6.19) for $|x| < 50c^{n/2}$ observe that $T_M f_n(x) = T_{M_n} g_n\left(\frac{M}{M_n}x\right)$ with $g_n(x) = \sqrt{\frac{M_n}{M}}\, f_n\left(\frac{M_n}{M}x\right)$, and $\left|\frac{M_n}{M} - 1\right| \leqq \frac{2}{M^2}c^{-n}$. Since

$$|g_n(x) - f_n(x)| \leqq \left|\sqrt{\frac{M_n}{M}} - 1\right| \left|f_n\left(\frac{M_n}{M}x\right)\right| + \left|f_n\left(\frac{M_n}{M}x\right) - f_n(x)\right| \leqq c^{-n}MC\exp\left(-\frac{\mu}{2}|x|\right)$$

and

$$\left|T_{M_n}g_n\left(\frac{M}{M_n}x\right) - T_{M_n}g_n(x)\right| \leqq \left|\frac{M}{M_n} - 1\right| |x| \sup_{x<y<\frac{M}{M_n}x} \left|\frac{d}{dx}T_{M_n}g_n(y)\right|$$

relation (6.19) can be proved with the help of Lemma 6.

To prove (6.20) observe that

$$|A_{n+1} - 1| = \left|\int_{-c^{n+1}M_{n+1}}^{\infty} [1 - \phi_{n+1}(x)]\cdot f_{n+1}(x)dx\right| \leqq c^{-n} \tag{7.1}$$

and

$$B_{n+1} = \frac{1}{A_{n+1}}\left|\int_{-c^{n+1}M_{n+1}}^{\infty} x\phi_{n+1}(x)f_{n+1}(x)dx\right| \leqq c^{-n}\frac{1}{M_{n+1}}.$$

Then we get relation (6.20) by estimating the expressions

$$f_{n+1}(x + B_{n+1}) - f_{n+1}(x) \quad \text{and} \quad (A_{n+1} - 1)f_{n+1}(x)$$

with the help of Proposition 3 and Lemma 10.

Let us finally remark that in the same way as we deduced relation (6.17) we get that

$$\left|\frac{d^j}{dx^j}\phi_n(f_n)(x)\right| \leqq CM^{j+1}\exp\left(-\frac{\mu M}{2}|x|\right), \quad j = 0, 1, 2, \quad x \in R^1 \tag{6.21}$$

7. ON THE FIXED POINT EQUATION $T_M g = g$

By formulas (2.16) and (2.17) the operator $T_M g$ can be written in the form

$$T_M g(x) = \frac{2}{c\sqrt{\pi}}\int e^{-v^2} g\left(\frac{x}{c} - \frac{1}{4M} + u + \frac{v^2}{2M}\right) g\left(\frac{x}{c} - \frac{1}{4M} - u - \frac{v^2}{2M}\right) du dv, \quad g \in \mathscr{A} \tag{7.1}$$

where the class of functions $\mathscr{A}$ is defined in (2.19). We prove the following.

LEMMA 12. — *The fixed point equation* $T_M g = g$ *has a unique solution in the class* $\mathscr{A}$ *beside the trivial one* $g(x) \equiv 0$. *It can be written in the form* $g_M(x) = Mg_1(Mx)$, *where* $g_1(x)$ *is the solution of the equation* $T_1 g_1 = g_1$. *The function* $g_1(x)$ *is the density function of the random variable*

$$\zeta = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \left(\frac{c}{2}\right)^{k+1} (1 - \eta_{j,k}^2), \tag{7.2}$$

where $\eta_{j,k}$, $k = 0, 1, 2, \ldots, j = 1, 2, \ldots, 2^k$ *are independent standard normal random variables. The solution of the equation (1.6) is* $g_1\left(x - \dfrac{c}{4(c-1)}\right)$.

Remark. — In the case of p dimensional models, $p \geq 2$, $g_1(x)$ is the density function of the random variable

$$\zeta = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \sum_{l=1}^{p-1} \left(\frac{c}{2}\right)^{k+1} (1 - \eta_{j,k,l}^2),$$

where $\eta_{j,k,l}$ are independent standard normal variables.

Proof of Lemma 12. — Relation (2.20)′ tells us that

$$\tilde{T}_M \tilde{g}(\xi) = \tilde{g}\left(\frac{c}{2}\xi\right)^2 \frac{\exp\left(i\dfrac{c}{4M}\xi\right)}{\sqrt{1 + i\dfrac{c}{2M}\xi}} \quad \text{if} \quad g \in \mathscr{A} \tag{7.3}$$

and this relation also holds if $\xi = t + is$, $|s| < s(f)$. If $T_M g_M = g_M$ then relation (7.3) with $\xi = 0$ implies that $\tilde{g}_M(0) = \tilde{g}_M(0)^2$, hence either $\tilde{g}_M(0) = 1$ or $\tilde{g}_M(0) = 0$. If $\tilde{g}_M(0) = 0$ then successive differentiation of the equation $\tilde{T}_M \tilde{g}_M(\xi) = \tilde{g}_M(\xi)$ yields that $\left.\dfrac{d^n}{dt^n} \tilde{g}_M(t)\right|_{t=0} = 0$ for all $n = 0, 1, 2, \ldots$, therefore $\tilde{g}_M(\xi) \equiv 0$, i. e. in this case we get the trivial solution $g_M(x) \equiv 0$.

Iterating the equation (7.3) we get that

$$\tilde{T}_M^k \tilde{g}(\xi) = \tilde{g}\left(\left(\frac{c}{2}\right)^k \xi\right)^{2^k} \prod_{j=1}^{k} \exp\left[\frac{(i/2M)(c/2)^j \xi}{\sqrt{1 + \dfrac{i}{M}\left(\dfrac{c}{2}\right)^j \xi}}\right]^{2^{j-1}} \quad \text{for all } k \geq 1. \tag{7.4}$$

If $g_M = T_M g_M$, $g_M \in \mathscr{A}$, and $\tilde{g}_M(0) = 1$ then we get differentiating (7.3) that $\tilde{g}'_M(0) = c\tilde{g}'_M(0)$, hence $\tilde{g}'_M(0) = 0$. On the other hand

$$|\tilde{g}''_M(\xi)| \leqq \int x^2 e^{sx} |g_M(x)| \, dx \leq A < \infty$$

with some appropriate $A > 0$ (which may depend on g_M) if

$$|\operatorname{Im} \xi| = |s| < \frac{s(g_M)}{2}.$$

Hence $\exp(-A|\xi|^2) \leq \tilde{g}_M(\xi) \leq \exp(A\xi^2)$ if $|\xi| < \varepsilon$, and $\tilde{g}_M\left(\left(\frac{c}{2}\right)^k \xi\right)^{2^k} \to 1$ if $k \to \infty$ for arbitrary ξ. $\left(\text{Observe that } \frac{c^2}{2} < 1.\right)$ Since $\tilde{g}_M(\xi) = \lim_{k\to\infty} \tilde{T}_M^k \tilde{g}_M(\xi)$ the above relation together with (7.4) imply that $\tilde{g}_M$ must be of the form

$$\tilde{g}_M(\xi) = \prod_{j=1}^{\infty} \exp\left[\frac{(i/2M)\left(\frac{c}{2}\right)^j \xi}{\sqrt{1 + \frac{i}{M}\left(\frac{c}{2}\right)^j \xi}}\right]^{2^{j-1}}. \tag{7.5}$$

Since $\dfrac{\exp\left(i\frac{1}{2M}\left(\frac{c}{2}\right)^j \xi\right)}{\sqrt{1 + \frac{i}{M}\left(\frac{c}{2}\right)^j \xi}}$ is the characteristic function of the random variable $\frac{1}{2M}\left(\frac{c}{2}\right)^j (1 - \eta^2)$, where η is a standard normal random variable, hence the function $\tilde{g}_M$ is the characteristic function of the random variable $\frac{1}{M}\zeta$, where ζ is defined in (7.2). If $g_M(x)$ denotes the density function of the random variable $\frac{1}{M}\zeta$ then $g_M \in \mathscr{A}$, and its Fourier transform is given in (7.5). Since this function $\tilde{g}_M$ satisfies the relation $\tilde{g}_M = \tilde{T}_M \tilde{g}_M$, as a simple calculation shows with the help of (7.3), the function g_M is the solution of the fixed point equation $g = T_M g$. The rest of Lemma 12 follows from simple calculation.

In the next lemma we prove the properties of the function g_M important for us

LEMMA 13. — *For any* $1 > \varepsilon > 0$ *the function* $g_1(x)$ *defined in Lemma 12 satisfies the relations*

a) $|\tilde{g}_1(t+is)| \leq \frac{C_j(\varepsilon)}{1+t^j}$ *for* $|s| < (1-\varepsilon)\cdot\frac{2}{c}$ *and arbitrary* $t, j = 1, 2, \ldots$

b 1) $\left|\frac{d^j}{dx^j} g_1(x)\right| < C_j(\varepsilon) \exp\left[-(1-\varepsilon)\frac{2}{c}|x|\right]$ *for arbitrary* x *and* $j = 0, 1, \ldots$

b 2) $\left|\frac{d^j}{dx^j} g_1(x)\right| \leq C_j \exp(-Ax^\alpha)$ *with* $\alpha = \frac{\log 2}{\log c}$ *for* $x > 0, j = 0, 1, \ldots$

with some appropriate positive constants $C_j(\varepsilon)$, C_j *and* A *which may depend on the parameter c.*

