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Spectral resonances for the Laplace-Beltrami operator

by

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ABSTRACT. — We study the Laplace-Beltrami operator $-\Delta_g$ on a class of spherically symmetric, complete Riemannian manifolds $M = \mathbb{R}^n$ with metrics g. We formulate conditions on g such that (M, g) has a suitably large set of bounded geodesics in a compact set. Under these conditions, we prove that $-\Delta_g$ has either spectral resonances or positive eigenvalues z_n. The $\operatorname{Re}(z_n)$ is approximately an eigenvalue of $-\Delta_g$ restricted to the region where the bounded orbits are concentrated. Sufficient conditions on g are given which guarantee that $\sigma_{pp}(-\Delta_g) \cap \mathbb{R}^+ = \phi$. When these are satisfied, $-\Delta_g$ has spectral resonances. In the proofs, the angular momentum quantum number ℓ plays the role of the semi-classical parameter. The widths of the resonances are proved to be exponentially small in ℓ. These results also apply to manifolds of the form $M = X \times N$ where $X = \mathbb{R}$ or $\mathbb{R}^+$, N is compact, and the metric has the form $ds^2 = dr^2 + h(r)^2 d\omega^2$. An application to surfaces of revolution is given.

RÉSUMÉ. — Nous étudions l'opérateur de Laplace-Beltrami $-\Delta_g$ sur une classe de variétés Riemanniennes complètes $M = \mathbb{R}^n$ munies d'une métrique g à symétrie rotationnelle. Nous formulons des conditions sur g pour que (M, g) ait une collection suffisamment large de géodésiques bornées dans un ensemble compact. Sous ces conditions, nous montrons que $-\Delta_g$ a des résonances spectrales ou des valeurs propres positives z_n. La partie réelle de z_n est proche d'une valeur propre de l'opérateur $-\Delta_g$ restreint à

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la région où les trajectoires bornées sont concentrées. Des conditions suffisantes sur g sont données pour assurer que $\sigma_{pp}(-\Delta_g) \cap \mathbb{R}^+ = \phi$. Si celles-ci sont satisfaisantes, $-\Delta_g$ a des résonances spectrales. Dans les démonstrations, le numéro quantique du moment angulaire ℓ joue le rôle de paramètre semi-classique. Nous montrons que les parties imaginaires des résonances sont de type exponentiel en ℓ. Ces résultats s'appliquent aussi aux variétés de la forme $M = X \times N$, où $X = \mathbb{R}$ ou $\mathbb{R}^+$, N est compacte et la métrique est de la forme $ds^2 = dr^2 + h(r)^2 d\omega^2$. Nous donnons une application aux surfaces de révolution.

1. INTRODUCTION

In this paper, we strengthen the connection between the occurrence of bounded orbits for classical Hamiltonian systems and of either bound states or spectral resonances for the corresponding quantum systems. More precisely, we prove that the Laplace-Beltrami operator $-\Delta_g$ on a class of complete Riemannian manifolds with sufficiently many bounded geodesics has either bound states or spectral resonances. The manifolds which we consider are of the form $(\mathbb{R}^n, g)$, $n \geq 2$, where the metric g is SO(n)-invariant. Hence, it is given by

$$ds^2 = f(u)^{-2} du^2 + k(u)^2 u^2 d\omega^2 \tag{1.1}$$

in radial coordinates where $d\omega^2$ is the usual metric on S^{n-1}.

We establish our results for metrics g on $\mathbb{R}^n$ satisfying the following conditions (they are given in detail in the text). We require that the metric is C^2 and give conditions on f such that it is geodesically complete. We can then change coordinates such that on $\mathbb{R}^n \setminus \{0\}$, the metric has the form

$$ds^2 = dr^2 + h(r)^2 d\omega^2 \tag{1.2}$$

where $h \in C^2(\mathbb{R}^+)$, $h > 0$, and h satisfies certain regularity conditions as $r \to 0$. We characterize those functions h for which the metric admits a sufficiently large number of geodesics lying entirely inside a compact set $\Omega \subset \mathbb{R}^n$, whereas all geodesics with initial conditions in a suitable neighborhood of infinity leave every bounded set. These conditions follow from studying the flow on $T^*M \approx \mathbb{R}^{2n}$ generated by the Hamiltonian vector field corresponding to the classical Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j = \frac{1}{2} p_r^2 + h^{-2} H_{S^{n-1}} \tag{1.3}$$

where $H_{S^{n-1}}$ is the Hamiltonian for S^{n-1} with metric $d\omega^2$ and is a constant of motion.

For such metrics, we consider $H_g \equiv -\Delta_g$ acting on $L^2(\mathbb{R}^n, d\mu_g)$,

$d\mu_g \equiv (\det g)^{1/2}dx$, as a quantization of the classical system described by (1.3) on T*M (we drop the factor of 1/2). We are interested in the effect of the large family of bounded geodesics on H_g. Because of the spherical symmetry, we can reduce the problem to a one-dimensional one. For angular momentum ℓ, the reduced Hamiltonian acting on $L^2(\mathbb{R}^+, dr)$ has the form

$$H_g(\ell) = p^2 + \lambda(\ell)h(r)^{-2} + V_2(r) \tag{1.4}$$

where V_2 is a bounded function away from the origin and $\lambda(\ell) = \ell(\ell + n - 2)$. We show that under the conditions on h outlined above, the potential $h(r)^{-2}$ is of the same form as a shape resonance potential. Hence, states with non-zero angular momentum will experience an effective trapping potential. By adapting the machinery developed in [*1*] to solve the shape resonance problem, we are able to prove that $H_g(\ell)$ and consequently $H_g = -\Delta_g$ has either spectral resonances or bound states. Moreover, these are approximately given by the energy levels of H_g restricted to the region $\Omega \subset \mathbb{R}^n$ where the bounded geodesics are concentrated.

We emphasize that although we have transformed the problem to a Schrödinger operator problem, the effective potential h^{-2} is purely of geometric origin. Moreover, the angular momentum ℓ plays the crucial role of a semi-classical parameter, although it is not explicit in H_g.

To discuss these aspects of the problem in more detail, we recall that the existence of resonances (in the absence of positive bound states) has been previously established for certain classes of Schrödinger operators $H(\lambda)$ of the form

$$H(\lambda) = p^2 + V(\lambda; x) \tag{1.5}$$

where $V(\lambda; x)$ is a potential depending explicitly on a parameter λ. Examples of these are Schrödinger operators for shape resonances [*1*] [*2*] [*3*]; the AC Stark effect [*4*]; the DC Stark effect [*5*]; and the Zeeman effect [*3*] [*6*]. In all of these cases, the corresponding classical system has a set of initial conditions with non-zero measure for which the orbits are bounded but to which no bound states of the corresponding Schrödinger operator can be associated. Instead, the bounded orbits give rise to resonances.

The confining of classical orbits in these systems arises from the conservation of energy together with the existence of a potential barrier formed by $V(\lambda; x)$ which is made large (in the sense that the width of the barrier in the Agmon metric is made large) by increasing the external parameter λ, which is, for example, Planck's constant or the inverse of the electric field strength. We call λ the *semi-classical parameter* and in the large λ regime the tunneling through the barrier is suppressed. In the semi-classical region of large λ, one expects to be able to see the influence of the bounded geodesics more easily, and this is indeed the case for the systems mentioned above.

In the systems we study, the trapping of orbits originates in the geometry through the form of the metric and in the dynamics through the conservation of angular momentum. The latter plays the role of the semi-classical parameter but, unlike the examples (1.5), it is an internal parameter not appearing explicitly in H_g. That resonances should be more easily discerned at high values of ℓ can be understood as follows. It is basically believed that in the limit of high energy, the behavior of quantum systems is partially determined by classical quantities and consequently, the influence of the bounded geodesics should be more readily detected at high energies E. On the other hand, it is seen from the classical equations of motion (see (2.14)) that for fixed E, but small values of ℓ, there are no bounded geodesics. Equivalently, the quantity $\ell h^{-1}(2E)^{-1/2}$ measures the sine of the angle between the geodesic and the radial direction. If this angle is too small, the geodesic will not be bounded. Consequently, we expect to be able to locate the resonances more readily in the large E and ℓ region (of course, the latter implies the former). This explains the role of ℓ as the semi-classical parameter.

When H_g has no positive eigenvalues, the proofs presented here imply that H_g has spectral resonances $z_n(\ell)$ with $\operatorname{Im} z_n(\ell) < 0$ for all ℓ sufficiently large. We give two conditions on the function h which are sufficient to prove that $\sigma_{pp}(H_g) \cap \mathbb{R}^+ = \phi$. One condition is due to Escobar [*7*] who found sufficient conditions on a spherically symmetric metric on $\mathbb{R}^n$ to insure the absence of positive eigenvalues for H_g (for more general results, see [*8*].) The $\operatorname{Im} z_n(\ell)$ can be estimated as in the shape resonance problem. Since the resonances arise because of tunneling, the width is exponentially small in $\lambda(\ell)^{1/2}$. We show that for any $\varepsilon > 0$, there exists a $c_n > 0$ such that for all ℓ sufficiently large

$$|\operatorname{Im} z_n(\ell)| \leq c_n e^{-\lambda(l)^{1/2}(\hat{\rho}_0 - \varepsilon)}. \tag{1.6}$$

Here, $\lambda(\ell)^{1/2}\hat{\rho}_0$ is the leading asymptotic contribution in ℓ to the width, in the Agmon metric, of the potential barrier through which the particle with energy $\operatorname{Re} z_n(\ell)$ must tunnel.

The proof of the existence of resonances or bound states for H_g follows the procedure developed in [*1*] with one major change. The potentials occurring in (1.4) are more general than those discussed in [*1*] and require a different method for obtaining uniform estimates on the resolvent of the distorted Hamiltonian restricted to the exterior of the potential well. We develop here a quantum non-trapping condition in the exterior region for the potential based upon the general results derived in [*9*]. We formulate a condition on the metric which guarantees that all geodesics with initial conditions in the exterior of a ball of radius R_{NT} leave every bounded set and prove that this insures that the effective potential in (1.4) satisfies the quantum non-trapping condition in $\mathbb{R}^n \backslash B_{R_{NT}}(0)$.

We mention that our results apply to the Laplace-Beltrami operator on manifolds M of the form $M = X \times N$ where $X = \mathbb{R}$ or $\mathbb{R}^+$, N is compact, and the metric has the form $ds^2 = dr^2 + h(r)^2 d\omega^2$. In this setting, the eigenvalues of the Laplace-Beltrami operator on N play the role of the semi-classical parameter.

This paper is organized as follow. In Section 2, we study spherically symmetric manifolds $\mathbb{R}^n$ with metrics as in (1.2) using the classical Hamiltonian (1.3). We make a preliminary study of the spectral properties of H_g in Section 3 under the conditions on h formulated in Section 2. We also give two different conditions on h which imply the absence of positive energy bound states. The proof of the existence of bound states or spectral resonances for H_g is given in Sections 4 and 5. The quantum non-trapping condition is applied in Section 5 to prove uniform (in ℓ) bounds on the resolvent of the distorted Hamiltonian. In Section 6, we sketch the proof of the upper bound on the width of the resonances in terms of the width of the classically forbidden region in the Agmon metric. An application of these results to surfaces of revolution is given in Section 7. We present a model which illustrates the dissolution of bound states and their subsequent reappearance as resonances as a parameter is varied. We conclude with some comments on generalizations and extensions of this work in Section 8. In Appendix A, we formulate a quantum non-trapping condition and prove that it implies a uniform lower bound on the distorted Hamiltonian restricted to the non-trapping region. In Appendix B, we prove that the classical non-trapping condition on the metric formulated in Section 2 implies that the quantum condition of Appendix A holds.

2. CLASSICAL BOUNDED ORBITS AND NON-TRAPPING CONDITIONS

We consider the most general C^2, spherically symmetric (i. e. $SO(n)$-invariant) Riemannian metric on $\mathbb{R}^n\,(n > 1)$. In spherical coordinates $(u, \omega) \in \mathbb{R}^+ \times S^{n-1}$, it is given by

$$ds^2 = f(u)^{-2}du^2 + k(u)^2u^2d\omega^2 \tag{2.1}$$

where $f, k \in C^2(\mathbb{R}^+)$ are strictly positive and $d\omega^2$ is the metric on the unit sphere in $\mathbb{R}^n$. By comparing (2.1) with its expression in Cartesian coordinates in a neighborhood of the origin, one sees that

$$0 < \lim_{u\to 0} f(u) < \infty \tag{2.2 a}$$

$$0 < \lim_{u\to 0} k(u) < \infty \tag{2.2 b}$$

and

$$\lim_{u\to 0} k(u)f(u) = 1\,. \tag{2.3}$$

We introduce a new radial variable r as follows:

$$r(u) = \int_0^u f(s)^{-1} ds \tag{2.4}$$

so that $r'(u) = f(u)^{-1}$ and $r \in (0, b)$, with $b \equiv r(\infty)$. The coordinates $(r, \omega) \in (0, b) \times S^{n-1}$ are called geodesic coordinates. In terms of them, (2.1) becomes

$$ds^2 = dr^2 + h(r)^2 d\omega^2 \tag{2.5}$$

with

$$h(r(u)) = k(u)u. \tag{2.6}$$

It follows from (2.6), (2.2)-(2.4) that $h \in C^2((0, b))$, $h > 0$ and

$$\lim_{r \to 0} h(r) = 0 \tag{2.7}$$

$$\lim_{r \to 0} h(r)r^{-1} = \lim_{r \to 0} h'(r) = 1. \tag{2.8}$$

Calculating the curvature R of (2.5), one finds

$$R(r) = -(n-1)\frac{h''}{h} + \frac{(n-1)(n-2)}{2h^2}(1 - h'^2). \tag{2.9}$$

Since we assume that (2.5) is the restriction to $\mathbb{R}^n \setminus \{0\}$ of a metric on $\mathbb{R}^n$, the limit as $r \to 0$ of $R(r)$ exists and is finite; using (2.7) and (2.8), we find

$$\lim_{r \to 0} R(r) = \lim_{r \to 0} \left[-\frac{n(n-1)}{2} \frac{h''}{h} \right]. \tag{2.10}$$

Hence we conclude that $h''h^{-1}$ is bounded on any interval $(0, b_0)$, $b_0 < b$.

We now turn to the study of the geodesic flow of (2.5). It is well-known [*10*] that the geodesics of a Riemannian manifold (M, g) can be found by integrating the Hamiltonian equations of motion on T^*M, obtained from the Hamiltonian

$$H : (x^i, p_i) \in T^*M \to 1/2 g^{ij}(x) p_i p_j \in \mathbb{R} \tag{2.11}$$

(in local coordinates). With the metric as in (2.5), this reads

$$H = 1/2 p_r^2 + h(r)^{-2} H_{S^{n-1}}(\omega, p_\omega) \tag{2.12}$$

where we write $(r, \omega, p_r, p_\omega) \in T^*((0, b) \times S^{n-1})$. $H_{S^{n-1}}$ is the Hamiltonian obtained from the Riemannian metric on S^{n-1} as in (2.11). Let $\{ \cdot, \cdot \}$ denote the Poisson bracket on $T^*\mathbb{R}^n$. As a result of the spherical symmetry of H, one finds

$$\{ H_{S^{n-1}}, H \} = 0 \tag{2.13}$$

so that $H_{S^{n-1}}$ is a constant of the motion. It is then a simple matter to verify that

$$\frac{dr}{dt} = \pm (2H - h^{-2}\ell^2)^{1/2} \tag{2.14}$$

where we set

$$H_{S^{n-1}} = \ell^2 \tag{2.15}$$

and H is constant. In the sequel, we shall only be interested in geodesically complete metrics. We have the following criterion:

LEMMA 2.1. — *The metric (2.1) is geodesically complete iff* $\int_0^\infty f(s)^{-1}ds$ *diverges (i. e.* $b = \infty$*).*

Proof.

1) From (2.14) it follows that the condition is necessary; to see this it suffices to take $\ell = 0$.

