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Note on a construction of unbounded measures on a nonseparable Hilbert space quantum logic

by

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ABSTRACT. — We present classifications and examples of measures with possible infinite values defined on a quantum logic of all closed subspaces of an infinite-dimensional Hilbert space. Moreover, it is shown that any singular bilinear form generates no Gleason measure. The necessary and sufficient conditions in order that a positive symmetric bilinear form generates a Gleason measure are given. For a separable Hilbert space these results are due to Lugovaja [*1*], [*2*].

RÉSUMÉ. — Nous présentons une classification et des exemples de mesures avec valeurs possibles infinies qui sont définies sur une logique quantique d'un Hilbert espace de dimension infinie. On montre qu'une forme sesquilinéaire singulière ne génère aucune mesure de Gleason. On présente les conditions nécessaires et suffisantes pour qu'une forme positive symétrique sesquilinéaire génère une mesure de Gleason. Pour l'espace de Hilbert séparable ces conditions sont dues à Lugovaja [*1*], [*2*].

1. INTRODUCTION

Let L(H) be a quantum logic of all closed subspaces of a (not necessarily separable) Hilbert space H over the field C of real or complex numbers.

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A measure on L(H) is a map $m : \mathrm{L(H)} \to [0, \infty]$ such that *i*) $m(0) = 0$; *ii*) $m(\bigoplus_{n=1}^{\infty} \mathrm{M}_n) = \sum_{n=1}^{\infty} m(\mathrm{M}_n)$ whenever $\mathrm{M}_n \perp \mathrm{M}_m$ for $n \neq m$. (Here by $\bigoplus_{t \in \mathrm{T}} \mathrm{M}_t$ we denote the join of mutually orthogonal subspaces $\{ \mathrm{M}_t : t \in \mathrm{T} \}$.)

The theorem of Gleason [*3*] says that any finite measure m on a separable Hilbert space, $\dim \mathrm{H} \neq 2$, is in one-to-one correspondence with positive Hermitian operators T on H of finite trace via

$$m(\mathrm{M}) = tr\,(\mathrm{TM}), \qquad \mathrm{M} \in \mathrm{L(H)}, \tag{1.1}$$

(we identify a subspace M with its orthoprojector $\mathrm{P^M}$). The Gleason formula (1.1) has been extended to a nonseparable Hilbert space by Eilers and Horst [*4*].

For measures with $m(\mathrm{H}) = \infty$ we need the following notions. By $\mathrm{T}r\,(\mathrm{H})$ we mean the class of all bounded operators T in H such that, for every orthonormal basis $\{ x_a : a \in \mathrm{I} \}$ of H, the series $\sum_{a \in \mathrm{I}} (\mathrm{T}x_a, x_a)$ converges and is independent of the basis used; the expression $\mathrm{tr}\,\mathrm{T} := \sum_{a \in \mathrm{I}} (\mathrm{T}x_a, x_a)$ is called the trace of T.

A bilinear form is a function $t : \mathrm{D}(t) \times \mathrm{D}(t) \to \mathrm{C}$, where $\mathrm{D}(t)$ is a submanifold (not necessarily dense or closed in H), called the domain of t, such that t is linear in both arguments, and $t(\alpha x, \beta y) = \alpha\bar{\beta}t(x, y)$, $x, y \in \mathrm{D}(t)$, $\alpha, \beta \in \mathrm{C}$. If $t(x, y) = \overline{t(y, x)}$ for all $x, y \in \mathrm{D}(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x, x) \geqslant 0$ for all $x \in \mathrm{D}(t)$, then t is said to be positive.

Let $\mathrm{P} \in \mathrm{L(H)}$. By $t \circ \mathrm{P}$ we mean a bilinear form with domain $\mathrm{D}(t \circ \mathrm{P}) = \{ x \in \mathrm{H} : \mathrm{P}x \in \mathrm{D}(t) \}$ such that $t \circ \mathrm{P}(x, y) = t(\mathrm{P}x, \mathrm{P}y)$, $x, y \in \mathrm{D}(t \circ \mathrm{P})$. If $t \circ \mathrm{P}$ is induced by an operator $\mathrm{T} \in \mathrm{T}r(\mathrm{H})$, that is, $t \circ \mathrm{P}(x, y) = (\mathrm{T}x, y)$ for any $x, y \in \mathrm{H}$, then we say $t \circ \mathrm{P} \in \mathrm{T}r(\mathrm{H})$ and we put $tr\, t \circ \mathrm{P} = tr\,\mathrm{T}$.

For any nonzero $x \in \mathrm{H}$, by P_x we denote the one-dimensional subspace of H spanned over x.

Let $\mathfrak{n}$ be a cardinal. We say that a measure m is $\mathfrak{n}$-finite if there is a set I whose cardinal is $\mathfrak{n}$ and a set of mutually orthogonal elements $\{ \mathrm{M}_a : a \in \mathrm{I} \}$ of L(H) such that $\bigoplus_{a \in \mathrm{I}} \mathrm{M}_a = \mathrm{H}$ and $m(\mathrm{M}_a) < \infty$ for any $a \in \mathrm{I}$. If, in particular, $\mathfrak{n} = \mathcal{N}_0$ ($\mathcal{N}_0$ denotes the cardinal of all integers), we say that m is σ-finite.

Let $\mathfrak{m}$ be a cardinal. A function $m : \mathrm{L(H)} \to [0, \infty]$ with $m(0) = 0$ is said to be *i*) $\mathfrak{m}$-additive if

$$m(\bigoplus_{a \in \mathrm{A}} \mathrm{M}_a) = \sum_{a \in \mathrm{A}} m(\mathrm{M}_a) \tag{1.2}$$

whenever card $\mathrm{A} \leqslant \mathfrak{m}$;

ii) totally additive, if (1.2) holds for any index set A. Hence, a measure is a σ-additive function.

