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QED on a Lattice:
I. Infrared asymptotic freedom for bounded fields

by

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ABSTRACT. — We study Euclidean quantum electrodynamics on a four dimensional unit lattice. The fermions are integrated out giving an effective photon interaction. For this interaction we give a formulation of the renormalization group which involves taking successive block field averages. In an approximation in which the fields are bounded and for sufficiently massive fermions we show that the flow of these transformations is toward a free electromagnetic field, i. e. that the theory is asymptotically free in the infrared.

RÉSUMÉ. — Nous étudions l'électrodynamique quantique euclidienne sur un réseau de dimension quatre. L'intégration sur les fermions fournit une interaction effective pour les photons que nous renormalisons en prenant des moyennes successives sur des blocs de spins. Dans l'approximation des champs bornés et des fermions lourds, nous montrons que l'itération de cette transformation converge vers un champ électromagnétique libre en d'autres termes nous montrons que la théorie est asymptotiquement libre dans l'infra-rouge.

1. INTRODUCTION

We consider quantum electrodynamics (QED) on a four dimensional Euclidean spacetime lattice with unit lattice spacing. The theory is formulated in terms of integrals over fermion and photon fields at each point in the lattice, and is well-defined as long as the lattice is finite. We are interested in taking the limit as the lattice becomes infinite and in studying the long distance behavior of the correlation functions in this limit. The expectation is that this will have something to do with the long distance behavior of the putative continuum theory on a Lorentzian spacetime, i. e. the real world. Furthermore, the techniques we employ may be useful in the eventual construction of this theory.

Because the photon field is massless, interactions are inherently long range and standard techniques of constructive field theory such as the cluster expansion are not sufficient. Instead we use Wilson's renormalization group approach in which the interaction for a large lattice is replaced by an effective interaction on a somewhat smaller lattice by taking block field averages. This procedure is iterated until one obtains an effective theory on a lattice with as few sites as one wishes. The key issue for our purposes is whether these effective interactions are tending toward a free field interaction as the iterations increase, i. e. whether the theory is asymptotically free in the infrared. If it is true then one is in a position to solve long distance problems for the original theory by relating them to properties of the free field theory.

In this paper we take the first steps toward carrying out this program. The fermion variables are integrated out at the start yielding an effective photon interaction (Chapter 2). For this interaction (or any in a large class) we give a precise formulation of the process of taking block field averages (Chapter 3). Finally asymptotic freedom is established in an approximation in which the fields are bounded and assuming the fermions are sufficiently massive (Chapter 4).

Our formulation of the renormalization group combines the earlier work of Balaban [*1*]-[*3*], and Gawedzki-Kupiainen [*9*]-[*12*]. Balaban gave a mathematically precise formulation of block field transformations for gauge fields. By successively performing these transformations and doing estimates he was able to obtain some deep stability results. On the other hand, Gawedzki-Kupiainen working with scalar fields developed techniques for tracking the exact flow of the block field transformations and were thereby able to obtain quite extensive results. We find that these two approaches combine rather nicely.

An important point in establishing the asymptotic freedom is in showing that the terms in the interaction which are quadratic and quartic in the photon field A are respectively marginal and irrelevant. *A priori* they could be respectively relevant and marginal. However because of the gauge invariance of the interaction one can show that these terms are really functions of the field strength $F = dA$ which accounts for the improved performance. Introducing the field strength globally is a dangerous thing to do because of the non-locality involved. The key to making the whole thing work is to introduce the field strength only in terms which are already well-localized. (Coupling through F has proved useful elsewhere, for example [8], [13].)

We remark that it is a particularly strong form of infrared asymptotic freedom which holds for this problem: the effective interactions approach the fixed point exponentially fast. This corresponds to the fact that the model is superrenormalizable in the infrared. We also remark that our method works in any space-time dimension and holds for a general class of models of the form $\| dA \|^2 + V(A)$ where $V(A)$ is gauge invariant and a sum of local terms.

In a subsequent paper we intend to remove the restriction to bounded fields and establish asymptotic freedom for the full interaction. The bounded field results should be a major ingredient in the solution of this problem.

2. THE MODEL

General references for lattice gauge theories are Seiler [13] and Balaban and Jaffe [4]. We begin by defining the lattice. Let L be an odd positive integer and let $\Lambda = \Lambda(N) = (\mathbb{Z}/L^N\mathbb{Z})^4$ for some (large) integer N. Λ is a four dimensional toroidal lattice with unit lattice spacing and L^N sites in each direction. The oriented bonds in Λ are denoted Λ^*: these are pairs $b = b_\mu(x) = \langle x, x + e_\mu \rangle$ where $e_0, \ldots, e_3$ is the standard basis for $\mathbb{Z}^4$ and $x \in \Lambda$. The oriented plaquettes are denoted Λ^{**}: these are quadruples of the form $p = p_{\mu\nu}(x) = \langle x, x + e_\mu, x + e_\mu + e_\nu, x + e_\nu \rangle$ with $\mu < \nu$.

The photon field (gauge field) A is a function from Λ^* to $\mathbb{R}$ which we write as $A \in \mathbb{R}^{\Lambda^*}$ or $[\Lambda^* : \mathbb{R}]$. The definition is extended to non-oriented bonds by $A(-b) = -A(b)$ where $-b$ is the reverse of b. The corresponding field strength is $dA \in \mathbb{R}^{\Lambda^{**}}$ and is defined for $p \in \Lambda^{**}$ by

$$dA(p) = \sum_{b \in \partial p} A(b).$$

The action for the free photon field is

$$\frac{1}{2} \| dA \|^2 \equiv \frac{1}{2} \sum_{p \in \Lambda^{**}} | dA(p) |^2 .$$

The fermion field algebra is constructed as follows. Let W be a four dimensional complex vector space (e. g. $W = \mathbb{C}^4$) and let $\overline{W}$ be the dual space. We form the vector space $(W \oplus \overline{W})^\Lambda$ of all $W \oplus \overline{W}$ valued functions on Λ and then let $\mathcal{G}_\Lambda$ be the Grassman algebra (exterior algebra) generated by $(W \oplus \overline{W})^\Lambda$.

If $\{ \psi_\alpha \}$ is a basis for W and $\{ \bar{\psi}_\alpha \}$ is the dual basis for $\overline{W}$, then these determine a basis $\{ \psi_\alpha(x), \bar{\psi}_\alpha(x) \}$ for $(W \oplus \overline{W})^\Lambda$ and hence a basis for $\mathcal{G}_\Lambda$.

Note that expressions like $\bar{\psi}(x)\psi(x) \equiv \Sigma_\alpha \bar{\psi}_\alpha(x) \wedge \psi_\alpha(x)$ are independent of basis and commute with everything. Also let $\{ \gamma_\mu \}$, $\mu = 0, \ldots, 3$ be a representation of the Clifford algebra $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ on W and define for $\langle x, y \rangle \in \Lambda^*$ the operator γ_{xy} on W by $\gamma_{xy} = 1 \pm \gamma_\mu$ for $y = x \pm e_\mu$. Then we define the basis independent combination

$$\bar{\psi}(x)\gamma_{xy}\psi(y) = \sum_{\alpha,\beta} (\gamma_{xy})_{\beta\alpha}\bar{\psi}_\alpha(x) \wedge \psi_\beta(y)$$

where $(\gamma_{xy})_{\beta\alpha}$ are the matrix elements of γ_{xy} in the particular basis.

The free fermion action is defined by

$$\frac{\kappa}{2} \sum_{\langle x,y \rangle \in \Lambda^*} \bar{\psi}(x)\gamma_{xy}\psi(y) + \sum_{x \in \Lambda} \bar{\psi}(x)\psi(x).$$

For $\kappa > 0$ small this describes a theory with massive fermions, the mass becoming infinite as $\kappa \to 0$. We study the case $\kappa \ll 1$. Integration on $\mathcal{G}_\Lambda$ is the linear functional $\langle \cdot \rangle_\Lambda$ defined by

$$\left\langle \bigwedge_{x,\alpha} [\bar{\psi}_\alpha(x) \wedge \psi_\alpha(x)] \right\rangle_\Lambda = 1$$

for elements of maximal degree, and set equal to zero on elements of smaller degree. This definition is independent of basis.

For the combined interaction we introduce the parallel translation operator $U(A)_{xy} = \exp(ieA_{xy})$ where the charge e is a fixed constant. Then the total action for QED is

$$S(A, \bar{\psi}, \psi) = \frac{Z_0}{2} \| dA \|^2 + \frac{\kappa}{2} \sum_{x,y} \bar{\psi}(x)\gamma_{xy}U(A)_{xy}\psi(y) + \sum_x \bar{\psi}(x)\psi(x).$$

Correlation functions are defined by taking moments of $\exp(-S(A, \bar{\psi}, \psi))$ with respect to integration over $A, \bar{\psi}, \psi$.

It is simplest to concentrate on the gauge degrees of freedom and integrate out the fermions at the start. We have

$$\exp(-S(A)) \equiv \langle \exp(-S(A, \bar{\psi}, \psi)) \rangle_\Lambda = \exp\left(-\frac{Z_0}{2} \| dA \|^2\right) Z(A)$$

where

$$Z(A) = \left\langle \exp\left(-\frac{\kappa}{2} \sum_{x,y} \bar{\psi}(x)\gamma_{xy}U(A)_{xy}\psi(y) - \sum_x \bar{\psi}(x)\psi(x)\right)\right\rangle_\Lambda.$$

We devote the remainder of this section to a study of Z(A), before considering integrals over A in the balance of the paper. The analogous object for the pseudoscalar Yukawa model was studied in [7].

The function Z(A) can be evaluated as a determinant, but we will not make any use of this representation. We do note the well-known fact that Z(A) is real for A real. Furthermore Z(A) is gauge invariant. If $\chi \in \mathbb{R}^{\Lambda}$ and $d\chi \in \mathbb{R}^{\Lambda^*}$ is defined by $d\chi(\langle x, y \rangle) = \chi(y) - \chi(x)$ then $Z(A) = Z(A + d\chi)$. This can be understood by observing that the $d\chi$ induces $\bar{\psi}_\alpha(x) \to e^{ie\chi(x)}\bar{\psi}_\alpha(x)$, $\psi_\alpha(x) \to e^{-ie\chi(x)}\psi_\alpha(x)$ and this change makes no difference when we take the expectation $\langle \cdot \rangle_\Lambda$.

To estimate Z(A) we need a norm on $\mathcal{G}_\Lambda$. Suppose that the base vector space W has an inner product. Then this induces an inner product on $\overline{W}$ in such a way that if $\{ \psi_\alpha \}$ is an orthonormal basis for W, then the dual basis $\{ \bar{\psi}_\alpha \}$ is orthonormal in $\overline{W}$. Thus we get an inner product on $(W \oplus \overline{W})^\Lambda$ and hence an inner product on $\mathcal{G}_\Lambda$ in such a way that orthonormal $\{ \psi_\alpha(x), \bar{\psi}_\alpha(x) \}$ generate an orthonormal basis. Let $\|\cdot\|$ be the associated norm. If $F, G \in \mathcal{G}_\Lambda$ and one of them is simple (i. e. a product of single vectors) then $\| F \wedge G \| \leq \| F \| \| G \|$. However this inequality does not hold in general (contrary to Ref. [*13*]). Integration can be regarded as taking the inner product with $\bigwedge_{x,\alpha} (\bar{\psi}_\alpha(x) \wedge \psi_\alpha(x))$ and so by the Schwarz inequality we have $| \langle F \rangle_\Lambda | \leq \| F \|$.

We actually want to write $Z(A) = \exp(-V(A))$ where V(A) is a sum of localized pieces with good estimates. For this we use a cluster expansion very much as in Seiler [*13*]. For the cluster expansion we assume $N > N_0$ and divide Λ up into blocks Δ with length L^{N_0} on a side and centered on the points of $(L^{N_0}\mathbb{Z})^4$. We always assume that L, N_0 are sufficiently large, the choice of N_0 possibly depending on L. A paved set is a non-empty union of blocks Δ and is denoted X, Y, etc. For any paved set X, $| X |$ is the number of blocks Δ in X and $\mathscr{L}(X)$ is the length (in an l^1 metric) of the shortest tree graph on the centers of the blocks Δ in X.

THEOREM 2.1. — Given M, C, ε there is a $\underline{\kappa}$ so for $\kappa \leq \underline{\kappa}$ and $| \operatorname{Im} A(b) | \leq C$

$$Z(A) = \exp(-V(A)) = \exp\left(-\sum_Y V_Y(A)\right)$$

where $V_Y(A)$ is real for A real, gauge invariant, depends only on $A(b)$ for $b \subset Y$ and satisfies in this region:

$$| V_Y(A) | \leq \kappa^{1-\varepsilon} e^{-M\mathscr{L}(Y)}.$$

Proof. — Define

$$\begin{aligned} h_\Delta &= h_{1,\Delta} + h_{0,\Delta} \\ &= \frac{\kappa}{2} \sum_{\langle x,y \rangle \subset \Delta} \bar{\psi}(x)\gamma_{xy}U(A)_{xy}\psi(y) + \sum_{x \in \Delta} \bar{\psi}(x)\psi(x) \end{aligned}$$

and for a pair of adjacent blocks $B = \{ \Delta, \Delta' \}$:

$$h_B = \frac{\kappa}{2} \sum_{\langle x,y \rangle \text{ conn. } B} \bar{\psi}(x)\gamma_{xy} U(A)_{xy} \psi(y)$$

where the sum is over nearest neighbor pairs $\langle x, y \rangle$ intersecting Δ and Δ'. We then have:

$$Z(A) = \left\langle \exp\left(- \sum_{\Delta} h_\Delta - \sum_{B} h_B \right) \right\rangle_\Lambda .$$

Making a Mayer expansion gives

$$Z(A) = \sum_{\{B_i\}} \left\langle \exp\left(- \sum_{\Delta} h_\Delta \right) \prod_i (\exp(-h_{B_i}) - 1) \right\rangle_\Lambda .$$

where the sum is over all collections of adjacent blocks $\{ B_i \}$. Each $\{ B_i \}$ determines a partition of Λ into connected paved sets $\{ X_j \}$ and the integration $\langle \cdot \rangle_\Lambda$ factors over these sets. We have

$$Z(A) = \sum_{\{X_j\}} \prod_j \rho_{X_j}(A)$$

where

$$\rho_X(A) = \sum_{\{B_i\} \to X} \left\langle \exp\left(- \sum_{\Delta \subset X} h_\Delta \right) \prod_i (\exp(-h_{B_i}) - 1) \right\rangle_X$$

and the sum is over all $\{ B_i \}$ contained in and connecting X. If X is a single block Δ, then $\rho_\Delta(A) = \langle \exp(-h_\Delta) \rangle_\Delta$. Note that each $\rho_X(A)$ is gauge invariant and depends only on $\{ A(b) \}_{b \in X}$.