2) Conversely, suppose $b = \infty$. Let $\gamma:(c, d) \to T^*\mathbb{R}^n$ be a maximal integral curve of the Hamiltonian vector field X_H, with $H = 1/2g^{ij}p_ip_j$, as before; $\gamma(s) = (x(s), p(s))$:

$$\dot{\gamma}(s) = X_H(\gamma(s)). \tag{2.16}$$

Compute the length $\ell(c_0, d)$ of the projected curve $x:(c_0, d) \to \mathbb{R}^n$: $(c < c_0 < d)$

$$\begin{aligned} \ell(c_0, d) &= \int_{c_0}^{d} \sqrt{g(\dot{x}(s), \dot{x}(s))}ds \\ &= (2H)^{1/2}(d - c_0) \end{aligned} \tag{2.17}$$

where we used the constancy of H. Now, if $(d - c_0) < \infty$, then we show that $x(s)$ lies in a compact subset of $\mathbb{R}^n$. Indeed, let $y:(\alpha, \beta) \to \mathbb{R}^n$ be a curve that leaves every ball of radius U in $\mathbb{R}^n$.

Then

$$\begin{aligned} \ell(\alpha, \beta) &= \int_\alpha^\beta \sqrt{g(\dot{y}(s), \dot{y}(s))}ds \\ &\geq \int_\alpha^\beta \left|\frac{du}{ds}\right| f(u(s))^{-1}ds \\ &\geq \int_I \frac{du}{ds}(s) f(u)^{-1}ds \end{aligned}$$

where I is the open subset of (α, β) on which $\dot{u} > 0$. Changing variables, we have

$$\ell(\alpha, \beta) \geq \int_{u(I)} f(u)^{-1}du .$$

Since $\overline{\lim}_{s\to\beta} u(s) = \infty$, we know that $u(I) = (\inf u(I), \infty)$ and hence

$$\ell(\alpha, \beta) \geq \int_{\inf u(I)}^{\infty} f(u)^{-1}du \tag{2.18}$$

so that $\ell(\alpha, \beta)$ is infinite by assumption. Consequently $x(s)$ lies in a compact set.

3) Since X_H is a C^1 vector field (because the metric is C^2) we have that

$$\lim_{s\to d} x(s) \equiv x_1 \in \mathbb{R}^n$$
$$\lim_{s\to d} \dot{x}(s) \equiv v_1 \in \mathbb{R}^n$$

where we used (2.16). Using (x_1, v_1) as a new set of initial conditions in the geodesic equations of motion, the existence and uniqueness of solutions to ordinary differential equations guarantees that we can extend γ to some interval $(c, d + \varepsilon)$, $\varepsilon > 0$. But that contradicts our assumption that γ is maximal. So $(d - c_0) = \infty$ and hence $d = \infty$. Similarly one proves $c = -\infty$. □

As a result of Lemma 2.1, the metric in (2.5), with h defined on $(0, b)$ and satisfying (2.7)-(2.8), is geodesically complete iff $b = \infty$. From now on we always work with geodesically complete, SO(n)-invariant metrics on $\mathbb{R}^n$, expressed in geodesic coordinates as in (2.5). We summarize the conditions on the allowed functions h as follows:

CONDITIONS A′. — *(Conditions on the metric-classical case).*

i) $$h \in C^2(\mathbb{R}^+) \tag{2.19}$$

ii) $$h > 0 \tag{2.20}$$

iii) $$\lim_{r\to 0} h(r)r^{-1} = \lim_{r\to 0} h'(r) = 1 \tag{2.21}$$

iv) $$-\infty < \lim_{r\to 0} h''h^{-1} < \infty . \tag{2.22}$$

We now determine under what additional conditions on h there exist geodesics of (2.5) that lie inside a compact subset of $\mathbb{R}^n$. We will refer to these as *bounded orbits.* We have the following result:

THEOREM 2.2. — *The initial conditions in* $T^*\mathbb{R}^n$ *leading to bounded orbits under the flow of (2.12) are a set of non-zero measure iff* $\exists 0 < r_1 < r_2$ *such that*

$$h'(r_1) = 0 = h'(r_2) \tag{2.23 a}$$
$$\forall r \in (r_1, r_2), \qquad h'(r) < 0 . \tag{2.23 b}$$

Proof. — Suppose (2.23) is satisfied; consider initial conditions $(r, p_r, \omega, p_\omega)$ such that

$$r_3 \le r \le r_2 \tag{2.24 a}$$
$$1/2\ell^2 h(r_1)^{-2} \le H \le 1/2\ell^2 h(r_2)^{-2} \tag{2.24 b}$$

where $r_3 \equiv \sup\{ r < r_1 \mid h(r) = h(r_2) \}$; r_3 certainly exists as a result of (2.21). Then the initial conditions (2.24) lead to bounded orbits in view of (2.14); moreover, they clearly form a set of non-zero measure. Conversely, suppose (2.23) does not hold; then it follows from (2.21) that h is monotonically non-decreasing. It follows from (2.14) that bounded orbits are then of the form

$$t \in \mathbb{R} \to (r, \omega(t)) \in \mathbb{R}^+ \times S^{n-1} \tag{2.25}$$

where r is such that $h'(r) = 0$. It follows that

$$H = 1/2\ell^2 h(r)^{-2} \tag{2.26}$$

so that initial conditions leading to bounded orbits are a set of measure zero. □

We remark that if $n = 1$, it follows from (2.11) and the analog of (2.14) that there are no bounded geodesics; this is why we exclude that case. Note also that it is possible for r_2 in Theorem 2.2 to be infinite. However, the *non-trapping* condition below will guarantee that this possibility does not occur.

The metrics we wish to consider will have bounded orbits in some compact region of $\mathbb{R}^n$, but outside a (possibly larger) compact set we want almost all initial conditions to lead to orbits that move out to infinity. In light of Theorem 2.2 we have

DEFINITION 2.3. — *We say a metric (2.5) on* $\mathbb{R}^n$, *satisfying Conditions* A′, *is non-trapping for* $r > R$ *iff*

$$h'(r) \geq 0 \tag{2.27}$$

for all $r > R$.

We can paraphrase the non-trapping condition (2.27) by saying that there are almost no bounded orbits for $r > R$. We illustrate a typical function h with trapped orbits and satisfying (2.27) as well as A′, together with the effective potential $\ell^2 h^{-2}$, in Figure 2.1.

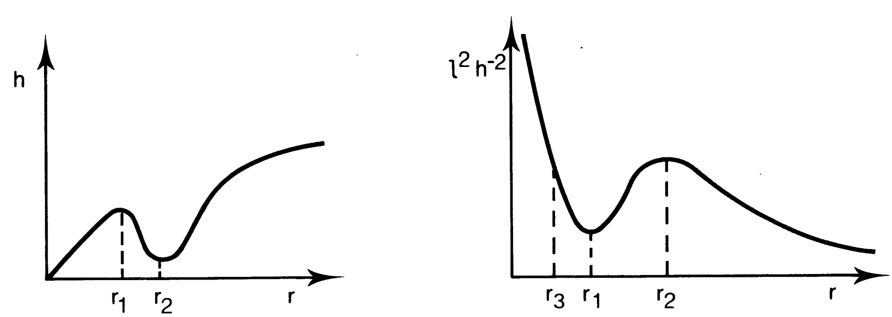


FIG. 2.1. — Typical h and $\ell^2 h^{-2}$.

The bounded orbits stay within the region $r_3 \leq r \leq r_2$. Note that the depth of the potential well behaves as $\mathcal{O}(\ell^2)$ for large ℓ, so that it becomes deeper as $\ell \to \infty$.

3. PRELIMINARIES ON THE LAPLACE-BELTRAMI OPERATOR

In this Section, we make a preliminary study of the Laplace-Beltrami operator $H_g \equiv -\Delta_g$. We show that under Conditions A (given below), H_g is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Spherical symmetry is used to reduce the problem to a one-dimensional problem. It is shown that H_g restricted to each angular momentum subspace is unitarily equivalent to a Schrödinger operator $H_g(\ell)$ on the half-line. We show that under the hypotheses of Theorem 2.2, the potential V_l is similar to the type of potential studied in the shape resonance problem. In the present problem, ℓ plays the role of the semi-classical parameter. We formulate a separate condition on h which we prove implies the absence of positive eigenvalues for H_g.

We will always assume that the function h, which determines the metric g on $\mathbb{R}^n$ as in (2.5), satisfies the following Conditions A, which are an extension of the Conditions A′ given in Section 2.

CONDITIONS A. — *(Conditions on the metric-quantum case).*

i) $h \in C^2(\mathbb{R}^+)$ and $h > 0$ on $\mathbb{R}^+$;

ii) $\lim_{r \to 0} r^{-1} h(r) = 1$;

iii) $h' h^{-1}$ is bounded outside of any neighborhood of the origin; $h'' h^{-1}$ is everywhere bounded;

iv) $\exists \delta > 0$ such that $\underline{\lim}_{r \to \infty} h(r) > \delta > 0$.

REMARK 3.1. — The curvature R, defined in (2.9), is everywhere bounded under Conditions A.

In addition, we will at times impose the following conditions:

CONDITION B. — *(Exterior non-trapping.)* There exists an $R_{NT} > 0$ (R_{NT} must satisfy an additional condition specified in (4.2) below) such that for $r > R_{NT}$, $h'(r) \geq 0$ (see Definition 2.3), and $h'(r) > 0$ for $r \in [R_{NT}, \tilde{r}]$ where $\tilde{r} \equiv \inf \{ r > R_{NT} \mid V_1(r) = \min_{s \in [0, R_{NT}]} V_1(s) \}$.

CONDITION C. — *(Existence of bounded geodesics.)*

i) h satisfies the hypotheses of Theorem 2.2 for some (and at most finitely many) pairs of points (r_1, r_2);

ii) for any such pair (r_1, r_2),

$$\underline{\lim}_{r \to \infty} h(r) > h(r_1)$$

$$h''(r_1) < 0$$

Condition C *ii)* implies that the effective potential given below is a shape resonance potential of non-threshold type in the terminology of *[1]*.

Two other technical conditions, D and E, will be given below. To simplify the proof of the existence of bound states or resonances, we will assume that if h satisfies Condition C, it does so for only one pairs of points (r_1, r_2). This amounts to treating a shape resonance potential with a single well. We comment on the general case in Section 8.

Note that as in the shape resonance problem, Condition C *i)* that $h''(r_1) < 0$, is required for the harmonic approximation. This is not essential and can be replaced by any condition which guarantees that the embedded eigenvalues of an approximate Hamiltonian H_0, given below, do not move together at an exponentially fast rate as the angular momentum ℓ increases.

The significance of Conditions B and C for the geodesic flow was discussed in Section 2. We will explore the quantum consequences of these conditions in this and the following sections. We will show that they imply that H_g has either bound states or spectral resonances. We assume that H_g represents the appropriate quantization of the generator of the geodesic flow (however, see Section 8).

We begin by showing that the dynamics generated by H_g is well-defined. This reflects the fact that the manifold is geodesically complete. We have the following standard result, essentially due to Chernoff [*11*].

LEMMA 3.2. — *Let h satisfy Condition* A. *The Laplace-Beltrami operator* H_g *is essentially self-adjoint* $C_0^\infty(\mathbb{R}^n)$.

Proof. — By Theorem 2.4, Condition A on h implies that $(\mathbb{R}^n, g)$ is geodesically complete. As $h \in C^2$, then by the Hopf-Rinow Theorem [*12*], $(\mathbb{R}^n, g)$ is a complete metric space. It now follows by a theorem of Chernoff that H_g is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. □

We now proceed to the analysis of H_g. By the spherical symmetry of g, we reduce questions about H_g to ones about the restriction of H_g to SO(n)-invariant subspaces. We have

$$H_g \equiv -\Delta_g = -(\bar{g})^{-1/2}\partial_i g^{ij}(\bar{g})^{1/2}\partial_j \tag{3.1}$$

where $[g^{ij}]$ is the inverse of $[g_{ij}]$ and $\bar{g} \equiv \det [g_{ij}]$. H_g acts on $\mathcal{H} \equiv L^2(\mathbb{R}^n, (\bar{g})^{1/2}dx)$. From (2.5) and (3.1), it is a simple calculation to show that

$$H_g = h(r)^{-(n-1)}\frac{\partial}{\partial r}h(r)^{n-1}\frac{\partial}{\partial r} + h(r)^{-2}L^2 \tag{3.2}$$

where L^2 is the Laplace-Beltrami operator on S^{n-1}. We have that

$$\sigma(L) = \{\, \ell(\ell + n - 2) \mid \ell \in \mathbb{Z}_+ \,\}.$$

The Hilbert space $\mathcal{H}$ affords a direct sum decomposition $\mathcal{H} = \bigoplus_\ell \mathcal{H}_\ell$, where $\mathcal{H}_\ell \simeq L^2(\mathbb{R}^+, h^{n-1}dr) \otimes \mathcal{V}_\ell$, $\mathcal{H}_\ell$ is an SO(n)-invariant subspace and, for each ℓ, $\mathcal{V}_\ell$ is a (finite-dimensional) carrier space for inequivalent representations of SO(n).

Since $L^2 \mid \mathscr{V}_\ell = \ell(\ell + n - 2)$, we define

$$\tilde{H}_g(\ell) \equiv - \Delta_g \mid \mathscr{H}_\ell = T_g + h(r)^{-2}(\ell(\ell + n - 2)) \qquad (3.3\,a)$$

where

$$T_g = - h(r)^{-(n-1)} \frac{d}{dr} h(r)^{n-1} \frac{d}{dr}. \qquad (3.3\,b)$$

Let $U_h : L^2(\mathbb{R}^+, h(r)^{n-1}dr) \to L^2(\mathbb{R}^+, dr)$ be the transformation defined by

$$(U_h f)(r) \equiv h(r)^{(n-1)/2} f(r).$$

Then U_h is a unitary operator. We define $H_g(\ell)$ acting on $L^2(\mathbb{R}^+, dr)$ by

$$H_g(\ell) \equiv U_h \tilde{H}_g(\ell) U_h^{-1} = - \frac{d^2}{dr^2} + V_\ell \qquad (3.4\,a)$$

where

$$\begin{aligned} V_\ell(r) &\equiv \lambda(\ell) V_1(r) + V_2(r) \\ V_1(r) &\equiv h(r)^{-2} \\ V_2(r) &\equiv (1/4)(n-1)(n-3)(h'h^{-1})^2(r) + (1/2)(n-1)(h''h^{-1})(r) \end{aligned} \qquad (3.4\,b)$$

and $\lambda(\ell) \equiv \ell(\ell + n - 2)$.

LEMMA 3.3. — *Let h satisfy Conditions* A *and let* $H_g(\ell)$ *be the symmetric operator defined in (3.4) with domain* $C_0^\infty(\mathbb{R}^+)$. *For* $n \geq 4$, *and any* $\ell \geq 0$, $H_g(\ell)$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^+)$. *For* $n = 3$ *(for* $n = 2$*)*, $H_g(\ell)$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^+)$ *for* $\ell \geq 1$ *(for* $\ell \geq 2$, *respectively).*

Proof. — This follows from a simple application of Weyl's limit point-limit circle criterion [13]. Let $\tilde{V}_\ell \equiv \lambda(\ell)h^{-2} + (1/4)(n-1)(n-3)(h'h^{-1})^2$. It is easy to check that $\tilde{V}_\ell$ is in the limit point case at infinity by verifying that $\tilde{V}_\ell' \tilde{V}_\ell^{-3/2}$ is bounded in a neighborhood of infinity and that

$$\int_1^\infty [\tilde{V}_{\ell,0} - \tilde{V}_\ell]^{-1/2} dr = \infty \qquad \text{where} \qquad \tilde{V}_{\ell,0} \equiv \sup_{r>1} |\tilde{V}_\ell(r)|$$

(this latter condition guarantees that the classical motion is complete at infinity). As for the behavior at $r = 0$, for $n \geq 4$, $\tilde{V}_\ell(r) \geq (3/4)r^{-2}$ in a small neighborhood of zero, for all $\ell \geq 0$ as $\lim_{r \to \infty} r^{-1}h(r) = 1$. For $n = 3$, the same result holds provided $\ell \geq 1$; for $n = 2$, we must take $\ell \geq 2$. Hence, $- D_r^2 + \tilde{V}_\ell$ is in the limit point case at zero and infinity and, consequently it is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+)$, under the restrictions on ℓ. Since $h''h^{-1}$ is bounded, this perturbation does not change the domain of essential self-adjointness. □

This lemma suffices to define $H_g(\ell)$ as a self-adjoint operator for our purposes since we are concerned with high angular momentum ℓ. However, for completeness, we note that $H_g(\ell)$ should be defined by the Friedrichs extension of the symmetric operator $-D_r^2 + V_\ell$ on $C_0^\infty(\mathbb{R}^+)$ for the cases (when $n = 2, 3$) not covered by Lemma 3.2.