Lugovaja and Sherstnev [5] proved that for any σ-finite measure m on L(H), with $m(\mathrm{H}) = \infty$, of a separable Hilbert space H, there exists a unique positive symmetric bilinear form t defined on a dense domain such that

$$m(\mathrm{P}) = \begin{cases} tr\, t \circ \mathrm{T} & \text{iff} \quad t \circ \mathrm{P} \in Tr(\mathrm{H}) \\ \infty & \text{otherwise.} \end{cases} \tag{1.3}$$

This result has been extended by the author [6] to any σ-finite measure on L(H) of a Hilbert space H whose dimension is a non-measurable cardinal $\neq 2$. We recall, according to Ulam [7], that the cardinal I is non-measurable if there is no non-trivial finite measure v on the power set 2^{A} of a set A, whose cardinal is I, such that $v(\{a\}) = 0$ for any $a \in \mathrm{A}$. For example, all finite cardinals and $\mathcal{N}_0$ are nonmeasurable cardinals. So $\mathfrak{c}$ ($\mathfrak{c}$ is the cardinal of all reals) is (under the continuum hypothesis or under a more weaker form, see [7]).

2. THE LUGOVAJA-SHERSTNEV PROPERTY

Let m be a measure on L(H). We say that m has *i*) the Lugovaja-Sherstnev property (L-S property in short) if there is a two-dimensional subspace $\mathrm{Q} \in \mathrm{L(H)}$ with $m(\mathrm{Q}) < \infty$; *ii*) the density property if

$$\mathrm{D}(m) = \{ x \in \mathrm{H} : m(\mathrm{P}_x) < \infty \} \cup \{ 0 \} \tag{2.1}$$

is dense in H; *iii*) the L-S density property if *i*) and *ii*) holds.

LEMMA 2.1. — ([8]) Let $\infty > \dim \mathrm{H} \geqslant 3$ and let m be a measure on L(H) with $m(\mathrm{H}) = \infty$. If there are P and Q such that $\dim \mathrm{Q}^{\perp} = 1 = \dim \mathrm{P}$, $m(\mathrm{Q}) + m(\mathrm{P}) < \infty$, then $\mathrm{P} \subset \mathrm{Q}$.

LEMMA 2.2. — If m has the Lugovaja-Sherstnev property, then D(m) is a linear submanifold of H.

Proof. — It is clear that $0 \in \mathrm{D}(m)$, and if $x \in \mathrm{D}(m)$, then $\alpha x \in \mathrm{D}(m)$ for any $\alpha \in \mathrm{C}$.

Now let $x, y \in \mathrm{D}(m)$, and let x and y be linearly independent vectors. Choose a two-dimensional $\mathrm{Q} \subset \mathrm{D}(m)$. We assert that $m(\mathrm{Q} \cup \mathrm{P}_x) < \infty$; if not, then, due to Lemma 2.1, $\mathrm{P}_x \subset \mathrm{Q}$. Applying once more Lemma 2.1 to $m(\mathrm{Q} \cup \mathrm{P}_x \cup \mathrm{P}_y)$, we obtain $\mathrm{Q} \cup \mathrm{P}_x \cup \mathrm{P}_y \subset \mathrm{D}(m)$. Q. E. D.

It is evident that any σ-finite or $\mathfrak{n}$-finite measure has the L-S property. The converse is not true, in general, as we show in Section 3. If m has the L-S density on L(H), $2 \leqslant \dim \mathrm{H} < \infty$, then m is finite.

REMARK 2.3. — Let m be a measure on L(H) with a not dense domain

$D(m)$. Put $m_0 := m \mid L(\overline{D(m)})$, where $\overline{D(m)}$ is the closure of $D(m)$ in H. If we put

$$m_1(M) = \begin{cases} m_0(M) & \text{if } \quad M \subset \overline{D(m)} \\ \infty & \text{otherwise.} \end{cases}$$

then $m = m_1$, and $D(m_0)$ is dense in $\overline{D(m)}$.

Therefore, without loss of generality we limit ourselves only to measures with the density property.

We now introduce some definitions.

An element $M \in L(H)$ is a support of a measure m if $m(P) = 0$ iff $M \perp P$. Let $S(H) = \{ x \in H : \| x \| = 1 \}$. A map $f : S(H) \rightarrow [0, \infty]$ is a frame function if $i)$ $f(\alpha x) = f(x)$ for any $x \in S(H)$ and any scalar $\alpha \in C, |\alpha| = 1$; $ii)$ there is a constant W (may be $+\infty$, too), called the weight of f, such that, for any orthonormal basis $\{ x_a : a \in A \}$ of H, $\sum_{a \in A} f(x_a) = W$. A frame function f has a finiteness property if $\sum_{i \in I} f(x_i) < \infty$, for some orthonormal systems of vectors $\{ x_i : i \in I \} \subset H$, implies $f \mid S(G)$ is a frame function with a finite weight, where $G = \bigoplus_{i \in I} P_{x_i}$. A frame function f is regular if there is a positive symmetric bilinear form t with $D(t) = \{ x \in H : x \neq 0, f(x/\| x \|) < \infty \} \cup \{ 0 \}$ such that $f(x) = t(x, x)$ for any $x \in S(H) \cap D(t)$. An example of a regular frame function without the finiteness property is given in Example 5.2.

Similarly as in [*6*], we may show that if f is a frame function on a Hilbert space H of dimension $\neq 2$ such that there are two orthonormal vectors x and y with $f(x) + f(y) < \infty$, then f is regular.