Now we estimate for any $F \in \mathcal{G}_\Lambda$ and κ sufficiently small

$$\begin{aligned} \| h_{1,\Delta} F \| &\leq (\kappa/2) \sum_{\langle x,y \rangle \subset \Delta} \| \bar{\psi}_x \gamma_{xy} U(A)_{xy} \psi_y F \| \\ &\leq \mathcal{O}(1) \kappa L^{4N_0} e^{\varepsilon C} \| F \| \\ &\leq \kappa^{1-\varepsilon} \| F \| . \end{aligned}$$

If F is absent we have just $\| h_{1,\Delta} \| \leq \kappa^{1-\varepsilon}$. Similarly:

$$\| h_{0,\Delta} F \| \leq 2L^{4N_0} \| F \|$$
$$\| h_B F \| \leq \kappa^{1-\varepsilon} \| F \|$$

Since $\langle \exp(-h_{0,\Delta}) \rangle_\Delta = \det 1 = 1$ we have

$$\begin{aligned} | \rho_\Delta(A) - 1 | &= | \langle (\exp(-h_{1,\Delta}) - 1) \exp(-h_{0,\Delta}) \rangle_\Delta | \\ &\leq (\exp(\kappa^{1-\varepsilon}) - 1) \exp(2L^{4N_0}) \\ &\leq \kappa^{1-2\varepsilon} . \end{aligned}$$

Also for $|X| \geq 2$

$$|\rho_X(A)| \leq \sum_{\{B_i\}\to X} \exp(4L^{4N_0}|X|) \prod_i (\exp(\kappa^{1-\varepsilon}) - 1).$$

Each $\{B_i\}$ has at least as many terms as the number of lines in a nearest neighbor tree graph joining the blocks in X, and the latter is bounded below by $L^{-N_0}\mathscr{L}(X)$. Since $(\exp(\kappa^{1-\varepsilon}) - 1) \leq \kappa^{1-2\varepsilon} \leq \exp(\varepsilon \log \kappa)\kappa^{1-3\varepsilon}$ we have $\Pi_i(...) \leq \kappa^{1-3\varepsilon} \exp(\varepsilon \log \kappa L^{-N_0}\mathscr{L}(X))$. In addition there are fewer than $2^4|X| \leq \exp(2^4|X|)$ terms in the sum over $\{B_i\}$ and $|X| \leq 2L^{-N_0}\mathscr{L}(X)$. Thus for κ sufficiently small:

$$\begin{aligned}|\rho_X(A)| &\leq \kappa^{1-3\varepsilon} \exp((2^5L^{-N_0} + 8L^{3N_0} + \varepsilon(\log \kappa)L^{-N_0})\mathscr{L}(X)) \\ &< \kappa^{1-3\varepsilon} \exp(-8M\mathscr{L}(X)).\end{aligned}$$

Now define for $|X| \geq 2$

$$\tilde{\rho}_X(A) = \rho_X(A) \prod_{\Delta \subset X} \rho_\Delta(A)^{-1}.$$

Our estimate on ρ_Δ implies $|\log \rho_\Delta| \leq \kappa^{1-3\varepsilon}$ and so $\Pi_\Delta(...) \leq \exp(\kappa^{1-3\varepsilon}|X|)$. Combined with $|X| \leq 2L^{-N_0}\mathscr{L}(X)$ we have $|\tilde{\rho}_X(A)| \leq \kappa^{1-3\varepsilon}\exp(-4M\mathscr{L}(X))$.

From the theory of polymer gasses (see for examples [*6*], [*12*], [*13*]) we have

$$Z(A) = \exp\left(-\sum_Y V_Y(A)\right)$$

where for $|Y| = 1$

$$V_\Delta(A) = -\log(\rho_\Delta(A))$$

and for $|Y| \geq 2$

$$V_Y(A) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(X_1,\ldots,X_n):\cup X_i = Y} a(X_1, \ldots, X_n) \prod_{i=1}^{n} \tilde{\rho}_{X_i}(A).$$

Here the sum is over ordered collections of subsets of Λ and $a(X_1, \ldots, X_n)$ is the truncated correlation function for a hard core polymer gas.

As noted $|V_\Delta| \leq \kappa^{1-3\varepsilon}$ which is sufficient. Furthermore V_Y can be estimated by $Q(1 - Q)^{-1}|Y|$ [*6*] where

$$Q \equiv \sup_\Delta \left[\sum_{X:\Delta \subset X} |\tilde{\rho}_X| e^{|X|}\right].$$

But $|\tilde{\rho}_X e^{|X|}| \leq \kappa^{1-3\varepsilon}\exp(-M\mathscr{L}(X))$ and $\sum_{\Delta \subset X} \exp(-M\mathscr{L}(X)) \leq 2$ for N_0 sufficiently large (we assume $M \geq 1$). Then $Q \leq 2\kappa^{1-3\varepsilon}$ and

$|V_Y| \leq \kappa^{1-4\varepsilon}|Y|$. The same bound works if we first extract from $\tilde{\rho}_{X_i}$ a factor $\exp(-2M\mathscr{L}(X_i))$ and use $\Sigma_i \mathscr{L}(X_i) \geq \mathscr{L}(Y)$ ($a(X_1, \ldots, X_n) = 0$ unless the X_i overlap) to obtain $|V_Y| \leq \kappa^{1-4\varepsilon}|Y| \exp(-2M\mathscr{L}(Y))$ and hence $|V_Y| \leq \kappa^{1-4\varepsilon}\exp(-M\mathscr{L}(Y))$.

REMARK 2.2. — We expand V_Y in powers of A by $V_Y = V_{2,Y} + V_{4,Y} + V_{\geq 6,Y}$ where $V_{2,Y}(A) = (1/2)d^2/dt^2\,[V_Y(tA)]_{t=0}$, etc. $\left(\text{then } V_2 = \sum_Y V_{2,Y}, \text{ etc.}\right)$. Each term is gauge invariant. By the previous theorem and the Cauchy bounds we conclude that for $|A(b)| \leq C$ and κ sufficiently small we have

$$|V_{2,Y}(A)|, |V_{4,Y}(A)|, V_{\geq 6,Y}(A)| \leq \kappa^{1-\varepsilon}e^{-M\mathscr{L}(Y)}.$$

We may also write:

$$V_{2,Y}(A) = \sum_{b,b' \subset Y} A(b)\Pi_Y(b, b')A(b')$$

$$V_{4,Y}(A) = \sum_{b_1,\ldots,b_4 \subset Y} \Lambda_Y(b_1, \ldots, b_4)A(b_1) \ldots A(b_4)$$

where $\Pi_Y(b, b') = (1/2)[\partial^2 V_Y/\partial A(b)\partial A(b')](0)$ and

$$\Lambda_Y(b_1, \ldots, b_4) = (1/4!)[\partial^4 V_Y/\partial A(b_1) \ldots \partial A(b_4)](0).$$

Then by the Cauchy bounds

$$|\Pi_Y(b, b')| \leq C^{-2}\kappa^{1-\varepsilon}e^{-M\mathscr{L}(Y)}$$
$$|\Lambda_Y(b_1, \ldots, b_4)| \leq C^{-4}\kappa^{1-\varepsilon}e^{-M\mathscr{L}(Y)}.$$

REMARK 2.3. — $V_{2,Y}$ and $V_{4,Y}$ are localized gauge invariant polynomials in A. It is important to be able to change such polynomials to polynomials in dA of the same type. We now explain how this is done. (See also [*4*], § III.3.)

Given $x, y \in \Lambda$ let Γ_{yx} be the shortest rectilinear contour from y to x passing through the points $y = (y_0, y_1, y_2, y_3)$, (y_0, y_1, y_2, x_3), (y_0, y_1, x_2, x_3), (y_0, x_1, x_2, x_3) and $x = (x_0, x_1, x_2, x_3)$. (Since our lattice has an odd number of points in each direction there is a unique such contour.)

Now given $A \in \mathbb{R}^{\Lambda^*}$ and a reference point $y \in \Lambda$ we define $\chi \in \mathbb{R}^{\Lambda}$ by

$$\chi(x) = \sum_{b \in \Gamma_{yx}} A(b).$$

Then the gauge transformed field $A' = A - d\chi$ is given by

$$A'(b) = \sum_{b' \in C_{y,b}} A(b')$$

$$C_{y,b} = \Gamma_{y,b_-} \cup b \cup \Gamma_{b_+,y} \qquad b = \langle b_-, b_+ \rangle .$$

Note that $C_{y,b}$ is a closed curve. Let $\mathcal{O}_y \subset \Lambda^*$ be all bonds such that $C_{y,b}$ is topologically trivial. Most bonds are in $\mathcal{O}_y$. For such bonds we have either that $C_{y,b} = \emptyset$ (if b is on the tree $\cup_x \Gamma_{yx}$) or else $C_{y,b}$ consists of two rectilinear lines one unit apart and the bonds joining them at the end. In the latter case let $S_{y,b}$ be the oriented surface between these lines so that $\partial S_{y,b} = C_{y,b}$; see figure 1. Then by the lattice version of Stoke's theorem we have

$$A'(b) = \sum_{p \in S_{y,b}} dA(p) \qquad b \in \mathcal{O}_y .$$

There are also unfortunately exceptional bonds $b \notin \mathcal{O}_y$ for which $C_{y,b}$ is

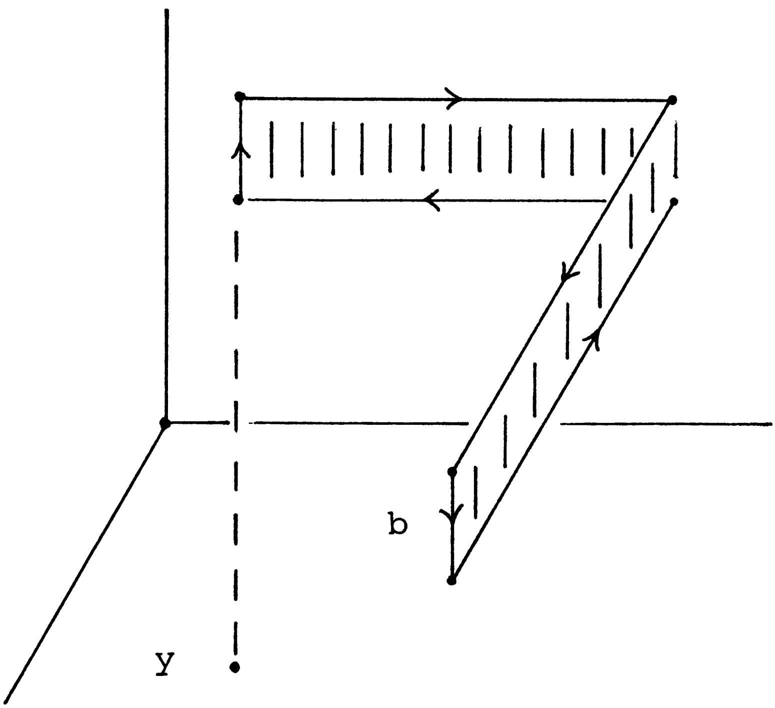

FIG. 1. — The contour $C_{y,b}$ (with arrows) and the surface $S_{y,b}$ (shaded) so $\partial S_{y,b} = C_{y,b}$. We have assumed both y and b are in the plane $x_0 = 0$.

topologically non-trivial. For this to occur the length of $C_{y,b}$ must at least L^N and so $d(y, b) \geq (L^N - 1)/2$. ($d(y, b)$ = distance from y to b in l^1 metric.) Roughly speaking b must be maximally distant from y.

These considerations are used in the following theorem. Let us make the convention that if X is a paved set then $X^* \subset \Lambda^*$ is all bonds $b = b_\mu(x)$ such that $x \in X$, and $X^{**} \subset \Lambda^{**}$ is all plaquettes $p = p_{\mu\nu}(x)$ such that $x \in X$. Also $(X \times Y)^{**} = X^{**} \times Y^{**}$, etc.

THEOREM 2.4. — Under the hypotheses of Theorem 2.1 we have

$$V_2(A) = \sum_{p,p'} dA(p)I(p,p')dA(p') + U_2(A)$$

$$V_4(A) = \sum_{p_1,\dots,p_4} J(p_1, \dots, p_4)dA(p_1) \dots dA(p_4) + U_4(A)$$

where I, J are invariant under lattice symmetries and satisfy:

$$\sum_{(p,p')\in(\Delta\times\Delta')^{**}} | I(p,p') | \leq \kappa^{1-\varepsilon} \exp(- M\mathscr{L}(\Delta \cup \Delta'))$$

$$\sum_{(p_1,\dots,p_4)\in(\Delta_1\times\dots\times\Delta_4)^{**}} | J(p_1, \dots, p_4) | \leq \kappa^{1-\varepsilon} \exp(- M\mathscr{L}(\Delta_1 \cup \dots \cup \Delta_4))$$

and U_2, U_4 are polynomials of the indicated degree and satisfy for $| A(b) | \leq C$

$$| U_2(A) |, | U_4(A) | \leq \kappa^{1-\varepsilon} | \Lambda |^{-1}.$$

Proof. — We first establish such a representation for $V_{2,Y}(A)$. Now $V_{2,Y}(A)$ can be regarded as a gauge invariant function on all of $\mathbb{R}^{\Lambda^*}$, even though it only depends on $A(b)$, $b \subset Y$. As such we may pick a reference point y in the center of some block $\Delta \subset Y$ and replace $A(b)$ by $A'(b)$ as above. For ordinary points $b \in \mathcal{O}_y$ we change to dA. This gives

$$V_{2,Y}(A) = \sum_{p,p'} dA(p)I_Y(p,p')dA(p') + \sum_{b_1,b_2} A(b_1)L_Y(b_1, b_2)A(b_2)$$

where

$$I_Y(p,p') = \sum_{b,b'\in\mathcal{O}_y} \chi_{S_{y,b}}(p)\chi_{S_{y,b'}}(p')\Pi_Y(b, b')$$

$$L_Y(b_1, b_2) = \sum_{b_1',b_2'\in\sim(\mathcal{O}_y\times\mathcal{O}_y)} \chi_{C_{y,b_1'}}(b_1)\chi_{C_{y,b_2'}}(b_2)\Pi_Y(b_1', b_2').$$

Note that I_Y, L_Y may depend on variables outside Y. If we now sum over Y we have the claimed representation for $V_2 = \sum_Y V_{2,Y}$ with $I = \sum_Y I_Y$, $L = \sum_Y L_Y$, and $U_2(A) = \Sigma_{b_1,b_2} A(b_1)L(b_1, b_2)A(b_2)$.