As a final topic of this Section, we consider some conditions on h which guarantee the absence of positive energy bound states for $H_g(\ell)$. We will see in Section 5 that when h satisfies this condition, $H_g(\ell)$ will have spectral resonances, i. e. the imaginary part of the resonance energy will be strictly negative. Our condition D1 on h is motivated by well-known results on the absence of positive energy bound states for Schrödinger operators [*14*]. The alternative condition D2 is a condition used by Escobar [*7*] in a recent paper.

CONDITION D. — *(Absence of positive energy bound states).*

(D1) For some $b > 0$, h satisfies $\int_b^\infty |h''h^{-1}| < \infty$ and $\lim_{r\to\infty} r^{-1/2+\varepsilon}h(r) = c$ for some small $\varepsilon > 0$ and $0 < c < \infty$.

(D2) Let K be the radial curvature defined by $K \equiv -h''h^{-1}$. Then either

i) $\lim_{r\to\infty} h(r) = \infty$ and $\exists R_+ > 0$ such that $K \geq 0$ on (R_+, ∞);

or,

ii) $K \leq 0$, $\exists R_+$ such that $h'(R_+) > 0$ and $\lim_{r\to\infty} K(r) = 0$.

THEOREM 3.4. — *Let h satisfy Conditions* A. *If h satisfies Condition* D1 *or Condition* D2, *then* $H_g(\ell)$ *(and hence* $-\Delta_g$*) has no positive energy bound states.*

Proof. — That D2 implies the absence of positive energy bound states is proved in [*7*] or [*8*]. We consider D1. Note that $\lim_{r\to\infty} r^{-1/2+\varepsilon}h(r) = c$ implies that $\lim_{r\to\infty} r^{1/2+\varepsilon}h'(r) = c'$, $0 < c' < \infty$ by l'Hopital's rule.Any eigenfunction u for $H_g(\ell)$ with eigenvalue $k^2 > 0$ must be an L^2-distributional solution of

$$-u'' + (V_\ell - k^2)u = 0. \tag{3.5}$$

Conditions D_1 imply that $V_\ell \in L^1([b, \infty))$ and Conditions A imply that $V_\ell \in L^2_{loc}(\mathbb{R}^+)$. Consequently, any solution of (3.5) must be a linear combination of Jost solutions [*14*]. As these solutions behave asymptotically as e^{ikr} $(\ell = 0)$ and $krj_\ell(kr)$ $(\ell > 0)$ (where j_ℓ is the ℓ^{th} spherical Bessel function), they are not L^2 at infinity. Consequently, (3.5) has no L^2-solutions. □

REMARKS 3.5.

1) The second alternative in Condition D2, $K \leq 0$, will be excluded when we impose Conditions C, since as a consequence of Conditions C, the case of $K \leq 0$ everywhere is ruled out. Note that, as discussed by Esco-

bar [7], in the case of negative radial curvature the absence of positive eigenvalues is unstable under even compact perturbations of the metric, unlike the non-negative curvature case.

2) Escobar [7] also mentions that S. T. Yau [15] proved that if M is a complete, non-compact Riemannian manifold, then there are no non-constant harmonic function in $L^p(M)$, $p \in (1, \infty)$. Consequently, 0 is not an eigenvalue of $-\Delta_g$ and Theorem 3.4 implies that $-\Delta_g$ has no eigenvalues.

4. APPROXIMATE HAMILTONIAN AND SPECTRAL DISTORTION

We continue our study of the Schrödinger operator $H_g(\ell)$ on $L^2(\mathbb{R}^+, dr)$ as defined in (3.4). We will assume that h satisfies Conditions A-C of Section 3. We will not impose Conditions D at this point. As a consequence, we will prove that $H_g(\ell)$ has either a bound state or a spectral resonance in a neighborhood of an eigenvalue of H_{01} (defined below) which describes a particle confined to the potential well of V_ℓ. This result is obtained by modifying the proof given in [1] for the shape resonance problem, as we have already noted that $\lambda(\ell)h(r)^{-2}$ has the form of a non-threshold shape resonance potential. The V_2 part of the potential is bounded on any set $[\varepsilon, \infty)$, $\varepsilon > 0$, and $O(\lambda(\ell)^{-1})$ relative to V_1. Hence, in the large ℓ regime, it will contribute negligibly to the calculations.

In this and the next sections, we outline the main steps of [1], using freely the material developed there. There is one major difference between the problem at hand and the shape resonance problems studied here. In [1], it was assumed that on the exterior of a ball of radius R, $B_R(0)$, the potential is a sum of homogeneous functions. This allowed one to obtain specific estimates on the resolvent of the distorted Hamiltonian restricted to $C_0^\infty(\mathbb{R}^n \setminus B_R(0))$. In the present case, we are dealing with more general potentials and we replace this argument with a non-trapping argument derived from [9] (see Section 5 and Appendix A). Consequently, Condition B on h will play a major role in these estimates.

A. The Approximate Hamiltonian.

Let $\{j_i\}_{i=1}^3$ be a C^∞ partition of unity for $\overline{\mathbb{R}^+}$, $0 \le j_i \le 1$, with $\sum_{i=1}^{3} j_i^2 = 1$ and $j_1(0) = 1$. Let $V_0(\ell) \equiv \min_{r<r_2} V_\ell(r) \equiv V_\ell(r_0)$, and let $S_0^+ > r_2$ be such that $V_\ell(S_0^+) = V_0(\ell)$. S_0^+ is the exterior exit point for energy $V_0(\ell)$. Note that $V_0(\ell) = O(\ell^2)$. Let $r_2 < R_2 < S_0^+$. We choose j_1 and j_2 such that

$\text{supp}(j_1) = [0, R_2] \equiv D_1$, $\text{supp}(j_2) = [R_2, \infty) \equiv D_2$ and $j_1(R_2) = j_2(R_2) = 0$. Choose $\varepsilon_1, \varepsilon_2 > 0$ small such that

$$r_2 < R_2 - \varepsilon_1 < R_2 < R_2 + \varepsilon_2 < S_0^+ \tag{4.1}$$

and take j_3 such that $\text{supp}(j_3) = [R_2 - \varepsilon_1, R_2 + \varepsilon_2]$. A partition of unity is sketched in Figure 4.1. We assume that the non-trapping radius R_{NT} described in Condition B satisfies

$$R_2 + \varepsilon_2 < R_{NT} < S_0^+ \tag{4.2}$$

By the IMS localization formula, we have

$$p^2 = \sum_{i=1}^{3} j_i p^2 j_i - \sum_{i=1}^{3} (j_i')^2 \tag{4.3}$$

so we define localized Hamiltonians H_{0i}, $i = 1, 2$, by

$$H_{0i} \equiv j_i p^2 j_i + [\lambda(\ell)V_1 + V_2 + J]\chi_i \tag{4.4}$$

where $J \equiv -\Sigma(j_i')^2$ and χ_i is the characteristic function for D_i, $i = 1, 2$. The approximate Hamiltonian is

$$H_0(\ell) = H_{01} \oplus H_{02} \tag{4.5}$$

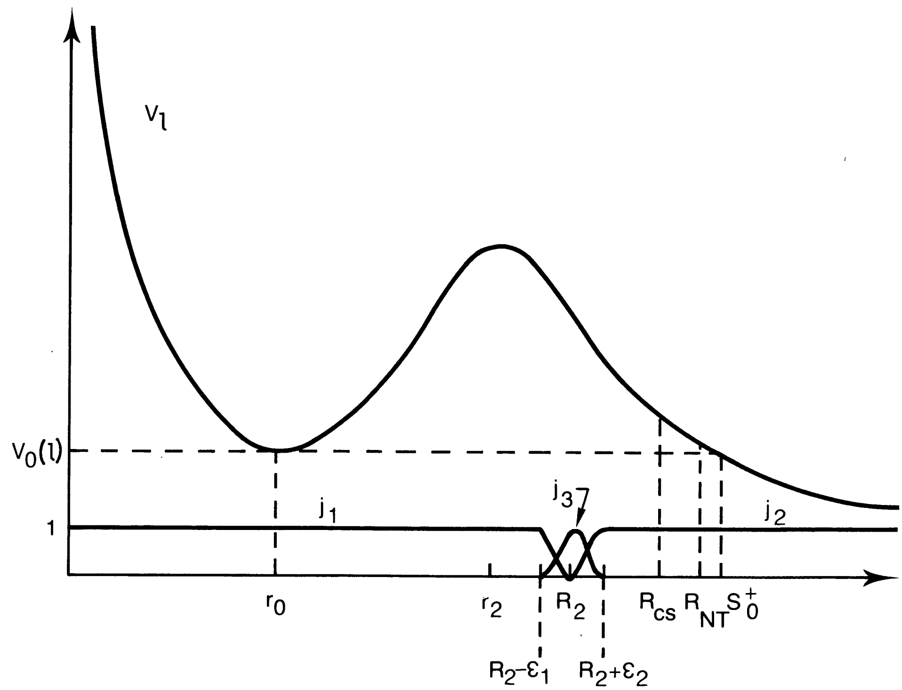


FIG. 4.1. — The Potential V_ℓ with a Partition of Unity.

acting on $L^2(\mathbb{R}^+, dr)$. The localized perturbation W is given by

$$W = H_g(\ell) - H_0(\ell) = j_3 p^2 j_3 . \tag{4.6}$$

H_{02} is essentially self-adjoint on $C_0^\infty(D_2)$. It follows from the proof of Lemma 3.4 that H_{01} is essentially self-adjoint on $C_0^\infty(D_1)$ for $\ell \geq 2$. For $\ell = 0, 1$, we take H_{01} to be the Friedrichs extension from the domain $C_0^\infty(D_1)$.

B. Spectral Preliminaries.

We now study the spectra of the self-adjoint operators $H_g(\ell)$ and H_{0i} under Conditions A and C. Since $H_g(\ell)$ is unitarily equivalent to $\tilde{H}_g(\ell) \geq 0$ (see (3.3)) we have $\sigma(H_g(\ell)) \subset [0, \infty)$. Moreover, we will prove below that (under an additional Condition E), $H_g(\ell)$ extends to an analytic type A family of operators. It is proved in [1] that $\sigma_{ess}(H_g(\ell, \theta)) = e^{-2\theta}\mathbb{R}_+$. Then, by a standard argument [14], these two facts imply that $\sigma_{sc}(H_g(\ell)) = \phi$.

Because we have defined H_{0i} using smooth cut-off functions, the operator H_{01} may have essential spectrum. The spectra of H_{0i} are characterized in the following theorem. The proof follows along the same lines as that of Theorem 3.3 in [1] so we only comment on part of it.

THEOREM 4.1. — *Assume Conditions* A *and* C *hold.*

1) $\sigma_{ess}(H_{01}) \subset [v_1(\ell), \infty)$, *where* $v_1(\ell) \equiv \min_{R_2 - \varepsilon_1 \leq r \leq R_2} (V_\ell + J)(r)$,

so $v_1(\ell) = O(\lambda(\ell))$; $\sigma_d(H_{01}) \neq \phi$ *and finite and if* $e_n(\ell) \in \sigma_d(H_{01})$, *then* $e_n(\ell) - V_0(\ell) = O(\ell)$.

2) $\sigma(H_{02}) = [v_2(\ell), \infty)$ *where* $v_2(\ell) \equiv \min(\sigma(\ell), \lim_{r\to\infty} V_\ell(r))$ *and* $\sigma(\ell) \equiv \min_{R_2 \leq r \leq R_2 + \varepsilon_2} (V_\ell + J)(r)$.

REMARKS 4.2.

1) As in [1], the $\sigma_{ess}(H_{0i})$ comes from the regions where $j_i' \neq 0$. The proof follows from local compactness and Weyl criteria arguments as given in [1]. We present details of a similar local compactness argument in this setting in Section 7, Lemma 7.2.

2) The discrete spectrum of H_{01} is established using the harmonic approximation and the Min-max Principle. The comparison harmonic oscillator Hamiltonian is:

$$K(\ell) = p^2 + \lambda(\ell) V_1''(r_1)(r - r_1)^2 + V_0(\ell) . \tag{4.7}$$

The minimum of V_1 at r_1 is non-degenerate by Condition C. Consider the related oscillator

$$K = p^2 + V_1''(r_1) r^2 \tag{4.8}$$

acting on $L^2(\mathbb{R})$. If $\{ e_n \}$ is a listing (in increasing order) of $\sigma(K)$, then

$$e_n = (1/2 + n)(V_1''(r_1))^{1/2}. \tag{4.9}$$

Note that all of the eigenvalues are simple. If $E_n(\ell) \in \sigma(K(\ell))$, it is easy to see that:

$$E_n(\ell) = \lambda(\ell)^{1/2} e_n + V_0(\ell) \tag{4.10}$$

from (4.7)-(4.9). Then, the full statement of the harmonic approximation is:

$$e_n(\ell) = \lambda(\ell)^{1/2} e_n + V_0(\ell) + O(\lambda(\ell)^{\frac{2}{5}}). \tag{4.11}$$

The proof of (4.11) follows from [1] or [16] (see also Theorem 7.2). Note that for large ℓ, $e_n(\ell) < \inf \sigma_{ess}(H_{01})$.

3) For any $e_n(\ell) \in \sigma_d(H_{01}(\ell))$, let $F_n(\ell)$ denote the classically forbidden region for energy $e_n(\ell)$, i. e.

$$F_n(\ell) \equiv \{ r \mid V_\ell(r) - e_n(\ell) > 0 \} \tag{4.12}$$

and let $S_n^\pm$ be the classical turning points for energy $e_n(\ell)$, i. e. $S_n^- < R_2 < S_n^+$ and $V_\ell(S_n^\pm) = e_n(\ell)$. Then, $F_n(\ell) = (S_n^-, S_n^+)$. Note that the condition $S_n^- < R_2$ and $S_n^+ > R_2$ is a restriction on n. We will always suppose that n is small enough such that this condition holds.

C. **Spectral Distortion.**

We use the method of smooth vector fields developed in [1] (see also [9]). In order to construct analytic families of operators associated with H_0 and $H_g(\ell)$, we must make additional assumptions on the function h.

CONDITIONS E. — *(Analyticity).*

i) $\exists R_{cs} > 0$ such that $R_2 + \varepsilon_2 < R_{cs} < S_0^+$, and $\exists \delta_0 > 0$ such that h is the restriction to $[R_{cs} - \delta_0, \infty)$ of a function $\tilde{h}$ analytic in a sector $\mathcal{S}_h$ with vertex at $R_{cs} - \delta_0$ and angle θ_h;

ii) $\exists \delta > 0$ such that $\underline{\lim}_{z \in \mathcal{S}_h} | \tilde{h}(z) | > \delta > 0$;

iii) $\tilde{h}^{(i)} \tilde{h}^{-1}$, $i = 1, 2, 3, 4$, are bounded on the same sector $\mathcal{S}_h$.

Note that given R_{NT} of Condition B, we can always find $\tilde{R}_{NT}$ such that $R_{cs} < \tilde{R}_{NT} < S_0^+$, so without loss of generality we will assume that $R_{cs} < R_{NT} < S_0^+$. As a consequence of Condition E, V_ℓ in (3.4), when restricted to $[R_{cs}, \infty)$, extends to a bounded, analytic, operator-valued function on $\mathcal{S}_h$. We note that functions $\tilde{h}$ with polynomial and exponential growth satisfy Conditions E.