THEOREM 2.4. — (Generalized Maeda's Theorem) Let L(H) be a quantum logic of a real or complex Hilbert space H of dimension $\neq 2$. Let m be a measure with the L-S density property. The following assertions are equivalent.

i) There exists a unique positive symmetric bilinear form t with a dense domain such that (1.3) holds.

ii) m has a support.

iii) m is totally additive.

iv) For any $M \in L(H)$, $f \mid S(M)$ is a frame function with a weight $m(M)$, where $f(x) = m(P_x)$, $x \in S(H)$.

Proof. — The equivalence of *i*)-*iii*) may be proved using Lemma 2.2 and repeating the proof of Theorem 5 in [*6*]. The equivalence of *iii*) and *iv*) is easily verifiable. Q. E. D.

If m satisfies the assertion i) of Theorem 2.4, m is called a Gleason measure. It is evident that then m has the L-S density property. It will be shown below that the converse is not true, in general.

3. CLASSIFICATIONS AND EXAMPLES OF MEASURES

Let H be an infinite-dimensional Hilbert space and let $\mathfrak{n}$ be a cardinal. by $\mathcal{M}_{\mathfrak{n}}(\mathrm{H})$ we denote the set of all $\mathfrak{n}$-finite measures (with the L-S density property). In particular, if $\mathfrak{n} = \mathcal{N}_0$, we write $\mathcal{M}_\sigma(\mathrm{H})$. By $\mathcal{M}_f(\mathrm{H})$, $\mathcal{M}_{\mathrm{G}}(\mathrm{H})$ and $\mathcal{M}(\mathrm{H})$ we denote the sets of all finite measures, Gleason measures and measures with the L-S density property.

If $\mathfrak{n}$ and $\mathfrak{m}$ are two infinite cardinals with $\mathfrak{n} < \mathfrak{m}$, then

$$\mathcal{M}_f(\mathrm{H}) \subsetneqq \mathcal{M}_\sigma(\mathrm{H}) \subseteq \mathcal{M}_{\mathfrak{n}}(\mathrm{H}) \subseteq \mathcal{M}_{\mathfrak{m}}(\mathrm{H}) \subseteq \mathcal{M}_{\mathrm{G}}(\mathrm{H}) \subseteq \mathcal{M}(\mathrm{H}). \qquad (3.1)$$

The aim of the present section is to give examples of Hilbert spaces and measures when some of the inclusions in (3.1) are proper.

EXAMPLE 3.1. — If $\dim \mathrm{H} = \mathcal{N}_0$, then $\mathcal{M}_\sigma(\mathrm{H}) = \mathcal{M}(\mathrm{H})$.

Proof. — The assertion will be proved if we show that any dense submanifold D in H contains at least one orthonormal basis of H. Indeed, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of H. For any n, there is $x_n \in \mathrm{D}$ such that $\| e_n - x_n \|^2 < 1/2^n$, so that $\sum_{n=1}^{\infty} \| e_n - x_n \|^2 < 1$. Now we claim to show that $\{x_n\}_{n=1}^{\infty}$ generates H. Define a linear transformation A, say, from H into the submanifold D via

$$\mathrm{A}e_n = x_n .$$

If $f = \sum_{j \in \mathrm{J}} \xi_j e_j$, where J is a finite subset of integers, then

$$\| f - \mathrm{A}f \|^2 = \left\| \sum_{j\in \mathrm{J}} \xi_j e_j - \sum_{j\in \mathrm{J}} \xi_j x_j \right\|^2 = \left\| \sum_{j\in \mathrm{J}} \xi_j (e_j - f_j) \right\|^2 \leqslant$$

$$\leqslant \sum_{j\in \mathrm{J}} |\xi_j|^2 \sum_{j\in \mathrm{J}} \| e_j - f_j \|^2 \leqslant \| f \|^2 \sum_{j=1}^{\infty} \| e_j - f_j \|^2 < 1 .$$

Hence, the linear transformation $1-\mathrm{A}$ may be extended uniquely to a bounded linear operator defined on whole H (we denote it also as $1-\mathrm{A}$). According to Halmos [9, p. 52], A is invertible. Therefore, the closed linear submanifold generated by $\{x_n\}_{n=1}^{\infty}$ coincides with H. If not, then there is $f \neq 0$, $f \perp x_n$ $(n \geqslant 1)$. Moreover, there exists $g \in \mathrm{H}$ such that $\mathrm{A}g = f$ and

$$\| f \|^2 = (\mathrm{A}g, f) = \sum_{n=1}^{\infty} (g, e_n)(x_n, f) = 0,$$

which implies a contradiction.

Now it suffices to apply the Gram-Schmidt orthogonalization process to $\{ x_n \}_{n=1}^{\infty}$, and Theorem 2.4 gives the desired result. Q. E. D.

PROPOSITION 3.2. — Let $\dim \mathrm{H} = \mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal with $\mathfrak{m}^{\mathcal{N}_0} > \mathfrak{m}$. Let $\mathscr{H}$ be any Hamel basis of H. Then card $\mathrm{H} = \operatorname{card} \mathscr{H} = \mathfrak{m}^{\mathcal{N}_0}$.

Proof. — Let B be any orthonormal basis in H and let $\mathrm{X} = \{ \mathrm{A} \subset \mathrm{B} : \operatorname{card} \mathrm{A} = \mathcal{N}_0 \}$. Let P_2 be the set of all sequences of square-summable numbers from C. Assuming the axion of choice, we may well order the set B. For any $f \in \mathrm{H}$ we denote by $\mathrm{A}_f = \{ b \in \mathrm{B} : (f, b) \neq 0 \}$, $\bar{f} = \{ (f, b) : b \in \mathrm{A}_f \} \in \mathrm{P}_2$. Due to [*11*, p. 291], card $\mathrm{X} = \mathfrak{m}^{\mathcal{N}_0}$. The set H may be splitted onto two disjoint subsets H_0 and H_∞, where $\mathrm{H}_0 = \{ f \in \mathrm{H} : (f, b) \neq 0$ for finitely many $b \in \mathrm{B} \}$, $\mathrm{H}_\infty = \{ f \in \mathrm{H} : (f, b) \neq 0$ for infinitely many $b \in \mathrm{B} \}$. Hence, card $\mathrm{H} = \operatorname{card} \mathrm{H}_\infty$.