Proof of Lemma 13. — Proof of Part *a*). Relation (7.5) with $M = 1$ implies that

$$|\tilde{g}_1(t+is)| = \exp\left\{-\sum_{k=1}^{\infty} 2^{k-2}\left(\left(\frac{c}{2}\right)^k s + \frac{1}{2}\log\left[\left(1-\left(\frac{c}{2}\right)^k s\right)^2 + \left(\frac{c}{2}\right)^{2k} t^2\right]\right)\right\} =$$

$$= \exp\left\{-\sum_{k=1}^{\infty} 2^{k-2}\left[\left(\frac{c}{2}\right)^k s + \log\left(1-\left(\frac{c}{2}\right)^k s\right)\right]\right\}$$

$$\exp\left\{-\sum_{k=1}^{\infty} 2^{k-3}\log\left[1+\frac{\left(\frac{c}{2}\right)^{2k} t^2}{\left(1-\left(\frac{c}{2}\right)^k s\right)^2}\right]\right\} = I_1(s)\cdot I_2(s,t). \tag{7.6}$$

For $|s| < (1-\varepsilon)\ \frac{2}{c}\left|\left(\frac{c}{2}\right)^k s\right| < 1-\varepsilon$ for all $k = 1, 2, \ldots$, hence the relation $|x - \log(1-x)| \leqq \frac{x^2}{1-|x|}$ for $|x| < 1$ implies that

$$I_1(s) \leqq \exp\left\{-\frac{1}{\varepsilon}\sum_{k=1}^{\infty}\frac{1}{4}\left(\frac{c^2}{2}\right)^k s^2\right\} < \exp\left(-\frac{1}{\varepsilon(2-c^2)}\right) \text{ if } |s| < \frac{2}{c}(1-\varepsilon). \tag{7.7}$$

On the other hand, since each term in the sum in the expression $I_2(s, t)$ is non-negative

$$I_2(s, t) \leq 1 \tag{7.8}$$

and for any $k = 1, 2, \ldots$

$$I_2(s,t) \leqq \exp\left\{-2^{k-3}\log\left[1+\frac{\left(\frac{c}{2}\right)^{2k} t^2}{\left(1-\left(\frac{c}{2}\right)^k s\right)^2}\right]\right\} \leqq$$

$$\leqq \exp\left\{-2^{k-3}\log\left[t^2\frac{\left(\frac{c}{2}\right)^{2k}}{\left[1-\left(\frac{c}{2}\right)^k s\right]^2}\right]\right\} \leqq |t|^{-2^{k-2}} A_k(s) \tag{7.8'}$$

with $A_k(s) = \left\{\left(\frac{2}{c}\right)^{2k}\left[1-\left(\frac{c}{2}\right)^k s\right]^2\right\}^{2^{k-3}}$.

Since $A_k(s) < A_k$ with some A_k for $|s| < \frac{2}{c}(1-\varepsilon)$ relations (7.6), (7.7), (7.8) and (7.8)' imply Part *a*).

Since

$$\frac{d^j}{dx^j} g_1(x) = \frac{1}{2\pi} \int (-it+s)^j \exp\{-i(t+is)x\} \tilde{g}_1(t+is)dt$$

Part *a*) gives with the choice $s = -(1-\varepsilon)\frac{2}{c}\,\mathrm{sign}\, x$ that

$$\left|\frac{d^j}{dx^j} g_1(x)\right| \leqq \frac{1}{2\pi} C_{j+2}(\varepsilon) \exp\left[-(1-\varepsilon)\frac{2}{c}|x|\right] \int \left(|t|+\frac{2}{c}\right)^j \frac{1}{1+|t|^{j+2}} dt \leqq$$
$$\leqq C'(j) \exp\left[-(1-\varepsilon)\frac{2}{c}|x|\right].$$

To prove *b* 2) first we give a better estimate on $I_1(s)$ for $s < 0$. Take some integer $L > 0$ to be defined later. Since $x - \log(1+x) < \frac{x^2}{2}$ and $x - \log(1+x) < x$ we can write for $s < 0$

$$I_1(s) = \exp\left\{\sum_{k=1}^{L} 2^{k-2}\left[\left(\frac{c}{2}\right)^k |s| - \log\left(1+\left(\frac{c}{2}\right)^k |s|\right)\right]\right.$$
$$\left. + \sum_{k=L+1}^{\infty} 2^{k-2}\left[\left(\frac{c}{2}\right)^k |s| - \log\left(1+\left(\frac{c}{2}\right)^k |s|\right)\right]\right\} \leqq$$
$$\leqq \exp\left\{\frac{1}{4}\left[\frac{c^{L+1}-c}{c-1}|s| + \frac{1}{1-\frac{c^2}{2}}\left(\frac{c^2}{2}\right)^{L+1} s^2\right]\right\} \leqq$$
$$\leqq \exp\left\{\mathrm{const.}\left\{c^L |s| + \left(\frac{c^2}{2}\right)^L s^2\right\}\right\}. \tag{7.9}$$

Choose L in such a way that $\left(\frac{c}{2}\right)^L |s| \geq 1$ and $\left(\frac{c}{2}\right)^{L+1} |s| < 1$. (Let $L = 1$ if $\frac{c}{2}|s| < 1$.) Then relation (7.9) with this L implies that $I_1(s) < \exp(A|s|^{\bar{\alpha}})$ for $s < 0$ with some $A > 0$ and $\bar{\alpha} = \frac{\log 2}{\log 2 - \log c}$.

Relations (7.8), (7.8)' and the inequality $A_k(s) < A_k$ hold for all $s < 0$, hence $I_2(s,t) \leqq \frac{C_k}{1+|t|^k}$. The estimates given for I_1 and I_2 together with (7.6) yield that $|\tilde{g}(t+is)| \leqq C(j)\frac{\exp(A|s|^{\bar{\alpha}})}{1+|t|^j}$ for $s < 0$ and arbitrary $j \geq 0$.

Hence, by applying the inverse Fourier transformation, we get that for arbitrary $s < 0$

$$\left| \frac{d^j}{dx^j} g_1(x) \right| \leqq \int (|t|+|s|)^j e^{sx} |\tilde{g}(t+is)| \, dt \leqq$$
$$\leqq C(j+2) \exp(sx + A|s|^{\bar{\alpha}}) \cdot \int \frac{(|t|+|s|)^j}{1+|t|^{j+2}} dt . \quad (7.10)$$

Given some $x > 0$ choose $s = -\left(\frac{x}{\bar{\alpha}A}\right)^{\frac{1}{\bar{\alpha}-1}}$. Then

$$\exp(sx + A|s|^{\bar{\alpha}}) = \exp\left\{ -\left(1 - \frac{1}{\bar{\alpha}}\right)(\bar{\alpha}A)^{-\frac{1}{\bar{\alpha}-1}} x^{\frac{\bar{\alpha}}{\bar{\alpha}-1}} \right\},$$

and $C(j+2) \int \frac{(|t|+|s|)^j}{1+|t|^{j+2}} dt \leqq C(j)(1+|s|^j) \leqq C(j,\varepsilon) \exp\{ \varepsilon x^{\frac{\bar{\alpha}}{\bar{\alpha}-1}} \}$ for arbitrary $\varepsilon > 0$. Since $1 - \frac{1}{\bar{\alpha}} > 0$ the above relations together with (7.10) imply that

$$\left| \frac{d^j}{dx^j} g_1(x) \right| \leqq C_j \exp\left(-\bar{A} x^{\frac{\bar{\alpha}}{\bar{\alpha}-1}}\right) = C \exp(-\bar{A} x^{\alpha}).$$

Lemma 13 is proved.

8. THE PROOF OF THEOREMS 1 AND 1′

The following lemma can be considered as a rigorous version of the heuristic argument at the end of Section 2.

LEMMA 14. — *If $\hat{M}_0$ is sufficiently large, $0 < T < \frac{1}{10}$, then there exist some threshold $n_0 = n_0(\hat{M}_0)$ and constant $L = L(\hat{M}_0, c)$ such that*

$$|\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| < L\left[c^{-n} + \left(\frac{c^2}{2}\right)^n\right] n \quad \text{if} \quad n > n_0$$

and

$$|t| < \frac{M\mu^{3/2}}{4\sqrt{C}} = A_0, \qquad t \in R^1,$$

where C, $C > 0$ is an absolute constant, and μ, M and the threshold n_0 are the same as in Lemmas 10 and 11

Proof of Lemma 14. — First we show that for $n \geq n_0$

$$\left| \frac{d^2}{dt^2} \log \tilde{\phi}_n(f_n)(t) \right| \leqq \frac{20C}{M^2\mu^3} \quad \text{if} \quad |t| < A_0 \qquad (8.1)$$

and

$$\left| \frac{d^2}{dt^2} \log \tilde{\phi}_{n+1}(f_{n+1})(t) - \frac{d^2}{dt^2} \log \tilde{T}_M \tilde{\phi}_n(f_n)(t) \right| < \frac{500C}{M^2\mu^3} c^{-n} \quad \text{if} \quad |t| < A_0. \tag{8.2}$$

By relation (6.21)

$$\left| \frac{d^2}{dt^2} \tilde{\phi}_n(f_n)(t) \right| \leqq \int x^2 \phi_n(f_n)(x) dx \leqq CM \int x^2 \exp\left(- \frac{\mu M}{2} |x| \right) dx = \frac{8C}{M^2\mu^3}, \tag{8.3}$$

and since $\frac{d}{dt} \tilde{\phi}_n(f_n)(0) = 0$, $\tilde{\phi}_n(f_n)(0) = 1$

$$\left| \frac{d}{dt} \tilde{\phi}_n(f_n)(t) \right| \leqq \frac{8C|t|}{M^2\mu^3} \leqq \frac{2\sqrt{C}}{M} \mu^{-3/2} \quad \text{if} \quad |t| < A_0 \tag{8.3$'$}$$

and

$$|1 - \tilde{\phi}_n(f_n)(t)| \leqq \frac{4Ct^2}{M^2\mu^3} \leqq \frac{1}{4} \quad \text{if} \quad |t| < A_0 \tag{8.3$''$}$$

Hence $\log \tilde{\phi}_n(f_n(t)$ exists for $|t| < A_0$, and since

$$\frac{d^2}{dt^2} \log \tilde{\phi}_n(f_n)(t) = \frac{\tilde{\phi}_n(f_n)(t) \frac{d^2}{dt^2} \tilde{\phi}_n(f_n)(t) - \left[\frac{d}{dt} \tilde{\phi}_n(f_n)(t) \right]^2}{\tilde{\phi}_n(f_n)(t)^2} \tag{8.4}$$

the estimates (8.3)-(8.3)″ imply (8.1).