We consider flows on $\mathbb{R}$ (and $\mathbb{C}$) generated by C^k vector fields f, $k \geq 2$ with f' and f'' bounded and which are linear at infinity, i. e.

$$\lim_{r \to \infty} r^{-1} f(r) = 1 .$$

We require that f, together with its first derivative, be monotonically increasing. In addition, f has the form

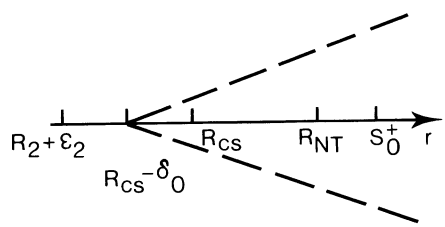
$$f(r) = \begin{cases} 0 & r \leq R_{cs} \\ \tilde{f}(r) & r \geq R_{cs}. \end{cases}$$

Here, $\tilde{f}$ is the restriction to $[R_{cs} - \delta_1, \infty)$, for some $\delta_1 > 0$, of a function analytic on a sector $\mathscr{S}_f$ with vertex at $R_{cs} - \delta_1$ and angle $0 < \theta_f < \pi$. An example of such an $\tilde{f}$ is

$$\tilde{f}(z) = [(z - R_{NT}) + (R_{NT} - R_{cs})e^{-(z-R_{cs})^2}], \qquad (4.13)$$

see Figure 4.2.

a) Sector $\mathscr{S}_h$



b) Vector field f

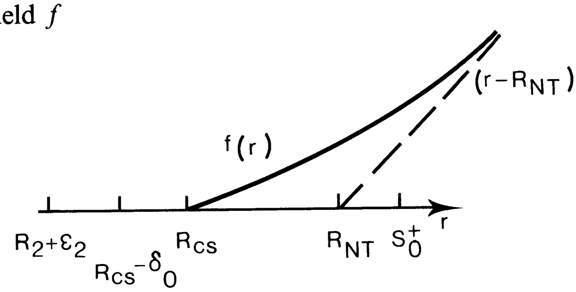


FIG 4.2. — Analyticity region of $\tilde{h}$ and a Vector field f.

It follows from Section 4 of [*1*] that any vector field f as above generates a global flow ϕ_θ, $\theta \in \mathbb{R}$, which has an analytic continuation in θ onto a strip $S_\eta \equiv \{ \theta \in \mathbb{C} \mid |\operatorname{Im} \theta| < \eta \}$ for some $\eta > 0$ determined by the sector angle θ_f. Note that the segment $[0, R_{cs}]$ consists of equilibrium points for the flow ϕ_θ.

We proceed to construct the analytic families $H_0(\ell, \theta)$ and $H_g(\ell, \theta)$ using a flow ϕ_θ as described above. Given the functions h and $\tilde{h}$ satisfying Conditions A and E, we choose the vector field f satisfying the conditions above. Moreover, we require that there exists $\tilde{\eta}, 0 < \tilde{\eta} \leq \eta$, such that

$$\{ \phi_\theta(r) \mid r \geq R_{cs}, \theta \in S_{\tilde{\eta}} \} \subset \mathcal{S}_h$$

(If $\theta_h > \theta_f$, we take $\eta = \tilde{\eta}$, otherwise we can always restrict $\tilde{f}$ to a smaller sector.) Then $V_\ell \circ \phi_\theta$ is analytic in θ on $S_{\tilde{\eta}}$. We assume that this has been done and write η for $\tilde{\eta}$ below.

Let U_θ, $\theta \in \mathbb{R}$, be the unitary group defined by

$$(U_\theta f)(r) = J_\theta(r)^{1/2} f(\phi_\theta(r)) \tag{4.14}$$

where $J_\theta(r) \equiv [\partial\phi_\theta(r)/\partial r]$ is the Jacobian of the flow. By an elementary calculation we have

$$p_\theta^2 \equiv U_\theta p^2 U_\theta^{-1} = J_\theta(r)^{-1/2} p J_\theta(r)^{-1} p J_\theta(r)^{-1/2} \tag{4.15}$$

where $p \equiv -id/dr$. This can be expressed in the following useful form:

$$p_\theta^2 = p J_\theta^{-2} p + F_\theta \tag{4.16 a}$$

where F_θ is the function

$$F_\theta(r) \equiv (1/2)(J_\theta^{-5/2}(r) J'_\theta(r))' J_\theta^{-1/2}(r). \tag{4.16 b}$$

Note that by construction, U_θ has a natural action on $L^2(D_1) \oplus L^2(D_2)$. Defining $H_0(\ell, \theta)$ and $H_g(\ell, \theta)$ by $U_\theta H_0(\ell) U_\theta^{-1}$ and $U_\theta H_g(\ell) U_\theta^{-1}$, $\theta \in \mathbb{R}$, respectively, we have:

THEOREM 4.3. — *Let $S_\eta = \{ \theta \mid |\operatorname{Im}\theta| < \eta \}$, where η is determined by the vector field f, as above. Then there exists θ_0, $0 < \theta_0 < \eta$, such that*

1) *$H_0(\ell, \theta) \equiv H_{01}(\ell) \oplus H_{02}(\ell, \theta)$ is an analytic family of type A on the strip S_{θ_0} with domain $D(H_{01}) \oplus D(j_2 p^2 j_2)$.*

2) *$H_g(\ell, \theta)$ is an analytic family of type A on the strip S_{θ_0} with domain $D(H_g(\ell))$ for $\ell \geq 2$.*

Proof.

1) By construction of U_θ, we have for $\theta \in \mathbb{R}$:

$$U_\theta H_0 U_\theta^{-1} = H_{01}(\ell) \oplus H_{02}(\ell, \theta) \tag{4.17}$$

and $U_\theta D(j_2 p^2 j_2) \subset D(j_2 p^2 j_2)$. Since $j_2 \mid [R_{cs}, \infty) = 1$,

$$H_{02}(\ell, \theta) = j_2 p_\theta^2 j_2 + [\lambda(\ell) V_1(\theta) + V_2(\theta) + J]\chi_2 \tag{4.18}$$

where p_θ^2 is defined in (4.16 *a*). From (4.16 *a*), it is clear that for $u \in D(j_2 p^2 j_2)$,

$$\theta \in S_\eta \mapsto j_2 p_\theta^2 j_2 u$$

is analytic on S_η. As in Section 4 of [*1*], one shows that $j_2 p^2 j_2$ and $j_2 p_\theta^2 j_2$ are mutually, relatively bounded on some strip $S_{\theta_1}, 0 < \theta_1 < \eta$. This establishes the type A property for $j_2 p_\theta^2 j_2$ on $D(j_2 p^2 j_2)$. By Condition E, $\chi_2 V_i(\theta)$, $i = 1, 2$, extend to bounded, operator-valued analytic functions on S_η. Hence, by relative boundedness, $H_{02}(\ell, \theta)$, and consequently $H_0(\ell, \theta)$ by (4.17), is a type A analytic family on the strip S_{θ_1} with domain $D(j_2 p^2 j_2)$.

2) For $\ell \geq 2$, $H_g(\ell)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+)$ by Lemma 3.3, and $U_\theta C_0^\infty(\mathbb{R}^+) \subset C_0^2(\mathbb{R}^+)$, $\theta \in \mathbb{R}$. Let χ be a smoothed characteristic function with support $[R_{cs} - \varepsilon, \infty)$, some $\varepsilon > 0$, and $\chi \mid [R_{cs}, \infty) = 1$. Then $\| \chi(V_\ell - V_\ell(\theta)) \| < C(\ell)$, some $0 < C(\ell) < \infty$, for $\theta \in S_\eta$. We compute for $u \in C_0^\infty(\mathbb{R}^+)$:

$$\| (H_g(\ell, \theta) - H_g(\ell))u \| \leq \| \chi(p^2 - p_\theta^2)u \| + C(\ell) \| u \| . \qquad (4.19)$$

By (4.16 *a*),

$$p^2 - p_\theta^2 = (1 - J_\theta^{-2})p^2 + 2iJ_\theta^{-3}J'_\theta p + F_\theta . \qquad (4.20)$$

It suffices to consider $\theta = i\beta$. By continuity and boundedness of f' and f'', we have that $\| F_{i\beta} \| < b$ and

$$\lim_{\beta \to 0} \| \tilde{\chi}(1 - J_{i\beta}^{-2}) \|_\infty = 0 \qquad (4.21)$$

$$\lim_{\beta \to 0} \| \tilde{\chi}(J_{i\beta}^{-3} J'_{i\beta}) \|_\infty = 0 \qquad (4.22)$$

where $\tilde{\chi}$ is the characteristic function for $[R_{cs}, \infty)$. Moreover, as V_ℓ is bounded on supp χ:

$$\| \chi p^2 (H_g(\ell) + 1)^{-1} \| \leq 1 + \| \chi V_\ell \| < \infty \qquad (4.23)$$

$$\| \chi p (H_g(\ell) + 1)^{-1} \| \leq d, \quad 0 < d < \infty . \qquad (4.24)$$

The bounds (4.24) follows from (4.23), for let $\xi \geq 0$, supp $\xi = [R_{cs} - 2\varepsilon, \infty)$ and $\xi \mid \text{supp}\, \chi = 1$. Then with $R = (H_g(\ell) + 1)^{-1}$,

$$\chi p R = \chi p (p^2 + 1)^{-1} (p^2 + 1) \xi R = \chi p (p^2 + 1)^{-1} [p^2, \xi] R + \chi p (p^2 + 1)^{-1} \xi (p^2 + 1) R$$

and each term is bounded; the second one by (4.23). Combining (4.21)-(4.24) in the right side of (4.19) yields

$$\| (H_g(\ell, i\beta) - H_g(\ell))u \| \leq (\varepsilon_1(\beta) + \varepsilon_2(\beta)) \| H_g(\ell)u \| + \\ + (\varepsilon_1(\beta) + \varepsilon_2(\beta) + C(\ell) + d) \| u \| \qquad (4.25)$$

where $\varepsilon_i(\beta) \to 0$ as $\beta \to 0$ due to (4.21)-(4.24). Hence for $0 < | \beta | < \eta$, it follows from (4.25) that:

$$\| H_g(\ell, i\beta)u \| \leq c_1 \| H_g(\ell)u \| + c_2 \| u \| \qquad (4.26)$$

on $D(H_g(\ell))$. Likewise, there exists θ_2, $0 < \theta_2 < \eta$, such that $0 < | \beta | < \theta_2$:

$$\frac{1}{2} \| H_g(\ell)u \| \leq (1 - \varepsilon_1(\beta) - \varepsilon_2(\beta)) \| H_g(\ell)u \| \leq \| H_g(\ell, i\beta)u \| + c_3 \| u \| \qquad (4.27)$$

on $D(H_g(\ell, i\beta))$. By (4.26)-(4.27) and the analyticity of $H_g(\ell, \theta)u$ on S_{θ_2} for $u \in D(H_g(\ell))$, we obtain that $H_g(\ell, \theta)$ is type A on S_{θ_2} with domain $D(H_g(\ell))$. Setting $\theta_0 = \min(\theta_1, \theta_2)$, we obtain the theorem. □

5. EXISTENCE OF SPECTRAL RESONANCES OR BOUND STATES

We sketch the proof that $H_g(\ell, \theta)$, $\theta \in S_{\theta_0}^+ \equiv \{ \theta \in S_{\theta_0} \mid \operatorname{Im} \theta > 0 \}$ has an eigenvalue in any small complex neighborhood of an eigenvalue $e_n(\ell)$ of the approximate Hamiltonian $H_0(\ell)$ for all ℓ sufficiently large. Throughout this section we assume that h satisfies Conditions A, B, C and E. If, in addition, h satisfies Condition D1 or D2, the imaginary part of this eigenvalue is negative and an upper bound for it is given in Section 6. The major difference between the existence proof in this case and the one presented in [*1*] for the nonthreshold case is the use of a non-trapping condition in the proof of the bound on the resolvent of $H_g(\ell, \theta)$. This appears in the proof of Theorem 5.1 below.

The non-trapping condition used here differs from those used in [*2*] and [*17*] and is based upon the more general formulation given in [*9*]. We say that a potential V is *non-trapping* with respect to energy E if there exists a vector field f and $\exists \delta_0 > 0$ such that for all $u \in C_0^\infty([R, \infty))$ (for some $R > 0$):

$$\langle S_f(V, E) \rangle_u \equiv \langle [-fV' + 2(E - V) \langle pf'p \rangle_u \langle p^2 \rangle_u^{-1}] \rangle_u \geq \delta_0 \|u\|^2 \quad (5.1)$$

where $\langle A \rangle_u \equiv \langle u, Au \rangle$. The proof of the fact that (5.1) implies the absence of spectrum for $H_g(\ell, \theta)$, restricted to the exterior region $[R, \infty)$, in a neighborhood of $e_n(\ell)$, is given in Appendix A. In Appendix B, we prove that the classical non-trapping Condition B on h implies that the potential V_ℓ is non-trapping with respect to the energy $e_n \approx \lambda(\ell)^{-1/2}(e_n(\ell) - V_0(\ell))$ and vector fields of the type considered in Section 4 on the exterior region $[\tilde{R}_{NT}, \infty)$, for some $R_{NT} < \tilde{R}_{NT} < S_n^+$.

Because of the expansion for the eigenvalue $e_n(\ell)$ given in (4.11) it is convenient to first rescale all Hamiltonians. Let $H_\mu(\ell, \theta)$, $\mu = 0, 1$, denote $H_0(\ell, \theta)$ or $H_g(\ell, \theta)$, respectively. Define operators

$$\tilde{H}_{0i}(\ell, \theta) \equiv H_{0i}(\ell, \theta) - V_0(\ell)\chi_i, \qquad i = 1, 2 \quad (5.2)$$

and

$$\tilde{H}_\mu(\ell, \theta) \equiv H_\mu(\ell, \theta) - V_0(\ell), \qquad \mu = 0, 1 \quad (5.3)$$

where $V_0(\ell) \equiv \min_{r < r_2} V_\ell(r) \equiv V_\ell(r_0)$, as above. (Note that $r_0 = r_1 + O(\lambda(\ell)^{-\varepsilon})$ for some $\varepsilon > 0$). Let U_λ be the unitary defined by

$$(U_\lambda g)(r) = \lambda(\ell)^{-1/8} g(\lambda(\ell)^{-1/4} r) \quad (5.4)$$

and define

$$\mathrm{H}_{\mu,\ell}(\theta) \equiv \lambda(\ell)^{-1/2}\mathrm{U}_\lambda \tilde{\mathrm{H}}_\mu(\ell,\theta)\mathrm{U}_\lambda^{-1} \tag{5.5}$$

$$\mathrm{H}_{0i}^{\ell}(\theta) \equiv \lambda(\ell)^{-1/2}\mathrm{U}_\lambda \tilde{\mathrm{H}}_{0i}(\ell,\theta)\mathrm{U}_\lambda^{-1} \tag{5.6}$$

where $\mathrm{H}_{0i}^{\ell}(\theta)$ acts on the scaled spaces $\mathrm{L}^2(\lambda(\ell)^{-1/2}\mathrm{D}_i)$. We note that

$$\mathrm{H}_{\mu,\ell}(\theta) = p_{\theta,\ell}^2 + \lambda(\ell)^{1/2}[\mathrm{V}_1(\theta)_\ell - \mathrm{V}_1(r_0)] + \lambda(\ell)^{-1/2}[\mathrm{V}_2(\theta)_\ell - \mathrm{V}_2(r_0)] \tag{5.7}$$

where $f(r)_\ell \equiv f(\lambda(\ell)^{-1/4}r)$ and

$$p_{\theta,\ell}^2 \equiv p\mathrm{J}_{\theta,\ell}^{-2}p + \mathrm{F}_{\theta,\ell}\,. \tag{5.8}$$

Because of this scaling, if $\tilde{e}_n(\ell) \in \sigma_d(\mathrm{H}_{01}^{\ell})$, then

$$\tilde{e}_n(\ell) = e_n + \mathrm{O}(\lambda(\ell)^{-1/10}) \tag{5.9}$$

where e_n is defined in (4.9).