Define a mapping $\Phi : \mathrm{H}_\infty$ into $\mathrm{X} \times \mathrm{P}_2$ via $\phi(f) = (\mathrm{A}_f, \bar{f})$. It is easy to verify that ϕ is an one-to-one map from H_∞ onto $\mathrm{X} \times \mathrm{P}_2$. Therefore, card $\mathscr{H} = \operatorname{card} \mathrm{X} \times \mathrm{P}_2 = \mathfrak{m}^{\mathcal{N}_0}$. Q. E. D.

PROPOSITION 3.3. — Let $\mathfrak{m}$ be an infinite cardinal such that $\mathfrak{m}^{\mathcal{N}_0} > \mathfrak{m}$. Then in any Hilbert space of dimension $\mathfrak{n} = \mathfrak{m}^{\mathcal{N}_0}$ there is a linear submanifold dense in H containing no basis of H.

Proof. — The present proof follows the idea of Gudder's construction for $\mathfrak{n} = \mathfrak{c}$ [*11*].

Let E and F be two Hilbert spaces over the same field C of dimension $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Define the direct sum of E and F as $\mathrm{H} = \mathrm{E} \tilde{\oplus} \mathrm{F} = \{ \langle e, f \rangle : e \in \mathrm{E}, f \in \mathrm{F} \}$. Here an inner product is defined by

$$(\langle e_1, f_1 \rangle, \langle e_2, f_2 \rangle) = (e_1, e_2)_{\mathrm{E}} + (f_1, f_2)_{\mathrm{F}}.$$

It is easy to show that $\dim \mathrm{H} = \mathfrak{n}$.

Choose an orthonormal basis $\{ e_t : t \in \mathrm{T}_0 \}$ in E which is a part of a Hamel basis $\{ e_t : t \in \mathrm{T} \}$. According to our assumption and Proposition 3.1, card $\mathrm{T} = \mathfrak{n}$. Choose an orthonormal basis $\{ f_t : t \in \mathrm{T} - \mathrm{T}_0 \}$ in F (for simplicity we may put $\mathrm{T} - \mathrm{T}_0$ as the index set of all f_t's) and define a linear mapping : $\mathrm{E} \rightarrow \mathrm{F}$ via $\mathrm{B}e_t = 0$, if $t \in \mathrm{T}_0$, and $e_t = f_t$, if $t \in \mathrm{T} - \mathrm{T}_0$. Then the graph of B, that is the set $\mathrm{G} = \{ \langle e, \mathrm{B}e \rangle : e \in \mathrm{E} \}$, is a dense submanifold in $\mathrm{H} = \mathrm{E} \tilde{\oplus} \mathrm{F}$. Indeed, since $\langle e_t, 0 \rangle \in \mathrm{G}$ for each $t \in \mathrm{T}_0$, it follows that $\langle e, 0 \rangle \in \overline{\mathrm{G}}$ for each $e \in \mathrm{E}$, where $\overline{\mathrm{G}}$ is the closure of G in H. This implies that $\langle e, \mathrm{B}e \rangle - \langle e, 0 \rangle = \langle 0, \mathrm{B}e \rangle \in \overline{\mathrm{G}}$ for each $e \in \mathrm{E}$. Since the range of B is dense in F, we conclude that $\langle e, f \rangle = \langle e, 0 \rangle + \langle 0, f \rangle \in \overline{G}$ for all $e \in E$, $f \in F$, so that $\overline{G} = H$.

Now we establish that G contains no basis of H. Since $\langle e_t, 0 \rangle \perp \langle e_s, 0 \rangle$, $t \neq s$, $t, s \in T_0$ we assert that $\{ \langle e_t, 0 \rangle : t \in T_0 \}$ is a maximal orthonormal system in G. This follows from a simple observation that if $\langle e, Be \rangle \perp \langle e_t, 0 \rangle$ for any $t \in T_0$, then $e \perp e_t$. Hence $e = 0$ and $Be = 0$. Since all maximal orthonormal systems in any fixed linear submanifold has the same cardinality, we conclude that G contains no basis of H. Q.E.D.

We remark that the following alephs $\aleph_0, \aleph_\omega, \aleph_{\omega+\omega}, \aleph_{\omega.n}$ ($n \geqslant 1$), $\aleph_{\omega_\omega}$, $\aleph_{\omega_{\omega \dots}}$, are (nonmeasurable) cardinals $\mathfrak{m}$ fulfilling Proposition 3.1. Moreover, any $\mathfrak{m}$ with $\aleph_1 \leqslant \mathfrak{m} < \mathfrak{c}$ also fulfils it; but we note that in this case the author does not know whether any linear dense submanifold contains some orthonormal basis of H, when $\dim H = \mathfrak{m}$.

Additionally, if $\mathfrak{m}$ has a countable cofinality, that is, it may be expressed as the sum of countably many cardinals less that $\mathfrak{m}$, then $\mathfrak{m}^{\aleph_0} > \mathfrak{m}$. There are arbitrarily large cardinals $\mathfrak{m}$ such that $\mathfrak{m}^{\aleph_0} > \mathfrak{m}$. Indeed, for each ordinal α, $\aleph_{\alpha+\omega}$ has a countable cofinality.