Turning now to the estimates we consider a term in the sum defining $I_Y(p, p')$. Since M is arbitrary the bound on $\Pi_Y(b, b')$ gives a factor $\exp(-8M\mathscr{L}(Y))$. But since $b, y \in Y$ we have $\mathscr{L}(Y) \geq d(b, y) - 4L^{N_0}$. Thus we may extract factors like $\exp(-3Md(b, y))$. In addition $p \in S_{y,b}$ implies that $d(b, y) \geq d(p, y)$. These considerations give:

$$| I_Y(p, p') | \leq \mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} \exp(-2M(\mathscr{L}(Y) + d(p, y) + d(p', y)))$$

where $\mathcal{O}(1)$ depends on L, N_0.

Now suppose $p \in \Delta^{**}$, $p' \in \Delta'^{**}$ and choose $\Delta_y \subset Y$ so $y \in \Delta_y$. Then $d(p, y) \geq \mathscr{L}(\Delta \cup \Delta_y) + \mathcal{O}(1)$ and so

$$| I_Y(p, p') | \leq \mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} \exp(-2M\mathscr{L}(Y \cup \Delta \cup \Delta')).$$

Using $\Sigma_Y \exp(-M\mathscr{L}(Y \cup \Delta \cup \Delta')) \leq \mathcal{O}(1)$ we get

$$| I(p, p') | \leq \mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} \exp(-M\mathscr{L}(\Delta \cup \Delta')).$$

This gives an estimate of the same form when we sum over $(\Delta \times \Delta')^{**}$. Since $\mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} \leq \kappa^{1-2\varepsilon}$ for κ large, and since ε is arbitrary we have the estimate on I.

Proceeding similarly we have the bound

$$| L_Y | \leq \mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} \exp(-8M\mathscr{L}(Y)).$$

However L_Y is non-zero only if there exists a bond b in $Y \cap \sim \mathcal{O}_y$. Since b, $y \in Y$ and $d(b, y) \geq L^N/2 + \mathcal{O}(1)$ we must have $\mathscr{L}(Y) \geq L^N/2 + \mathcal{O}(1)$. This leads to the bound $| L(b_1, b_2) | \leq \mathcal{O}(1)C^{-2}\kappa^{1-\varepsilon} | \Lambda | \exp(-4ML^N)$ and hence $| U_2(A) | \leq \mathcal{O}(1)\kappa^{1-\varepsilon} | \Lambda |^3 \exp(-4ML^N)$. Since, $| \Lambda | = L^{4(N-N_0)}$ the bound $| U_2(A) | \leq \kappa^{1-2\varepsilon} | \Lambda |^{-1}$ follows.

Now I, U_2 may not be invariant under lattice symmetries, but since V_2 is invariant we may average over all lattice symmetries and get invariant objects which have bounds of the same form.

This completes the analysis of V_2, and V_4 is treated similarly.

Remarks. — We rewrite the second order term in a tensor notation by defining $A_\mu(x) = A(b_\mu(x))$ and $(dA)_{\mu\nu}(x) = (dA)(p_{\mu\nu}(x)) = (\partial_\mu A_\nu)(x) - (\partial_\nu A_\mu)(x)$ where $(\partial_\mu \chi)(x) = \chi(x + e_\mu) - \chi(x)$. With a similar definition for I we have (with summation convention and $\mu < \nu$, $\rho < \sigma$)

$$V_2(A) = \sum_{x,y} (dA)_{\mu\nu}(x) I_{\mu\nu\rho\sigma}(x, y)(dA)_{\rho\sigma}(y).$$

Define a constant k_0 satisfying $|k_0| \leq \kappa^{1-\varepsilon}$ by

$$\frac{1}{2} k_0 \delta_{\mu\rho} \delta_{\nu\sigma} = \sum_y I_{\mu\nu\rho\sigma}(x, y).$$

The sum must be independent of x and have the indicated form because of the invariance under lattice symmetries. We replace $(dA)_{\rho\sigma}(y)$ by $(dA)_{\rho\sigma}(x)$ in $V_2(A)$ we get $k_0/2 \| dA \|^2$ and then choose Z_0 so $Z_0 + k_0 = 1$ to get $\frac{1}{2} \| dA \|^2$. There is of course a correction term left over.

Our results to this point can be summarized by saying that the effective photon action has the form $S(A) = \frac{1}{2} \| dA \|^2 + V(A)$ where now $V = V' + U'$

$$V'(A) = \sum_{x,y} dA_{\mu\nu}(x) I_{\mu\nu\rho\sigma}(x, y)((dA)_{\rho\sigma}(y) - (dA)_{\rho\sigma}(x))$$
$$+ \sum_{p_1, \ldots, p_4} J(p_1, \ldots p_4) dA(p_1) \ldots dA(p_4) + V_{\geq 6}(A)$$

and $U' = U_2 + U_4$.

In the renormalization group analysis to which we now turn we will see that all the terms in V' are irrelevant. This is not strictly true for U' which comes from the global loops. But this term is so small that it does not matter if it grows or not. This term would be completely absent if we had chosen a lattice with a trivial topology. We have not done this because it would mean giving up translation invariance.

3. RENORMALIZATION GROUP TRANSFORMATIONS

We consider the measure $\exp(-S(A))dA$ on $[\Lambda^* : \mathbb{R}]$ where dA is Lebesgue measure. The study of this measure is reduced to the study of a sequence of more tractable measures following the analysis of Balaban [1].

The analysis is based on an averaging operator Q defined as follows. Let $\Lambda_n = \Lambda(N - n)$ be the unit toroidal lattice with $L^{4(N-n)}$ sites; we have $\Lambda_0 = \Lambda$. Then $Q = Q(L)$ is defined from $[\Lambda_n^* : \mathbb{R}]$ to $[\Lambda_{n+1}^* : \mathbb{R}]$ by:

$$(3.1) \qquad (QA)(b) = L^{-4} \sum_{|x_\mu| < L/2} A(Lb + x).$$

Here for $b \in \Lambda_{n+1}^*$, $x \in \mathbb{Z}^4$, $Lb + x$ is a contour in Λ_n and for any such contour γ we define $A(\gamma) = \sum_{b \in \gamma} A(b)$. We also consider $Q_k \equiv (Q)^k = Q(L^k)$.

We also need some gauge fixing procedure to have convergent integrals. For $y \in L\Lambda_{n+1} \subset \Lambda_n$ let $B(y)$ be the block

$$B(y) = \{ x \in \Lambda_n : | x_\mu - y_\mu | < L/2 \} . \tag{3.2}$$

These blocks partition Λ_n. Now for $x \in B(y)$ let τ_{yx} be the symmetrized rectilinear path $\tau_{yx} = \cup_\pi \Gamma^\pi_{yx}$ where π is a permutation of (3, 2, 1, 0) and in Γ^π_{yx} the coordinates are changed in the order π. The symmetrization (as in [3]) is necessary to preserve lattice symmetries. Then define

$$A(\tau_{yx}) = (4!)^{-1} \sum_\pi A(\Gamma^\pi_{yx}) .$$

Our symmetrized axial gauge fixing consists in setting $A(\tau_{yx}) = 0$ by inserting

$$\delta_{\mathscr{T}}(A) = \sum_{y \in L\Lambda_{n+1}} \sum_{x \in B(y)} \delta(A(\tau_{yx}))$$

where $\mathscr{T}$ denotes all the τ_{yx} in Λ_n.

The renormalization group transformation consists in defining effective actions S_n on $[\Lambda^*_n : \mathbb{R}]$ by $S_0 = S$ and by

$$\exp(-S_{n+1}(A)) = \int_{[\Lambda^*_n : \mathbb{R}]} \delta(A - QA')\delta_{\mathscr{T}}(A') \exp(-S_n(A'))dA' . \tag{3.3}$$

One can show that these integrals are well-defined. For large n, S_n will carry information about the expectations of large scale observables.

With this definition S_n is gauge invariant. To see this for given $\lambda \in [\Lambda_{n+1} : \mathbb{R}]$ define $\lambda' \in [\Lambda_n : \mathbb{R}]$ so $\lambda'(Lx) = \lambda(x)$ and λ' is constant on L-blocks. Then $d\lambda = d(Q_0\lambda') = Qd\lambda'$ where Q_0 is a block averaging operator on scalars. The result now follows by a change of variables and $\delta_{\mathscr{T}}(A' + d\lambda') = \delta_{\mathscr{T}}(A')$.

In the following it will be convenient to modify (3.3) somewhat to isolate the constant terms and allow for a field strength renormalization. Thus we replace (3.3) by

$$\exp(-S_{n+1}(A)) = \left(\int \delta(A - \zeta_{n+1}^{-\frac{1}{2}} QA')\delta_{\mathscr{T}}(A') \exp(-S_n(A'))dA' \right) \Big/ [A = 0] \tag{3.4}$$

where $[A=0]$ denotes the numeration with $A=0$ so that $S_n(0)=0$ for all n. The constants ζ_n are to be chosen for later convenience. The idea is that if one can control (3.4) with some reasonable choice of ζ_n, then one can control (3.3).

First consider (3.4) with $\zeta_n = 1$ and $S_0(A) = \frac{1}{2} \| dA \|^2 = \frac{1}{2}(A, \Delta^T_0 A)$

where $\Delta_0^T = \delta d$, $\delta = (d)^*$. We assert that $S_n(A) = (A, \Delta_0^T A)$ for some operator Δ_0^T on $[\Lambda_n^* : \mathbb{R}]$. Indeed, suppose it is true for n. Define

$$H^{(n+1)} : [\Lambda_{n+1}^* : \mathbb{R}] \to [\Lambda_n^* : \mathbb{R}]$$

by putting $H^{(n+1)}A = A'$ where A' is the minimizer of $(A', \Delta_n^T A')$ on the manifold $QA' = A$, $A'_{\mathcal{T}} = 0$. Then making the change of variables $A' \to A' + H^{(n+1)}A$ in (3.4) we find $S_{n+1}(A) = (A, \Delta_{n+1}^T A)$ with $\Delta_{n+1}^T = H^{(n+1)*}\Delta_n^T H^{(n+1)}$. If we define $H_n : [\Lambda_n^* : \mathbb{R}] \to [\Lambda^*, \mathbb{R}]$ by

$$H_n = H^{(1)}H^{(2)} \dots H^{(n)}$$

then

$$\Delta_n^T = H_n^* \Delta_0^T H_n. \tag{3.5}$$

One can also characterize $H_n A$ as the minimizer for $\| dA' \|^2$ subject to the constraints $Q_n A' = A$; $A'_{\mathcal{T}} = 0$, $(QA')_{\mathcal{T}} = 0, \dots, (Q_{n-1}A')_{\mathcal{T}} = 0$.

In the general case we make the change of variable $A' \to \zeta_{n+1}^{\frac{1}{2}} H^{(n+1)}A + A'$ and find that

$$S_n(A) = \frac{1}{2}(A, \Delta_n^T A) + V_n(A) \tag{3.6}$$

where $V_0 = V$ and

$$\exp(-V_{n+1}(A)) = \exp\left(\frac{1}{2}(1 - \zeta_{n+1})(A, \Delta_{n+1}^T A)\right). \tag{3.7}$$

$$\int \delta(QA')\delta_{\mathcal{T}}(A') \exp(-V_n(\zeta_{n+1}^{\frac{1}{2}} H^{(n+1)}A + A') - (A'. \Delta_n^T A'))dA' / [A = 0].$$

Since $S_n(A)$ and $(A, \Delta_n^T A)$ are gauge invariant we have that $V_n(A)$ is gauge invariant. Similarly, $V_n(A)$ is invariant under lattice symmetries.