The primary technical estimate required for the existence proof is given in Theorem 5.1. All the details of the proof (except for the non-trapping part) are given in [*1*]. Below, we will sketch the proof.

THEOREM 5.1. — *Let $\tilde{e}_n(\ell) \in \sigma_d(\mathrm{H}_{01}^{\ell})$ be chosen such that $\mathrm{R}_{\mathrm{NT}} \in \mathrm{F}_n(\ell)$. Let β_0, β_1 be sufficiently small such that Theorem* A.1 *holds and* $0<\beta_0<\beta_1$. *For any* $\varepsilon>0$ *define* $\mathrm{A}_{n,\varepsilon} \equiv \{z \mid \varepsilon < |z-e_n| < 2\varepsilon\}$ *where* $\tilde{e}_n(\ell)=e_n+\mathrm{O}(\lambda(\ell)^{-1/10})$. *Then* $\exists\varepsilon > 0$ *and constants* $\delta_\mu > 0$, $\mu = 0, 1$, *such that for all* ℓ *sufficiently large, for all* $\theta \in \mathrm{S}_{\theta_0}^{+}$ *with* $\beta_0 < \mathrm{Im}\,\theta < \beta_1$, *for all* $u \in \mathrm{D}(\mathrm{H}_{\mu,\ell})$ *and for all* $z \in \mathrm{A}_{n,\varepsilon}$:

$$\|(z - \mathrm{H}_{\mu,\ell}(\theta))u\| \geq \delta_\mu \|u\|, \qquad \mu = 0, 1\,. \tag{5.10}$$

The proof of Theorem 5.1 is based upon the following localization formula which is a slight extension of the IMS formula. Let $\{g_k\}_{k=1}^{\mathrm{N}}$ be a smooth partition of unity for $\mathbb{R}^+$ with $\sum_{k=1}^{\mathrm{N}} g_k^2 = 1$. Then (neglecting domain considerations):

$$\|(z - \mathrm{A})u\|^2 \geq (1/2)\sum_{k=1}^{\mathrm{N}} \|(z - \mathrm{A})g_k u\|^2 - \mathrm{R}(u) \tag{5.11}$$

where the remainder R is given by

$$\mathrm{R}(u) = \sum_{k=1}^{\mathrm{N}} \|[\mathrm{A}, g_k]u\|^2\,. \tag{5.12}$$

We apply this to $\mathrm{H}_{\mu,\ell}(\theta)$. We choose the partition of unity $\{g_k\}_{k=1}^{3}$ as follows; see Figure 5.1.

1) Let $\tilde{\mathrm{R}}_{\mathrm{NT}}$ be as defined in Appendix A, $\mathrm{R}_{cs} < \tilde{\mathrm{R}}_{\mathrm{NT}} < \mathrm{S}_n^{+}$, and let $\varepsilon_3 > 0$ (small) be such that $\tilde{\mathrm{R}}_{\mathrm{NT}} + \varepsilon_3 < \mathrm{S}_n^{+}$. Then choose g_3 such that $\mathrm{supp}\, g_3 = [\tilde{\mathrm{R}}_{\mathrm{NT}}, \infty)$ and $\mathrm{supp}\,(g_3') = [\tilde{\mathrm{R}}_{\mathrm{NT}}, \tilde{\mathrm{R}}_{\mathrm{NT}} + \varepsilon_3]$.

2) Choose g_1 such that $j_1 \mid \mathrm{supp}\, g_1 = 1$ and, for example, $\mathrm{supp}\,(g_1') = [\mathrm{R}_2 - \varepsilon_4, \mathrm{R}_2 - \varepsilon_1]$ for some $\varepsilon_4 > \varepsilon_1$ and small such that $\mathrm{supp}\,(g_1')$ lies in the forbidden region for $e_n(\ell)$.

3) Choose g_2 such that $\Sigma g_k^2 = 1$. Note that $\mathrm{supp}\, g_2 = [\mathrm{R}_2 - \varepsilon_4, \tilde{\mathrm{R}}_{\mathrm{NT}} + \varepsilon_3]$ lies entirely in the classically forbidden region for $e_n(\ell)$.

We write g_k^ℓ for $g_k(\lambda(\ell)^{-1/4}r)$. Note that $\mathrm{H}_{\mu,\ell}(\theta) \mid \mathrm{supp}\, g_1^\ell = \mathrm{H}_{01}^\ell$ and that $\mathrm{H}_{\mu,\ell}(\theta) \mid \mathrm{supp}\, g_3^\ell = \mathrm{H}_\ell(\theta)$.

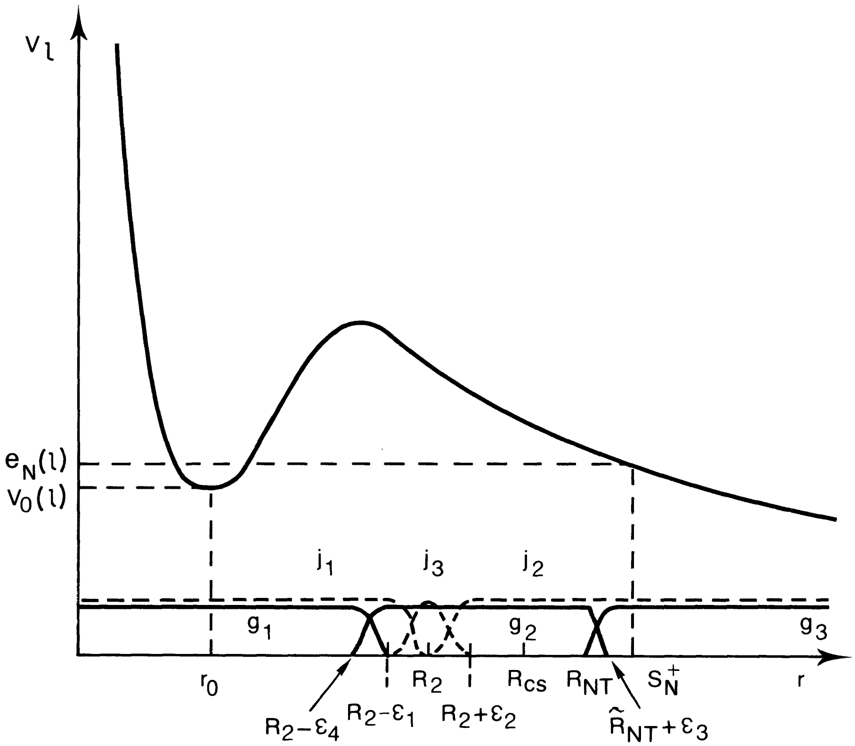


FIG. 5.1. — Partition of Unity $\{ g_k \}_{k=1}^3$ for Theorem 5.1.

Sketch of the Proof of Theorem 5.1.

1) We use formula (5.11) with the scaled partition of unity $\{ g_k^\ell \}_{k=1}^3$ defined above. The remainder term is estimated in [*1*] under an additional assumption on the rate of decay of g_k at the boundary of their support.

With this assumption, we find that there exist constants $c_{j,\mu} > 0$, $\mu = 0, 1$; $j = 1, 2, 3$ such that for all ℓ sufficiently large:

$$R_\mu(u) \leq c_{1,\mu}\lambda(\ell)^{-1/2} \|(H_{\mu,\ell}(\theta)-z)u\|^2 + c_{2,\mu}\lambda(\ell)^{-1/8}\|u\|^2 + c_{3,\mu}\lambda(\ell)^{5/8}\|g_2^\ell u\|^2 \quad (5.13)$$

The first term on the right in (5.13) is brought to the left side in (5.11) and the last term will be combined with the $g_2^\ell u$-term in the main sum in (5.11) which will be shown to be $O(\lambda(\ell))$.

2) For the $g_1^\ell u$-term, we have $(H_{\mu,\ell}(\theta) - z)g_1^\ell u = (H_{01}^\ell - z)g_1^\ell u$. Since $\tilde{e}_n(\ell) < \inf \sigma_{\text{ess}}(H_{01}^\ell)$, by Theorem 4.1, $\exists \varepsilon_1 > 0$ such that $A_{n,\varepsilon_1} \subset \rho(H_{01}^\ell)$ for all large ℓ by the expansion (5.9). Consequently, we have for all ℓ sufficiently large:

$$\|(H_{\mu,\ell}(\theta) - z)g_1^\ell u\| \geq \text{dist}(A_{n,\varepsilon_1}, e_n)\|g_1^\ell u\|. \quad (5.14)$$

Turning to the $g_2^\ell u$-term, we use the fact that $\text{supp}\, g_2 \subset F_n(\ell)$ and, consequently $\text{Re}\, V(\theta) - e_n > \kappa > 0$ on $\text{supp}\, g_2$. To implement this idea, we write:

$$\|(H_{\mu,\ell}(\theta) - z)g_2^\ell u\| \geq \|g_2^\ell u\|^{-1}\,\text{Re}\,(g_2^\ell u, (H_{\mu,\ell}(\theta) - z)g_2^\ell u) \quad (5.15)$$

using the Schwarz inequality. For $\text{Im}\,\theta$ sufficiently small, it follows from (5.8) that:

$$\text{Re}\,(g_2^\ell \tilde{u}, p_{\theta,\ell}^2 g_2^\ell \tilde{u}) \geq 0 \quad (5.16)$$

where $\tilde{u} = u$ for $\mu = 1$ and $\tilde{u} \equiv f_2^\ell u$ for $\mu = 0$. (Note that this is sufficient for $\mu = 0$ since one trivially has a lower bound on $(H_{01}^\ell - z)$). As for the potential energy contribution to (5.15), we use the facts that $[V_2(\theta) - V_2(r_0)]$ is uniformly bounded on $\text{supp}\, g_2$, that $V_1(i\beta) = V_1 + i\beta f V_1'$, and that there exists $\kappa > 0$ such that $[V_1(r) - V_1(r_0)] \mid \text{supp}\, g_2 > \kappa > 0$. It now follows from (5.7) that the potential energy part of (5.15) (with $\theta = \theta_1 + i\beta$) is bounded below by:

$$\lambda(\ell)^{1/2}[\kappa - \beta^2 c_1 - \lambda(\ell)^{-1}c_2]\,\|g_2^\ell u\| \quad (5.17)$$

so for β sufficiently small and ℓ sufficiently large, the contribution is bounded by $\lambda(\ell)^{1/2}c_3\,\|g_2^\ell u\|$. Combining (5.17) with (5.16) in (5.15), it is clear that $\exists c > 0$ such that

$$\|(H_{\mu,\ell}(\theta) - z)g_2^\ell u\| \geq c\lambda(\ell)^{1/2}\,\|g_2^\ell u\|. \quad (5.18)$$

3) The $g_3^\ell u$-term is bounded using the non-trapping condition (5.1). In Appendix B, it is shown that, under Condition B on h, the potential V_1 is non-trapping at energy e_n for any vector field f satisfying the conditions of Section 4 on $[\tilde{R}_{NT}, \infty)$. This result is then used in Theorem A.1 to prove that $z = e_n - i\tilde{\Gamma} \in \rho(H_{\mu,\ell}(\theta) \mid \text{supp}\, g_3^\ell)$ for $\theta = \theta_1 + i\beta$, $\beta_0 < \beta < \beta_1$, and $\tilde{\Gamma}$ sufficiently small ($\tilde{\Gamma} < (1/4)\beta_0\delta_0$). We now show that, in fact, the set

$$\rho_0 \equiv \{ z \mid z = E + i\tilde{\Gamma}, |\tilde{\Gamma}| < (1/16)\beta_0\delta_0, |E - e_n| < (1/8)(\delta_0\delta_1^{-1}) \} \quad (5.19)$$

(where δ_0, δ_1 are defined in Appendix A) lies in $\rho(H_{\mu,\ell}(\theta) \mid \text{supp}\, g_3^\ell)$. It follows from the proof in Appendix A that if $\tilde{\Gamma} < 0$ (i. e. $z \in$ upper half plane) then $E + i|\tilde{\Gamma}| \in \rho(H_{\mu,\ell}(\theta) | \operatorname{supp} g_3^\ell)$. As for E near e_n, we see that we get an additional term of the form

$$- 2 | E - e_n | \delta_1 \| u \|^2$$

on the right side of (A.16). Tracing through the argument to (A.17) we see that ρ_0 in (5.19) is in $\rho(H_{\mu,\ell}(\theta) | \operatorname{supp} g_3^\ell)$ provided $| E - e_n |$ is small. From (A.2), therefore, if ε is sufficiently small, for all $z \in A_{n,\varepsilon}$:

$$\| (H_{\mu,\ell}(\theta) - z) g_3^\ell u \| \geq c_3 \beta_0 \delta_0 \| g_3^\ell u \| . \tag{5.20}$$

Collecting together the estimates (5.13), (5.14), (5.18) and (5.20) and using the identity (5.11), we obtain the result. □

Given Theorem 5.1, it is now a straightforward task to apply the results of Section 6 of [1] to prove the existence of resonances. Let $R_{\mu,\ell}(z) \equiv (H_{\mu,\ell}(\theta) - z)^{-1}$ be the resolvents and set

$$F_\ell(z, \theta) \equiv R_{0,\ell}(z) - R_\ell(z) .$$

By the second resolvent formula,

$$F_\ell(z, \theta) = - R_{0,\ell}(z) W_\ell R_\ell(z) . \tag{5.21}$$

As a result of Theorem 5.1, F is jointly analytic on $A_{n,\varepsilon} \times S_\beta$ for some $\varepsilon > 0$ and $S_\beta \equiv \{ \theta \in S_{\theta_0} | \beta_0 < \operatorname{Im} \theta < \beta_1 \}$.

LEMMA 5.2. — *Let $z \in A_{n,\varepsilon}$, where n, ε are as in Theorem 5.1.*

1) $\| j_3^\ell R_{\mu,\ell}(z)^{\#} \| \leq c_\mu \lambda(\ell)^{-1/2}$, *for all ℓ sufficiently large and some $c_\mu > 0$, where $A^{\#} = A$ or A^*, $\mu = 0, 1$.*

2) $\| W_\ell R_\ell(z) \| \leq c$, *for some $0 < c < \infty$, uniformly on $A_{n,\varepsilon}$, for all ℓ sufficiently large.*

Sketch of the Proof.

1) Since $\operatorname{supp} j_3 \subset \operatorname{supp} g_2$ it is sufficient to prove the first statement for g_2. By the remainder estimate (5.13), the lower-bound (5.18) and the expansion (5.11), we have

$$\| (H_{\mu,\ell}(\theta) - z)u \|^2 \geq (1/2) \| (H_{\mu,\ell}(\theta) - z) g_2^\ell u \|^2 - R(u)$$

or, for all ℓ sufficiently large:

$$\| (H_{\mu,\ell}(\theta) - z)u \|^2 \geq \lambda(\ell) c_\mu \| g_2^\ell u \|^2 . \tag{5.22}$$

Letting $u = R_{\mu,\ell}(z)v$, the result follows.