We recall that there are also arbitrarily large cardinals $\mathfrak{m}$ such that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. In fact. Let $\mathfrak{a}_{\alpha+1}$ be an exponent cardinal, that is $\mathfrak{a}_{\alpha+1} = 2^{\mathfrak{a}_\alpha}$. If α is a limit ordinal, then, due to [10, p. 300] $\aleph_{\alpha+1} \leqslant \mathfrak{a}_{\alpha+1}$. Hence, $\mathfrak{a}_{\alpha+1}^{\aleph_0} = (2^{\mathfrak{a}_\alpha})^{\aleph_0} = 2^{\mathfrak{a}_\alpha \cdot \aleph_0} = 2^{\mathfrak{a}_\alpha} = \mathfrak{a}_{\alpha+1}$. Therefore,

$$\aleph_\alpha < \aleph_{\alpha+\omega} < \aleph_{\alpha+\omega+1} \leqslant \mathfrak{a}_{\alpha+\omega+1} = 2^{\mathfrak{a}_{\alpha+\omega}}.$$

If we put $\mathfrak{m} = \aleph_{\alpha+\omega+1}$ we conclude the assertion.

EXAMPLE 3.4. — If $\dim H = \mathfrak{m}^{\aleph_0}$, where $\mathfrak{m}$ is an infinite cardinal with $\mathfrak{m}^{\aleph_0} > \mathfrak{m}$, then

$$\mathcal{M}_{\mathfrak{n}}(H) \subseteq \mathcal{M}_G(H)$$

for any cardinal $\mathfrak{n}$.

Proof. — According to Proposition 3.3, we may choose a linear submanifold G dense in H containing no basis of H. We define a map f: $S(H) \to [0, \infty]$ via

$$f(x) = \begin{cases} 1 & \text{if } x \in G, \\ \infty & \text{if } x \notin G. \end{cases}$$

We assert that f is a frame function with the finiteness property and with a weight $W = \infty$. It determines a measure m (see Lemma 5.1) via $m(M) = \sum_i f(x_i)$, where $\{ x_i \}$ is an orthonormal basis in M. It is clear that f is regular; it is determined by a positive symmetric bilinear form $t(.,.) = (.,.) \mid G$. Since $m(M) < \infty$, iff $\dim M < \infty$, we conclude that $m \in \mathcal{M}_G(H) - \mathcal{M}_{\mathfrak{n}}(H)$ for any $\mathfrak{n}$. Q.E.D.

EXAMPLE 3.5. — If $\dim H \geqslant \mathfrak{m} > \mathfrak{n} \geqslant \mathcal{N}_0$, then

$$\mathcal{M}_n(H) \subsetneqq \mathcal{M}_{\mathfrak{m}}(H).$$

Proof. — Let $P \in L(H)$ and $\dim P = \mathfrak{m}$. Let t be a symmetric bilinear form defined via $t(x, y) = (Px, Py)$, $x, y \in H$. Then t determines some measure m by (1.3) (see also the first part of Theorem 5.3). Due to Theorem 2.4, m has a carrier which is equal to P, and $m \in \mathcal{M}_{\mathfrak{m}}(H)$. If, in addition, $m \in \mathcal{M}_n(H)$, then, by Theorem 6 [*6*], $\dim P \leqslant \max\{\mathcal{N}_0, \mathfrak{n}\}$ which is a contradiction. Q.E.D.

Let $\mathfrak{n}$ be a cardinal. Denote by $\mathcal{M}^{\mathfrak{n}}(H)$ the set of all $\mathfrak{n}$-additive measures on L(H) with the L–S density property. By $\mathcal{M}^f(H)$, $\mathcal{M}^\sigma(H)$ and $\mathcal{M}^t(H)$ we denote the classes of all finitely additive, σ-additive and totally additive measures on L(H). If $\mathcal{N}_0 \leqslant \mathfrak{n} < \mathfrak{m}$, then

$$\mathcal{M}^t(H) = \mathcal{M}_G(H) \subseteq \mathcal{M}^{\mathfrak{m}}(H) \subseteq \mathcal{M}^{\mathfrak{n}}(H) \subseteq \mathcal{M}^\sigma(H) \subseteq \mathcal{M}^f(H).$$

The equality $\mathcal{M}^t(H) = \mathcal{M}_G(H)$ follows from Theorem 2.4. There are simple examples which show that all inclusions may be proper.

4. SINGULAR BILINEAR FORMS

A positive symmetric bilinear form t is said to be closed if $t(x_n - x_m, x_n - x_m) \to 0$ and $\| x_n - x \| \to 0$ ($\{ x_n \} \subset D(t)$) imply $x \in D(t)$ and $t(x_n - x, x_n - x) \to 0$. Given two positive symmetric bilinear forms t_1 and t_2 we write $t_1 \leqslant t_2$ if and only if $D(t_1) \supseteq D(t_2)$ and $t_1(x, x) \leqslant t_2(x, x)$ for all $x \in D(t_2)$. The closure $\bar{t}$ of t is the minimal closed extension of t (if it exists). According to Simon [*12*], for any positive symmetric bilinear form t, there is a unique pair of positive symmetric bilinear forms t_r and t_s with

$$t = t_r + t_s, \qquad D(t) = D(t_r) = D(t_s), \tag{4.1}$$

such that t_r is the largest (in the above ordering) closable positive symmetric bilinear form less than t and $t_s = t - t_r$. t_r and t_s are called the regular and singular part of t. If $t = t_r$ ($t = t_s$), then t is said to be regular (singular). We recall that $D(t)$ must not be dense in H.

It is a straightforward verification that t with a dense domain $D(t)$ in H is singular iff, for any $0 \neq g \in H$, there is $f \in D(t)$ with

$$t(f, f) \leqslant |(f, g)|^2. \tag{4.2}$$

We remark that in any infinite-dimensional Hilbert space H there is a singular bilinear form, even with a domain equal to H. Indeed, let $\{ e_t : t \in T_0 \}$ be an orthonormal basis in H which is a part of a Hamel basis $\{ e_t : t \in T \}$.