Similarly, we get with the help of (6.18) and the relations

$$\frac{d}{dt} [\tilde{\phi}_{n+1}(f_{n+1})(0) - \tilde{T}_M \tilde{\phi}_n(f_n)(0)] = \tilde{\phi}_{n+1}(f_{n+1})(0) - \tilde{T}_M \tilde{\phi}_n(f_n)(0) = 0$$

that

$$\left| \frac{d^2}{dt^2} [\tilde{\phi}_{n+1}(f_{n+1})(t) - \tilde{T}_M \tilde{\phi}_n(f_n)(t)] \right| \leqq \frac{8C}{M^2\mu^3} c^{-n} \tag{8.5}$$

$$\left| \frac{d}{dt} [\tilde{\phi}_{n+1}(f_{n+1})(t) - \tilde{T}_M \tilde{\phi}_n(f_n)(t)] \right| \leqq \frac{2\sqrt{C}}{M\mu^{3/2}} c^{-n}, \quad \text{if} \quad t \leq A_0 \tag{8.5$'$}$$

$$|\tilde{\phi}_{n+1}(f_{n+1})(t) - \tilde{T}_M \tilde{\phi}_n(f_n)(t)| \leqq \frac{1}{4} c^{-n}, \quad \text{if} \quad |t| < A_0. \tag{8.5$''$}$$

By expressing $\frac{d^2}{dt^2} \log \tilde{T}_M \tilde{\phi}_n(f_n)(t)$ similarly to (8.4) and applying (8.3)-(8.3)″ (for the function $\tilde{\phi}_{n+1}(f_{n+1})(t)$) and (8.5)-(8.5)″ we get (8.2).

We claim that

$$\left| \frac{d^2}{dt^2} \log \tilde{g}_M(t) \right| \leqq \frac{9}{M^2\mu^2}. \tag{8.6}$$

Indeed, by (7.5)

$$\frac{d^2}{dt^2}\log \tilde{g}_M(t) = -\frac{1}{4M^2}\sum_{j=1}^{\infty}\left(\frac{c^2}{2}\right)^j\left[1 + i\left(\frac{c}{2}\right)^j\frac{t}{M}\right]^{-2}$$

and

$$\left|\frac{d^2}{dt^2}\log \tilde{g}_M(t)\right| \leqq \frac{1}{4M^2}\sum_{j=1}^{\infty}\left(\frac{c^2}{2}\right)^j\left[1 + \left(\frac{c}{2}\right)^{2j}\frac{t^2}{M^2}\right]^{-1} \leqq \frac{1}{4M^2}\sum_{j=1}^{\infty}\left(\frac{c^2}{2}\right)^j \leqq \frac{1}{4M^2}\cdot\frac{1}{1-\frac{c^2}{2}}$$

as we claimed. Since $\tilde{T}_M\tilde{g}_M = \tilde{g}_M$ relation (7.3) yields that

$$\frac{d^2}{dt^2}\log \tilde{T}_M\tilde{\phi}_n(f_n)(t) - \frac{d^2}{dt^2}\log \tilde{g}_M(t) = \frac{d^2}{dt^2}\{\log \tilde{T}_M\phi_n(f_n)(t) - \log \tilde{T}_M\tilde{g}_M(t)\} =$$

$$= \frac{c^2}{2}\left\{\left[\log \tilde{\phi}_n(f_n)\left(\frac{c}{2}t\right)\right]'' - \left[\log \tilde{g}_M\left(\frac{c}{2}t\right)\right]''\right\} \quad \text{if} \quad |t| < \frac{M\mu^{3/2}}{4\sqrt{C}}. \tag{8.7}$$

Put $\delta_n = \sup\limits_{|t|<A_0}\left|\frac{d^2}{dt^2}\log \tilde{\phi}_n(f_n)(t) - \frac{d^2}{dt^2}\log \tilde{g}_M(t)\right|$. By relations (8.1) and (8.6)

$$\delta_{n_0} \leqq \frac{30C}{M^2\mu^3}. \tag{8.8}$$

and by (8.2) and (8.7)

$$\delta_{n+1} \leqq \frac{c^2}{2}\delta_n + \frac{500}{M^2\mu^3}c^{-n}. \tag{8.8$'$}$$

The quantity δ_n can be estimated with the help of (8.8) and the recursive estimate (8.8)′, and we get

$$\delta_n \leqq \frac{L}{M^2}\left[\left(\frac{c^2}{2}\right)^n + c^{-n}\right]n \qquad \text{for} \qquad n \geq n_0$$

with some $L = L(c)$. (The multiplying factor n appears only in the case $\frac{c^2}{2} = \frac{1}{c}$.) Since $\frac{d}{dt}\log \tilde{\phi}_n(f_n)(0) = \log \tilde{\phi}_n(f_n)(0) = \frac{d}{dt}\log \tilde{\phi}_M(0) = \log \tilde{g}_M(0) = 0$ this inequality implies that

$$|\log \tilde{\phi}_n(f_n)(t) - \log \tilde{g}_M(t)| \leqq \frac{Lt^2}{2M^2}\left[\left(\frac{c^2}{2}\right)^n + c^{-n}\right]n \leqq L'\left[\left(\frac{c^2}{2}\right)^n + c^{-n}\right]n$$

for $t < A_0$. As $|\tilde{\phi}_n(f_n)(t)| \leq 1$ and $|\tilde{g}_M(t)| \leq 1$ for all t and

$$|e^A - e^B| \leqq |A - B| \max\{e^{|A|}, e^{|B|}\}$$

for all $A, B \in \mathbf{C}$ the last relation implies Lemma 14.

In the next lemma we enlarge the domain where we can give the same estimate as in Lemma 14.

LEMMA 15. — *If* $\hat{M}_0$ *is sufficiently large,* $0 < T < \frac{1}{10}$, *then there exist some* $n_0 = n_0(\hat{M}_0, c)$, $\eta = \eta(\hat{M}_0, c)$ *and* $L = L(\hat{M}_0, c)$ *constants such that for all* $n > n_0$

$$|\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| < Ln\left[c^{-n} + \left(\frac{c^2}{2}\right)^n\right] \quad \text{if} \quad |t| < \eta \cdot \left(\frac{2}{c}\right)^n, \quad t \in R^1. \tag{8.9}$$

Proof of Lemma 15. — It follows from (6.18) that

$$|\tilde{\phi}_{n+1}(f_{n+1})(t) - \tilde{T}_M\tilde{\phi}_n(f_n)(t)| \leqq$$
$$\leqq \int |\phi_{n+1}(f_{n+1})(x) - T_M\phi_n(f_n)(x)|\,dx < \frac{2C}{\mu}c^{-n} \tag{8.10}$$

for $n > n_0$ and all t. On the other hand by (7.3) and the relation $\tilde{T}_M\tilde{g}_M = \tilde{g}_M$

$$|\tilde{T}_M\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| = \frac{\left|\tilde{\phi}_n(f_n)\left(\frac{c}{2}t\right)^2 - \tilde{g}_M\left(\frac{c}{2}t\right)^2\right|}{\left|\sqrt{1 + i\frac{c}{2M}t}\right|} \leqq$$

$$\leqq 2\left(1 + \frac{c^2t^2}{4M^2}\right)^{-1/2}\left|\tilde{\phi}_n(f_n)\left(\frac{c}{2}t\right) - \tilde{g}_M\left(\frac{c}{2}t\right)\right|. \tag{8.11}$$

Define the sets $I_k = \left\{t, \left(\frac{2}{c}\right)^{k-1}A_0 < |t| < \left(\frac{2}{c}\right)^k A_0\right\}$, $k = 1, 2, \ldots$ and $I_0 = \{t, |t| < A_0\}$, where A_0 is the same as in Lemma 14, and put

$$\delta_{k,n} = \sup_{t \in I_k} |\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)|.$$

Then

$$\delta_{0,n} \leq Ln\left[c^{-n} + \left(\frac{c^2}{2}\right)^n\right] \quad \text{for} \quad n \geq n_0$$

by Lemma 14, and relations (8.10) and (8.11) imply that

$$\delta_{k+1,n+1} \leqq \frac{2}{\sqrt{1 + A_1(2/c)^{2k}}}\delta_{k,n} + \frac{2C}{\mu}c^{-n}, \qquad n \geq n_0. \tag{8.12}$$

with $A_1 = \frac{c^2}{4M^2}A_0^2 = \frac{c^2\mu^3}{64C}$.

Define $B_{j,k} = \frac{2C}{\mu} \prod_{l=j}^{k} \frac{2}{\sqrt{1 + A_1\left(\frac{2}{c}\right)^{2l}}}$ for all $0 \leq j \leq k$. Since there is some l_0 such that $2\left[1 + A_1\left(\frac{2}{c}\right)^{2l}\right]^{-1/2} \leqq \frac{1}{2}$ if $l \geq l_0$, the inequality $B_{j,k} \leqq K \cdot 2^{-(k-j)}$ holds for any $j, k \geq 0$ with some appropriate $K = K(c)$. Hence relation (8.12) implies that

$$\delta_{k,n+k} \leqq \delta_{0,n} \cdot B_{1,k} + \sum_{j=1}^{k} c^{-(n+j)} B_{j,k} \leqq K'(2^{-k}\delta_{0,n} + c^{-(n+k)}).$$

The last inequality together with (8.11) imply that

$$\delta_{k,n+k} \leqq L(n+k)\left[\left(\frac{c^2}{2}\right)^{n+k} + c^{-(n+k)}\right] \quad \text{if} \quad n > n_0, \qquad k \geq 0.$$

Observe that the right hand side in the last relation depends on n and k only through $n + k$. It implies Lemma 15 with $\eta = \left(\frac{c}{2}\right)^{n_0} A_0$, where A_0 is the same as in Lemma 14.

