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**An explicit determination of the space-times  
on which the conformally invariant scalar wave equation  
satisfies Huygens' principle.**

**Part III: Petrov type III space-times**

by

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**ABSTRACT.** — It is shown that the validity of Huygens' principle for the conformally invariant scalar wave equation, Maxwell's equations or Weyl's neutrino equation on a Petrov type III space-time, implies that the space-time is conformally related to one on which the repeated principal null vector field of the Weyl tensor, is recurrent. Further, it is proven that if a certain mild assumption which is shown to be suggested by the necessary conditions, is imposed on the covariant derivative of the Weyl tensor, then there are no Petrov type III space-times on which any of the above equations satisfies Huygens' principle.

**RÉSUMÉ.** — Nous prouvons que la validité du principe de Huygens pour l'équation d'onde scalaire invariante conforme, les équations de Maxwell ou l'équation de Weyl pour les neutrinos sur un espace-temps de Petrov de type III impliquent que cet espace-temps est conformément relié à un espace où le noyau principal répété du tenseur de Weyl est récurrent. Nous montrons de plus sous une hypothèse faible sur la dérivée covariante du tenseur de Weyl suggérée par la condition nécessaire qu'il n'existe pas d'espace-temps de Petrov de type III où les équations ci-dessus satisfont le principe de Huygens.

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## 1. INTRODUCTION

This paper is the third in a series devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation, Maxwell's equations and Weyl's neutrino equation on a curved space-time. These equations may be written respectively as

$$\square u + \frac{1}{6} Ru = 0, \quad (1.1)$$

$$d\omega = 0, \quad \delta\omega = 0, \quad (1.2)$$

$$\nabla_A{}^B \phi_B = 0, \quad (1.3)$$

where  $\square$  denotes the Laplace-Beltrami operator corresponding to the metric  $g_{ab}$  of the space-time  $V_4$ ,  $u$  the unknown scalar function,  $R$  the curvature scalar,  $d$  the exterior derivative,  $\delta$  the exterior co-derivative,  $\omega$  the Maxwell 2-form,  $\nabla_A{}^B$  the covariant derivative on 2-spinors, and  $\phi_A$  a 1-spinor. Our conventions are those of McLenaghan [18]. All considerations in this paper are entirely local.

According to Hadamard [14] *Huygens' principle* (in the strict sense) is valid for Eq. (1.1) if and only if for every Cauchy initial value problem and every  $x_0 \in V_4$ , the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of  $S \cap C^-(x_0)$  where  $S$  denotes the initial surface and  $C^-(x_0)$  denotes the past null conoid from  $x_0$ . Analogous definitions of the validity of the principle for Maxwell's equations (1.2) and Weyl's equation (1.3) have been given by Günther [12] and Wunsch [28] respectively, in terms of the appropriate formulations of the initial value problems for these equations. *Hadamard's problem* for Eqs. (1.1), (1.2) or (1.3), originally posed only for scalar equations, is that of determining all space-times for which Huygens' principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens' principle, the determination may only be effected up to a conformal transformation of the metric

$$\tilde{g}_{ab} = e^{2\phi} g_{ab}, \quad (1.4)$$

where  $\phi$  is an arbitrary function.

Huygens' principle is valid for (1.1), (1.2) and (1.3) on any conformally flat space-time and also on any space-time conformally related to the exact plane wave space-time [11] [15] [29], the metric of which has the form

$$ds^2 = 2dv \{ .du + [D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z}]dv \} - 2dzd\bar{z}, \quad (1.5)$$

in a special co-ordinate system, where  $D$  and  $e$  are arbitrary functions. These are the only *known* space-times on which Huygens' principle is

valid for these equations. Furthermore, it has been shown [16] [13] [29] that these are the only conformally empty space-times on which Huygens' principle is valid.

More recently, the authors have outlined a program [2] for the solution of Hadamard's problem based on the conformally invariant Petrov classification [22] [9], of the Weyl conformal curvature tensor. This involves the consideration of five disjoint cases which exhaust all the possibilities for non-conformally flat space-times. This program has now been completed in the cases of Petrov types N and D. In the former case we proved that *every Petrov type N space-time on which the conformally invariant scalar wave equation (1.1) satisfies Huygens' principle is conformally related to an exact plane wave space-time (1.5), ([3] and Theorem 2 of [4] denoted by CM1 in the sequel).* This result when combined with Günthers [11] solves Hadamard's problem in this case. In the latter case we proved that *there exist no Petrov type D space-times on which the conformally invariant scalar wave equation (1.1) satisfies Huygens' principle* (Theorem 1 of [5] and Theorem 6 of [6] denoted by CM2 in the sequel).