Fix an element $e_{t_0} \in \{ e_t : t \in \mathrm{T} - \mathrm{T}_0 \}$ and define an unbounded linear operator $\mathrm{S} : \mathrm{H} \to \mathrm{H}$ via

$$\mathrm{S}\left(\sum_{t \in \mathrm{A}_0} \alpha_t e_t\right) = \alpha_{t_0} e_{t_0}, \tag{4.3}$$

where A_0 is an arbitrary finite subset of T containing t_0. Then t (with $\mathrm{D}(t) = \mathrm{H}$) defined by

$$t(f, g) = (\mathrm{S}f, \mathrm{S}g), \quad f, g \in \mathrm{H}, \tag{4.4}$$

is a positive symmetric singular bilinear form. Hence, in particular, t determines no Gleason measure.

Lugovaja [*1*] proved that on a separable Hilbert space any singular bilinear form with a dense domain determines no σ-finite Gleason measure via (1.3).

In this section we extend this result to a nonseparable Hilbert space. Although the proof of this assertion is similar to that of Lugovaja, here it depends on Theorem 2.4, and we present it here.

First of all we show the following lemma whose proof is identical to that of Lugovaja [*1*].

LEMMA 4.1. — Let t be a singular bilinear form with a domain dense in H. Let $\mathrm{P} \in \mathrm{L}(\mathrm{H})$ be a subspace such that

i) $\mathrm{P} \subset \mathrm{D}(t)$;

ii) there is a constant $c > 0$ with $t(f, f) \leqslant c$ for any $f \in \mathrm{S}(\mathrm{P})$. Then $t \mid \mathrm{D}$, where $\mathrm{D} = \mathrm{P}^{\perp} \cap \mathrm{D}(t)$, is a singular form with a domain dense in the Hilbert space $\mathrm{P}^{\perp}$.

The main result of the present sections:

THEOREM 4.2. — Let t be a singular bilinear form with a domain dense in H. Then a mapping $m: \mathrm{L}(\mathrm{H}) \to [0, \infty]$ defined by (1.3) is not a measure on the quantum logic L(H) of the infinite-dimensional Hilbert space H.

Proof. — Suppose that m given by (1.3) is a measure. Since t is unbounded, there exists $x \in \mathrm{S}(\mathrm{H}) \cap \mathrm{D}(t)$ such that $t(x, x) > 1$. The singularity of t implies, by (4.2), that there exists $e_1 \in \mathrm{S}(\mathrm{H}) \cap \mathrm{D}(t)$ such that $t(e_1, e_1) < |(e_1, x)|^2$. Then

$$\| x - (x, e_1)e_1 \|^2 = 1 - |(x, e_1)|^2 < 1 - t(e_1, e_1).$$

According to Lemma 4.1, $t_1 := t \mid \mathrm{D}(t_1)$, where $\mathrm{D}(t_1) = \mathrm{P}_{e_1}^{\perp} \cap \mathrm{D}(t)$, is singular. Due to the Zorn lemma and above, there exists a non-empty system of orthonormal vectors, $\{ e_a : a \in A \} \subset D(t)$, such that, for any finite B, $\emptyset \neq B \subset A$, we have

i) $\displaystyle\sum_{a \in B} t(e_a, e_a) < 1$;

ii) $\displaystyle\left\| x - \sum_{a \in B} (x, e_a) e_a \right\|^2 < 1 - \sum_{a \in B} t(e_a, e_a)$;

iii) $t_B := t \mid (1 - \bigoplus_{a \in A} P_{e_a}) \cap D(t)$ is singular.

We assert that $x = \displaystyle\sum_{a \in A} (x, e_a) e_a$. Suppose the converse:

$$y = x - \sum_{a \in A} (x, e_a) e_a \neq 0 .$$

The condition *ii)* implies

$$\sum_{a \in A} t(e_a, e_a) < 1 . \tag{4.5}$$

Put $P = \bigoplus_{a \in A} P_{e_a}$. Since, due to the assumption, m is a Gleason measure, Theorem 2.4 entails that m is totally additive. By (4.5) we have $m(P) < 1$ and $P \subset D(t)$. Due to Lemma 4.1, $t_A := t \mid P^{\perp} \cap D(t)$ is singular. Hence, there is $e \in S(P^{\perp}) \cap D(t)$ such that $t(e, e) < |(e, y)|^2$.

For any finite subset $B \subset A$ we have

$$\left\| x - \sum_{a \in B} (x, e_a) e_a - (x, e) e \right\|^2 = \left\| x - \sum_{a \in B} (x, e_a) e_a \right\|^2 -$$

$$- |(e, y)|^2 < 1 - \sum_{a \in B} t(e_a, e_a) - t(e, e) .$$

Therefore, for a system $\{ e_a : a \in A \} \cup \{ e \}$, the conditions *i)*-*iii)* are fulfilled, which contradict the maximality of $\{ e_a : a \in A \}$. Consequently, $x = \displaystyle\sum_{a \in A} (x, e_a) e_a$ and $x \in P \subset D(t)$. But on the other hand,

$$1 < t(x, x) = m(P_x) \leqslant m(P) < 1$$

which is a contradiction. The theorem is proved. Q.E.D.

5. BILINEAR FORMS GENERATING MEASURES

In this section we give the necessary and sufficient condition for a positive symmetric bilinear form with a dense domain to generate a Gleason measure via (1.3). This assertion is a generalization of that for a separable Hilbert space, due to Lugovaja [2]. Her result is given in [2] without proof.

First of all we present two assertions on frame functions. The proof of the first one is straightforward and therefore it is omitted.