The proof of Theorem 1'. — It follows from Lemma 15 and (7.3) that for $n > n_0$

$$\begin{aligned} |\tilde{T}_M \tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| &\leqq 2\, |\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)| \leqq \\ &\leqq 2Ln\left[c^{-n} + \left(\frac{c^2}{2}\right)^n\right] \quad \text{if} \quad |t| < \eta\left(\frac{2}{c}\right)^n \end{aligned} \tag{8.13}$$

On the other hand we get by applying (3.17) in Proposition 2 with $f = f_n$, $\beta = \frac{4}{M^2\mu^2}$ (this can be done, see (6.17)) the inequality

$$\left|\frac{d^4}{dx^4} T_M \phi_n(f_n)(x)\right| < C^2 M^5 \exp\left(-\frac{\mu M}{2}|x|\right).$$

Hence integration by parts yields that

$$|\tilde{T}_M \tilde{\phi}_n(f_n)(t)| \leqq \frac{1}{t^4} \int \left|\frac{d^4}{dx^4} T_M \phi_n(f_n)(x)\right| dx \leqq \frac{\overline{C}}{t^4}. \tag{8.13$'$}$$

By part *a*) of Lemma 13 and the relation $\tilde{g}_M(t) = \tilde{g}_1\left(\frac{t}{M}\right)$ also

$$|\tilde{g}_M(t)| < \overline{C} t^{-4}. \tag{8.13$''$}$$

Relations (8.13)-(8.13)'' imply that we have with arbitrary $\eta\left(\frac{2}{c}\right)^n > B > 0$

$$\left|\frac{d^j}{dx^j}[T_M\phi_n(f_n)(x) - g_M(x)]\right| \leqq \int |t|^j |\tilde{T}_M\tilde{\phi}_n(f_n)(t) - \tilde{g}_M(t)|\, dt =$$

$$= \int_{|t|<B} + \int_{|t|>B} \leqq C'\left\{B^{j+1}n\left[c^{-n} + \left(\frac{c^2}{2}\right)^n\right] + \frac{1}{B^{3-j}}\right\} \text{for } j=0,1,2. \quad (8.14)$$

Put $\bar{q} = \max\left(\frac{1}{\sqrt{c}}, \frac{c}{\sqrt{2}}\right)$, and choose $B = \bar{q}^{-n}$. Then (8.14) with this B implies that

$$\left|\frac{d^j}{dx^j}[T_M\phi_n(f_n)(x) - g_M(x)]\right| \leqq C'q^n, \quad j=0,1,2, \quad n > n_0, \quad \text{if} \quad \bar{q} < q < 1\,. \quad (8.15)$$

We get, similarly to the proof of (6.19), the strengthened form

$$\left|\frac{d^j}{dx^j}[T_M\phi_n(f_n)(x) - T_{M_n}\phi_n(f_n)(x)]\right| \leqq c^{-n}CM^{j+1}\exp\left(-\frac{\mu M}{2}|x|\right). \quad (8.15)'$$

Since $g_M(x) = Mg\left(M\left(x + \frac{c}{4(c-1)}\right)\right)$ by Lemma 12, where g is the solution of the equation (1.6), relations (8.15), (8.15)' and (3.13) imply formula (2.9) in Theorem 1'. The remaining statements of Theorem 1' are contained in Lemmas 10 and 11.

In the proof of Theorem 1 we need the following

LEMMA 16. — *If $n > n_0(\hat{M}_0, c)$, $\varepsilon > 0$ then there exists some $C(\varepsilon) = C(\varepsilon, c)$ such that*

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leq C(\varepsilon)M^{j+1}\exp\left(-\frac{(1-\varepsilon)M}{c}\left|2x + \frac{x^2}{c^nM_n}\right|\right)$$

for $x > -c^nM_n$, $\quad j = 0, 1, 2\,.$

Proof of Lemma 16. — By Property I(n) and the behaviour of the sequence β_n there is some $\mu > 0$ and $m \geq 0$ such that

$$\left|\frac{d^j}{dx^j} f_m(x)\right| \leqq CM^{j+1}\exp\left(-\mu M\left|2x + \frac{x^2}{c^mM_m}\right|\right),$$

$$j = 0, 1, 2, \qquad x > -c^mM_m\,. \quad (8.16)$$

We can improve the constant μ in (8.16) by successive application of Proposition 2. Choose $\eta = \frac{\varepsilon}{5}$ and the integer k in such a way that

$$\frac{\mu^2}{3}\left[\frac{2}{c}(1-\eta)\right]^{k-1} \leqq \frac{1-\varepsilon/2}{4} < \frac{\mu^2}{3}\left[\frac{2}{c}(1-\eta)\right]^k.$$

Define $\bar{\mu}$, $\frac{\mu^2}{3} \leq \bar{\mu}^2 < \mu^2$ so that $\bar{\mu}^2\left[\frac{2}{c}(1-\eta)\right]^{k-1} = \frac{1-\varepsilon/2}{4}$. Put $m=n-k-1$. Since $f_n(x) = Q_{n-1,M_{n-1}} \cdots Q_{m,M_m}f_m(x)$ we get Lemma 16 by applying formula (3.16) successively for $Q_{m+j,M_{m+j}}f_{m+j}(x)$, $0 \leq j \leq k-1$, with $\varepsilon = \eta$ and $\beta = \frac{1}{M^2\bar{\mu}^2}\left(\frac{c}{2(1-\eta)}\right)^j$, and finally for f_{n-1} with $\beta = \frac{4}{(1-\varepsilon/2)M^2}$ and $\varepsilon = \eta$. This is possible since for all $0 \leq j \leq k-1$

$$100 > \beta \geq \frac{1}{M^2\bar{\mu}^2}\left(\frac{c}{2(1-\eta)}\right)^{k-1} = \frac{4}{(1-\varepsilon/2)M^2}.$$

(Since the number of iterations k does not depend on n neither the constant $C(\varepsilon)$ does.)

The proof of Theorem 1. — Simple calculation yields that

$$c^{-n}\bar{p}_n(x, T) = \frac{1}{B_n}\exp\left(-\frac{a_0}{2T}c^n x^2\right)f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x - M_n\right), T\right), \quad (8.17)$$

where the norming constant B_n is determined by the equation

$$\int_{R^2} p_n(x, T)dx_1dx_2 = \int \bar{p}_n(\sqrt{x_1^2 + x_2^2}, T)dx_1dx_2 = 1, \quad x=(x_1, x_2). \quad (8.17)'$$

By relations (2.9) and (2.8)′

$$f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x - M_n\right), T\right) = Mg\left(c^n M\left(\sqrt{\frac{a_1}{T}}x - M_n + \frac{c^{1-n}}{4(c-1)M}\right)\right) + r_n^{(1)}(x) =$$

$$= \sqrt{\frac{a_1}{T}}\,\overline{M}g\left(\frac{a_1c^n}{T}\overline{M}(x-\overline{M})\right) + r_n^{(2)}(x) \quad \text{for} \quad x > 0 \quad (8.18)$$

with $\overline{M} = \sqrt{\frac{T}{a_1}}M$ and errors terms $r_n^{(1)}(x)$, $r_n^{(2)}(x)$ satisfying the inequality $\left|\frac{d^j}{dx^j}r_n^{(l)}(x)\right| \leqq K\bar{q}^n c^{nj}$ with some $K > 0$, $0 < \bar{q} < 1$, $j=0, 1, 2$, $l=1, 2$.

We claim that

$$\left|\frac{d^j}{dx^j}\left[\exp\left(-\frac{a_0}{2T}c^n x^2\right)f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x - M_n\right), T\right) - \right.\right.$$

$$\left.\left. - \exp\left\{-\left(\frac{a_0c^n}{2T}2\overline{M}(x-\overline{M}) + \overline{M}^2\right)\right\}\cdot\sqrt{\frac{a_1}{T}}\,\overline{M}g\left(\frac{a_1c^n}{T}\overline{M}(x-\overline{M})\right)\right]\right| \leqq$$

$$\leqq K\bar{q}^{n/2}c^{nj}\exp\left(-\frac{a_0}{2T}c^n\overline{M}^2\right), \quad j = 0, 1, 2, \quad |x-\overline{M}| < \delta n c^{-n} \quad (8.19)$$

if $\delta > 0$ is sufficiently small. Indeed, in this case

$$\left|\frac{d^j}{dx^j}\left[\exp\left(-\frac{a_0}{2T}c^n(x^2-\overline{M}^2)\right)\left(f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x-M_n\right),T\right)-\right.\right.\right.$$
$$\left.\left.\left.-\sqrt{\frac{a_1}{T}}\overline{M}g\left(\frac{a_1c^n}{T}\overline{M}(x-\overline{M})\right)\right)\right]\right| \leqq K\overline{q}^{-n/2}c^{nj} \quad (8.20)$$

because of (8.18) and the inequality $\left|\frac{d^j}{dx^j}\exp\left[\frac{a_0}{2T}c^n(x^2-\overline{M}^2)\right]\right| \leqq K\overline{q}^{-n/2}c^{nj}$ if $|x-\overline{M}| < \delta nc^{-n}$, and $\delta=\delta(\overline{M},c) > 0$ is sufficiently small. On the other hand

$$\left|\frac{d^j}{dx^j}\left[\exp\left(-\frac{a_0}{2T}c^n(x^2-M^2)\right)-\right.\right.$$
$$\left.\left.-\exp\left(-\frac{a_0}{T}c^nM(x-M)\right)\right]\sqrt{\frac{a_1}{T}}\overline{M}g\left(\frac{a_1c^n}{T}\overline{M}(x-\overline{M})\right)\right| \leqq$$
$$\leqq K\overline{q}^{n/2}c^{nj} \quad \text{if} \quad |x-\overline{M}| < \delta nc^{-n}, \quad j=0,1,2 \qquad (8.20)'$$

since

$$\left|\frac{d^j}{dx^j}\left[\exp\left(-\frac{a_0}{2T}c^n(x^2-\overline{M}^2)\right)-\exp\left(-\frac{a_0}{T}c^n\overline{M}(x-\overline{M})\right)\right]\right| =$$
$$=\left|\frac{d^j}{dx^j}\exp\left(-\frac{a_0}{T}c^n\overline{M}(x-\overline{M})\right)\left[\exp\left\{-\frac{a_0}{2T}c^n(x-\overline{M})^2\right\}-1\right]\right| \leqq$$
$$\leqq K\overline{q}^{-n/2}c^{nj}(x-\overline{M}^2) \leqq K\overline{q}^{n/2}c^{nj}.$$

Relation (8.19) follows from (8.20) and (8.20)′.