2) This estimate is based upon the first one and the identity

$$\| (H_\ell(\theta) - z)u \| \geq \alpha \| (H_\ell(\theta) - z) g_2^\ell u \| \geq$$
$$\geq \alpha_1 \| p^2 g_2^\ell u \| - \alpha_2 \| (\lambda(\ell)^{1/2} V_{1,\ell} + V_{2,\ell} - z) g_2^\ell u \| \tag{5.23}$$

for some $0 < \alpha_1, \alpha_2 < \infty$, which follows from (5.11). (We have neglected the remainder term as it contributes a lower-order term). We let $u \equiv R_\ell(z)v$. Then, by part (1), the last term in (5.23) is bounded above by

$$\| (\lambda(\ell)^{1/2} V_{1,\ell} + V_{2,\ell} - z) g_2^\ell u \| \leq c \| v \|$$

for some constant $c > 0$. Using this in (5.23), we get:

$$(1 + c) \| v \| \geq \| p^2 g_2^\ell R_\ell(z) v \| \tag{5.24}$$

and the result follows from this and the definition of W_ℓ. □

THEOREM 5.3. — (Existence of Resonances or bound states). *Let* $\{ e_m(\ell) \}$ *be the distinct positive eigenvalues of* $H_0(\ell)$ *below* $\inf \sigma_{ess}(H_0(\ell))$ *arranged in increasing order*: $e_1(\ell) < e_2(\ell) < \ldots$. *Let* $e_n(\ell)$ *be such that* $R_{NT} \subset F_n(\ell)$ *for all large* ℓ. *Then for any* $\varepsilon > 0$ *sufficiently small, there exists* $\ell_{n,\varepsilon}$ *such that for all* $\ell > \ell_{n,\varepsilon}$, $H_g(\ell)$ *(and hence* $H_g \equiv -\Delta_g$*) has a spectral resonance or bound state* $z_n(\ell)$ *(with multiplicity 1) satisfying*:

$$| z_n(\ell) - e_n(\ell) | < \varepsilon$$

and

$$\operatorname{Im} z_n(\ell) \leq 0 .$$

Proof. — Choose any $\varepsilon > 0$ such that Theorem 5.1 holds. By Lemma 5.2 and (5.21), $\lim_{\ell \to \infty} \| F_\ell(z, \theta) \| = 0$, uniformly on $A_{n,\varepsilon}$, for any $\theta \in S_\beta$. Let $P_0(\ell)$ be the projector for $H_{0,\ell}(\theta)$ for the eigenspace $\tilde{e}_n(\ell) = \lambda^{-1/2}(e_n(\ell) - V_0(\ell))$, i.e.

$$P_0(\ell) = (2\pi i)^{-1} \oint_{\gamma_\varepsilon} R_{0,\ell}(z) dz$$

where γ_ε is a simple closed contour in $A_{n,\varepsilon}$. By Theorem 5.2, the integral

$$P(\ell) \equiv (2\pi i)^{-1} \oint_{\gamma_\varepsilon} R_\ell(z) dz$$

exists and $\lim_{\ell \to \infty} \| P(\ell) - P_0(\ell) \| = 0$. Hence, for ℓ sufficiently large, $H_\ell(\theta)$ has an eigenvalue in

$\{ z \mid | z - E_n(\ell) | < \varepsilon \}$ and $\dim(\operatorname{Ran} P(\ell)) = \dim(\operatorname{Ran} P_0(\ell)) = 1$. □

COROLLARY 5.4. — *Suppose h satisfies Condition* D1 *or* D2. *Then* $\operatorname{Im} z_n(\ell) < 0$, *i.e.* $H_g(\ell)$ *has spectral resonances for* ℓ *sufficiently large.*

Proof. — Conditions D guarantee that $\sigma_{pp}(H_g(\ell)) \cap (0, \infty) = \phi$, as proved in Theorem 3.4. The corollary now follows from the stability, with respect to θ, of real eigenvalues of $H_\ell(\theta)$. □

REMARK 5.5. — Theorem 5.3 establishes the existence of positive energy bound states or spectral resonances for the Laplace-Beltrami operator $H_g = -\Delta_g$ when the manifold $(\mathbb{R}^n, g)$ has sufficiently many bounded geodesics. Note that the angular momentum quantum number provides an « internal » semi-classical parameter which does not appear in H_g (unlike the shape resonance problem). We have used this parameter to prove the existence of spectral resonances or bound states associated with high angular momentum subspaces.

6. LOCATION OF THE RESONANCES OR BOUND STATES

We assume Conditions A-C and E in this section. The purpose of this section is to derive a precise estimate on the location of the resonance or bound state $z_n(\ell)$ of $H_g(\ell)$ with respect to be eigenvalue $e_n(\ell)$ of the approximate Hamiltonian $H_0(\ell)$. Specifically, it is shown that $|z_n(\ell) - e_n(\ell)|$ is exponentially small in $\lambda(\ell)^{1/2} = O(\ell)$. If, in addition, we impose Condition D, then by Corollary 5.4, $|\operatorname{Im} z_n(\ell)| > 0$ and this result shows that $\operatorname{Im} z_n(\ell)$ is exponentially small. The exponentially small width of the resonance reflects the fact that it is due to tunneling through the potential barrier formed by the metric for sufficiently high angular momentum states. As in the shape resonance problem, the difference $|z_n(\ell) - e_n(\ell)|$ is given in terms of the exponential of the distance, in the Agmon metric [*18*] [*19*], between the classical turning points S_n^- and S_n^+ for the potential V_ℓ and energy $e_n(\ell)$.

DEFINITION 6.1. — *For any* $x, y \in \overline{F_n(\ell)}$, $x \leq y$, *define*:

$$\rho_A(x, y) \equiv \int_x^y [\lambda(\ell)V_1(r) + V_2(r) - e_n(\ell)]^{1/2} dr . \tag{6.1}$$

For any $r \in \overline{F_n(\ell)}$, *define* $\rho_n(r)$ *to be the distance in the metric* ρ_A *from* $S_n^-(\ell)$ *to* r:

$$\rho_n(r) \equiv \rho_A(S_n^-(\ell), r) \tag{6.2}$$

and let $\rho_n(\ell) \equiv \rho_n(S_n^+(\ell))$, *the distance between the classical turning surfaces for energy* $e_n(\ell)$.

We are interested in the leading asymptotic behaviour of $\rho_n(\ell)$ as a function of ℓ. Note that V_2 is bounded over $F_n(\ell)$ and that

$$e_n(\ell) = \lambda(\ell)^{1/2} e_n + \lambda(\ell)V_1(r_0) + V_2(r_0) + O(\lambda(\ell)^{2/5}),$$

for $e_n \in \sigma(K)$ as in (4.9). Consequently, we can write for large ℓ:

$$\rho_n(\ell) = \lambda(\ell)^{1/2}\hat{\rho}_0 + O(\lambda(\ell)^{-1/4}) \tag{6.3}$$

where

$$\hat{\rho}_0 \equiv \int_{r_1}^{S_0^+} [V_1(r) - V_1(r_1)]^{1/2} dr \tag{6.4}$$

and S_0^+ satisfies $V_0(\ell) = \lambda(\ell)V_1(S_0^+) + V_2(S_0^+)$ (S_0^+ is the exterior turning point for the bottom of the well $V_0(\ell)$ and is asymptotically independent of ℓ). Note that $\hat{\rho}_0$ is the width of the classically forbidden region for the potential V_1 and energy $V_1(r_1)$, its local minimum (recall that $r_0 = r_1 + O(\lambda(\ell)^{-\varepsilon})$).

We can now state the main result on the resonance width.

THEOREM 6.2. — *Let $z_n(\ell)$ be a spectral resonance or bound state of $H_g(\ell)$ in a neighbourhood of $e_n(\ell) \in \sigma(H_0(\ell))$ as described in Theorem 5.3 for all large ℓ. Then for any $\varepsilon > 0$ there exists a constant $d_{n,\varepsilon} > 0$ such that for all ℓ sufficiently large:*

$$0 \leq | z_n(\ell) - e_n(\ell) | \leq d_{n,\varepsilon} e^{-2\lambda(\ell)^{1/2}(\hat{\rho}_0 - \varepsilon)}. \tag{6.5}$$

(In the case that Condition D holds, the inequality on the left in (6.5) is strict.)

Sketch of the Proof.

1) Fix n as in Theorem 5.3 and let ψ_n be an eigenfunction of $H_{01}(\ell)$ such that $H_{01}(\ell)\psi_n = e_n(\ell)\psi_n$. Let $\eta \in C^\infty$ be such that $\eta > 0$, supp $\eta = [S_n^-(\ell) + \varepsilon, S_n^+(\ell)]$, supp $\eta' \cap D_1 \subset \overline{F}_n(\ell)$ with $\eta \mid [S_n^-(\ell) + 2\varepsilon, R_2] = 1$. We take $\varepsilon = O(\ell^{-1})$. Using the Agmon technique [*18*], one proves that there exists a constant $c_n > 0$ such that for all ℓ sufficiently large:

$$\| e^{\rho_n} \eta \psi_n \| \leq c_n \lambda(\ell) \tag{6.6}$$

where ρ_n is defined in (6.2).

2) We next obtain an expression for $\Delta_n \equiv | z_n(\ell) - e_n(\ell) |$ in terms of the norm of the eigenfunction ψ_n of $H_{01}(\ell)$ localized to supp (j_3). We use the Feshbach method [*20*] to decompose $H_g(\ell)$ with respect to the projection $P_0^n \equiv \tilde{P}_0^n \oplus 0$, where $\tilde{P}_0^n$ is the projection on $L^2(D_1)$ corresponding to the eigenspace of $H_{01}(\ell)$ with eigenvalue $e_n(\ell)$. The major technical part of this step consists in obtaining uniform bounds in ℓ on the resolvent of the reduced Hamiltonian $\overline{P}_0^n H_g(\ell) \overline{P}_0^n$, where $\overline{P}_0^n \equiv 1 - P_0^n$, at the spectral resonance or bound state $z_n(\ell)$. The proof of this fact is similar to the proof of Theorem 5.1 (see [*1*]). The result of this analysis is as follows. Let ξ_n be an eigenfunction of $H_g(\ell, \theta)$ for the eigenvalue $z_n(\ell)$. Let χ_3 be a smoothed characteristic function for supp (j_3). Then there exists a constant d_n such that for all ℓ sufficiently large

$$\Delta_n \leq d_n \lambda(\ell)^{1/2} \| \chi_3 P_0^n \xi_n \|^2. \tag{6.7}$$

3) We now combine (6.6) and (6.7) to obtain a preliminary estimate on Δ_n as supp $(\chi_3) \cap D_1 \subset \{ r \mid \eta(r) = 1 \}$ for all ℓ large. There exists a constant $a_n > 0$ such that for all large ℓ:

$$\Delta_n \leq a_n \lambda(\ell)^{3/2} e^{-2\tilde{\rho}_n} \tag{6.8}$$

where $\tilde{\rho}_n \equiv \rho_n(R_2 - \varepsilon_2)$ is the infimum of $\rho_n(r)$ on supp $(j_3) \cap D_1$. In the last step, we show that we can replace $\tilde{\rho}_n$ by $\hat{\rho}_0 \lambda(\ell)^{1/2}$ with a small error. Let $\varepsilon > 0$ be given. By the triangle inequality for the Agmon metric

$$\lambda(\ell)^{1/2} \hat{\rho}_0 \leq \tilde{\rho}_n + \rho_A(R_2 - \varepsilon_2, S_n^+) \tag{6.9}$$

so we must show that $\rho_A(R_2 - \varepsilon_2, S_n^+) \equiv \rho_A(R_2) \leq \lambda(\ell)^{1/2}\varepsilon$. This we arrange by taking R_2 sufficiently close to S_0^+, ε_2 sufficiently small, and noting that S_0^+ is asymptotically independent of ℓ. Then, for all large ℓ:

$$\rho_A(R_2) \leq c\lambda(\ell)^{1/2}(S_0^+ - R_2 - \varepsilon_2)$$

for some constant $c > 0$. (Note that as we take R_2 close to S_0^+, we must take lowlying eigenvalues of $H_{01}(\ell)$ so that the relation $R_2 < S_n^+(\ell) < S_0^+$ is maintained.) Consequently, for R_2 such that $S_0^+ - \varepsilon + \varepsilon_2 < R_2 < S_0^+$, we have $\rho_A(R_2) < \varepsilon\lambda(\ell)^{1/2}$ and from (6.8) and (6.9):

$$\Delta_n = | z_n(\ell) - e_n(\ell) | \leq d_n e^{-2\lambda(\ell)^{1/2}(\hat{\rho}_0 - \varepsilon)} \tag{6.10}$$

for some $d_n > 0$ and all ℓ sufficiently large. If we impose Condition D the rough lower bound on the left in (6.5) follows from Theorem 3.4 and the fact that the real positive eigenvalues of $H_g(\ell)$ and $H_g(\ell, \theta)$ are the same. □

7. AN APPLICATION TO SURFACES OF REVOLUTION

In this section, we construct a family of surfaces of revolution M_ρ for which the metric depends on a parameter $\rho > 0$. We study the Laplace-Beltrami operator $-\Delta_\rho$ on M_ρ and shown that for $\rho > U_1$ (a value specified below), $-\Delta_\rho$ has positive eigenvalues located near the energy levels $e_n(\ell)$ of an effective potential well. For $\rho < U_1$, these eigenvalues dissolve and $-\Delta_\rho$ has spectral resonances $z_n(\ell)$ with $\operatorname{Re} z_n(\ell) \cong e_n(\ell)$ (for ℓ sufficiently large). In fact, it follows from Section 6 that $| e_n(\ell) - z_n(\ell) |$, $z_n(\ell)$ the eigenvalues or resonances of $-\Delta_\rho$, is exponentially small in $\lambda(\ell)^{1/2}$.

Let $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ with coordinates (x, u, θ^i), where $x \in \mathbb{R}$ and (u, θ^i) are spherical coordinates on $\mathbb{R}^n$. We consider surfaces of revolution M in $\mathbb{R}^{n+1}$ of the following from. Let $i: \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a C^2 injective immersion such that, in spherical coordinates on $\mathbb{R}^n$:

$$\begin{aligned} i: \mathbb{R}^+ \times S^{n-1} \subset \mathbb{R}^n &\to \mathbb{R} \times \mathbb{R}^n \\ (\tau, \theta) &\xrightarrow{i} (x(\tau), u(\tau), \theta) \end{aligned} \tag{7.1}$$

with $\lim_{\tau \to 0} x(\tau) = 0 = \lim_{\tau \to 0} u(\tau)$. Then $M = i(\mathbb{R}^n)$; see Figure 7.1.

In coordinates (τ, θ), the metric is

$$ds^2 = (x'(\tau)^2 + u'(\tau)^2)d\tau^2 + u(\tau)^2 d\omega^2. \tag{7.2}$$

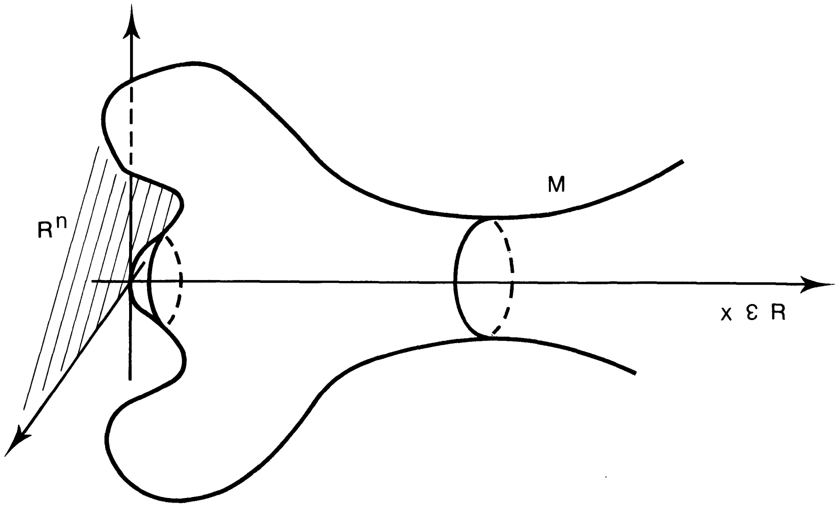


FIG. 7.1. — Typical Surface of Revolution in $\mathbb{R}^{n+1} = \mathbb{R}_x \times \mathbb{R}^n$.

As in Section 2, we introduce a new variable

$$r(\tau) \equiv \int_0^\tau (x'(t)^2 + u'(t)^2)^{1/2} dt \tag{7.3}$$

and write:

$$ds^2 = dr^2 + h(r)^2 d\omega^2 \tag{7.4}$$

with $h(r(\tau)) = u(\tau)$. That h satisfies Conditions A' of Section 2 follows as in Section 2 because we assumed that (7.2) is the restriction to $\mathbb{R}^+ \times S^{n-1}$ of a metric on $\mathbb{R}^n$. Rather than listing in detail all of the conditions on $(x(\tau), u(\tau))$ that guarantee that h satisfies Conditions A-E, we explicitly construct a one-parameter family M_ρ of surfaces of revolution satisfying Conditions A-E and show that they exhibit the properties described above.