We next parametrize the subspace $QA = 0$, $A_{\mathcal{T}} = 0$ in $[\Lambda_n^* : \mathbb{R}]$. For $y \in L\Lambda_{n+1}$ let $\mathcal{B}_y \subset \Lambda_n^*$ be all bonds contained in $B(y)$ ($\mathcal{B}_y \neq B(y)^*$) and let $\mathcal{E}_y \subset \Lambda_n^*$ be the edge bonds connecting $B(y)$ to some other block. Then $[\Lambda_n^* : \mathbb{R}]$ is a direct sum over y of $[\mathcal{B}_y : \mathbb{R}] \oplus [\mathcal{E}_y : \mathbb{R}]$. We specify coordinates on $[\mathcal{B}_y : \mathbb{R}]$ by giving a basis for this space as follows. The set of characteristic functions of the τ_{yx}, denoted $\mathcal{T}_y$, are independent and span a subspace $[\mathcal{T}_y] \subset [\mathcal{B}_y : \mathbb{R}]$. Let $[\Sigma_y]$ be the orthogonal subspace and let $\Sigma_y = \{\sigma_{y,i}\}$ be an orthonormal basis for this space such that the translation of $\sigma_{y,i}$ by $\underline{L}e_\mu$ is $\sigma_{y+Le_\mu,i}$. A basis for $[\mathcal{B}_y : \mathbb{R}]$ is then $\mathcal{T}_y \cup \Sigma_y$. For $[\mathcal{E}_y : \mathbb{R}]$ we let $\overline{\mathcal{E}}_y$ be the edge bonds with the central bond excluded and use standard coordinates on $[\overline{\mathcal{E}}_y : \mathbb{R}]$. The central bond coordinates are replaced by QA. Thus with $\mathcal{T} = \cup_y \mathcal{T}_y$, $\Sigma = \cup_y \Sigma_y$, $\overline{\mathcal{E}} = \cup_y \overline{\mathcal{E}}_y$ we may define a non-singular operator

$$T : [\Lambda_n^* : \mathbb{R}] \to [\mathcal{T} : \mathbb{R}] \oplus [\Sigma : \mathbb{R}] \oplus [\overline{\mathcal{E}} : \mathbb{R}] \oplus [\Lambda_{n+1}^* : \mathbb{R}]$$

by

$$TA = ((A(\tau_{yx})), (A(\sigma_{y,i})), (A(e)), Q(A))$$

where $A(\sigma) = \Sigma_b A(b)\sigma(b)$. Integrals over $QA = 0$, $A_{\mathcal{T}} = 0$ then may be expressed as integrals over $\{0\} \oplus [\Sigma : \mathbb{R}] \oplus [\overline{\mathscr{E}} : \mathbb{R}] \oplus \{0\}$ which we identify with $[\overline{\Sigma} : \mathbb{R}]$ where $\overline{\Sigma} = \Sigma \cup \overline{\mathscr{E}}$ also written $\overline{\Sigma}_n$. Let C be T^{-1} restricted to this subspace. Then C is a strictly local operator satisfying $QC\overline{A} = 0$ and $(C\overline{A})_{\mathscr{T}} = 0$.

Then (3.7) becomes

$$\exp(-V_{n+1}(A)) = \exp\left(\frac{1}{2}(1 - \zeta_{n+1})(A, \Delta_{n+1}^{T} A)\right). \int_{[\overline{\Sigma}_n:\mathbb{R}]} \exp(-V_n(\zeta_{n+1}^{\frac{1}{2}} H^{(n+1)} A + C\overline{A}) - (C\overline{A}, \Delta_n^{T} C\overline{A})) d\overline{A} / [A = 0]. \tag{3.8}$$

Let $\Gamma_n = C^* \Delta_n^T C$. Then Γ_n is a positive operator and we have after the change of variable $\overline{A} \to \Gamma_n^{-\frac{1}{2}} \overline{A}$

$$\exp(-V_{n+1}(A)) = \exp\left(\frac{1}{2}(1 - \zeta_{n+1})(A, \Delta_{n+1}^{T} A)\right) \int \exp(-V_n(\zeta_{n+1}^{\frac{1}{2}} H^{(n+1)} A + C\Gamma_n^{-\frac{1}{2}} \overline{A}) d\mu(\overline{A}) / [A = 0] \tag{3.9}$$

where μ is the Gaussian measure on $[\overline{\Sigma}_n : \mathbb{R}]$ with mean zero and unit covariance.

The potentials $V_n(A)$ actually depend on A only through $\hat{A} \equiv H_n A$. In fact suppose we define $\hat{V}_n$ on $[\Lambda^* : \mathbb{R}]$ by $\hat{V}_0 = V$ and

$$\exp(-\hat{V}_{n+1}(\hat{A})) = \exp\left(\frac{1}{2}(1 - \zeta_{n-1}) \| d\hat{A} \|^2 - \delta \hat{V}_{n+1}(\hat{A})\right). \int \exp(-\hat{V}_n(\zeta_{n+1}^{\frac{1}{2}} \hat{A} + M_n \overline{A})) d\mu(\overline{A}) / [A = 0] \tag{3.10}$$

where $M_n = H_n C \Gamma^{-\frac{1}{2}}$ and $\delta \hat{V}_n$ is any function which vanishes on Ran (H_n). Then $V_n(A) = \hat{V}_n(H_n A)$. The $\delta \hat{V}_{n+1}$ will be chosen later for convenience.

The final modification is to scale down to lattices $L^{-n}\Lambda$ with lattice spacing L^{-n}. This is useful for estimates. Accordingly we introduce the scaling operator σ_L mapping $[L^{-n-1}\Lambda^* : \mathbb{R}] \to [L^{-n}\Lambda^* : \mathbb{R}]$ defined by

$$(\sigma_L \mathscr{A})(b) = L^{-1} \mathscr{A}(L^{-1} b).$$

Similarly we use $\sigma_{L^k} = (\sigma_L)^k$. Let d be the exterior derivative defined on $\mathscr{A} \in [L^{-n}\Lambda^* : \mathbb{R}]$ by

$$d\mathscr{A}(p) = L^{2n} \int_{b \in \partial p} \mathscr{A}(b) \equiv L^n \sum_{b \in \partial p} \mathscr{A}(b).$$

Then $(d(\sigma_L \mathscr{A})(p) = L^{-2} d\mathscr{A}(L^{-1} p)$. If we define

$$\| d\mathscr{A} \|^2 = \int_P | d\mathscr{A}(p) |^2 \equiv L^{-4n} \sum_{p \in L^{-n}\Lambda^{**}} | d\mathscr{A}(p) |^2$$

then we have $\| d(\sigma_L \mathscr{A}) \|^2 = \| d\mathscr{A} \|^2$.

Now define $\mathscr{H}_n = \sigma_{L^{-n}} \circ H_n$ mapping $[\Lambda_n^* : \mathbb{R}]$ to $[L^{-n}\Lambda^* : \mathbb{R}]$. We then define $\mathscr{V}_n$ on $[L^{-n}\Lambda^* : \mathbb{R}]$ by $\mathscr{V}_0 = V$ and

$$(3.11)\quad \exp(-\mathscr{V}_{n+1}(\mathscr{A})) = \exp\left(\frac{1}{2}(1-\zeta_{n+1})\,\|d\mathscr{A}\|^2 - \delta\mathscr{V}_{n+1}(\mathscr{A})\right).$$
$$\int \exp(-\mathscr{V}_n(\zeta_{n+1}^{\frac{1}{2}}\sigma_L\mathscr{A} + \mathscr{M}_n\overline{A}))d\mu(\overline{A})/[\mathscr{A}=0]$$

where $\mathscr{M}_n = \sigma_{L^{-n}} \circ M_n = \mathscr{H}_n C\Gamma^{-\frac{1}{2}}$ and where $\delta\mathscr{V}_n(\mathscr{A}) = \delta\hat{V}_n(\sigma_{L^n}\mathscr{A})$. Then we have $\mathscr{V}_n(\mathscr{A}) = \hat{V}_n(\sigma_{L^n}\mathscr{A})$ and so

$$(3.12)\qquad V_n(A) = \mathscr{V}_n(\mathscr{H}_n A).$$

Thus to study the flow of V_n (and hence S_n) it suffices to study the flow of $\mathscr{V}_n$ as defined by (3.11). We have the freedom to choose ζ_n and to choose $\delta\mathscr{V}_n$ be any function which vanishes on Ran $(\mathscr{H}_n)$.

At this point we make some remarks on notation. Define

$$\mathscr{Q}_n : [L^{-n}\Lambda^* : \mathbb{R}] \to [\Lambda_n^* : \mathbb{R}]$$

by $\mathscr{Q}_n = Q_n \circ \sigma_{L^n}$. Then it is $\mathscr{Q}_n$ which Balaban calls Q_n (after his rescaling). Our $\mathscr{H}_n$ can be characterized by saying that $\mathscr{A} = \mathscr{H}_n A$ is the minimizer for $\|d\mathscr{A}\|^2$ subject to $\mathscr{Q}_n\mathscr{A} = A$, $(\sigma_{L^n}\mathscr{A})_{\mathscr{T}} = 0$ $(\mathscr{Q}\sigma_{L^{n-1}}\mathscr{A})_{\mathscr{T}} = 0$, etc. and thus it is our $\mathscr{H}_n$ which is Balaban's H_n.

For future reference we note that $\mathscr{Q}_n$ can be written

$$(\mathscr{Q}_n\mathscr{A})(b) = \int_{x:|x_\mu|<1/2} \mathscr{A}(b+x)$$

where $\int_x = L^{-4n}\Sigma_x$ and where for any contour γ,

$$\mathscr{A}(\gamma) = \int_{b\in\gamma} \mathscr{A}(b) = L^{-n}\Sigma_{b'\in\gamma}\mathscr{A}(b').$$

So far we have worked exclusively in an axial gauge. However, we could have changed to another gauge such as the Feynmann gauge. This would have given rise to new minimizers $\mathscr{H}_n'$, which however are related to the axial minimizers by a gauge transformation $\mathscr{H}_n = \mathscr{H}_n' + d(\cdot)$ [5]. Since $\mathscr{V}_n$ is gauge invariant we can make the replacement $\mathscr{H}_n \to \mathscr{H}_n'$ in (3.11), (3.12). Hereafter we assume this has been done, but keep the same notation. The point is that one has better estimates for the Feynman gauge minimizers.

The estimates we need are the following:

LEMMA 3.1. — There is a constant β (depending on L) such that for all n:

a) $|\Gamma_n^{-\frac{1}{2}}(\sigma,\sigma')| \le \mathcal{O}(1)\exp(-\beta d(\sigma,\sigma'))\qquad \sigma,\sigma' \in \overline{\Sigma}_n$

b) $|\mathscr{H}_{n,\mu\nu}(x,y)| \leq \mathcal{O}(1)\exp(-\beta|x-y|) \quad x\in \mathrm{L}^{-n}\Lambda, y\in\Lambda_n$

$|\partial_\rho\mathscr{H}_{n,\mu\nu}(x,y)| \leq \mathcal{O}(1)\exp(-\beta|x-y|)$

$$|\partial_\rho\mathscr{H}_{n,\mu\nu}(x,y) - \partial_\rho\mathscr{H}_{n,\mu\nu}(x',y)| \leq \mathcal{O}(1)|x-x'|^{\frac{1}{2}}(\exp(-\beta|x-y|)+\exp(-\beta|x'-y|)).$$

c) Similar bounds for $\mathscr{M}_n = \mathscr{H}_n\mathrm{C}\Gamma_n^{-\frac{1}{2}}$.

Proof. — To prove *a*) we use a result of Balaban and Jaffe [4] which says that if T is a positive operator on l^2 of some lattice with $\mathrm{T}\geq m$ and $|\mathrm{T}(x,x')| \leq \exp(-\beta d(x,x'))$, then there exist constants K, β' depending only on m, β such that $|(\mathrm{T})^{-1}(x,x')| \leq \mathrm{K}\exp(-\beta' d(x,x'))$.

Now $\Gamma_n = \mathrm{C}^*\Delta_n^{\mathrm{T}}\mathrm{C}$ is an operator on $l^2(\overline{\Sigma}_n)$ which satisfies the hypothesis of this theorem with constants β, m independent of n [1]. The same is true of $(m+\lambda)^{-1}(\Gamma_n+\lambda)$ for any $\lambda\geq 0$ with constants independent of λ, n. It follows that for some $\beta>0$ and $\sigma,\sigma'\in\overline{\Sigma}_n$

$$|(\Gamma_n+\lambda)^{-1}(\sigma,\sigma')| \leq \mathcal{O}(1)(m+\lambda)^{-1}\exp(-\beta d(\sigma,\sigma')).$$

The result now follows from the representation

$$\Gamma_n^{-\frac{1}{2}}(\sigma,\sigma') = \pi^{-1}\int_0^\infty \lambda^{-\frac{1}{2}}(\Gamma_n+\lambda)^{-1}(\sigma,\sigma')d\lambda.$$

Part *b*) is also a result of Balaban [1], and *c*) follows from *a*), *b*) and the fact that C is strictly local.

4. BOUNDED FIELD APPROXIMATION

We now consider a modification of the recursion for $\mathscr{V}_n$ in which the fluctuation fields are bounded. This provides a relatively simple framework for introducing techniques which will be useful in the general proof. Our treatment follows that of Gawedski-Kupiainen [10], [11], [12].

The basic modification is to replace the unit Gaussian measure $d\mu(\overline{\mathrm{A}})$ by $d\mu_n(\overline{\mathrm{A}}) = \chi_n(\overline{\mathrm{A}})d\mu(\overline{\mathrm{A}})$ where $\chi_n(\overline{\mathrm{A}})$ is the characteristic function of $|\overline{\mathrm{A}}(\sigma)| \leq (n_0+n)^\nu$. Here ν is a fixed integer and n_0 will be chosen large. Thus for $\mathscr{A}\in[\mathrm{L}^{-n-1}\Lambda^* : \mathbb{R}]$ we have instead of (3.11)

$$(4.1)\quad \exp(-\mathscr{V}_{n+1}(\mathscr{A})) = \exp\left(\frac{1}{2}(1-\zeta_{n+1})\|d\mathscr{A}\|^2 - \delta\mathscr{V}_{n+1}(\mathscr{A})\right) \int \exp(-\mathscr{V}_n(\zeta_{n+1}^{\frac{1}{2}}\sigma_{\mathrm{L}}\mathscr{A} + \mathscr{M}_n\overline{\mathrm{A}}))d\mu_n(\overline{\mathrm{A}})/[\mathscr{A}=0]$$

with $\mathscr{V}_0 = \mathrm{V}$ from the end of section 2.

We study this recursion relation with restrictions on the magnitude and smoothness of $\mathscr{A}$, which however weaken as n increases. We define for some constant C

$$C_n = C(n_0 + n)^{\nu}.$$

and require for $\mathscr{A} \in [L^{-n}\Lambda^* : \mathbb{R}]$:

$$(4.2)\qquad \begin{aligned} &| \mathscr{A}(b) | \leq C_n \\ &| d\mathscr{A}(p) | \leq C_n \quad \text{or} \quad | (d\mathscr{A})_{\mu\nu}(x) | \leq C_n \\ &| (d\mathscr{A})_{\mu\nu}(x) - (d\mathscr{A})_{\mu\nu}(y) | \leq C_n | x - y |^{\frac{1}{2}} \end{aligned}$$

Here $(d\mathscr{A})_{\mu\nu}(x) = (d\mathscr{A})(p_{\mu\nu}(x))$ where

$$p_{\mu\nu}(x) = \langle x, x + \eta e_\mu, x + \eta e_\mu + \eta e_\nu, x + \eta e_\nu \rangle$$

with $\mu < \nu$ and $\eta = L^{-n}$. It will also be useful to allow $\mathscr{A}$ to be complex. For a paved set Y let $\mathscr{K}_n(Y)$ be all complex-valued functions $\mathscr{A}$ on bonds in Y satisfying (4.2). $\mathscr{K}_n(L^{-n}\Lambda)$ is denoted $\mathscr{K}_n$.