LEMMA 5.1. — Let m be a totally additive measure with not necessarily the density property of an arbitrary Hilbert space H. Then f defined by

$$f(x) = m(\mathrm{P}_x), \qquad x \in \mathrm{S(H)}, \tag{5.1}$$

is a frame function with the finiteness property.

Conversely, let f be a frame function with the finiteness property. Then a mapping m defined via

$$m(\mathrm{M}) = \sum_i f(x_i), \tag{5.2}$$

where $\{ x_i \}$ is an orthonormal basis in M, is a totally additive measure on L(H).

EXAMPLE 5.2. — Let $\mathfrak{m}$ be an infinite cardinal such that $\mathfrak{m}^{\mathcal{N}_0} > \mathfrak{m}$. Then in any Hilbert space H of dimension $\mathfrak{n} = \mathfrak{m}^{\mathcal{N}_0}$ there is a regular frame function without the finiteness property and with the set $\{ x \in \mathrm{S(H)} : f(x) < \infty \}$ dense in H.

Proof. — Let us assume that H is of the form $\mathrm{H} = \mathrm{E} \tilde{\oplus} \mathrm{F}$, where E and F are from the proof of Proposition 3.3. Let $\mathrm{B} : \mathrm{E} \to \mathrm{F}$ be the linear transformation and let B be the graph of B from Proposition 3.3. Define a positive symmetric bilinear form t with $\mathrm{D}(t) = \mathrm{G}$ via $t(\langle e, \mathrm{B}e \rangle, \langle e, \mathrm{B}e \rangle) = \| \mathrm{S}e \|^2$, where S is the unbounded operator from (4.3). Then a mapping f: $\mathrm{S(H)} \to [0, \infty]$ defined through

$$f(x) = \begin{cases} t(x, x) & \text{if } \|x\| = 1, \quad x \in \mathrm{G}, \\ \infty & \text{if } \|x\| = 1, \quad x \notin \mathrm{G}, \end{cases}$$

is a regular frame function. Q.E.D.

THEOREM 5.3. — A positive symmetric bilinear form t with a dense domain determines via (1.3) a Gleason measure on L(H) of an arbitrary infinite-dimensional Hilbert space H iff, for any $\mathrm{P} \in \mathrm{L(H)}$,

$$(\overline{t \circ \mathrm{P}})_r \in \mathrm{Tr(H)} \text{ implies } t \circ \mathrm{P} \in \mathrm{Tr(H)}, \tag{5.3}$$

where $(\overline{t \circ \mathrm{P}})_r$ is the regular part of the closure $\overline{t \circ \mathrm{P}}$.

Proof. — Suppose that (5.3) hold. Let $\mathcal{M}(t) = \{ \mathrm{P} \subseteq \mathrm{L(H)} : m(\mathrm{P}) < \infty \}$, where m is defined by (1.3). A straightforward calculation shows that *i)* $0 \in \mathcal{M}(t)$ and $m(0) = 0$; *ii)* if $\mathrm{N} \subset \mathrm{M} \in \mathcal{M}(t)$, then $\mathrm{N} \in \mathcal{M}(t)$ and

$m(\mathrm{N}) \subseteq m(\mathrm{M})$; *iii)* if $\mathrm{M}_1, \mathrm{M}_2 \in \mathscr{M}(t)$, $\mathrm{M}_1 \perp \mathrm{M}_2$, then $\mathrm{M}_1 \oplus \mathrm{M}_2 \in \mathscr{M}(t)$ and $m(\mathrm{M}_1 \oplus \mathrm{M}_2) = m(\mathrm{M}_1) + m(\mathrm{M}_2)$.

Now let $\mathrm{M} = \bigoplus_{n=1}^{\infty} \mathrm{M}_n$. We claim that

$$m(\mathrm{M}) = \sum_{n=1}^{\infty} m(\mathrm{M}_n). \tag{5.4}$$

It is evident that the series in (5.4) is less that $m(\mathrm{M})$. If at least one $\mathrm{M}_i \notin \mathscr{M}(t)$, then (5.4) is true. Suppose only $\sum_{n=1}^{\infty} m(\mathrm{M}_n) < \infty$. Then $\mathrm{D}(t \circ \mathrm{M}) = \{ x \in \mathrm{H} : \mathrm{M}x \in \mathrm{D}(t) \} = \mathrm{D}((t \circ \mathrm{M})_r)$. It is clear that $\mathrm{M}^{\perp}, \mathrm{M}_i \subset \mathrm{D}(t \circ \mathrm{M})$, $i \geqslant 1$. For the closure $(\overline{t \circ \mathrm{M}})_r$ there is a unique positive symmetric operator T such that $\mathrm{D}(\mathrm{T}^{1/2}) = \mathrm{D}((\overline{t \circ \mathrm{M}})_r)$ and $(\overline{t \circ \mathrm{M}})_r(x, x) = (\mathrm{T}^{1/2}x, \mathrm{T}^{1/2}x)$, $x \in \mathrm{D}(\mathrm{T}^{1/2})$, where $\mathrm{D}(\mathrm{T}^{1/2})$ is the domain of an operator $\mathrm{T}^{1/2}$.