The surfaces M_ρ are given as the level surfaces

$$u = f_\rho(x) \tag{7.5}$$

where the function $f_\rho \in C^2(\mathbb{R}^+)$ is given by:

$$f_\rho(x) = \begin{cases} g(x) & 0 \le x \le x_2 \quad (7.6\,a) \\ \rho + (U_2 - \rho)[b(x - x_2)^2 + 1]^{-k} & x \ge x_2 \quad (7.6\,b) \end{cases}$$

(see Figure 7.2). The function $g \in C^2([0, x_2])$ and satisfies $g(0) = 0$ and $g(x) > 0$ for $x > 0$; $g'(0) = \infty$; $g'(x_2) = 0$; $g''(x_2) > 0$. We define $U_2 \equiv g(x_2)$ and take $\rho > U_2$; $k > 1$ and $b \equiv (1/2)g''(x_2)(\rho - U_2)^{-1}k^{-1} > 0$. Moreover, we assume that g has a unique, non-degenerate maximum at $x_1 < x_2$, $g''(x_1) < 0$ and $g(x_1) \equiv U_1$.

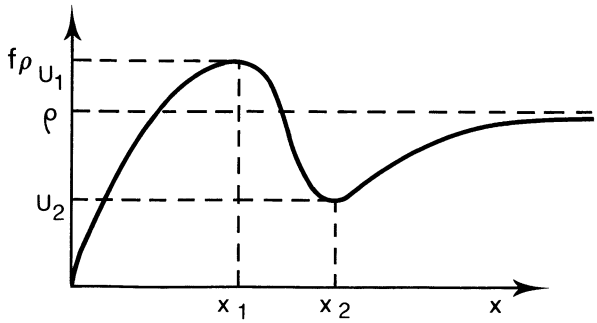


FIG. 7.2. — A typical shape for f_ρ.

Introducing a new coordinate (the distance along the curve):

$$r_\rho(x) \equiv \int_0^x [1 + f_\rho'(s)]^{1/2} ds \tag{7.7}$$

and defining

$$h_\rho(r_\rho(x)) \equiv f_\rho(x) \tag{7.8}$$

the metric on M_ρ is

$$ds^2 = dr_\rho^2 + h_\rho(r_\rho)^2 d\omega^2 \tag{7.9}$$

With x_1 and x_2 as above, we define

$$r_1 \equiv r_\rho(x_1) \qquad \text{and} \qquad r_2 \equiv r_\rho(x_2) \tag{7.10}$$

and note that from (7.7) and (7.6 *a*), r_1 and r_2 are independent of ρ.

Under the above assumptions, it is easily checked that h_ρ satisfies Condition A and Condition B with $R_{NT} = r_2$; h_ρ also satisfies Condition C provided $\rho > U_1$. The choice of f_ρ in (7.6) guarantees that the analyticity Condition E is satisfied by h_ρ.

As in Section 3, we have that $-\Delta_\rho$ restricted to a rotationally invariant subspace $\mathscr{H}_\ell$ is unitarily equivalent to

$$H_\rho(\ell) = p^2 + \lambda(\ell)V_1 + V_2 \tag{7.11 a}$$

on $L^2(\mathbb{R}^+)$ with

$$V_1 \equiv h_\rho^{-2}$$
$$V_2 \equiv (1/4)(n-1)(n-2)(h_\rho' h_\rho^{-1})^2 + (1/2)(n-1)h_\rho'' h_\rho^{-1} \tag{7.11 b}$$

and $\lambda(\ell) \equiv \ell(\ell + n - 2)$. We have from (7.10), (7.8) and (7.6) that

$$V_1(r_1) = U_1^{-2} \qquad \text{and} \qquad V_1(r_2) = U_2^{-2}. \tag{7.12}$$

A sketch of V_1 is given in Figure 7.3.

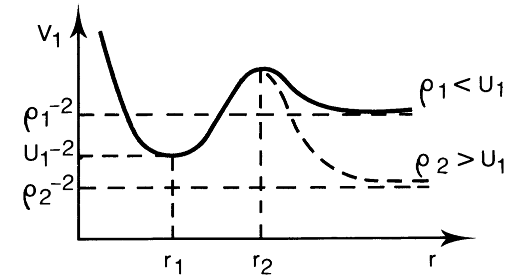


FIG. 7.3. — Typical shapes for V_1 with $\rho = \rho_1 < U_1$ and $\rho = \rho_2 > U_1$.

We note that it follows from (7.6)-(7.8) that

$$V_2(r) = O(r^{-2}) \quad \text{as} \quad r \to \infty$$

$$\lim_{r\to\infty} V_1(r) = \rho^{-2} \tag{7.13 a}$$

and that

$$(V_1(r) - \rho^{-2}) = O(r^{-4k}) \quad \text{as} \quad r \to \infty . \tag{7.13 b}$$

As a consequence of (7.13), it follows by a simple modification of the argument in the proof of Theorem 3.4 that

$$\sigma_{pp}(H_\rho(\ell)) \subset [0, \lambda(\ell)\rho^{-2}]$$

As a result of this we have:

PROPOSITION 7.1. — *If $\rho > U_1$ then for all ℓ sufficiently large $H_\rho(\ell)$ has spectral resonances $z_n(\ell)$ with* $\operatorname{Im} z_n(\ell) < 0$ *and* $|e_n(\ell) - z_n(\ell)|$ *bounded as in Theorem 6.2.*

Proof. — If $\rho > U_1$, Condition C is satisfied and by (7.13) $H_\rho(\ell)$ has no positive eigenvalues above $\lambda(\ell)\rho^{-2}$ which lies below $e_n(\ell)$. Hence Theorem 5.3 guarantees the existence of resonances. □

We now study the case $\rho < U_1$. Condition C now fails and the bottom of the well U_1^{-2} lies below the value of $\lambda(\ell)V_1 + V_2$ at infinity. We expect that there are bound states associated with the bottom of the well. To prove this, we first use a local compactness argument (for any $\rho > U_2$) to locate the bottom of $\sigma_{ess}(H_\rho(\ell))$ and we then use the Min-max Theorem [*14*] in conjunction with a semi-classical approximation [*1*] [*16*] to prove the existence of positive eigenvalues below in $\sigma_{ess}(H_\rho(\ell))$. (A similar local compactness argument is used in the proof of Theorem 4.1. For a general result on local compactness see [*21*].)

LEMMA 7.2. — *Let $\rho > U_2$. Then*

a) *$H_\rho(\ell)$ is locally compact, i. e. for any $\chi \in C^\infty([a, b])$, $0 < a < b < \infty$, $\chi(H_\rho(\ell) - z)^{-1}$ is compact for any z, $\operatorname{Re} z \notin \overline{\mathbb{R}}^+$;*

b) *for any* χ *as in* (*a*), $[p^2, \chi](\mathrm{H}_\rho(\ell) - z)^{-1}$ *is compact for any* z, $\mathrm{Re}\, z \notin \overline{\mathbb{R}}^+$.

Proof.

1) It suffices to consider $z = -1$ so let $\mathrm{R} \equiv (\mathrm{H}_\rho(\ell) + 1)^{-1}$. Choose $\chi \in \mathrm{C}^\infty([a, b])$ and $b' > b$, $b' < \infty$. Let p_D^2 be the Dirichlet Laplacian on $[0, b']$; p_D^2 is invertible and $(p_\mathrm{D}^2)^{-1}$ is compact. Then

$$\chi\mathrm{R} = (p_\mathrm{D}^2)^{-1} p_\mathrm{D}^2 \chi\mathrm{R} = (p_\mathrm{D}^2)^{-1} p^2 \chi\mathrm{R}, \tag{7.14}$$

so it suffices to show that $\chi' p\mathrm{R}$ is bounded since $(\lambda(\ell)\mathrm{V}_1 + \mathrm{V}_2)\chi$ is bounded. Let $\tilde{\chi} \in \mathrm{C}^\infty([a'', b''])$ for $0 < a'' < a < b < b'' < b'$ and $\tilde{\chi} \mid [a, b] = 1$. Then $\tilde{\chi}\chi' = \chi'$ so

$$\begin{aligned} \chi' p\mathrm{R} &= \chi' p \tilde{\chi}\mathrm{R} = \chi' p(p^2 + 1)^{-1}(p^2 + 1)\tilde{\chi}\mathrm{R} = \\ &= \chi' p(p^2 + 1)^{-1}\tilde{\chi}(p^2 + 1)\mathrm{R} + \chi' p(p^2 + 1)^{-1}[\tilde{\chi}, p^2]\mathrm{R} \end{aligned} \tag{7.15}$$

and it is easily seen that both terms on the right in (7.15) are bounded (here, p^2 is the Dirichlet Laplacian on $\mathbb{R}^+$).

2) By part *a*) it suffices to show that $\chi p\mathrm{R}$ is compact, for χ as above. Let $\mathrm{R}_\mathrm{D}^{1/2} \equiv |p_\mathrm{D}^2|^{-1/2}$ and $|p| \equiv |p_\mathrm{D}^2|^{1/2}$; $\mathrm{R}_\mathrm{D}^{1/2}$ is compact. Hence, since

$$\chi p\mathrm{R} = \mathrm{R}_\mathrm{D}^{1/2} |p| \chi p\mathrm{R} \tag{7.16}$$

and $|p| \chi p\mathrm{R}$ is bounded, the result follows. □

PROPOSITION 7.3. — *For* $\rho > \mathrm{U}_2$, $\sigma_{\mathrm{ess}}(\mathrm{H}_\rho(\ell)) \subset [\lambda(\ell)\rho^{-2}, \infty)$.

Proof. — Let $\omega \in \sigma_{\mathrm{ess}}(\mathrm{H}_\rho(\ell))$ and let $\{\psi_n\}$ be a Weyl sequence for ω and $\mathrm{H}_\rho(\ell)$, i. e. $\psi_n \in \mathrm{D}(\mathrm{H}_\rho(\ell))$, $\|\psi_n\| = 1$, $\psi_n \overset{w}{\to} 0$ and $\|(\mathrm{H}_\rho(\ell) - \omega)\psi_n\| \to 0$. Note that $(\mathrm{H}_\rho(\ell) + 1)\psi_n \overset{w}{\to} 0$. Let $\chi \in \mathrm{C}^\infty([a, b])$, $0 < a < b < \infty$ and let $\mathrm{R} \equiv (\mathrm{H}_\rho(\ell) + 1)^{-1}$, as above. Then $\|\chi\psi_n\| \to 0$ as

$$\chi\psi_n = \chi\mathrm{R}(\mathrm{H}_\rho(\ell) + 1)\psi_n$$

and $\chi\mathrm{R}$ is compact by Lemma 7.2. Consequently, $\phi_n \equiv (1-\chi)\psi_n$ satisfies $\|\phi_n\| \to 1$ so we take $\|\phi_n\| = 1$. We have that $\phi_n \overset{w}{\to} 0$ and $\|(\mathrm{H}_\rho(\ell) - \omega)\phi_n\| \to 0$ since

$$(\mathrm{H}_\rho(\ell) - \omega)\phi_n = (1 - \chi)(\mathrm{H}_\rho(\ell) - \omega)\psi_n + [\chi, p^2]\mathrm{R}(\mathrm{H}_\rho(\ell) + 1)\psi_n$$

and $[\chi, p^2]\mathrm{R}$ is compact by Lemma 7.2. Hence $\{\phi_n\}$ is a Weyl sequence for $\mathrm{H}_\rho(\ell)$ and ω. Now, by the Schwarz inequality, we have:

$$\|(\mathrm{H}_\rho(\ell) - \omega)\phi_n\| \geq \langle \phi_n, (\mathrm{V}_\rho - \omega)\phi_n \rangle \tag{7.17}$$

where $\mathrm{V}_\rho \equiv \lambda(\ell)\mathrm{V}_1 + \mathrm{V}_2$. We now choose $r_2 < b < b' < \infty$ and $a > 0$ small enough such that $\inf_{r \in \mathrm{supp}(1-\chi)} \mathrm{V}_\rho(r) = \lambda(\ell)\rho^{-2}$ (recall that $\chi \in \mathrm{C}^\infty([a, b])$). Then, we obtain a lower bound for (7.17):

$$\|(\mathrm{H}_\rho(\ell) - \omega)\phi_n\| \geq (\lambda(\ell)\rho^{-2} - \omega) \tag{7.18}$$

so if $\omega < \lambda(\ell)\rho^{-2}$, the right side of (7.18) is positive whereas the left side converges to zero. This proves the result. □

PROPOSITION 7.4. — *Let* $U_2 < \rho < U_1$. *For fixed n and all* ℓ *sufficiently large,* $H_\rho(\ell)$ *has at least n eigenvalues* $e_n(\ell)$ *below* $\inf \sigma_{ess}(H_\rho(\ell))$ *and*

$$e_n(\ell) = \lambda(\ell)^{1/2}e_n + \lambda(\ell)U_1^{-2} + O(\lambda(\ell)^{2/5})$$

where $e_n = (1/2 + n)(V_1''(r_1))^{1/2}$, $n \in \mathbb{N}$.

Proof. — Since V_1 has a non-degenerate minimum at r_1 and as

$$\lim_{r\to\infty} (\lambda(\ell)V_1(r) + V_2(r)) = \lambda(\ell)\rho^{-2} > \lambda(\ell)V_1(r_1)$$

the result follows from [*16*] or [*1*]. □

8. COMMENTS

1) Our results can be slightly generalized to manifolds of the form $M = X \times N$, where $X = \mathbb{R}^+$ or $\mathbb{R}$ and N is a compact, n-dimensional manifold $n \geq 1$, with the metric given by:

$$ds^2 = dr^2 + h(r)^2 d\omega^2 .$$

Here $d\omega^2$ corresponds to a fixed metric on N and $h \in C^2(X)$. In local coordinates on N we write

$$d\omega^2 = k_{ij}dx^i dx^j$$

so we can introduce, as in (2.11)-(2.12):

$$H_N \equiv 1/2k^{ij}p_ip_j$$

and the Hamiltonian

$$H = 1/2p_r^2 + h(r)^{-2}H_N .$$

The Hamiltonian H_N is a constant of the motion and the classical equation of motion (2.14) follows if we set $H_N = \sigma^2$. When $X = \mathbb{R}^+$, the manifold is not geodesically complete.

Turning to the Laplace-Beltrami operator $H_g \equiv -\Delta_g$, it is straightforward to determine the equivalent of Conditions A-C on h. When $X = \mathbb{R}$, conditions at $r = 0$ are unnecessary and we require Condition C *ii*), that $\lim_{|r|\to\infty} h(r) > h(r_1)$, hold at least in one direction and in that direction Condition B hold for some R_{NT}. When $X = \mathbb{R}^+$, M is not geodesically complete so we can define H_g with Dirichlet boundary conditions at $r = 0$.

As in Section 3, the fact that H_N is a constant of the motion means that we can express H_g as a direct sum of one-dimensional Schrödinger operators. Let $-\Delta_N$ denote the Laplace-Beltrami operator on N with spectrum

$\lambda_1 \leq \lambda_2 \leq \ldots$, which we denote by λ. Let $H_g(\lambda)$ denote the restriction of H_g to the invariant subspace indexed by λ. Then

$$H_g(\lambda) = -\frac{d^2}{dr^2} + V_\lambda$$

where V_λ is as in (3.4) with λ replacing $\lambda(\ell)$. We note that Condition E on the analyticity of h assumes the same form.

We conclude that, in this case as well, the existence of a sufficiently large set of bounded geodesics for M implies that H_g has either bound states on resonances in subspaces corresponding to sufficiently high eigenvalues λ of $-\Delta_N$.