We also claim that

$$\left|\frac{d^j}{dx^j}\exp\left(-\frac{a_0}{2T}c^nx^2\right)f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x-M_n\right),T\right)\right| \leqq$$
$$\leqq K\exp\left(-\frac{a_0}{2T}c^n\overline{M}^2\right)c^{nj}\exp(-\mu c^n\overline{M}\,|x-\overline{M}|) \quad (8.21)$$

for $j=0,1,2$ with some appropriate $K=K(\overline{M},c) > 0$ and $\mu=\mu(c) > 0$. For $x > \sqrt{\frac{T}{a_1}}M_n$ relation (8.21) immediately follows from the inequalities

$$\left|\frac{d^j}{dx^j}\exp\left(-\frac{a_0}{2T}c^nx^2\right)\right| \leqq Kc^{nj}\exp\left(-\frac{a_0}{2T}c^n\overline{M}^2\right) \text{ for } x > \sqrt{\frac{T}{a_1}}M_n \quad \text{and}$$

$$\left|\frac{d^j}{dx^j}f_n\left(c^n\sqrt{\frac{a_1}{T}}x-M_n,T\right)\right| \leqq Kc^{nj}\exp\left(-\mu c^nM\left|\sqrt{\frac{a_1}{T}}x-M_n\right|\right) \leqq$$
$$\leqq \overline{K}c^{nj}\exp\left(-\mu c^n\overline{M}\sqrt{\frac{T}{a_1}}|x-M|\right).$$

For $0 < x < \sqrt{\frac{T}{a_1}}\,M_n$ we need the sharper bound given in Lemma 16 for the function f_n. In this case the estimates

$$\left|\frac{d^j}{dx^j}\exp\left(-\frac{a_0}{2T}c^n(x^2-\overline{M}^2)\right)\right| \leqq Kc^{nj}\exp\left(-\frac{(1+\varepsilon)a_0}{2T}c^n(x^2-\overline{M}^2)\right)$$

$$\left|\frac{d^j}{dx^j}f_n\left(c^n\left(\sqrt{\frac{a_1}{T}}x-M_n\right),T\right)\right| \leqq$$

$$\leqq Kc^{nj}\exp\left\{\frac{(1-\varepsilon)M}{c}\left[2c^n\left(\sqrt{\frac{a_1}{T}}x-M_n\right)+\frac{c^{2n}\left(\sqrt{\frac{a_1}{T}}x-M_n\right)^2}{c^nM_n}\right]\right\}=$$

$$=Kc^{nj}\exp\left\{\frac{1-\varepsilon}{c}c^n\left(\frac{a_1}{T}\frac{M}{M_n}x^2-MM_n\right)\right\}\leqq\overline{K}c^{nj}\exp\left\{\frac{1-\varepsilon}{c}\frac{a_1}{T}c^n(x^2-\overline{M}^2)\right\}$$

imply (8.21), since $\mu = \frac{1-\varepsilon}{c}\frac{a_1}{T} - \frac{(1+\varepsilon)a_0}{2T} > 0$.

It is not difficult to see with the help of Part b 1) of Lemma 13

$$\left|\frac{d^j}{dx^j}\sqrt{\frac{a_1}{T}}\,\overline{M}g\left(\frac{a_1c^n}{T}\overline{M}(x-\overline{M})\right)\exp\left(-\frac{a_0c^n}{T}\overline{M}(x-\overline{M})\right)\right| \leqq$$

$$\leqq Kc^{nj}\exp(-\alpha c^n\overline{M}\,|\,x-\overline{M}\,|), \quad j=0,1,2 \quad (8.22)$$

with some $\alpha > 0$. We show with the help of (8.19), (8.21) and (8.22) that

$$\int_{\mathbf{R}^2}\exp\left\{-\frac{a_0}{2T}c^n(x_1^2+x_2^2)\right\}f_n\left(c^n\left(\sqrt{\frac{a_1}{T}(x_1^2+x_2^2)}-M_n\right),T\right)dx_1dx_2=$$

$$=\exp\left(-\frac{a_0}{2T}c^n\overline{M}^2\right)c^{-n}\overline{B}(M)(1+0(q^n)) \quad (8.23)$$

with

$$\overline{B}(M)=\sqrt{\frac{a_1}{T}}\,\overline{M}^2\int_{-\infty}^{\infty}\exp\left(-\frac{a_0}{T}\overline{M}r\right)g\left(\frac{a_1}{T}\overline{M}r\right)dr$$

and some appropriate $0 < q < 1$. Because of (8.17) and (8.17)′ relation (8.23) implies that the constant B_n defined in (8.17) satisfies the relation

$$B_n=\overline{B}(M)\exp\left(-\frac{a_0c^n}{2T}\overline{M}^2\right)(1+0(q^n)). \quad (8.23)'$$

To prove (8.23) let us first observe that we make an error of order

$0(\bar{q}^{n/2}c^{-n}n)\exp\left(-\frac{a_0}{2T}c^n\bar{M}^2\right)$ by replacing the integrand on the left-hand side of (8.23) by

$$J_n(x_1, x_2) = \sqrt{\frac{a_1}{T}}\,\bar{M}\exp\left\{-\frac{a_0c^n}{2T}2\bar{M}(\sqrt{x_1^2+x_2^2}-\bar{M})+\bar{M}^2\right\}$$
$$g\left(\frac{a_1c^n}{T}\bar{M}(\sqrt{x_1^2+x_2^2}-\bar{M})\right).$$

Indeed, by (8.19) the difference of the two integrals can be bounded by this error term if we integrate only in the domain $D_n = \{x = (x_1, x_2), \,||x| - \bar{M}| < \delta nc^{-n}\}$, and by (8.21) and (8.22) both integrals are bounded by this error term if we integrate outside the domain D_n. On the other hand, we get with the change of variables $r = c^n(\sqrt{x_1^2+x_2^2}-\bar{M})$ that

$$\int J_n(x_1, x_2)dx_1dx_2 = \frac{1}{c^n}\int_{-c^n\bar{M}}^{\infty}\left(\frac{r}{c^n}+\bar{M}\right)\exp\left(-\frac{a_0\bar{M}}{T}r\right)\sqrt{\frac{a_1}{T}}\,g\left(\frac{a_1\bar{M}}{T}r\right)dr =$$
$$= \bar{B}(M)(1+0(c^{-n})).$$

(In the last step we have applied again Part b1) of Lemma 13.) These estimations imply (8.23).

Relation (1.7) follows from (8.17), (8.23)′ and (8.21), relation (1.5) with the bound (1.8) on the error (and $B(\bar{M}, T) = \bar{B}(M)^{-1}$) follows from (8.17), (8.23)′ and (8.19) for $|x - \bar{M}| < \delta nc^{-n}$ and (8.21), (8.22) instead of (8.19) for $|x - \bar{M}) > \delta nc^{-n}$. The remaining statements of Theorem 1 are easy consequences of Lemmas 10, 12, 13 and the definition of $\bar{M}$. Theorem 1 is proved.

9. THE PROOF OF THEOREM 2. THE BEHAVIOUR OF THE DENSITY OF THE AVERAGE SPIN AT INFINITY

In this section we prove Theorem 2 with the help of two lemmas whose proofs are postponed to the next section. First we want to determine the typical region where the function $\bar{p}_n(x, T)$ is essentially concentrated. More precisely, we want to define an interval D_n in such a way that

$$\bar{p}_n(x, T) > \bar{q}^n \sup_x \bar{p}_n(x, T)$$

with some $0 < \bar{q} < 1$ if $x \in D_n$, and $[0, \infty] - D_n$ has exponentially small probability with respect to the probability measure with density function $K_n\bar{p}_n(x, T)$. (Here K_n is that norming constant with which $K_n\bar{p}_n(x, T)$

is a probability density function on $[0, \infty]$.) To solve this problem we need the following

LEMMA 17. — *Consider the function* $g_1(x)$ *defined in Lemma 12. There exist some* $A = A(c) > 0$ *and* $B = B(c) > 0$ *such that*

a) $g_1(x) > B \exp(-Ax^\alpha)$ *with* $\alpha = \dfrac{\log 2}{\log c}$ *for all* $x > 0$

b) $g_1(x) > \dfrac{B}{\sqrt{|x|+1}} \exp\left(-\dfrac{2}{c}|x|\right)$ *for all* $x < 0$.

Part $b\,2$) of Lemma 13 and Part *a*) of Lemma 17 imply together that

$$B_1 \exp(-A_1 x^\alpha) < g_1(x) < B_2 \exp(-A_2 x^\alpha) \quad \text{for} \quad x > 0 \qquad (9.1)$$

with some $B_1 > 0$, $B_2 > 0$ and $0 < A_2 < A_1$, and Part $b\,1$) of Lemma 13 together with Part *b*) of Lemma 17 yield that

$$B_1(\varepsilon) \exp\left(-(1+\varepsilon)\frac{2}{c}|x|\right) < g_1(x) < B_2(\varepsilon) \exp\left(-(1-\varepsilon)\frac{2}{c}|x|\right) \quad \text{for} \quad x < 0 \qquad (9.1)'$$

with arbitrary $\varepsilon > 0$ and appropriate $B_1(\varepsilon) > 0$, $B_2(\varepsilon) > 0$.