For the following theorem we suppose that the constants L, N_0, C, n_0 are sufficiently large and chosen in the indicated order. Thus $N_0 \geq \underline{N}_0(L)$, $C \geq \underline{C}(N_0, L)$, $n_0 \geq \underline{n}_0(C, N_0, L)$. We also let α be some fixed fraction of β from Lemma 3.1.

We further define:

$$(4.3)\qquad \delta_n = 2^{-(n_0+n)}.$$

In the following expressions like $\mathcal{O}(\delta_n^3)$ mean bounded by a constant times δ_n^3 where the constant can depend on L, N_0.

THEOREM 4.1. — Let κ be sufficiently small. Then there exists a choice for ζ_n with $| \zeta_n - 1 | < \delta_n^3$ and $\delta\mathscr{V}_n$ vanishing on Ran $\mathscr{K}_n$ so that $\mathscr{V}_n$ has the form $\mathscr{V}_n = \mathscr{V}_n' + \mathscr{U}_n'$ where $\mathscr{V}_n'$ is analytic on $\mathscr{K}_n$ with Taylor expansion $\mathscr{V}_n' = \mathscr{V}_{n,2} + \mathscr{V}_{n,4} + \mathscr{V}_{n,\geq 6}$ and where the various pieces have the form:

$$a)\ \mathscr{V}_{n,2}(\mathscr{A}) = \int_{x,y} d\mathscr{A}_{\mu\nu}(x)\mathscr{I}^n_{\mu\nu\rho\sigma}(x, y)((d\mathscr{A})_{\rho\sigma}(y) - (d\mathscr{A})_{\rho\sigma}(x))$$

$$\int_{x,y\in\Delta\times\Delta'} | \mathscr{I}^n_{\mu\nu\rho\sigma}(x, y) | | x - y |^{\frac{1}{2}} \leq C_n^{-2}\delta_n^3 \exp(-\alpha\mathscr{L}(\Delta \cup \Delta'))$$

$$b)\ \mathscr{V}_{n,4}(\mathscr{A}) = \int_{p_1,\ldots,p_4} d\mathscr{A}(p_1) \ldots d\mathscr{A}(p_4)\mathscr{I}_n(p_1, \ldots, p_4)$$

$$\int_{(p_1,\ldots,p_4)\in(\Delta_1 \times \ldots \times \Delta_4)^{**}} | \mathscr{I}_n(p_1, \ldots, p_4) | \leq C_n^{-4}\delta_n^4 \exp(-\alpha\mathscr{L}(\Delta_1 \cup \ldots \cup \Delta_4)).$$

c) $\mathscr{V}_{n,\geq 6} = \Sigma_Y \mathscr{V}_{n,\geq 6,Y}$ where $\mathscr{V}_{n,\geq 6,Y}$ depends only on $\mathscr{A}$ in Y, is analytic on $\mathscr{K}_n(Y)$, and satisfies there $| \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}) | \leq \delta_n^5 \exp(-\alpha\mathscr{L}(Y))$.

d) $\mathscr{U}'_n$ satisfies for $\mathscr{A} \in \mathscr{K}_n$, $\mathscr{A}$ real:

$$| \mathscr{U}'_n(\mathscr{A}) | \leq \delta_n^3 \, | L^{-n}\Lambda |^{-1} .$$

In addition each piece is gauge invariant.

Remarks. — The form of the small factors like $C_n^{-2}\delta_n^3$ has no fundamental significance and has been chosen for convenience. Now we write $\mathscr{V}_{n,2} = \sum_{Y} \mathscr{V}_{n,2,Y}$ where $\mathscr{V}_{n,2,Y}$ is an expression of the form *a)* but with the sum over μ, ν, ρ, σ and x, y restricted so that the smallest paved set containing $p_{\mu\nu}(x)$ and $p_{\rho\sigma}(y)$ is Y. Similarly we have $\mathscr{V}_{n,4} = \Sigma_Y \mathscr{V}_{n,4,Y}$ where $\mathscr{V}_{n,4,Y}$ is a localization of the expression in *b)*.

Note. — These localizations are distinct from those of Lemma 2.1 for $n = 0$. Then one can show using the estimates of the theorem that for $\mathscr{A} \in \mathscr{K}_n(Y)$

$$\{ | \mathscr{V}_{n,2,Y}(\mathscr{A}) |, | \mathscr{V}_{n,4,Y}(\mathscr{A}) |, | \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}) | \} \leq \{ \mathscr{O}(\delta_n^3), \mathscr{O}(\delta_n^4), \mathscr{O}(\delta_n^5) \} \exp(-\alpha\mathscr{L}(Y)). \tag{4.4}$$

We have arranged the bounds here so that the terms of higher order in $\mathscr{A}$ are smaller, but so that the product of any two $\mathscr{V}$'s is smaller still.

The estimates (4.4) and $\Sigma_Y \exp(-\alpha\mathscr{L}(Y)) \leq 2 | L^{-n}\Lambda |$ yield for $\mathscr{A} \in \mathscr{K}_n$: $| \mathscr{V}'_n(\mathscr{A}) | \, | L^{-n}\Lambda |^{-1} \leq \mathscr{O}(\delta_n^3)$. An even stronger estimate holds for $\mathscr{U}'_n(\mathscr{A})$ so we have

$$| \mathscr{V}_n(\mathscr{A}) | \, | L^{-n}\Lambda |^{-1} \leq \mathscr{O}(\delta_n^3). \tag{4.5}$$

Since $| L^{-n}\Lambda |$, the number of blocks in $L^{-n}\Lambda$, is proportional to the volume this says that the energy density (for bounded fields) goes to zero exponentially fast as $n \to \infty$. This is the infrared asymptotic freedom.

Proof. — It suffices to show that the results for n imply the results for $n + 1$. The case $n = 0$ follows from the results of § 2 if κ is sufficiently small.

Instead of studying $\mathscr{V}_{n+1}$ directly, we first modify (4.1) by setting $\zeta_{n+1} = 1$, $\delta\mathscr{V}_{n+1} = 0$, dropping the normalization, and replacing $\mathscr{V}_n$ by $\mathscr{V}'_n$. These omissions will be corrected later. Thus we define $\mathscr{W}$ (actually $\exp(-\mathscr{W})$) by

$$\exp(-\mathscr{W}(\mathscr{A})) = \int \exp(-\mathscr{V}'_n(\sigma_L\mathscr{A} + \mathscr{M}_n\overline{A}))d\mu_n(\overline{A}). \tag{4.6}$$

We analyze $\mathscr{W}$ for $\mathscr{A} \in L/2\mathscr{K}_{n+1}$ which is larger than the $\mathscr{K}_{n+1}$ we need at the end. For such $\mathscr{A}$ we have that $(\sigma_L\mathscr{A} + \mathscr{M}_n\overline{A})$ is in $\mathscr{K}_n$ so that the integrand is in the region where we have bounds. To see this note that by Lemma 3.1 we have for $|\overline{A}| \leq (n_0+n)^\nu$ that $|\mathscr{M}_n\overline{A}| \leq \mathscr{O}(1)(n_0+n)^\nu$.

Thus we have for C satisfying $\mathcal{O}(1) \leq C/4$ and n_0 sufficiently large

$$(4.7) \qquad |\sigma_L \mathcal{A} + \mathcal{M}_n \bar{A}| \leq 1/2C_{n+1} + \mathcal{O}(1)(n_0 + n)^v \leq 3/4C_n + 1/4C_n = C_n .$$

Similarly we establish the needed bounds on $d(\sigma_L \mathcal{A} + \mathcal{M}_n \bar{A})$, etc.

STEP 1. — The first step is to break $\exp(-\mathcal{W})$ into localized pieces. The decompositions $\mathcal{V}_{n,2} = \Sigma_Y \mathcal{V}_{n,2,Y}$, etc. give rise to a decomposition $\mathcal{V}'_n = \Sigma_Y \mathcal{V}'_{n,Y}$ where $\mathcal{V}'_{n,Y}$ depends on $\mathcal{A}$ only in Y. Correspondingly we make a Mayer expansion for $\exp(-\mathcal{V}'_n)$. With $\mathcal{A}^1 = \sigma_L \mathcal{A} + \mathcal{M}_n \bar{A}$ we have

$$(4.8) \qquad \exp(-\mathcal{W}(\mathcal{A})) = \sum_{\{Y_\beta\}} \int \prod_\beta (\exp(-\mathcal{V}'_{n,Y_\beta}(\mathcal{A}^1)) - 1) d\mu_n(\bar{A}) .$$

Here the sum is over all collections of paved subsets $\{ Y_\alpha \}$ of $L^{-n}\Lambda$.

Given $\{ Y_\beta \}$ let $\{ U_k \}$ be the finest partition of $L^{-n}\Lambda$ into unions of blocks $L\Delta$ so that each Y_β is in some U_k. Then each U_k is connected with respect to the Y_β's. For each distinct pair $\{ U_k, U_{k'} \}$ we introduce a variable $0 \leq s_{\{k,k'\}} \leq 1$ and let $s = (s_{\{k,k'\}})$. Let χ_k be the characteristic function of U_k^* or $\Sigma_{U_k} \equiv \cup_{y \in U_k} \Sigma_y$ and let

$$\mathcal{M}^s = \sum_k \chi_k \mathcal{M}_n \chi_k + \sum_{\{k,k'\}} s_{\{k,k'\}} (\chi_k \mathcal{M}_n \chi_{k'} + \chi_{k'} \mathcal{M}_n \chi_k) .$$

Then $\mathcal{M}^1 = \mathcal{M}$ while $\mathcal{M}^0$ decouples the U_k's. Now define

$$\mathcal{A}^s = \sigma_L \mathcal{A} + \mathcal{M}_n^s \bar{A} .$$

Then we have

$$(4.9) \quad \exp(-\mathcal{W}(\mathcal{A})) = \sum_{\{Y_\beta\}} \sum_\Gamma \int ds_\Gamma \partial/\partial s_\Gamma \left[\int \prod_\beta (\exp(-\mathcal{V}'_{n,Y_\beta}(\mathcal{A}^s)) - 1) d\mu_n(\bar{A}) \right] .$$

Here the sum is over all collections of pairs $\{ k, k' \}$ denoted Γ and $s_\Gamma = (s_{\{k,k'\}})$ with $\{ k, k' \} \in \Gamma$. The bracket $[\dots]$ is evaluated at $s_{\sim\Gamma} = 0$.

Each Γ divides the partition $\{ U_k \}$ into connected components. If we take the union of the U_k's in each connected component we get a partition of $L^{-n}\Lambda$ of the form $\{ LX_i \}$ where $\{ X_i \}$ is a partition of $L^{-(n+1)}\Lambda$ into paved sets. We classify the terms in our sum by the $\{ X_i \}$ they determine. Since $s_{\sim\Gamma} = 0$, $\mathcal{M}_n^s$ does not connect $\{ LX_i \}$ so for $Y_\beta \subset LX_i$ we have that $\mathcal{V}'_{n,Y_\beta}(\mathcal{A}^s)$ depends on $\bar{A}$ only in LX_i. Then the integrals factor into contributions from each LX_i denoted ρ_{X_i}. We have

$$(4.10) \qquad \exp(-\mathcal{W}(\mathcal{A})) = \sum_{\{X_i\}} \prod_i \rho_{X_i}(\mathcal{A})$$

where

$$(4.11) \quad \rho_X(\mathscr{A}) = \sum_{\{Y_\beta\}} \sum_{\Gamma} \int ds_\Gamma \partial/\partial s_\Gamma \left[\int \prod_\beta (\exp(-\mathscr{V}'_{n,Y_\beta}(\mathscr{A}^s)) - 1) d\mu_{n,LX}(\bar{A}) \right].$$

Here the sum is over collections of subsets $\{ Y_\alpha \}$ of LX. The $\{ Y_\beta \}$ determine a partition $\{ U_k \}$ of LX as above and the sum over Γ is over all collections of pairs $\{ k, k' \}$ so LX is connected.

Note that $\rho_X(\mathscr{A})$ depends on $\mathscr{A}$ only in X. Also the gauge invariance of ρ_X follows from the gauge invariance of $\mathscr{V}'_{n,Y}$.

STEP 2. — The major contribution to (4.10) comes from ρ_X for $| X | = 1$, i. e. $X = \Delta$ for some block Δ. In this case $\Gamma = \emptyset$ and doing the sum over $\{ Y_\beta \}$ we have

$$(4.12) \qquad \rho_\Delta(\mathscr{A}) = \int \exp(-\mathscr{V}^{\#}(\mathscr{A}^0)) d\mu_{n,L\Delta}(\bar{A})$$

where

$$\mathscr{V}^{\#}(\mathscr{A}^0) = \sum_{Y \subset L\Delta} \mathscr{V}'_{n,Y}(\mathscr{A}^0).$$

To estimate $\log \rho_\Delta$ we define

$$\langle F \rangle_t = \int F \exp(-t\mathscr{V}^{\#}(\mathscr{A}^0)) d\mu_{n,L\Delta}(\bar{A}) / [F = 0]$$

and then

$$(4.13) \quad \log(\rho_\Delta(\mathscr{A})) = -\langle \mathscr{V}^{\#}(\mathscr{A}^0) \rangle_0 + \int_0^1 (1-t) \langle \mathscr{V}^{\#}(\mathscr{A}^0), \mathscr{V}^{\#}(\mathscr{A}^0) \rangle_t^T dt - \log\left(\int d\mu_{n,L\Delta}(\bar{A}) \right)$$

where $\langle \cdot, \cdot \rangle^T$ is the truncated expectation.