The proof of the above theorems was obtained by solving the following sequence of necessary conditions for the validity of Huygens' principle for the equations (1.1), (1.2) and (1.3) [10] [25] [18] [17] [16] [28]:

$$\text{III}' \quad S_{abk}{}^k - \frac{1}{2} C^k{}_{ab}{}^l L_{kl} = 0. \quad (1.6)$$

$$\begin{aligned} \text{V}' \quad \text{TS}[k_1 C^k{}_{ab}{}^l{}^m C_{kcdl;m} + 2k_2 C^k{}_{ab}{}^l{}^c S_{kld} + 2(8k_1 - k_2) S_{ab}{}^k S_{cdk} \\ - 2k_2 C^k{}_{ab}{}^l S_{klc;d} - 8k_1 C^k{}_{ab}{}^l S_{cdk;l} + k_2 C^k{}_{ab}{}^l C_l{}^m{}_{ck} L_{dm} \\ + 4k_1 C^k{}_{ab}{}^l C^m{}_{cdl} L_{km}] = 0, \end{aligned} \quad (1.7)$$

where

$$C_{abcd} := R_{abcd} - 2g_{[a[d} L_{b]c]}, \quad (1.8)$$

$$L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab} \quad (1.9)$$

$$S_{abc} := L_{a[b;c]}. \quad (1.10)$$

In the above  $C_{abcd}$  denotes the Weyl tensor,  $R_{ab}$  the Ricci tensor and  $\text{TS}[\quad]$  the operator which takes the trace free symmetric part of the enclosed tensor. The quantities  $k_1$  and  $k_2$  appearing in Eq. (1.7) are constants whose values are given in the following table:

TABLE 1

Equation	$k_1$	$k_2$
Scalar	3	4
Maxwell	5	16
Weyl	8	13

It was also necessary to invoke the further necessary Condition VII, valid only for the scalar case, derived by Rinke and Wunsch [24], to complete the proofs. Some partial results for Maxwell's equations (1.2) and Weyl's equation (1.3) for both type N and type D are given in Theorem 1 of CM1 and Theorems 1, 2, 4 and 5 of CM2. However, Hadamard's problem still remains open for these equations on space-times of types N and D. The derivation of the analogue of Condition VII for these equations might settle the question as it did in the scalar case.

Our analysis has now been extended to include the case of Petrov type III space-times. We recall that such space-times are characterized by the existence of a null vector field  $l$  satisfying the following conditions [9]:

$$C_{abc[d}l_e]l^c = 0, \quad (1.11)$$

$$C_{abcd}l^d \neq 0, \quad (1.12)$$

Such a vector field, called a *repeated principal null vector* field of the Weyl tensor, is determined by  $C_{abcd}$  up to an arbitrary variable factor. Let  $\Psi_{ABCD}$  denote the symmetric 4-spinor, called the Weyl spinor, equivalent to  $C_{abcd}$ . The spinor equivalents of (1.11) and (1.12) are then given by

$$\Psi_{ABCD}o^C o^D = 0 \quad (1.13)$$

$$\Psi_{ABCD}o^D \neq 0, \quad (1.14)$$

where  $o^A$  denotes a 1-spinor field corresponding to  $l^a$ . Such a spinor field is called a *repeated principal spinor* of the Weyl spinor and is determined by the latter up to an arbitrary variable complex factor. Finally, let  $l^A$  be any spinor field satisfying

$$o_A l^A = 1. \quad (1.15)$$

The ordered set  $\{o_A, l_A\}$ , called a dyad, defines a basis for the 1-spinor fields on  $V_4$ .