Define the function $P_n(x) = Kc^n \exp\left(-\dfrac{a_1 c^n}{T}\overline{M}(x-\overline{M})\right) g\left(\dfrac{a_1 c^n \overline{M}}{T}(x-\overline{M})\right)$ with $\dfrac{1}{K} = \displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{a_1}{T}\overline{M}y\right) g\left(\frac{a_1 \overline{M}}{T} y\right) dy$. (The last integral is convergent because of Part $b\,1$) of Lemma 13.) Then $P_n(x)$ is a density function and a comparison of the functions $P_n(x)$ and $\bar{p}_n(x, T)$ gives with the help of Theorem 1 and Lemma 13 that

$$K_n = K + O(q^n) \qquad (9.2)$$

$$|P_n(x) - K_n \bar{p}_n(x, T)| < Cc^n q^n \quad \text{for} \quad x > 0 \qquad (9.2)'$$

and

$$\int_0^\infty |P_n(x) - K_n \bar{p}_n(x, T)|\, dx \leqq Cq^n \qquad (9.2)''$$

with some $0 < q < 1$, where K_n is defined so that $K_n \bar{p}_n(x, T)$ is a density function on $[0, \infty]$. It follows from Part *b*) of Lemma 13 and the identity $g(x) = g_1\left(x - \dfrac{c}{4(c-1)}\right)$ that for all $\varepsilon > 0$

$$\int_{x > \overline{M} + \varepsilon c^{-n} n^{1/\alpha}} P_n(x)\,dx < Cq^n$$

and

$$\int_{0 < x < \overline{M} - \varepsilon c^{-n} n} P_n(x)\,dx < Cq^n$$

with some $C = C(\varepsilon) > 0$, and $q = q(\varepsilon)$, $0 < q < 1$. These estimates together with (9.2)'' imply that

$$\int_{x > \overline{M} + \varepsilon c^{-n} n^{1/\alpha}} \overline{p}_n(x, T) dx < C q^n \tag{9.3}$$

and

$$\int_{0 < x < \overline{M} - \varepsilon c^{-n} n} \overline{p}_n(x, T) dx < C q^n . \tag{9.3'}$$

On the other hand, by the definition of $P_n(x)$, Lemma 17 and (9.2)' imply that

$$p_n(x, T) \geqq C_1 c^n (q_1^n - C q^n) > \frac{C_1}{2} c^n q_1^n, \qquad 0 < q < q_1 < 1 \tag{9.4}$$

for $\overline{M} - \varepsilon c^{-n} n < x < \overline{M} + \varepsilon c^{-n} n^{1/\alpha}$ if $\varepsilon > 0$ is a sufficiently small but fixed (depending only on c and $\overline{M}$) constant. Relations (9.2)' and (9.4) imply (1.10), and (9.3), (9.3)' explain why it is natural to consider the above interval.

We prove Part *a*) of Lemma 17 in the next section. Here we give a heuristic explanation for formulas (9.1) and (9.1)' and prove Part *b*).

It is natural to expect that $g_1(x) \sim B \exp(-A x^\alpha)$ with some $B > 0$, $A > 0$ for large x, $x > 0$. If we consider the expression $T_1 g_1(x)$ then for large x the main contribution to the integral is given when $u \sim 0$, $v \sim 0$, therefore $T_1 g_1(x) \sim B^2 \exp\left(-2A\left(\frac{x}{c}\right)^\alpha\right)$. Hence the identity $T_1 g_1(x) = g_1(x)$ suggests that $2A\left(\frac{x}{c}\right)^\alpha = A x^\alpha$, i. e. $\alpha = \frac{\log 2}{\log c}$. In (9.1) we have formulated a slightly weaker statement, since we have given only an upper and lower bound on the coefficient A inside the exponent. The above argument does not work for $x \overset{\vee}{<} 0$, since in this case the domain $u \sim 0$, $\frac{v^2}{2} \sim -x$ gives an essential contribution to the integral $T_1 g_1(x)$. This observation in the basis of the

Proof of Part b) of Lemma 17. — For $x < 0$ define the set

$$A = A(x) = \left\{ (u, v), |u| < 1, \frac{|x|}{c} + 1 < \frac{v^2}{2} < \frac{|x|}{c} + 2 \right\}.$$

Since $g_1(z) > C > 0$ with some C for $0 < z < 3$ (this will be proved in the proof of Part *a*)), $g_1(z) \geq 0$ for all z and $g_1(x) = T_1 g_1(x)$ hence

$$g_1(x) \geqq \int_A e^{-v^2} g_1\left(\frac{x}{c} + u + \frac{v^2}{2}\right) g_1\left(\frac{x}{c} - u + \frac{v^2}{2}\right) du dv \geqq$$

$$\geqq C^2 \int_A e^{-v^2} dv \geqq \frac{C'}{\sqrt{|x| + 1}} \exp\left(-\frac{2}{c}|x|\right).$$

It is natural to expect that the decrease of the function $\bar{p}_n(x, T)$ outside the typical region is similar to that of $P_n(x)$. For $x < \overline{M}$ we cannot expect a considerably better estimate than (1.7), but it is natural to expect that $\bar{p}_n(x, T) \leqq Cc^n \exp\{-A(c^n |x - \overline{M}|)^\alpha\}$ for $x > \overline{M}$ and sufficiently large n with some $C > 0$, $A > 0$ and the same α as in Lemma 17. However, this cannot hold without any further restriction, since the starting function $\bar{p}_0$ satisfies the inequality $\bar{p}_0(x, t) > C \exp(-Ax^4)$ for large x with $p_n(x, T)$. (This inequality holds for all $p_n(x, t)$ if x is sufficiently large depending on n.) Hence, in the case $\alpha > 4$ the above formulated conjecture does not hold. In the next section we prove the following weaker result:

LEMMA 18. — *Let $\bar{\alpha} < 4$ and $\bar{\alpha} < \alpha = \dfrac{\log 2}{\log c}$. Under the conditions of Theorem 1′ there is some $n_0 = n_0(c, t, T, \bar{\alpha})$ such that for $n > n_0$*

$$\left| \frac{d^j}{dx^j} f_n(x) \right| \leqq CM^{j+1} \exp(-AM^{\bar{\alpha}} x^{\bar{\alpha}}), \qquad x > 0, \quad j = 0, 1$$

with some $C > 0$ and $A > 0$ which depend only on $\bar{\alpha}$ and c.

It can be seen with the help of (8.17) and (8.23)′ that Lemma 18 implies the inequality $c^{-n}\bar{p}_n(x, T) \leqq K \exp(-A(c^n |x - M|)^{\bar{\alpha}})$ for $x > \overline{M}$ what is a considerable improvement of (1.7). However, since $\bar{\alpha} < \alpha$, even this estimate is not strong enough to imply (9.3). With the help of a simple trick we can prove a strengthened version of Lemma 18 (for $j = 0$) which is sharp enough to imply (9.3).

COROLLARY *of* LEMMA 18. — *Let the conditions of Lemma 18 be satisfied, and let some $\varepsilon > 0$ be given. Then there exists some $q = q(\varepsilon)$, $0 < q < 1$ and $K > 0$, $L > 0$ depending on c, T, t and $\bar{\alpha}$ such that*

$$c^{-n}\bar{p}_n(x, T) \leqq Kq^n \exp\{-L(c^n |x - \overline{M}|)^{\bar{\alpha}}\} \qquad \textit{for} \quad x > \overline{M} + \varepsilon n^{1/\alpha} c^{-n}.$$

Since $\dfrac{\log 2}{\log c} > 2$ for $1 < c < \sqrt{2}$ this Corollary implies (1.11)′, and thus completes the proof of Theorem 2.

Proof of the Corollary of Lemma 18.

We have

$$\left| \frac{d}{dx} \bar{p}_n(x, T) \right| \leqq Kc^{2n} \exp\{-AM^{\bar{\alpha}} |x - \overline{M}|^{\bar{\alpha}} c^{n\bar{\alpha}}\} \qquad \text{for} \quad x > \overline{M} \qquad (9.5)$$

with some appropriate $K > 0$ and $A > 0$. This follows from Lemma 18 relations (8.17), (8.23)′ the definition $\overline{M} = \sqrt{\dfrac{T}{a_1}} M$ and (2.8)′. (Observe that $\left|\frac{d^j}{dx^j}\exp\left(-\frac{a_0c^n}{2T}(x^2-\overline{M}^2)\right)\right| < Kc^{jn}$ for $x > \overline{M}$.) Given some $x > \overline{M}$ define $z = z(x, n) = \frac{1}{2K}c^{-2n}\exp\{A\overline{M}^{\bar\alpha}|x-\overline{M}|^{\bar\alpha}c^{n\bar\alpha}\}\bar{p}_n(x, T)$, where the constants K and A are the same as in relation (9.5). Then we have by (9.5)

$$\bar{p}_n(y, T) \geqq \bar{p}_n(x, T) - z \sup_{x<\xi<x+z}\left|\frac{d}{d\xi}\bar{p}_n(\xi, T)\right| > \frac{1}{2}\bar{p}_n(x, T) \text{ for } x < y < x+z. \tag{9.6}$$

If $x > \overline{M} + \varepsilon n^{1/\alpha}c^{-n}$ then relations (9.3) and (9.6) imply that

$$Cq^n > \int^{x+z}\bar{p}_n(y, T)dy \geqq \frac{1}{2}z\bar{p}_n(x, T)$$

or equivalently

$$\bar{p}_n(x, T)^2 < 2Kc^{2n}Cq^n\exp\{-A\overline{M}^{\bar\alpha}(c^n|x-\overline{M}|)\bar\alpha\}.$$

The last inequality implies the corollary with $\sqrt{q}$ instead of q.

10. THE PROOF OF LEMMAS 17 AND 18

Proof of Lemma 17. — We use the fact that $g_1(x)$ is the density function of the random variable ζ defined in (7.2). Introduce $\eta_k = \frac{1}{2}\left(\frac{c}{2}\right)^{k+1}\sum_{j=1}^{2^k}(1-\eta_{j,k}^2)$, $\zeta_k = \zeta - \eta_k$, $k = 0, 1, 2, \ldots$ where $\eta_{j,k}$ are the same as in (7.2). Let $p_k(x)$ denote the density function of η_k and $F_k(x)$ the distribution function of ζ_k.