For $\mathscr{A} \in L/2\mathscr{K}_{n+1}$ and $| \bar{A} | \leq (n_0 + n)^\nu$ we have that $\mathscr{A}^0$ (restricted to $L\Delta$) is in $\mathscr{K}_n(L\Delta)$ and so $| \mathscr{V}^{\#}(\mathscr{A}^0) | \leq \mathcal{O}(\delta_n^3)$. It follows that the truncated term in (4.13) is $\mathcal{O}(\delta_n^6)$. The last term is $\mathcal{O}(\exp(-1/4(n_0 + n)^\nu) \leq \mathcal{O}(\delta_n^6)$ as well (assuming $\nu > 1$).

The first term $\langle \mathscr{V}^{\#}(\mathscr{A}^0) \rangle_0$ is $\mathcal{O}(\delta_n^3)$ but if we isolate the low order terms in $\mathscr{A}$ we can do better. The constant term is $\langle \mathscr{V}^{\#}(\mathscr{M}_n(\cdot)) \rangle_0$ and is $\mathcal{O}(\delta_n^3)$. For the quadratic term put $\mathscr{A}_\lambda^0 = \lambda \sigma_L \mathscr{A} + \mathscr{M}^0 \bar{A}$ and we have

$$(4.14) \quad \frac{1}{2} \frac{d^2}{d\lambda^2} \langle \mathscr{V}^{\#}(\mathscr{A}_\lambda^0) \rangle_0 |_{\lambda=0} = \frac{1}{2} \frac{d^2}{d\lambda^2} \sum_{Y \subset L\Delta} [\langle \mathscr{V}_{n,2,Y}(\mathscr{A}_\lambda^0) \rangle_0 + \langle \mathscr{V}_{n,4,Y}(\mathscr{A}_\lambda^0) \rangle_0 + \langle \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}_\lambda^0) \rangle_0] |_{\lambda=0}.$$

The first term we keep and identify as

$$(4.15)\qquad \mathscr{W}^*_{2,\Delta}(\mathscr{A}) = \sum_{Y\subset L\Delta} \mathscr{V}_{n,2,Y}(\sigma_L\mathscr{A}).$$

For the second term we note that $\sum_{Y\subset L\Delta} \langle \mathscr{V}_{n,4,Y}(\mathscr{A}^0_\lambda)\rangle_0$ is analytic in the disc $|\lambda| \leq 1$ and bounded by $\mathscr{O}(\delta_n^4)$. It follows by the Cauchy bounds that this term is $\mathscr{O}_2(^4_n)$. Similarly the third term is $\mathscr{O}_2(\delta_n^5)$. (Here the subscript n in $\mathscr{O}_n(\cdot)$ denotes the order in $\mathscr{A}$.) The quartic term is $1/4!\, d^4/d\lambda^4 \langle \mathscr{V}^\#(\mathscr{A}^0_\lambda)\rangle_0|_{\lambda=0}$.

The leading term comes from $\mathscr{V}_{n,4,Y}$ and is

$$(4.16)\qquad \mathscr{W}^*_{4,\Delta}(\mathscr{A}) = \sum_{Y\subset L\Delta} \mathscr{V}_{n,4,Y}(\sigma_L\mathscr{A}).$$

The other contribution is $1/4!\, d^4/d\lambda^4 \sum_{Y\subset L\Delta} \langle \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}^0_\lambda)\rangle_0 |_{\lambda=0}$ and is $\mathscr{O}_4(\delta_n^5)$. Finally the terms of sixth order and higher are given by

$$(4.17)\qquad \mathscr{W}^*_{\geq 6,\Delta}(\mathscr{A}) \equiv 1/5!\int_0^1 (1-\lambda)^5 d^6/d\lambda^6\left[\sum_{Y\subset L\Delta} \langle \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}^0_\lambda)\rangle_0\right]d\lambda.$$

In summary if we define $\mathscr{W}^*_\Delta = \mathscr{W}^*_{2,\Delta} + \mathscr{W}^*_{4,\Delta} + \mathscr{W}^*_{\geq 6,\Delta}$ we have

$$(4.18)\qquad \log\rho_\Delta(\mathscr{A}) = -\mathscr{W}^*_\Delta(\mathscr{A}) + \mathscr{O}_0(\delta_n^3) + \mathscr{O}_2(\delta_n^4) + \mathscr{O}_4(\delta_n^5) + \mathscr{O}_{\geq 6}(\delta_n^6).$$

STEP 3. — Next we estimate ρ_X for $|X| \geq 2$. We introduce another parameter $r \geq 1$ for comparison purposes. We will choose r sufficiently large depending on L, N_0, but then allow C, n_0 to depend on r. Thus the order of choice is L, N_0, r, C, n_0.

In the definition of $\mathscr{M}^s_n$ we allow s to be complex and satisfy:

$$(4.19)\qquad |s_{\{k,k'\}}| \leq 2r\exp(\beta/2\, d(U_k, U'_k)).$$

Then we have

$$(4.20)\qquad \begin{aligned} |\mathscr{M}^s_{n,\mu}(x,\sigma)| &\leq \mathscr{O}(1) r\exp(-\beta/2\, d(x,\sigma)) \\ |\partial_\rho \mathscr{M}^s_{n,\mu}(x,\sigma)| &\leq \mathscr{O}(1) r\exp(-\beta/2\, d(x,\sigma)) \end{aligned}$$

$$|\partial_\rho \mathscr{M}^s_{n,\mu}(x,\sigma) - \partial_\rho \mathscr{M}^s_{n,\mu}(y,\sigma)| \leq \mathscr{O}(1) r|x-y|^{\frac{1}{2}}(\exp(-\beta/2\, d(x,\sigma)) + \exp(-\beta/2\, d(y,\sigma)).$$

Here we assume the relevant variables lie in a single U_k so that χ_k is not differentiated. With the same restriction we have for $\mathscr{A} \in L/2\mathscr{K}_{n+1}$ and $|\bar{A}| \leq (n_0+n)^\nu$ that $\mathscr{A}^s = \sigma_L\mathscr{A} + \mathscr{M}^s_n\bar{A}$ satisfies the bounds (4.2) (this follows as in (4.7) for C sufficiently large) and so restriction of $\mathscr{A}^s$ to each

Y_β is in $\mathscr{K}_n(Y_\beta)$. Thus $\mathscr{V}_{n,2,Y_\beta}(\mathscr{A}^s)$, etc. are analytic for s in the polydisc (4.19) and satisfy the bounds (4.4) there. We also have

$$|\exp(\mathscr{V}'_{n,Y_\beta}(\mathscr{A}^s)) - 1| \leq \mathcal{O}(\delta_n^3)\exp(-\alpha\mathscr{L}(Y_\beta))$$

(provided n_0 is sufficiently large).

With these estimates in mind we have

$$(4.21)\quad |\rho_X(\mathscr{A})| \leq \sum_{\{Y_\beta\}}\sum_{\Gamma}\Big[\prod_{\{k,k'\}\in\Gamma} r^{-1}\exp(-\beta/2\, d(U_k, U_{k'})\Big]\Big[\prod_\beta \mathcal{O}(\delta_n^3)\exp(-\alpha\mathscr{L}(Y_\beta))\Big].$$

Here the first bracketed expression comes from using the Cauchy bounds to estimate the derivatives $\partial/\partial s_\Gamma$.

To estimate ρ_X we use the bound $\mathscr{L}(Y) \geq \eta L\mathscr{L}(L^{-1}\tilde{Y}) + \mathcal{O}(1)$ where $\tilde{Y}$ is the union of all $L\Delta$ blocks intersecting Y and η is small and independent of L [*9*], [*10*]. Then

$$(4.22)\quad \frac{\beta}{4}\sum_{\{k,k'\}\in\Gamma} d(U_k, U_{k'}) + \frac{\alpha}{2}\sum_\beta \mathscr{L}(Y_\beta)$$
$$\geq \frac{\alpha\eta L}{2}\Big[\sum_{\{k,k'\}\in\Gamma} d(L^{-1}U_k, L^{-1}U_{k'}) + \sum_k \mathscr{L}(L^{-1}U_k)\Big] + \mathcal{O}(1)\,|\{Y_\beta\}|$$
$$\geq 12\alpha\mathscr{L}(X) + \mathcal{O}(1)\,|\Gamma| + \mathcal{O}(1)\,|\{Y_\beta\}|.$$

The last step follows by taking L sufficiently large and noting that Γ connects the partition $\{L^{-1}U_k\}$ of X. (We also use that $d(L^{-1}U_k, L^{-1}U_{k'})$ differs from a distance to block centers by $\mathcal{O}(1)$.) Thus we may extract a factor $\exp(-12\alpha\mathscr{L}(X))$ from (4.21) at a cost $\exp(\mathcal{O}(1)\,|\Gamma|)$ which we absorb into $\Pi_r\mathcal{O}(r^{-1})$ and a cost $\exp(\mathcal{O}(1)\,|\{Y_\beta\}|)$ which we absorb into $\prod_\beta \mathcal{O}(\delta_n^3)$.

The sum over graphs Γ connecting the U_k's is estimated by a sum over all graphs Γ on the U_k's and gives for r large

$$(4.23)\quad \sum_\Gamma \prod_{\{k,k'\}\in\Gamma} \mathcal{O}(r^{-1})\exp(-\beta/4\, d(U_k, U_{k'}))$$
$$\leq \prod_{\{k,k'\}} (1 + \mathcal{O}(r^{-1})\exp(-\beta/4\, d(U_k, U_{k'})))$$
$$\leq \exp\Big(\sum_{\{k,k'\}} \mathcal{O}(r^{-1})\exp(-\beta/4\, d(U_k, U_{k'}))\Big)$$
$$\leq \exp\Big(\mathcal{O}(r^{-1})\sum_k |U_k|\Big)$$
$$\leq \exp(|LX|).$$

The sum over $\{ Y_\beta \}$ is estimated by

$$(4.24) \qquad \sum_{\{Y_\beta\}} \prod_\beta \mathcal{O}(\delta_n^3) \exp(-\alpha\mathcal{L}(Y_\beta)) \leq \mathcal{O}(\delta_n^3) \exp(2 \mid LX \mid).$$

Here we use the fact that $\{ Y_\beta \} = \emptyset$ only contributes for $| X | = 1$ to extract the factor $\mathcal{O}(\delta_n^3)$. Combining these we have for $| X | \geq 2$

$$(4.25) \qquad | \rho_X(\mathcal{A}) | \leq \mathcal{O}(\delta_n^3) \exp(-10\alpha\mathcal{L}(X)).$$

We can get sharper bounds by considering separately the terms with $| \{ Y_\beta \} | = 1$. Terms with $| \{ Y_\beta \} | \geq 2$ are estimated exactly as above but now we may extract a factor $\mathcal{O}(\delta_n^6)$. For the terms with a single Y_β we expand $\exp(-\mathcal{V}) - 1 = -\mathcal{V} + \mathcal{O}(\mathcal{V}^2)$ and estimate the $\mathcal{O}(\mathcal{V}^2)$ as above to get another $\mathcal{O}(\delta_n^6)$ term. We are left with

$$(4.26) \qquad -\sum_{Y \subset LX} \sum_\Gamma \int ds_\Gamma \partial/\partial s_\Gamma \left[\int \mathcal{V}'_{n,Y}(\mathcal{A}^s) d\mu_{n,LX}(\bar{A}) \right].$$

We distinguish between $\Gamma = \emptyset$ which occurs when $\tilde{Y} = LX$ and $\Gamma \neq \emptyset$. If $\Gamma \neq \emptyset$ we again estimate as above taking a factor $r^{-\frac{1}{2}}$ from the $\partial/\partial s_\Gamma$. Treating the contributions from $\mathcal{V}_{n,2,Y}, \mathcal{V}_{n,4,Y}$ and $\mathcal{V}_{n,\geq 6,Y}$ separately we obtain a term of the form:

$$[\mathcal{O}_{\leq 2}(r^{-\frac{1}{2}}\delta_n^3) + \mathcal{O}_{\leq 4}(r^{-\frac{1}{2}}\delta_n^4) + \mathcal{O}(r^{-\frac{1}{2}}\delta_n^5)] \exp(-10\alpha\mathcal{L}(X)).$$

Terms with $\Gamma = \emptyset$ may be analyzed as the $X = \Delta$ terms in Step 2. Now we use

$$(4.27) \qquad \sum_{\tilde{Y} = LX} \exp(-\alpha\mathcal{L}(Y)) \leq \mathcal{O}(1) \exp(-10\alpha\mathcal{L}(X))$$

and obtain

$$-\mathcal{W}_X^*(\mathcal{A}) + [\mathcal{O}_0(\delta_n^3) + \mathcal{O}_2(\delta_n^4) + \mathcal{O}_4(\delta_n^5)] \exp(-10\alpha\mathcal{L}(x))$$

where $\mathcal{W}_X^*$ is a sum of the following

$$(4.28) \qquad \mathcal{W}_{2,X}^*(\mathcal{A}) = \sum_{Y:\tilde{Y} = LX} \mathcal{V}_{n,2,Y}(\sigma_L\mathcal{A})$$

$$\mathcal{W}_{4,X}^*(\mathcal{A}) = \sum_{Y:\tilde{Y} = LX} \mathcal{V}_{n,4,Y}(\sigma_L\mathcal{A})$$

$$\mathcal{W}_{\geq 6,X}^*(\mathcal{A}) = 1/5! \int_0^1 (1-\lambda)^5 d^6/d\lambda^6 \left[\sum_{Y:\tilde{Y} = LX} \int \mathcal{V}_{n,\geq 6,Y}(\mathcal{A}_\lambda^0) d\mu_{n,LX}(\bar{A}) \right].$$

Combining all the above and using $\delta_n \leq r^{-\frac{1}{2}}$ we have

$$(4.29) \quad \rho_X(\mathscr{A}) = -\mathscr{W}_X^*(\mathscr{A}) + \left[\mathcal{O}_0(\delta_n^3) + \mathcal{O}_2(r^{-\frac{1}{2}}\delta_n^3) + \mathcal{O}_4(r^{-\frac{1}{2}}\delta_n^4) + \mathcal{O}_{\geq 6}(r^{-\frac{1}{2}}\delta_n^5)\right] \exp(-10\alpha\mathscr{L}(X)).$$

STEP 4. — Now we take the logarithm of the expansion (4.10) for $\exp(-\mathscr{W})$ to define $\mathscr{W}$ and obtain a local expansion for it. As in Theorem 2.1 we have $\mathscr{W} = \Sigma_Y \mathscr{W}_Y$ where $\mathscr{W}_\Delta = -\log \rho_\Delta$ and for $|Y| \geq 2$

$$(4.30) \quad \mathscr{W}_Y(\mathscr{A}) = \sum_{n=1}^{\infty} 1/n! \sum_{(X_1,\ldots,X_n):\cup X_i = Y} a(X_1, \ldots, X_n) \prod_{i=1}^{n} \tilde{\rho}_{X_i}(\mathscr{A})$$

where $\tilde{\rho}_X = \rho_X(\Pi_{\Delta \subset X}\rho_\Delta)^{-1}$.