The main results of this paper may now be stated as follows:

**THEOREM 1.** — *The validity of Huygens' principle for the conformally invariant scalar wave equation (1.1), or Maxwell's equations (1.2), or Weyl's neutrino equation (1.3) on any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated principal spinor field  $o_A$  of the Weyl spinor is recurrent, that is*

$$o_A;B\dot{B} = o_A I_{B\dot{B}}, \quad (1.16)$$

where  $I_{B\dot{B}}$  is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}}l^A l^B l^C o^D o^E \bar{o}^{\dot{E}} = 0, \quad (1.17)$$

$$R = 0, \quad \Phi_{AB\dot{A}\dot{B}}o^A o^B = 0. \quad (1.18)$$

Since the equation (1.16) is equivalent to the equation

$$l_{a;b} = l_a p_b, \quad (1.19)$$

where  $p_b$  is equivalent to  $I_{BB}$ , it follows that *every repeated principal null vector field of the Weyl tensor is recurrent*. This is the form in which Theorem 1 was stated without proof in [5].

**THEOREM 2.** — *If any one of the following three conditions*

$$\Psi_{ABCD;E\dot{E}} l^A l^B l^C o^D l^E \bar{o}^{\dot{E}} = 0, \quad (1.20)$$

$$\Psi_{ABCD;E\dot{E}} l^A l^B l^C o^D o^E \bar{l}^{\dot{E}} = 0, \quad (1.21)$$

$$\Psi_{ABCD;E\dot{E}} l^A l^B l^C l^D o^E \bar{o}^{\dot{E}} = 0, \quad (1.22)$$

*is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation (1.1) or Maxwell's equations (1.2), or Weyl's equation (1.3) satisfies Huygens' principle.*

There is considerable evidence to suggest that the conditions (1.20) to (1.22) imposed in Theorem 2 are superfluous in that they themselves are consequences of the necessary conditions III', V' and VII in the scalar case. Indeed the invariant quantities appearing in Eqs. (1.20) to (1.22) satisfy, in a special spinor dyad  $\{o_A, l_A\}$ , a system of at least the equivalent of three complex polynomial equations. It appears that this system combined with the other equations of the problem have only the null solution. However, we have been unable to prove this. Attempts to use the Gröbner basis methods [1] which were successfully employed in the case of Petrov type D, have to date been unsuccessful due to the size and complexity of the system. It should also be noted that the Eqs. (1.16) to (1.18) have the *same form* in any spinor dyad  $\{o_A, l_A\}$  where  $o_A$  satisfies Eqs. (1.13) and (1.14) and  $l_A$  satisfies (1.15).

The results obtained thus far for the Petrov types N, D and III lend weight to the conjecture that every space-time on which the conformally invariant scalar wave equation satisfies Huygens' principle, is conformally related to the plane wave space-time (1.5) or is conformally flat [2] [3] [4].

The plan of the remainder of the paper is as follows. In Section 2, the formalisms used are briefly described. The proofs of the theorems are given in Sections 3 and 4 and some concluding remarks are given in Section 5.

## 2. FORMALISMS

We use the two-component spinor formalism of Penrose [21] [23] and the spin coefficient formalism of Newman and Penrose (NP) [20] whose conventions we follow. In the spinor formalism, tensor and spinor

indices are related by the complex connection quantities  $\sigma_a^{A\dot{A}}$  ( $a = 1, \dots, 4$ ;  $A = 0, 1$ ) which are Hermitian in the spinor indices  $A\dot{A}$ . Spinor indices are lowered by the skew symmetric spinors  $\varepsilon_{AB}$  and  $\varepsilon_{\dot{A}\dot{B}}$  defined by  $\varepsilon_{01} = \varepsilon_{\dot{0}\dot{1}} = 1$ , according to the convention

$$\zeta_A = \zeta^{\dot{B}} \varepsilon_{\dot{B}A}, \quad (2.1)$$

where  $\zeta_A$  is an arbitrary 1-spinor. Spinor indices are raised by the respective inverses of these spinors denoted by  $\varepsilon^{AB}$  and  $\varepsilon^{\dot{A}\dot{B}}$ . The spinor equivalents of the Weyl tensor (1.8) and the tensor  $L_{ab}$  defined by (1.9) are given respectively by

$$C_{abcd} \sigma^a_{A\dot{A}} \sigma^b_{B\dot{B}} \sigma^c_{C\dot{C}} \sigma^d_{D\dot{D}} = \Psi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{DC}, \quad (2.2)$$