Since

$$E\zeta_k^2 \le E\zeta^2 = A^2 \quad \text{with} \quad A = \frac{c}{2\sqrt{2-c^2}} \quad P(|\zeta_k| > 2A) \le \frac{1}{4} \quad \text{for all } k,$$

and since

$$g_1(x) = \int p_k(x-y)\,F_k(dy) \geqq \int_{-2A}^{2A} p_k(x-y)F_k(dy) \geqq$$
$$\geqq \inf_{|x-y|<2A} p_k(y)P(|\zeta_k| < 2A)$$

we have

$$g_1(x) \ge \frac{3}{4}\inf_{|x-y|<2A} p_k(y) \quad \text{for arbitrary } x \text{ and } k. \tag{10.1}$$

In order to estimate $p_k(y)$ we recall that the local large deviation theorem for partial sums of independent identically distributed random variables implies the following estimate: If $P_n(y)$ denotes the density function of the random variable $\sum_{j=1}^{n}(1-\eta_j^2)$, where η_j are independent standard normal random variables then there exist some $B > 0$, $\gamma > 0$, $C > 0$ such that

$$P_n(y) > \frac{C}{\sqrt{n}} \exp\left(-\frac{B}{n} y^2\right) \quad \text{if} \quad |y| < n\gamma. \tag{10.2}$$

Let us first consider the case $x > 4A$, $x > \gamma$, and define the integer $k = k(x)$ by the relation $\frac{\gamma}{2} c^{k-1} \leq x < \frac{\gamma}{2} c^k$, where γ is the same constant which appears in (10.2). We have $p_k(y) = 2 \cdot \left(\frac{2}{c}\right)^{k+1} P_{2^k}\left(2 \cdot \left(\frac{2}{c}\right)^{k+1} y\right)$. Hence for $|y - x| < 2A$ and $k = k(x)$

$$p_k(y) > 2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)^{k+1} \exp\left\{-8B\left(\frac{2}{c^2}\right)^{k+1} y^2\right\} > \bar{C} \exp\left(-\left(\frac{2}{c^2}\right)^k Bx^2\right)$$

with some $\bar{C} > 0$. This implies, because of the definition of $k(x)$ that

$$p_k(y) \geqq \bar{C} \exp(-\bar{B}x^\alpha) \quad \text{if} \quad |x-y| < 2A, \quad k=k(x), \quad \alpha = \frac{\log 2}{\log c} \tag{10.3}$$

with some $\bar{C} > 0$, $\bar{B} > 0$ if $x > \max(4A, \gamma)$. Relation (10.3) also holds for $0 < x < \max(4A, \gamma)$ with $k = 0$. Lemma 17 follows from (10.1) and (10.3).

To prove Lemma 18 first we introduce some notations. Let some α be given, $2 \leq \alpha < \frac{\log 2}{\log c}$, and choose some $\varepsilon = \varepsilon(\alpha, c) > 0$ such that $2c^{-\alpha} > (1+\varepsilon)/(1-\varepsilon)^2$, $\left(\frac{c^2}{2}\right)^\alpha < (1-\varepsilon)^2$. (Here we write α instead of the number denoted by $\bar{\alpha}$ in the previous section.) Define the sequence γ_n, $n = 0, 1, \ldots$

$$\gamma_0 = KM^{\alpha/2}, \qquad \gamma_{n+1} = (1-\varepsilon)\gamma_n + KM^{-\alpha} \tag{10.4}$$

with some $K > 0$ to be defined later. Clearly

$$\gamma_n = (1-\varepsilon)^n KM^{\alpha/2} + [1-(1-\varepsilon)^n]\frac{1}{\varepsilon}KM^{-\alpha}. \tag{10.4)'$$

(Here M is the same number as in Theorem 1'.) Let us fix some positive integer N and real number C so that Proposition 3 hold for large $\hat{M}_0$ with these parameters. Define β_n by (3.9) and (3.9)' for $n \geq N$ and let $\beta_n = \left(\frac{c^2}{2}\right)^n$ for $n < N$. We shall prove the following

LEMMA 18'. — *If* $\hat{M}_0$ *and also (in dependence on c, α and ε) the number* K *defined in (10.4) are sufficiently large,* $2 \leq \alpha < \min\left(4, \frac{\log 2}{\log c}\right)$ *then*

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left(-\frac{x^\alpha}{\gamma_n}\right) \quad \textit{for} \quad x > 0, \quad j = 0, 1 \qquad (10.5)$$

where C *is the same constant as in Proposition 3.*

Lemma 18 is an immediate consequence of Lemma 18' since by (10.4)' $\gamma_n > \text{const } M^{-\alpha}$ and also $\beta_n > \text{const } \frac{1}{M}$ for sufficiently large n. We shall prove Lemma 18' with the help of the following

LEMMA 19. — *Let some* α, $2 < \alpha \leq \frac{\log 2}{\log c}$, *be given, and define the sequence* γ_n *by (10.4). If* $\hat{M}_0$ *is sufficiently large, and the function* f_n *satisfies the inequality*

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left(-\frac{x^\alpha}{\gamma_n}\right) \quad \textit{for} \quad x > 0, \quad j = 0, 1$$

with the same constant $C > 0$ *as in Proposition 3 then*

$$\left|\frac{d^j}{dx^j} \overline{Q}_{n,M_n} f_n(x)\right| \leqq C_1 \frac{C^2 \gamma_n^{1/\alpha}}{\beta_n^{j/2+1}} \exp\left(-\frac{2x^\alpha}{c^\alpha \gamma_n}\right) \quad \textit{for } x > 0, \quad j = 0, 1, \qquad (10.7)$$

where C_1 *is an absolute constant. (Especially, it does not depend on the number* K *in the definition of* γ_n.)

Proof of Lemma 18' with the help of Lemma 19. — Relation (10.5) holds for $n = 0$. Indeed, we have an explicit expression for $f_0(x)$ with implies that $f_0(x) \leqq \exp\left(-\frac{1}{16}\frac{x^4}{M^2}\right) \leqq 3 \exp\left(-\left(\frac{x}{2\sqrt{M}}\right)^\alpha\right) \leqq C \exp\left(-\frac{x}{\gamma_0}\right)$ if $K \geq 16$. The derivative $\frac{df_0}{dx}$ can be estimated similarly.

If $0 < x < \frac{(2\gamma_n)^{1/(\alpha-1)}}{\beta_n^{1/2(\alpha-1)}} = B_n$ then by Property I(n)

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left(-\frac{2x}{\sqrt{\beta}}\right) \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left(-\frac{x^\alpha}{\gamma_n}\right),$$

therefore it is enough to prove (10.5) for $x > B_n$ with the help of (10.7). We show at the end of the proof that for arbitrary large $L > 0$ there is some $K = K(\varepsilon, \alpha, c)$ such that

$$\frac{\beta_n^{\alpha/2}}{2\gamma_n} < \frac{1}{L} \quad \text{for all } n \qquad (10.8)$$

if γ_n is defined with this K. Now we prove Lemma 17 if (10.8) holds with a sufficiently large L. In that case we have by (3.12)

$$|M_{n+1}-M_n|<Rc^{-n}\sqrt{\beta_{n+1}}=c^{-n}RB_{n+1}\cdot\left(\frac{\beta_{n+1}^{\alpha/2}}{2\gamma_{n+1}}\right)^{1/(\alpha-1)}<RB_{n+1}\left(\frac{1}{L}\right)^{1(\alpha-1)},$$

and for

$$x>B_{n+1} \quad |x+M_{n+1}-M_n|^\alpha>\left(1-\frac{\varepsilon}{2}\right)x^\alpha-C(\varepsilon)|M_n-M_{n+1}|^\alpha>(1-\varepsilon)x^\alpha.$$

Hence by (10.7)

$$\left|\frac{d^j}{dx^j}f_{n+1}(x)\right|<C_1\frac{C^2\gamma_n^{1/\alpha}}{\beta_n^{j/2+1}}\exp\left(-\frac{2|x+M_{n+1}-M_n|^\alpha}{c^\alpha\gamma_n}\right)\leqq$$

$$\leqq C_2\frac{C^2\gamma_{n+1}^{1/\alpha}}{\beta_{n+1}^{j/2+1}}\exp\left\{-(1-\varepsilon)^2\frac{2}{c^\alpha}\frac{x^\alpha}{\gamma_{n+1}}\right\}\quad\text{if}\quad x>B_{n+1}.$$

Here we have exploited that $\gamma_{n+1}>(1-\varepsilon)\gamma_n$ and $\beta_n>\frac{1}{3}\beta_{n+1}$. Since $(1-\varepsilon)^2 2c^{-\alpha}>1+\varepsilon$ the last relation implies that

$$\left|\frac{d^j}{dx^j}f_{n+1}(x)\right|<A_n\exp\left(-\frac{x^\alpha}{\gamma_{n+1}}\right)\quad\text{for}\quad x>B_{n+1}$$

with

$$A_n=C_1\frac{C^2\gamma_{n+1}^{1/\alpha}}{\beta_{n+1}^{j/2+1}}\exp\left(-\varepsilon\frac{B_{n+1}^\alpha}{\gamma_{n+1}}\right).$$

Simple calculation shows that

$$A_n<\frac{C_2C^2}{\beta_{n+1}^{(j+1)/2}}\left(\frac{\gamma_{n+1}}{\beta_{n+1}^{\alpha/2}}\right)^{1/\alpha}\left(\frac{\gamma_{n+1}}{B_{n+1}^\alpha}\right)^{(\alpha^2-1)/\alpha}<C_3\frac{C^2}{\beta_{n+1}^{(j+1)/2}}\frac{1}{L}<C\beta_{n+1}^{-(j+1)/2}$$

if L is sufficiently large. Hence it is enough to demonstrate (10.8) to complete the proof of Lemma 18'.