Using our bound (4.25) on ρ_X and $|\log \rho_\Delta| \leq \mathcal{O}(\delta_n^3)$ we obtain as before $|\tilde{\rho}_X| \leq \mathcal{O}(\delta_n^3)\exp(-9\alpha\mathscr{L}(X))$ and hence $|\mathscr{W}_Y| \leq \mathcal{O}(\delta_n^3)\exp(-8\alpha\mathscr{L}(X))$. If we only consider terms with $n \geq 2$ in (4.30) this can be improved to $\mathcal{O}(\delta_n^6)\exp(-8\alpha\mathscr{L}(Y))$. The $n = 1$ term is just $\tilde{\rho}_Y$ which is ρ_Y plus $\mathcal{O}(\delta_n^6)\exp(-8\alpha\mathscr{L}(X))$. Thus using our expression (4.29) for ρ_Y we have (still for $\mathscr{A} \in L/2\mathscr{K}_{n+1}$)

$$(4.31) \quad \mathscr{W}_Y(\mathscr{A}) = \mathscr{W}_Y^*(\mathscr{A}) + \left[\mathcal{O}_0(\delta_n^3) + \mathcal{O}_2(r^{-\frac{1}{2}}\delta_n^3) + \mathcal{O}_4(r^{-\frac{1}{2}}\delta_n^4) + \mathcal{O}_{\geq 6}(r^{-\frac{1}{2}}\delta_n^5)\right] \exp(-8\alpha\mathscr{L}(Y)).$$

This is established for $|Y| \geq 2$, but the same result holds for $Y = \Delta$ by (4.18). Note that $\mathscr{W}_Y$ depends on $\mathscr{A}$ only in Y, and is gauge invariant.

STEP 5. — We now work on the quadratic term $\mathscr{W}_2$ and rearrange it into the desired form. We have $\mathscr{W}_{2,Y} = \mathscr{W}_{2,Y}^* + \delta\mathscr{W}_{2,Y}$ where

$$|\delta\mathscr{W}_{2,Y}| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3)\exp(-8\alpha\mathscr{L}(Y))$$

on $L/2\mathscr{K}_{n+1}$. Summing over Y gives $\mathscr{W}_2 = \mathscr{W}_2^* + \delta\mathscr{W}_2$. It is straightforward to make the identification

$$(4.32) \quad \mathscr{W}_2^*(\mathscr{A}) = \int_{x,y} d\mathscr{A}_{\mu\nu}(x)(L^4\mathscr{S}_{\mu\nu\rho\sigma}^n(Lx, Ly))(d\mathscr{A}_{\rho\sigma}(y) - d\mathscr{A}_{\rho\sigma}(x)).$$

We need a similar expression for $\delta\mathscr{W}_2$.

It will be sufficient to obtain the representation assuming $\mathscr{A} = \mathscr{H}_{n+1}A$ for a unit lattice field A; in the general case we will enforce it by using the freedom to pick $\delta\mathscr{V}_{n+1}$. Accordingly we define for $A \in [\Lambda_{n+1}^* : \mathbb{C}]$

$$(4.33) \quad \delta W_{2,Y}(A) = \delta\mathscr{W}_{2,Y}(\mathscr{H}_{n+1}A).$$

Note that $\delta W_{2,Y}(A)$ is only approximately localized in Y. We adopt in this section the convention that unit lattice variables are capitalized: $B \in \Lambda_{n+1}^*$, $P \in \Lambda_{n+1}^{**}$, etc. If

$$(4.34) \quad |A(B)| \leq C_n \exp(4\alpha d(B, Y))$$

then $\mathscr{H}_{n+1}A$ restricted to Y is in $\mathcal{O}(1)\mathscr{H}_{n+1}(Y)$ and we have

$$(4.35)\qquad |\delta W_{2,Y}(A)| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3)\exp(-8\alpha\mathscr{L}(Y)).$$

Now let $\delta\Pi_Y$ be the kernel of $\delta W_{2,Y}$ so

$$(4.36)\qquad \delta W_{2,Y}(A) = \sum_{B,B'} A(B)\delta\Pi_Y(B,B')A(B').$$

By analyticity in the region (4.34) and the Cauchy bounds:

$$(4.37)\quad |\delta\Pi_Y(B,B')| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3 C_n^{-2})\exp(-8\alpha(d(B,Y)+d(B',Y)+\mathscr{L}(Y))).$$

We need the fact that $\delta W_{2,Y}$ is gauge invariant. This follows from the gauge invariance of $\delta\mathscr{W}_{2,Y}$ and the fact that $\mathscr{H}_{n+1}(d\lambda) = d\tilde{\lambda}$ for some $\tilde{\lambda}$ on $L^{-(n+1)}\Lambda$.

Because of the gauge invariance we may change from A to dA in $\delta W_{2,Y}(A)$ as in Theorem 2.4. Making this change and then summing over Y gives for $\delta W_2 = \Sigma_Y \delta W_{2,Y}$:

$$(4.38)\qquad \delta W_2(A) = \sum_{P,P'} dA(P)\delta I(P,P')dA(P') + \delta U_2(A)$$

where $\delta I = \Sigma_Y \delta I_Y$ and

$$(4.39)\qquad \delta I_Y(P,P') = \sum_{B,B'\in\mathcal{O}_y} \chi_{S_{y,B}}(P)\chi_{S_{y,B'}}(P')\delta\Pi_Y(B,B').$$

To estimate this we use the bound on $\delta\Pi_Y$ and $d(B,Y)+\mathscr{L}(Y) \geq d(B,y)+\mathcal{O}(1)$ and $d(B,y) \geq d(P,y)$ to obtain as in the proof of Theorem 2.4 that for $P \in \Delta^{**}$, $P \in \Delta'^{**}$

$$(4.40)\qquad |\delta I(P,P')| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3 C_n^{-2})\exp(-2\alpha\mathscr{L}(\Delta\cup\Delta')).$$

Similarly we obtain for say $|A| \leq 2C_{n+1}$

$$(4.41)\qquad |\delta U_2(A)| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3)\,|\Lambda_{n+1}|^{-1}.$$

One can argue that δW_2 is invariant under lattice symmetries so that averaging over lattice symmetries give symmetric δI, δU_2 satisfying bounds of the same form.

Now we return to $\delta\mathcal{W}_2$. Since $\mathcal{Q}_n \circ \mathcal{H}_n = id$ we have for $\mathcal{A} = \mathcal{H}_{n+1}\mathrm{A}$ that $\delta\mathcal{W}_2(\mathcal{A}) = \delta\mathrm{W}_2(\mathcal{Q}_n\mathcal{A})$. Also then for $\mathrm{P} \in \Lambda^{**}_{n+1}$, $p \in \mathrm{L}^{-(n+1)}\Lambda^{**}$

$$(d(\mathcal{Q}_{n+1}\mathcal{A}))(\mathrm{P}) = \sum_{\mathrm{B}\in\partial\mathrm{P}} \int_{|x_\mu|<1/2} \mathcal{A}(\mathrm{B}+x) \tag{4.42}$$
$$= \int_{|x_\mu|<1/2} \int_{b\in\partial(\mathrm{P}+x)} \mathcal{A}(b)$$
$$= \int_{|x_\mu|<1/2} \int_{p\in\mathrm{P}+x} d\mathcal{A}(p)$$
$$= \int_p h(\mathrm{P},p)d\mathcal{A}(p)$$

where

$$h(\mathrm{P},p) = \int_{|x_\mu|<1/2} \chi_{\mathrm{P}+x}(p) \tag{4.43}$$

and χ_P is the characteristic function the set of plaquettes contained in P. Note that:

$$\sum_{\mathrm{P}:\mathrm{P}||p} h(\mathrm{P},p) = 1 \qquad \int_{p:\mathrm{P}||p} h(\mathrm{P},p) = 1\,. \tag{4.44}$$

Thus we have

$$\delta\mathcal{W}_2(\mathcal{A}) = \int d\mathcal{A}_{\mu\nu}(x)\delta\mathcal{I}_{\mu\nu\rho\sigma}(x,y)d\mathcal{A}_{\rho\sigma}(y) + \delta\mathcal{U}_2(\mathcal{A}) \tag{4.45}$$

where

$$\delta\mathcal{I}(p,p') = \sum_{\mathrm{P},\mathrm{P}'} \delta\mathrm{I}(\mathrm{P},\mathrm{P}')h(\mathrm{P},p)h(\mathrm{P}',p') \tag{4.46}$$

and $\delta\mathcal{U}_2(\mathcal{A}) = \delta\mathrm{U}_2(\mathcal{Q}_{n+1}\mathcal{A})$. Since h is strictly localized $\delta\mathcal{I}$ satisfies the same bound (4.40) as $\delta\mathrm{I}$. Also if $|\mathcal{A}| \le 2\mathrm{C}_{n+1}$ then $|\mathcal{Q}_n\mathcal{A}| \le 2\mathrm{C}_{n+1}$ and so the bound (4.41) holds for $\delta\mathcal{U}_2(\mathcal{A})$.

Using the symmetry of $\delta\mathrm{I}$ and (4.44) we may define a constant k_{n+1} by

$$\int_y \delta\mathcal{I}_{\mu\nu\rho\sigma}(x,y) = \sum_y \delta\mathrm{I}_{\mu\nu\rho\sigma}(x,y) = \frac{1}{2}k_{n+1}\delta_{\mu\rho}\delta_{\nu\sigma}\,. \tag{4.47}$$

This enables us to extract the relevant part of $\mathcal{W}_2$. Combining (4.32), (4.45), and (4.47) we have

$$\mathcal{W}_2(\mathcal{A}) = \frac{1}{2}k_{n+1}\,\|\, d\mathcal{A}\,\|^2 \tag{4.48}$$
$$+ \int_{x,y} d\mathcal{A}_{\mu\nu}(x)\mathcal{I}^{0,n+1}_{\mu\nu\rho\sigma}(x,y)(d\mathcal{A}_{\rho\sigma}(y) - d\mathcal{A}_{\rho\sigma}(x)) + \delta\mathcal{U}_2(\mathcal{A})$$

where

$$\mathcal{I}^{0,n+1}_{\mu\nu\rho\sigma}(x, y) = L^4 \mathcal{I}^n_{\mu\nu\rho\sigma}(Lx, Ly) + \delta\mathcal{I}_{\mu\nu\rho\sigma}(x, y). \tag{4.49}$$

We complete this stage of the analysis with some estimates. Changing variables and using the estimate on $\mathcal{I}_n$ gives

$$\begin{aligned}(4.50)\quad \int_{x,y\in\Delta\times\Delta'} & | L^4 \mathcal{I}^n_{\mu\nu\rho\sigma}(Lx, Ly) | \, | x - y |^{\frac{1}{2}} \\ & \leq C_n^{-2}\delta_n^3 L^{-1/2}\left[L^{-4} \sum_{\delta\times\delta'\subset L\Delta\times L\Delta'} \exp(-\alpha\mathcal{L}(\delta\cup\delta'))\right] \\ & \leq \frac{1}{4} C_{n+1}^{-2}\delta_{n+1}^3 \exp(-\alpha\mathcal{L}(\Delta\cup\Delta')).\end{aligned}$$

Here δ, δ' are blocks like Δ, Δ'. The last step requires some explanation. If $\Delta = \Delta'$ then the bracketed expression is bounded by 2 and for L sufficiently large the $L^{-1/2}$ gives the small factor needed to get the last line. For $\Delta \neq \Delta'$ then there are fewer than L^3 terms in the sum over δ, δ' with $\mathcal{L}(\delta\cup\delta') = \mathcal{L}(\Delta\cup\Delta')$. Otherwise $\mathcal{L}(\delta\cup\delta') \geq \mathcal{L}(\Delta\cup\Delta' + L^{N_0}$. From this we get that the bracketed expression is bounded by

$$L^{-1}\exp(-\alpha\mathcal{L}(\Delta\cup\Delta'))$$

which is sufficient to get the last line.

We also have by the estimate from (4.40) for $\delta\mathcal{I}$ and for r sufficiently large

$$\begin{aligned}(4.51)\quad \int_{\Delta\times\Delta'} |(\delta\mathcal{I})_{\mu\nu\rho\sigma}(x, y)| \, | x - y |^{\frac{1}{2}} & \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3 C_n^{-2})\exp(-\alpha\mathcal{L}(\Delta\cup\Delta')) \\ & \leq \frac{1}{4} C_{n+1}^{-2}\delta_{n+1}^3 \exp(-\alpha\mathcal{L}(\Delta\cup\Delta')).\end{aligned}$$

Combining these estimates gives

$$(4.52)\quad \int_{\Delta\times\Delta'} |\mathcal{I}^{0,n+1}_{\mu\nu\rho\sigma}(x, y)| \, |x-y|^{\frac{1}{2}} \leq \frac{1}{2} C_{n+1}^{-2}\delta_{n+1}^3 \exp(-\alpha\mathcal{L}(\Delta\cup\Delta')).$$

We also easily establish from (4.40), (4.47) for r large

$$| k_{n+1} | \leqslant \mathcal{O}(r^{-\frac{1}{2}}\delta_n^3 C_n^{-2}) \leq \frac{1}{2}\delta_{n+1}^3. \tag{4.53}$$

STEP 6. — The analysis of the quartic part of $\mathcal{W}$ is quite similar to the analysis of the quadratic part. We have $\mathcal{W}_4 = \mathcal{W}_4^* + \delta\mathcal{W}_4$ where

$$\mathcal{W}_4^*(\mathcal{A}) = \int_{p_1,\ldots,p_4} L^8 \mathcal{G}_n(Lp_1, \ldots, Lp_4) d\mathcal{A}(p_1) \ldots d\mathcal{A}(p_4) \tag{4.54}$$

and where $\delta\mathcal{W}_4 = \Sigma_Y \delta\mathcal{W}_{4,Y}$ with $|\delta\mathcal{W}_{4,Y}| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta_n^4)\exp(-8\alpha\mathcal{L}(Y))$.