$$L_{ab} \sigma^a_{A\dot{A}} \sigma^b_{B\dot{B}} = 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda \varepsilon_{AB} \varepsilon_{\dot{A}\dot{B}}), \quad (2.3)$$

where  $\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})}$  denotes the Hermitian trace-free Ricci spinor and where

$$\Lambda = (1/24)R. \quad (2.4)$$

The covariant derivative of spinors is denoted by « ; » and satisfies

$$\sigma_a^{A\dot{A}}{}_{;b} = \varepsilon_{AB;b} = 0. \quad (2.5)$$

It will be necessary in the sequel to express spinor equations in terms of a spinor dyad  $\{o_A, \iota_A\}$  defined in Sec. 1. Associated to the spinor dyad is a null tetrad  $\{l, n, m, \bar{m}\}$  defined by

$$l^a = \sigma^a_{A\dot{A}} o^A \bar{o}^{\dot{A}}, \quad n^a = \sigma^a_{A\dot{A}} \iota^A \bar{\iota}^{\dot{A}}, \quad m^a = \sigma^a_{A\dot{A}} o^A \bar{\iota}^{\dot{A}}, \quad (2.6)$$

whose only non-zero inner products are

$$l_a n^a = -m_a \bar{m}^a = 1. \quad (2.7)$$

The metric tensor may be expressed in terms of the null tetrad by

$$g_{ab} = 2l_{(a} n_{b)} - 2m_{(a} \bar{m}_{b)}. \quad (2.8)$$

The NP spin coefficients associated with the dyad are defined by the equations

$$o_{A;\dot{B}\dot{B}} = o_A \mathbf{I}_{\dot{B}\dot{B}} + \iota_A \mathbf{II}_{\dot{B}\dot{B}}, \quad (2.9)$$

$$\iota_{A;\dot{B}\dot{B}} = o_A \mathbf{III}_{\dot{B}\dot{B}} - \iota_A \mathbf{I}_{\dot{B}\dot{B}}, \quad (2.10)$$

where

$$\mathbf{I}_{\dot{B}\dot{B}} := \gamma o_{\dot{B}} \bar{o}_{\dot{B}} - \alpha o_{\dot{B}} \bar{\iota}_{\dot{B}} - \beta \iota_{\dot{B}} \bar{o}_{\dot{B}} + \varepsilon \iota_{\dot{B}} \bar{\iota}_{\dot{B}}, \quad (2.11)$$

$$\mathbf{II}_{\dot{B}\dot{B}} := -\tau o_{\dot{B}} \bar{o}_{\dot{B}} + \rho o_{\dot{B}} \bar{\iota}_{\dot{B}} + \sigma \iota_{\dot{B}} \bar{o}_{\dot{B}} - \kappa \iota_{\dot{B}} \bar{\iota}_{\dot{B}}, \quad (2.12)$$

$$\mathbf{III}_{\dot{B}\dot{B}} := \nu o_{\dot{B}} \bar{o}_{\dot{B}} - \lambda o_{\dot{B}} \bar{\iota}_{\dot{B}} - \mu \iota_{\dot{B}} \bar{o}_{\dot{B}} + \pi \iota_{\dot{B}} \bar{\iota}_{\dot{B}}. \quad (2.13)$$

The NP components of the Weyl spinor and trace-free Ricci spinor are defined as follows:

$$\Psi_{ABCD} = \Psi_0 l_{ABCD} - 4\Psi_1 o_{(A} l_{BCD)} + 6\Psi_2 o_{(AB} l_{CD)} - 4\Psi_3 o_{(ABC} l_{D)} + \Psi_4 o_{ABCD}, \quad (2.14)$$

$$\begin{aligned} \Phi_{AB\dot{A}\dot{B}} = & \Phi_{22} o_{AB} \bar{o}_{\dot{A}\dot{B}} - 2\Phi_{21} o_{AB} \bar{o}_{(\dot{A}\dot{B})} - 2\Phi_{12} o_{(A} l_{B)} \bar{o}_{\dot{A}\dot{B}} + \Phi_{20} o_{AB} \bar{l}_{\dot{A}\dot{B}} \\ & + \Phi_{02} l_{AB} \bar{o}_{\dot{A}\dot{B}} + 4\Phi_{11} o_{(A} l_{B)} \bar{o}_{(\dot{A}\dot{B})} - 2\Phi_{10} o_{(A} l_{B)} \bar{l}_{\dot{A}\dot{B}} - 2\Phi_{01} l_{AB} \bar{o}_{(\dot{A}\dot{B})} + \Phi_{00} l_{AB} \bar{l}_{\dot{A}\dot{B}}. \end{aligned} \quad (2.15)$$