We have $\beta_n<B^2\left(\left(\frac{c^2}{2}\right)^n+\frac{1}{M^2}\right)$, hence $\beta_n^{\alpha/2}<\overline{B}\left[\left(\frac{c^2}{2}\right)^{n\alpha/2}+\frac{1}{M^\alpha}\right]$ with some $B=B(c)$ and $\overline{B}=\overline{B}(c)$. Since $\left(\frac{c^2}{2}\right)^\alpha<(1-\varepsilon)^2$ we have

$$\beta_n^{\alpha/2}<\overline{B}[(1-\varepsilon)^n+M^{-\alpha}].$$

On the other hand by (10.4)'

$$\gamma_n=(1-\varepsilon)^nKM^{\alpha/2}+[(1-(1-\varepsilon)^n)]\frac{K}{\varepsilon M^\alpha}\geqq K\left((1-\varepsilon)^n+\frac{1}{M^\alpha}\right),$$

and these relations imply (10.8) if $K>\overline{B}L$.

Proof of Lemma 19. — Let us introduce the functions

$$P_n(x, u) = \int e^{-v^2} f_n(l^+_{n,M_n}(x, u, v)) f_n(l^-_{n,M_n}(x, u, v)) dv$$

and

$$P_n(x) = P_n(x, 0),$$

where the functions $l^{\pm}_{n,M}$ are defined in (4.7). Then

$$\overline{Q}_{n,M_n} f_n(x) = 2 \int_0^\infty \exp\left(-\frac{u^2}{c^n}\right) P_n(x, u) du, \qquad (10.9)$$

and by the Cauchy- Schwartz inequality

$$P_n(x, u) \leqq [P_n(x + cu) P_n(x - cu)]^{1/2}. \qquad (10.10)$$

On the other hand

$$P_n(x) = \int e^{-v^2} f_n^2(l^+_{n,M_n}(x, 0, v)) dv \leqq \int e^{-v^2} \frac{C^2}{\beta_n} \exp\left(-\frac{2x^\alpha}{c^\alpha \gamma_n}\right) dv = \\ = \frac{C^2}{\beta_n} \sqrt{\pi} \exp\left(-\frac{2x^\alpha}{c^\alpha \gamma_n}\right) \quad \text{for} \quad x > 0, \qquad (10.11)$$

since in this case $l^+_{n,M}(x, 0, v) \geqq l^+_{n,M}(x, 0, 0) = \frac{x}{c}$, and

$$P_n(x) \leqq \frac{C^2}{\beta_n} \sqrt{\pi} \quad \text{for all } x \qquad (10.11)'$$

since $f_n(z) < \frac{C}{\sqrt{\beta_n}}$ for all z.

It follows from (10.10), (10.11) and (10.11)' that

$$P_n(x, u) \leqq \frac{C^2}{\beta_n} \sqrt{\pi} \exp\left\{-\frac{|x + cu|^\alpha}{c^\alpha \gamma_n} - \frac{|x - cu|^\alpha}{c^\alpha \gamma_n}\right\} \quad \text{for} \quad 0 < u < \frac{x}{c}$$

and

$$P_n(x, u) \leqq \frac{C^2}{\beta_n} \sqrt{\pi} \exp\left\{-\frac{|x + cu|^\alpha}{c^\alpha \gamma_n}\right\} \quad \text{for} \quad u > \frac{x}{c} > 0.$$

Hence by (10.9)

$$\overline{Q}_{n,M_n} f_n(x) \leqq 2 \frac{C^2}{\beta_n} \sqrt{\pi} \left[\int_0^{x/c} \exp\left(-\frac{|x + cu|^\alpha}{\gamma_n c^\alpha} - \frac{|x - cu|^\alpha}{\gamma_n c^\alpha}\right) du + \\ + \int_{x/c}^\infty \exp\left(-\frac{|x + cu|^\alpha}{\gamma_n c^\alpha}\right) du\right] = 2\sqrt{\pi} \frac{C^2}{\beta_n} (I_1 + I_2) \qquad (10.12)$$

To estimate I_1 we show that

$$|x + cu|^\alpha + |x - cu|^\alpha \geq 2|x|^\alpha + |cu|^\alpha \quad \text{if} \quad x > 0, \quad \alpha \geq 2. \qquad (10.13)$$

Indeed, (10.13) is equivalent to the inequality

$$h(u) = |1 + u|^\alpha + |1 - u|^\alpha - 2 - |u|^\alpha \geq 0,$$

and this inequality holds, since $h(0) = 0$, $h'(0) = 0$ and

$$h''(u) = \alpha(\alpha - 1)\{|u + 1|^{\alpha-2} + |u - 1|^{\alpha-2} - |u|^\alpha\} \geq 0.$$

By (1.13)

$$I_1 \leqq \int_0^{x/c} \exp\left\{-\frac{2x^\alpha}{c^\alpha\gamma_n} - \frac{u^\alpha}{\gamma_n}\right\} du \leqq K \exp\left(-\frac{2}{c^\alpha}\frac{x^\alpha}{\gamma_n}\right). \quad (10.14)$$

On the other hand

$$I_2 \leqq \int_{x/c}^\infty \exp\left(-\frac{|x + cu|^\alpha}{\gamma_n c^\alpha}\right) du = \gamma_n^{1/\alpha} \int_{\frac{2x}{\gamma_n^{1/2}c}}^\infty \exp(-u^\alpha) du \leqq$$

$$\leqq K\gamma_n^{1/\alpha} \exp\left(-\frac{2\alpha x^\alpha}{c^\alpha\gamma_n}\right) \leqq K\gamma_n^{1/\alpha} \exp\left(-\frac{2}{c^\alpha}\frac{x^\alpha}{\gamma_n}\right). \quad (10.15)$$

Relations (10.12), (10.14) and (10.15) together imply Lemma 19 for $j=0$. The proof for $j = 1$ is similar. The difference is that in the estimation of $\frac{d}{dx}\overline{Q}_{n,M_n}f_n(x)$ we have to work beside the function $P_n(x)$ also with

$$\overline{P}_n(x) = \int e^{-v^2} f_n'(l_{n,M_n}^+(x, 0, v))^2 dv.$$

It can be estimated in the same way as $P_n(x)$, the only difference is that now a multiplicative factor $\frac{1}{\beta_n}$ appears. Lemma 19 is proved.

APPENDIX A

THE PROOF OF THE BASIC RECURSIVE RELATIONS (2.1) AND (2.1') IN PART I

Formula (2.1)' immediately follows from (1.4) in Part I with $n = 0$. To prove (2.1) let us first observe that the recursive relation

$$\text{(A1)}\quad \mathcal{H}_{n+1}(x_1, \ldots, x_{2^{n+1}}) = \mathcal{H}_n(x_1, \ldots, x_{2^n}) + \mathcal{H}_n(x_{2^n+1}, \ldots, x_{2^{n+1}}) - c^n\left(2^{-n}\sum_{i=1}^{2^n} x_i\right)\left(2^{-n}\sum_{j=2^n+1}^{2^{n+1}} x_j\right)$$

holds for $n \geq 0$, where

$$\mathcal{H}_n(x_1, \ldots, x_{2^n}) = -\sum_{i=1}^{2^n}\sum_{j=i+1}^{2^n} \mathrm{U}(i, j)x_i x_j,$$

and $\mathrm{U}(i, j)$ is defined by (1.1) and (1.2)' in Part I. By relation (1.4) in Part I.

$$\text{(A2)}\quad p_{n+1}(x, \mathrm{T}) = \frac{1}{\mathrm{Z}_{n+1}(\mathrm{T}, t)}\int \exp\left\{-\frac{1}{\mathrm{T}}\mathcal{H}_n(x_1, \ldots, x_{2^{n+1}})\right\} \delta\left(2^{-(n+1)}\sum_{i=1}^{2^{n+1}} x_i - x\right)\prod_{i=1}^{2^{n+1}} p(x_i)dx_i,$$

where $\mathrm{Z}_{n+1}(\mathrm{T}, t)$ is an appropriate norming constant, and $\delta(2^{-(n+1)}\Sigma_{i=1}^{2^{n+1}} x_i - x)$ means that integration in (A2) is taken on the hyperplane $2^{-(n+1)}\Sigma_{i=1}^{2^{n+1}} x_i = x$ with respect to the Lebesgue measure. Let us fix some number u, and calculate the integral on the right-hand side of (A2) by integrating first on the hyperplane defined by the relations $2^{-n}\Sigma_{i=1}^{2^n} x_i = x + u$ and $2^{-n}\Sigma_{i=2^n+1}^{2^{n+1}} x_i = x - u$ and then by integrating by u. We get with the help of relations (A1) and (A2) that

$$p_n(x, \mathrm{T}) = \frac{1}{\mathrm{Z}_{n+1}(\mathrm{T}, t)}\int \exp\left\{\frac{c^n}{\mathrm{T}}(x+u)(x-u)\right\}$$
$$\left[\int \exp\left\{-\frac{1}{\mathrm{T}}\mathcal{H}_n(x_1, \ldots, x_{2^n})\right\}\delta\left(2^{-n}\sum_{i=1}^{2^n} x_i - (x+u)\right)\prod_{i=1}^{2^n} p(x_i)dx_i\right]$$
$$\left[\int \exp\left\{-\frac{1}{\mathrm{T}}\mathcal{H}_n(x_{2^n+1}, \ldots, x_{2^{n+1}})\right\}\delta\left(2^{-n}\sum_{i=2^n+1}^{2^{n+1}} x_i - (x-u)\right)\prod_{i=2^n+1}^{2^{n+1}} p(x_i)dx_i\right]du =$$
$$= \mathrm{C}_n\int \exp\left\{\frac{c^n}{\mathrm{T}}(x^2 - u^2)\right\} p_n(x+u)p_n(x-u)du,$$

as we have have claimed.

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