2) It is sometimes argued [22] [23] that the correct quantization of the classical Hamiltonian (2.11) is not the Laplace-Beltrami operator H_g but rather

$$H(\xi) \equiv -\Delta_g + \xi R$$

where R is the scalar curvature on M, (2.19), and ξ is a constant. Since under assumptions A-C the scalar curvature R is bounded, we can incorporate ξR into V_2 given in (3.4 *b*). Hence, all the results of this paper hold for $H(\xi)$ as well.

3) We comment on the situation when there are finitely many pairs of points (r_1, r_2) at which h satisfies Condition C. This corresponds to the case of multiple minima in the shape resonance problem; see Figure 8.1. In this case, we simply define the partition of unity $\{j_i\}_{i=1}^3$ with respect to the last well. The potential of the internal Hamiltonian $H_{01}(\ell)$ has

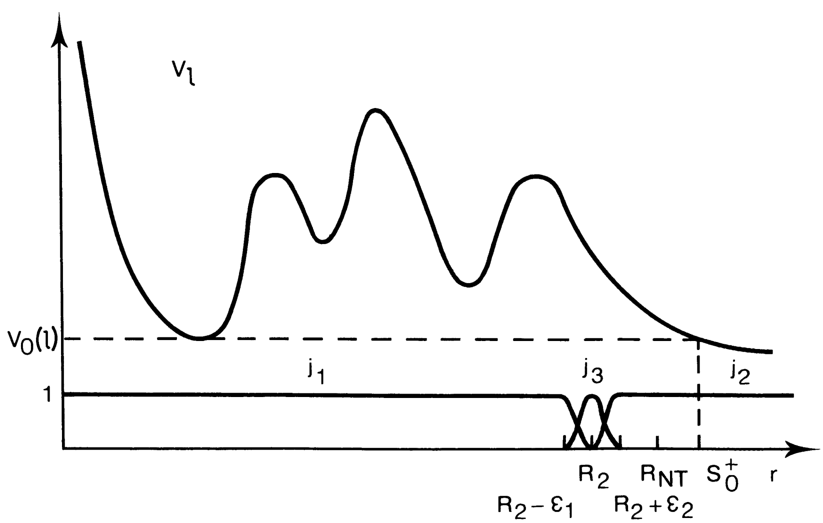


FIG. 8.1. — A partition of unity for multiple minima.

several non-degenerate minima and consequently $\sigma_d(H_{01}(\ell))$ is richer. The spectrum of a multi-well Hamiltonian has been studied and it is known [*19*] that the degeneracy of eigenvalues from different wells may be broken by tunneling. The splitting is exponentially small in $\lambda(\ell)^{-1/2}$, however, so $\sigma_d(H_{01}(\ell))$ will consist of groups of eigenvalues whose asymptotic behavior in ℓ is similar to that given in (4.11). Hence, the groups of eigenvalues are separated by $O(\lambda(\ell))$ and the analysis presented here applies.

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APPENDIX A

RESOLVENT ESTIMATE IN THE NON-TRAPPING REGION

In this appendix, we prove that if the quantum non-trapping condition (5.1) holds for energy E, then there is a neighborhood of $\lambda(\ell)E$ which is in the resolvent set of $H_g(\ell, \theta)$ for all ℓ sufficiently large and $\operatorname{Im}\theta$ sufficiently small. This theorem is based on ideas presented in [9]. We use the notation of the text. We assume Conditions A, B and E.

THEOREM A.1. — *Suppose* $\exists E > 0$ *such that the following holds*: $\exists \tilde{R}_{NT} > 0$ *with* $R_{cs} < \tilde{R}_{NT} < S_E^+$ *and* $\exists \delta_0 > 0$ *such that for all* $u \in C_0^\infty([\tilde{R}_{NT}, \infty))$:

$$\langle [-fV_1' + 2\langle pf'p\rangle_u \langle p^2\rangle_u^{-1}(E - V_1)]\rangle_u \geq \delta_0 \|u\|^2, \tag{A.1}$$

where $V_1(r) \equiv h(r)^{-2}$, $\langle A\rangle_u \equiv \langle u, Au\rangle$, *and* f *is a vector field as described in Section 4. Then for all* $u \in C_0^\infty([\tilde{R}_{NT}, \infty))$, *for all* ℓ *sufficiently large, for all* $\beta \equiv \operatorname{Im}\theta > 0$ *sufficiently small, for all* $\Gamma < (1/4)\lambda(\ell)\beta\delta_0$, *and for all* $z = \lambda(\ell)E - i\Gamma$:

$$\|(H_g(\ell, \theta) - z)u\| \geq (1/8)\lambda(\ell)\beta\delta_0 \|u\| \tag{A.2}$$

Proof.

1) By the Schwarz inequality,

$$\|(z - H_g(\ell, \theta))u\| \geq \|u\|^{-1}\{\operatorname{Im} e^{-i\phi}\langle u, (z - H_g(\ell, \theta))u\rangle\} \tag{A.3}$$

for any $\phi \in \mathbb{R}$. Upon expanding the right side of (A.3), we obtain

$$\operatorname{Im} e^{-i\phi}\langle u, (z - H_g(\ell, \theta))u\rangle \geq \cos\phi\{-\Gamma\|u\|^2 + \tan\phi \operatorname{Re}\langle u, (H_\theta - \lambda(\ell)E)u\rangle - \operatorname{Im}\langle u, H_\theta u\rangle\} \tag{A.4}$$

where we write $H_\theta \equiv H_g(\ell, \theta)$. From (3.4) and (4.16), we have

$$H_\theta = p_\theta^2 + V_\ell(\theta) = pJ_\theta^{-2}p + \lambda(\ell)\tilde{V}_\ell(\theta) + F_\theta \tag{A.5}$$

where

$$\tilde{V}_\ell(\theta) = V_1(\theta) + \lambda(\ell)^{-1}V_2(\theta) \tag{A.6}$$

and $V_i(\theta) \equiv V_i \circ \phi_\theta$; V_2 is defined in (3.4). Note that on $[\tilde{R}_{NT}, \infty)$, $V_2(\theta)$ is uniformly bounded and of lower order. Likewise, F_θ is uniformly bounded. Without loss of generality, we take $\theta = i\beta$, $\beta > 0$. We first analyse the kinetic energy contribution to the right side of (A.4):

$$K_\beta \equiv \langle u, p[\tan\phi \operatorname{Re} J_{i\beta}^{-2} - \operatorname{Im} J_{i\beta}^{-2}]pu\rangle + \langle u, [\tan\phi \operatorname{Re} F_{i\beta} - \operatorname{Im} F_{i\beta}]u\rangle \tag{A.7}$$

We choose the real phase ϕ such that

$$\tan\phi = \frac{1}{1 + \beta/2}\left[\frac{\operatorname{Im}\langle pJ_{i\beta}^{-2}p\rangle_u}{\operatorname{Re}\langle pJ_{i\beta}^{-2}p\rangle_u}\right] \tag{A.8}$$

where $\langle A\rangle_u \equiv \langle u, Au\rangle$. Next, we compute $\tan\phi$ to $O(\beta^3)$. We have

$$J_{i\beta}(r) = 1 + i\beta f' + O(\beta^2)$$

so

$$J_{i\beta}(r)^{-2} = 1 - 2i\beta f' + \beta^2[(f')^2 + f''] + O(\beta^3)$$

and consequently,

$$\operatorname{Re}\langle pJ_{i\beta}^{-2}p\rangle_u = \langle p^2\rangle_u + O(\beta^2)\langle p^2\rangle_u \geq 0$$

$$\operatorname{Im}\langle pJ_{i\beta}^{-2}p\rangle_u = -2\beta\langle pf'p\rangle_u + O(\beta^3)\langle p^2\rangle_u$$

and

$$\tan\phi = -2\beta(1+\beta/2)^{-1}\langle pf'p\rangle_u\langle p^2\rangle_u^{-1} + O(\beta^3) \tag{A.9}$$

Now, the first term on the right side of (A.7) is

$$-\beta(2+\beta)^{-1}\operatorname{Im}\langle pJ_{i\beta}^{-2}p\rangle_u = 2\beta^2(2+\beta)^{-1}\langle pf'p\rangle_u + O(\beta^4)\langle p^2\rangle_u$$
$$\geq 2\beta^2(2+\beta)^{-1}\delta_1\langle p^2\rangle_u + O(\beta^4)\langle p^2\rangle_u \geq 0$$

since $f' \mid [\tilde{R}_{NT}, \infty) \geq \delta_1 > 0$, so this term is non-negative. As for the second term on the right in (A.7), it follows from (A.9) and the boundedness of $F_{i\beta}$ that it is $O(\beta)\|u\|^2$. Consequently, K_β contributes only $O(\beta)\|u\|^2$ to (A.4).

2) We next turn to the potential energy contribution to (A.4). It has the form

$$P_\beta \equiv \lambda(\ell)\,[\tan\phi\,\operatorname{Re}\langle V_1(i\beta) - E\rangle_u - \operatorname{Im}\langle V_1(i\beta)\rangle_u$$
$$+ \lambda(\ell)^{-1}\langle \tan\phi\,\operatorname{Re} V_2(i\beta) - \operatorname{Im} V_2(i\beta)\rangle_u]. \tag{A.10}$$

We expand each V_k, $k = 1, 2$, in β:

$$V_k(i\beta) = V_k + i\beta V_k' f + O(\beta^2), \qquad k = 1, 2 \tag{A.11}$$

(The uniform boundedness of the remainder follows from Condition E *iii*). The second term in (A.10) is easily seen to be bounded below by

$$-\beta\lambda(\ell)^{-1}c_1\|u\|^2 + O(\beta^2)\lambda(\ell)^{-1}\|u\|^2 \tag{A.12}$$

for some constant $c_1 > 0$, as follows from (A.9) and (A.11) and the boundedness of V_2. As for the first term in (A.10), we first derive an estimate for $\tan\phi$. Since $M_1 = \sup\{ f'(r) \mid \tilde{R}_{NT} \leq r < \infty\} < \infty$, it follows from (A.9) that

$$|\tan\phi + 2\beta\langle pf'p\rangle_u\langle p^2\rangle_u^{-1}| \leq \beta^2 M_1. \tag{A.13}$$

Let $\kappa \equiv \sup\{V_1(r) - E \mid \tilde{R}_{NT} \leq r < \infty\}$. It follows from (A.13) and (A.11) that the first term of P_β in (A.10) is bounded below by:

$$\beta\lambda(\ell)\langle 2(E - V_1)\langle pf'p\rangle_u\langle p^2\rangle_u^{-1} - V_1 f'\rangle_u - \beta^2\lambda(\ell)M_1\kappa\|u\|^2. \tag{A.14}$$

Let $\langle S_f(V, E)\rangle_u \equiv \langle -V'f + 2(E - V)\langle pf'p\rangle_u\langle p^2\rangle_u^{-1}\rangle_u$; then condition (A.1) is that

$$\langle S_f(V_1, E)\rangle_u \geq \delta_0\|u\|^2. \tag{A.15}$$

Combining (A.12), (A.14) and (A.15) in (A.10), we obtain:

$$P_\beta \geq \beta\lambda(\ell)[\delta_0 - \beta M_1\kappa - \lambda(\ell)^{-1}c_1]\|u\|^2. \tag{A.16}$$

3) To complete the proof, we combine (A.16) with the result of part (1) to obtain a lower bound for (A.4):

$$\|(z - H_g(\ell, \theta))u\| \geq \cos\phi\{\lambda(\ell)\beta[\delta_0 - \beta M_1\kappa - \lambda(\ell)^{-1}c] - \Gamma\}\|u\|. \tag{A.17}$$

By (A.9), $\phi \to 0$ as $\beta \to 0$ so for all β small ($\beta \ll \delta_0(4M_1\kappa)^{-1}$) and ℓ sufficiently large ($\lambda(\ell) > 4\delta_0^{-1}c$)

$$\|(z - H_g(\ell, \theta))u\| \geq (1/4)[\lambda(\ell)\beta\delta_0 - 2\Gamma]\|u\|.$$

So if $\Gamma < \lambda(\ell)\beta\delta_0/4$, we have

$$\|(z - H_g(\ell, \theta))u\| \geq (1/8)\lambda(\ell)\beta\delta_0\|u\|$$

which verifies (A.2). □

APPENDIX B

QUANTUM NON-TRAPPING

With the notations of the text, we prove (see Figure B.1).

LEMMA B.1. — *Let f be a vector field as defined in Section 4 and $V_1 = h^{-2}$ with h satisfying conditions A i) and A iv) and B. Let $V_1(R_{NT}) > E > \min_{r \in [0, R_{NT}]} V_1(r) > \lim_{r \to \infty} V_1(r)$. Then $\exists R_0 > R_{NT}$ and $\exists \delta_0 > 0$ such that $\forall R$ for which $R_0 < R < S_E^+$ and $\forall u \in C_0^\infty([R, \infty))$, we have*

$$\langle S_f(V_1, E) \rangle_u \equiv \langle -fV_1' + 2 \langle pf'p \rangle_u \langle p^2 \rangle_u^{-1} (E - V_1) \rangle_u \geq \delta_0 \| u \|^2 \qquad (B.1)$$

where $\langle A \rangle_u \equiv \langle u, Au \rangle$.

Proof. — Choose $R_{NT} < R < S_E^+$. Introduce a smooth partition of unity χ_1, χ_2 for $[R, \infty)$ $\chi_1 + \chi_2 = 1$, such that $\operatorname{supp} \chi_2 \subset [S_E^+ + \varepsilon, \infty)$ and $\operatorname{supp} \chi_1 \subset [R, S_E^+ + 2\varepsilon)$, for some $\varepsilon > 0$. Hence $(E - V_1) \mid \operatorname{supp} \chi_2 > \eta > 0$; moreover $V_1' \mid [R_{NT}, \infty) < 0$ as a result of condition B. Then

$$\langle S_f(V_1, E) \rangle_u = \langle \chi_2 u, -fV_1' u \rangle + 2 \langle pf'p \rangle_u \langle p^2 \rangle_u^{-1} \langle u, (E - V_1)\chi_2 u \rangle + \langle \chi_1 S_f(V_1, E) \rangle_u . \qquad (B.2)$$

Since the first term in the right hand side of (B.2) is positive and f' is bounded on $[R, \infty)$ (i.e. $\exists \delta_1, \delta_2 > 0$ such that $\delta_2 > f' > \delta_1 > 0$ on $[R, \infty)$), we find, for some $C > 0$

$$\langle S_f(V_1, E) \rangle_u \geq C \langle u, \chi_2 u \rangle + \langle \chi_1 u, -fV_1' u - 2\delta_2 \kappa(R) u \rangle \qquad (B.3)$$

where $\kappa(R) = V_1(R) - E$. Equation (B.3) holds $\forall R \in (R_{NT}, S_E^+)$. Now $-fV_1' \mid [R_{NT}, S_E^+ + 2\varepsilon)$ is bounded below by a constant $D > 0$ and $\kappa(R)$ can be made arbitrarily small by choosing R close to S_E^+. Hence we get from (B.3), $\exists R_0 > R_{NT}$ and $C' > 0$ such that $\forall R > R_0$,

$$\langle S_f(V_1, E_0) \rangle_u \geq C \langle u, \chi_2 u \rangle + C' \langle u, \chi_1 u \rangle \geq \delta_0 \| u \|^2 \qquad \square \qquad (B.4)$$

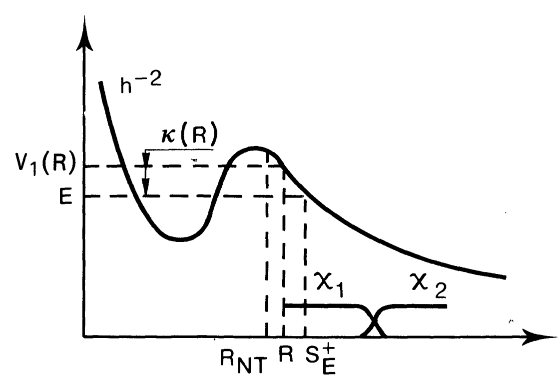

FIG. B.1. — A partition of unity used in the proof.

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