The NP differential operators are defined by

$$D := l^a \frac{\partial}{\partial x^a}, \quad \Delta := n^a \frac{\partial}{\partial x^a}, \quad \delta := m^a \frac{\partial}{\partial x^a}. \quad (2.16)$$

The equations relating the curvature components to the spin coefficients, the commutation relations satisfied by the above differential operators, and the Bianchi identities may be found in NP and [23].

The subgroup of the proper orthochronous Lorentz group  $L^\dagger$  preserving the direction of the vector  $l$  is given by

$$\begin{aligned} l' &= e^a l, & m' &= e^{ib}(m + \bar{q}l), \\ n' &= e^{-a}(n + qm + \bar{q}\bar{m} + q\bar{q}l) \end{aligned} \quad (2.17)$$

where  $a$  and  $b$  are real-valued and  $q$  is complex valued. The corresponding transformation of the spinor dyad is given by

$$\begin{aligned} o' &= e^{w/2} o, \\ l' &= e^{-w/2}(l + qo), \end{aligned} \quad (2.18)$$

where  $w = a + ib$ . These transformations induce the following transformations:

$$\begin{aligned} D' &= e^{(w+\bar{w})/2} D, \\ \delta' &= e^{(w-\bar{w})/2}(\delta + \bar{q}D), \\ \Delta' &= e^{-(w+\bar{w})/2}(\Delta + q\delta + \bar{q}\bar{\delta} + q\bar{q}D), \\ I'_{AA'} &= I_{AA'} + \frac{1}{2} w_{;AA'} - q\Pi_{AA'}, \\ \Pi'_{AA'} &= e^w \Pi_{AA'}, \\ \text{III}'_A &= e^{-w} [\text{III}_{AA'} + 2qI_{AA'} - q^2\Pi_{AA'} + q_{;AA'}], \end{aligned} \quad (2.19)$$

from which one may deduce via Eqs. (2.11) to (2.13) and (2.18) the transformation laws for the spin coefficients. The NP components of the Weyl tensor transform as follows:

$$\begin{aligned} \Psi'_0 &= e^{2w}\Psi_0, \\ \Psi'_1 &= e^w(\Psi_1 + q\Psi_0), \\ \Psi'_2 &= \Psi_2 + 2q\Psi_1 + q^2\Psi_0, \\ \Psi'_3 &= e^{-w}(\Psi_3 + 3q\Psi_2 + 3q^2\Psi_1 + q^3\Psi_0), \\ \Psi'_4 &= e^{-2w}(\Psi_4 + 4q\Psi_3 + 6q^2\Psi_2 + 4q^3\Psi_1 + q^4\Psi_0). \end{aligned} \quad (2.21)$$

We shall also need the following transformation of the null tetrad

$$\tilde{l}_a = e^{-\phi} l_a, \quad \tilde{n}_a = e^{3\phi} n_a, \quad \tilde{m}_a = e^{\phi} m_a, \quad (2.22)$$

which induces by (2.8) the conformal transformation of the metric (1.4). Some of the transformation formulas for the spin coefficients induced by (2.22) are as follows:

$$\begin{aligned} \tilde{\kappa} &= e^{-5\phi} \kappa, & \tilde{\rho} &= e^{-3\phi} (\rho - D\phi), \\ \tilde{\sigma} &= e^{-3\phi} \sigma, & \tilde{\tau} &= e^{-\phi} (\tau - \delta\phi), \\ \tilde{\varepsilon} &= e^{-3\phi} \left( \varepsilon - \frac{1}{2} D\phi \right). \end{aligned} \quad (2.23)$$