Now we treat $\delta\mathscr{W}_{4,Y}$ as we treated $\delta\mathscr{W}_{2,Y}$: change to unit lattice variables A, make the transformation from A to dA as in Theorem 2.4 and then change back to $d\mathscr{A}$. This yields

$$(4.55)\quad \delta\mathscr{W}_4(\mathscr{A}) = \int_{p_1,\dots,p_4} \delta\mathscr{J}(p_1, \dots, p_4) d\mathscr{A}(p_1) \dots d\mathscr{A}(p_4) + \delta\mathscr{U}_4(\mathscr{A})$$

where for $p_1 \in \Delta_1^{**}, \dots, p_4 \in \Delta_4^{**}$:

$$(4.56)\quad |\delta\mathscr{J}(p_1, \dots, p_4)| \leq \mathscr{O}(r^{-\frac{1}{2}}\delta_n^4 C_n^{-4}) \exp(-\alpha\mathscr{L}(\Delta_1 \cup \dots \cup \Delta_4))$$

and where for $|\mathscr{A}| \leq 2C_{n+1}$

$$(4.57)\quad |\delta\mathscr{U}_4(\mathscr{A})| \leq \mathscr{O}(r^{-\frac{1}{2}}\delta_n^4) |\Lambda_{n+1}|^{-1}.$$

Combining these gives

$$(4.58)\quad \mathscr{W}_4(\mathscr{A}) = \int_{p_1,\dots,p_4} \mathscr{J}_{n+1}^0(p_1, \dots, p_4) d\mathscr{A}(p_1) \dots d\mathscr{A}(p_4) + \delta\mathscr{U}_4(\mathscr{A})$$

$$\mathscr{J}_{n+1}^0(p_1, \dots, p_4) = L^8 \mathscr{J}_n(Lp_1, \dots, Lp_4) + \delta\mathscr{J}(p_1, \dots, p_4).$$

As in the estimate on $\mathscr{J}_{n+1}^0$ one can establish

$$(4.59)\quad \int_{(\Delta_1 \times \dots \times \Delta_4)^{**}} |\mathscr{J}_{n+1}^0(p_1, \dots, p_4)| \leq \frac{1}{2} C_{n+1}^{-4} \delta_{n+1}^4 \exp(-\alpha\mathscr{L}(\Delta_1 \cup \dots \cup \Delta_4)).$$

STEP 7. — We next estimate higher order terms in $\mathscr{W}$. We have $\mathscr{W}_{\geq 6,Y} = \mathscr{W}_{\geq 6,Y}^* + \delta\mathscr{W}_{\geq 6,Y}$ where $\mathscr{W}_{\geq 6,Y}^*$ is given by (4.28) (even for $Y = \Delta$) and where $|\delta\mathscr{W}_{\geq 6,Y}(\mathscr{A})| \leq \mathscr{O}(r^{-\frac{1}{2}}\delta_n^5) \exp(-8\alpha\mathscr{L}(Y))$ for $\mathscr{A} \in L/2\mathscr{K}_{n+1}$.

To get a sharp estimate on $\mathscr{W}_{\geq 6,Y}^*$ note that for $\mathscr{A} \in L/4\mathscr{K}_{n+1}$ and $|\lambda| \leq 2$ we have that $\mathscr{A}_\lambda^0$ restricted to Y is in $\mathscr{K}_n(Y)$ and so

$$|\mathscr{V}_{n,\geq 6,Y}(\mathscr{A}_\lambda^0)| \leq \delta_n^5 \exp(-\alpha\mathscr{L}(Y)).$$

By the Cauchy bounds we have for $|\lambda| \leq 1$ that

$$|d^6/d\lambda^6 \mathscr{V}_{n,\geq 6,Y}(\mathscr{A}_\lambda^0)| \leq 6!\delta_n^5 \exp(-\alpha\mathscr{L}(Y)).$$

We also use instead of (4.27) the estimate for $|X| \geq 2$

$$(4.60)\quad \sum_{Y:\tilde{Y}=LX} \exp(-\alpha\mathscr{L}(Y)) \leq (L^3 + 1) \exp(-\alpha\mathscr{L}(X)).$$

This follows (after some work) from the fact that there are less than or equal to L^3 choice of Y in this sum with $\mathscr{L}(Y) = \mathscr{L}(X)$ and for the others $\mathscr{L}(Y) \geq \mathscr{L}(X) + L^{N_0}$. For $|X| = 1$ the same sum is bounded by $L^4 + 1$. Thus we have for $\mathscr{A} \in L/4\mathscr{K}_{n+1}$ and $|Y| \geq 1$

$$(4.61)\quad |\mathscr{W}_{\geq 6,Y}^*(\mathscr{A})| \leq (L^4 + 1)\delta_n^5 \exp(-\alpha\mathscr{L}(Y)).$$

Since $\mathscr{W}^*_{\geq 6,\mathrm{Y}}$ is of order six or larger, if we cut down to $\mathscr{A} \in 2\mathscr{K}_{n+1}$ we gain a factor of $(8/\mathrm{L})^6$. This is enough to dominate the $\mathrm{L}^4 + 1$ and give a small factor and so for L sufficiently large

$$| \mathscr{W}^*_{\geq 6,\mathrm{Y}}(\mathscr{A}) | \leq \frac{1}{2} \delta^5_{n+1} \exp(-\alpha\mathscr{L}(\mathrm{Y})).$$

This same bound holds for $\delta\mathscr{W}_{\geq 6,\mathrm{Y}}$ for r sufficiently large and so we have for $\mathscr{A} \in 2\mathscr{K}_{n+1}$

$$(4.62) \qquad | \mathscr{W}_{\geq 6,\mathrm{Y}}(\mathscr{A}) | \leq \delta^5_{n+1} \exp(-\alpha\mathscr{L}(\mathrm{Y})).$$

STEP 8. — The analysis of $\mathscr{W}$ is complete and we now look at what it says about $\mathscr{V}_{n+1}$. First define

$$(4.63) \quad \exp(-\hat{\mathscr{V}}_{n+1}(\mathscr{A})) = \exp\left(\frac{1}{2}(1-\zeta_{n+1}) \| d\mathscr{A} \|^2 - \delta\mathscr{V}_{n+1}(\mathscr{A})\right) \int \exp(-\mathscr{V}'_n(\zeta^{\frac{1}{2}}_{n+1}\sigma_{\mathrm{L}}\mathscr{A} + \mathscr{M}_n\overline{\mathrm{A}}))d\mu_n(\overline{\mathrm{A}}) / [\mathscr{A} = 0].$$

This differs from $\exp(-\mathscr{V}_{n+1}(\mathscr{A}))$ only by the $\mathscr{V}'_n$ under the integral instead of $\mathscr{V}_n$, i. e. $\mathscr{U}'_n$ is missing. Then we have

$$(4.64) \quad \hat{\mathscr{V}}_{n+1}(\mathscr{A}) = -\frac{1}{2}(1-\zeta_{n+1}) \| d\mathscr{A} \|^2 + \delta\mathscr{V}_{n+1}(\mathscr{A}) + \mathscr{W}(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) - [\mathscr{A} = 0].$$

If $\mathscr{A} = \mathscr{H}_n\mathrm{A}$ we have by (4.48), (4.58)

$$(4.65) \quad \mathscr{W}_2(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) = \frac{1}{2} k_{n+1} \| d\mathscr{A} \|^2 + \zeta_{n+1} \int d\mathscr{A}_{\mu\nu}(x)\mathscr{I}^{0,n+1}_{\mu\nu\rho\sigma}(x,y)(d\mathscr{A}_{\rho\sigma}(y) - d\mathscr{A}_{\rho\sigma}(x)) + \delta\mathscr{U}_2(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) \equiv \mathrm{F}(\mathscr{A})$$

$$\mathscr{W}_4(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) = \int \zeta^2_{n+1}\mathscr{I}^0_{n+1}(p_1, \ldots, p_4)d\mathscr{A}(p_1) \ldots d\mathscr{A}(p_4) + \delta\mathscr{U}_4(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) \equiv \mathrm{G}(\mathscr{A}).$$

Accordingly we may define for general $\mathscr{A}$

$$(4.66) \quad \delta\mathscr{V}_{n+1}(\mathscr{A}) = (\mathrm{F}(\mathscr{A}) - \mathscr{W}_2(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A})) + (\mathrm{G}(\mathscr{A}) - \mathscr{W}_4(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A})).$$

This vanishes for $\mathscr{A} = \mathscr{H}_n\mathrm{A}$ as required.

We define ζ_{n+1} so $-(1-\zeta_{n+1}) + k_{n+1}\zeta_{n+1} = 0$, i.e. $\zeta_{n+1} = (1+k_{n+1})^{-1}$. We also make the following definitions:

$$\mathscr{I}^{n+1} = \zeta_{n+1}\mathscr{I}^{0,n+1}, \ \mathscr{I}_{n+1} = \zeta^2_{n+1}\mathscr{I}^0_{n+1}, \ \mathscr{V}_{n+1,\geq 6}(\mathscr{A}) = \mathscr{W}_{\geq 6}(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A})$$

and

$$\delta\mathscr{U}_{n+1}(\mathscr{A}) = \delta\mathscr{U}_2(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}) + \delta\mathscr{U}_4(\zeta^{\frac{1}{2}}_{n+1}\mathscr{A}).$$

Then we have $\hat{\mathcal{V}}_{n+1} = \mathcal{V}'_{n+1} + \delta\mathcal{U}_{n+1}$ where $\mathcal{V}'_{n+1}$ has the form required by the theorem, i. e.

$$(4.67)\quad \mathcal{V}'_{n+1}(\mathcal{A}) = \int d\mathcal{A}_{\mu\nu}(x)\mathcal{I}^{n+1}_{\mu\nu\rho\sigma}(x, y)(d\mathcal{A}_{\rho\sigma}(y) - d\mathcal{A}_{\rho\sigma}(x)) + \int \mathcal{J}_{n+1}(p_1, \ldots, p_4)d\mathcal{A}(p_1) \ldots d\mathcal{A}(p_4) + \mathcal{V}_{n+1,\geq 6}(\mathcal{A}).$$

The bounds of the theorem are also satisfied. By (4.53), $|\zeta_{n+1} - 1| \leq \delta^3_{n+1}$. The estimates on $\mathcal{I}_{n+1}$, $\mathcal{J}_{n+1}$ then follow from (4.52) and (4.59). If $\mathcal{A} \in \mathcal{K}_{n+1}$ then $\zeta^{\frac{1}{2}}_{n+1}\mathcal{A} \in 2\mathcal{K}_{n+1}$ and so the bound on $\mathcal{V}_{n+1,\geq 6,\mathrm{Y}}(\mathcal{A})$ follows from (4.62).

Now define $\hat{\mathcal{U}}_{n+1}$ by $\mathcal{V}_{n+1} = \hat{\mathcal{V}}_{n+1} + \hat{\mathcal{U}}_{n+1}$. Then we have the required $\mathcal{V}_{n+1} = \mathcal{V}'_{n+1} + \mathcal{U}'_{n+1}$ if we define $\mathcal{U}'_{n+1} = \hat{\mathcal{U}}_{n+1} + \delta\mathcal{U}_{n+1}$. We must estimate $\mathcal{U}'_{n+1}$ for $\mathcal{A} \in \mathcal{K}_{n+1}$ and real.

The assumed bound on $\mathcal{U}'_n$ says that for $\mathcal{A} \in \mathcal{K}_n$ we have

$$|\mathcal{V}_n(\mathcal{A}) - \mathcal{V}'_n(\mathcal{A})| \leq \delta^3_n |\Lambda_n|^{-1} \qquad (|\Lambda_n| = |\mathrm{L}^{-n}\Lambda|)$$

which implies for $\mathcal{A}$ real

$$\exp(-\delta^3_n|\Lambda_n|^{-1} - \mathcal{V}'_n) \leq \exp(-\mathcal{V}_n) \leq \exp(\delta^3_n|\Lambda_n|^{-1} - \mathcal{V}'_n).$$

By (4.1), (4.63) it follows that for $\mathcal{A} \in \mathcal{K}_{n+1}$

$$\exp(-2\delta^3_n|\Lambda_n|^{-1} - \hat{\mathcal{V}}_{n+1}) \leq \exp(-\mathcal{V}_{n+1}) \leq \exp(2\delta^3_n|\Lambda_n|^{-1} - \hat{\mathcal{V}}_{n+1})$$

and hence $|\hat{\mathcal{U}}_{n+1}| = |\mathcal{V}_{n+1} - \hat{\mathcal{V}}_{n+1}| \leq 2\delta^3_n|\Lambda_n|^{-1}$. Since $|\Lambda_{n+1}|/|\Lambda_n|^{-1} = \mathrm{L}^{-4}$ this gives $|\hat{\mathcal{U}}_{n+1}| \leq 1/2\ \delta^3_{n+1}|\Lambda_{n+1}|^{-1}$. We also have by (4.41) for $\delta\mathcal{U}_2$ and (4.57) for $\delta\mathcal{U}_4$ that for $\mathcal{A} \in \mathcal{K}_{n+1}$, $|\delta\mathcal{U}_{n+1}(\mathcal{A})| \leq \mathcal{O}(r^{-\frac{1}{2}}\delta^3_n)|\Lambda_{n+1}|^{-1}$. For r sufficiently large this is bounded by $1/2\ \delta^3_{n+1}|\Lambda_{n+1}|^{-1}$. Combining these two bounds we have the required $|\mathcal{U}'_{n+1}(\mathcal{A})| \leq \delta^3_{n+1}|\Lambda_{n+1}|^{-1}$.

We have now established all the bounds for $\mathcal{V}_{n+1}$. The gauge invariance of the various pieces is evident and so we are done.

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