We claim to show that if $x \in \mathrm{S}(\mathrm{M})$, then $x \in \mathrm{D}(\mathrm{T}^{1/2})$. It is evident that $x = \sum_{n=1}^{\infty} x_n$, where $x_n = \mathrm{M}_n x, n \geqslant 1$. Denote $y_n = x_1 + \ldots + x_n \to x$. Then, for any $n < m$,

$$\begin{aligned} \| \mathrm{T}^{1/2}(y_n - y_m) \|^2 &= (\overline{t \circ \mathrm{M}})_r(y_n - y_m, y_n - y_m) = (t \circ \mathrm{M})_r(y_n - y_m, y_n - y_m) \leqslant \\ &\leqslant t \circ \mathrm{M}(y_n - y_m, y_n - y_m) = (t \circ \bigoplus_{i=n+1}^{m} \mathrm{M}_i)(y_n - y_m, y_n - y_m) = \\ &= (\mathrm{T}_n^m(y_n - y_m), y_n - y_m), \end{aligned}$$

where T_n^m is a trace operator corresponding to $t \circ \bigoplus_{i=n+1}^{m} \mathrm{M}_i$. Then

$$\begin{aligned} (\mathrm{T}_n^m(y_n - y_m), y_n - y_m) &= \| x_{n+1} + \ldots + x_m \|^2 (\mathrm{T}_n^m x^0, x^0) \leqslant (\mathrm{T}_n^m x_0, x_0) \leqslant \\ &\leqslant tr \mathrm{T}_n^m = \sum_{i=n+1}^{m} m(\mathrm{M}_i) \to 0; \end{aligned}$$

here without loss of generality we assume that $x_i \neq 0$ for any $i \geqslant 1$ and $x^0 = (x_{n+1} + \ldots + x_m)/\| x_{n+1} + \ldots + x_m \|^2$. Therefore $\mathrm{T}^{1/2}y_n \to z$ and, the closedness of $\mathrm{T}^{1/2}$ implies $x \in \mathrm{D}(\mathrm{T}^{1/2})$ and $\mathrm{T}^{1/2}y_n \to \mathrm{T}^{1/2}x$. In other words $\mathrm{M} \subset \mathrm{D}(\mathrm{T}^{1/2})$. Therefore $\mathrm{D}(\mathrm{T}^{1/2}) = \mathrm{H}$ and T is a Hermitian operator.

Now we show $\mathrm{T} \in \mathrm{T}r(\mathrm{H})$. Choose an orthonormal basis $\{ x_n^i : i \in \mathrm{I}_n \}$ in M_n, for any $n \geqslant 1$, and let $\{ x_0^i : i \in \mathrm{I}_0 \}$ be an orthonormal basis in $\mathrm{M}^{\perp}$. Then

$$0 \leqslant \sum_{i,n} (\mathrm{T}x_n^i, x_n^i) = \sum_{n=1}^{\infty} \sum_{i \in \mathrm{I}_n} (\mathrm{T}x_n^i, x_n^i) = \sum_{n=1}^{\infty} m(\mathrm{M}_n) < \infty .$$

Hence, $(\overline{t \circ \mathrm{M}})_r \in \mathrm{T}r(\mathrm{H})$, consequently, $t \circ \mathrm{M} \in \mathrm{T}r(\mathrm{H})$ and (5.4) holds.

Conversely, suppose that m is a Gleason measure on L(H). Then, due to Theorem 2.4, there is a unique t such that (1.3) holds. We claim to establish that (5.3) is true.

Suppose the converse. Then there is $M \in L(H)$ such that $m(M) = \infty$ and $(\overline{t \circ M})_r \in Tr(H)$. So that there is a unique positive Hermitian operator T such that $D((\overline{t \circ M})_r) = D(T) = H$ and $(\overline{t \circ M})_r(x, x) = (Tx, x)$ for each $x \in H$.

The domain $D(t \circ M)$ is an essential region for T, that is, for any $x \in H$, there is $\{ x_n \} \subset D(t \circ M)$ such that $x_n \to x$ and $t \circ M(x_n - x_m, x_n - x_m) \to 0$. Therefore $D(t \circ M)$ is dense in H. Moreover, M reduces T.

Define a function $f: S(M) \to [0, \infty]$ via

$$f(x) = \begin{cases} (t \circ M)_s(x, x) & \text{if} \quad x \in D(t \circ M) \cap M, \\ \infty & \text{otherwise}. \end{cases}$$

We show that f is a frame function with infinite weight and the finiteness property in a Hilbert space M. Choose an orthonormal basis $\{ x_i : i \in I \}$ in M. Then either $f(x_i) = \infty$ for at least one $i \in I$, or $f(x_i) < \infty$ for any $i \in I$. The later case implies, due to the total additivity of m,

$$m(M) = \sum_i m(P_{x_i}) = \sum_i t(x_i, x_i) = \sum_i (((t \circ M)_r)(x_i, x_i) + (t \circ M)_s(x_i, x_i)) = \\ = tr\, T + \sum_i f(x_i),$$

which gives $\sum_i f(x_i) = \infty$.

Now let $\sum_{j \in J} f(y_j) < \infty$ for some orthonormal basis $\{ y_j : j \in J \}$ of $N \subset M$. Then $m(N) = tr\,(TN) + \sum_{j \in J} f(y_j) < \infty$, and $N \subset D(t)$. Hence, $N \subset D(t \circ M)$, therefore, for any basis $\{ z_j, j \in J \}$ of N we have

$$\sum_{j \in J} f(z_j) = m(N) - tr(TN).$$

Lemma 5.1 implies that m_M defined by $m_M(P) = \sum_i f(x_i)$, where $\{ x_i \}$ is an orthonormal basis in $P \subset M$, is a totally additive measure on L(M). It is simply to verificate that $m_M(P) < \infty$ iff $(t \circ M)_s \circ P \in Tr(M)$, and in this case $m_M(P) = tr(t \circ M)_s \circ P$.

On the other hand $(t \circ M)_s \mid M$ is, due to Lemma 4.1, a singular bilinear form. Repeating the proof of Theorem 4.2 (m_M is totally additive), we conclude that $(t \circ M)_s \mid M$ determines no measure on L(M) via (1.3).

Consequently, our assumption that (5.3) does not hold, is false, and the proof of the Theorem is finished. Q.E.D.

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