### 3. PROOF OF THEOREM 1

We recall from CM1 that the spinor equivalent of the conditions (1.6) and (1.7) are given by

$$\text{III's} \quad \Psi_{ABKL}; \overset{K}{A} \overset{L}{B} + \overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}; \overset{\dot{K}}{A} \overset{\dot{L}}{B} + \Psi_{AB}{}^{KL} \Phi_{KL\dot{A}\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}} \overline{\Phi}_{\dot{K}\dot{L}AB} = 0, \quad (3.1)$$

$$\begin{aligned} \text{V's} \quad & k_1 \Psi_{ABCD}; \overset{K}{A} \overset{L}{B} \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}; \overset{\dot{K}}{A} \overset{\dot{L}}{B} + k_2 \Psi^K{}_{(ABC;D)} \overline{\Psi}_{\dot{B}\dot{C}\dot{D}}; \overset{\dot{L}}{K} \\ & + k_2 \overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})} \Psi_{BCD} \overset{L}{L}; \overset{L}{K} - 2(8k_1 - k_2) \Psi_{(ABC|K|}; \overset{K}{A} \overline{\Psi}_{\dot{B}\dot{C}\dot{D}}; \overset{\dot{K}}{K} \overset{\dot{D}}{D} \\ & - k_2 \Psi^K{}_{(ABC} \overline{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|}; \overset{\dot{L}}{K} |\overset{\dot{L}}{K}| \overset{\dot{D}}{D}) - k_2 \overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}} \Psi_{(ABC|L|}; \overset{L}{L} |\overset{\dot{K}}{K}| \overset{\dot{D}}{D}) \\ & + 4k_1 \Psi^K{}_{(ABC} \overline{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|}; \overset{\dot{L}}{D} \overset{\dot{K}}{D}) + 4k_1 \overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}} \Psi_{(ABC|L|}; \overset{L}{D}) \overset{\dot{K}}{K} \\ & + 2(k_2 - 4k_1) \Psi^K{}_{(ABC} \overline{\Psi}_{\dot{B}\dot{C}\dot{D}}; \overset{\dot{K}}{A} \overline{\Psi}_{\dot{B}\dot{C}\dot{D}}) \overset{\dot{K}}{K} - 2(4k_1 + k_2) \Lambda \Psi_{ABCD} \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0. \end{aligned} \quad (3.2)$$

We now make the hypothesis that the space-time is of Petrov type III. These space-times are characterized by the existence of a spinor  $o^A$  satisfying (1.13) and (1.14). The Weyl spinor of a type III space-time also admits a second principal spinor  $l^A$  satisfying (1.15) and

$$\Psi_{ABCD} l^{ABCD} = 0, \quad (3.3)$$

$$\Psi_{ABCD} l^{ABC} \neq 0, \quad (3.4)$$

where the notation  $l^{A_1 \dots A_p} := l^{A_1} \dots l^{A_p}$ , has been used. If  $o_A$  and  $l_A$  are appropriately rescaled using (2.18), the Weyl spinor may be expressed as

$$\Psi_{ABCD} = 4o_{(ABC} l_{D)}, \quad (3.5)$$

which implies

$$\Psi_3 = -1, \quad \Psi_4 = 0. \quad (3.6)$$

In view of the transformation formulas (2.21) the choice (3.6) uniquely determines the spinor dyad  $\{o_A, l_A\}$ . It is important to note that the conformal transformation (2.22) preserves (3.6) for any choice of the function  $\phi$ .

We proceed by substituting for  $\Psi_{ABCD}$  in Eqs. (3.1) and (3.2) from (3.5). The covariant derivatives of  $o_A$  and  $\iota_A$  that appear are eliminated using Eqs. (2.9) and (2.10) respectively. The dyad form of the resulting equations is obtained by contracting them with appropriate products of  $o^A$  and  $\iota^A$  and their complex conjugates. In view of the conformal invariance of conditions III's and V's [18] [27], it follows that each dyad equation must be individually invariant under the conformal transformations (2.22). The first contraction to consider is  $o^{ABC} \iota^D \bar{o}^{\dot{A}\dot{B}} \bar{\iota}^{\dot{C}\dot{D}}$  with Condition V's which yields the equation

$$k_1 \kappa \bar{\kappa} = 0, \tag{3.7}$$

and consequently

$$\kappa = 0, \tag{3.8}$$

since  $k_1 \neq 0$ , by assumption, while the  $o^{ABCD} \bar{\iota}^{\dot{A}\dot{B}} \bar{\iota}^{\dot{C}\dot{D}}$  contraction with Condition V's implies

$$\sigma = 0. \tag{3.9}$$

The conditions (3.8) and (3.9), which are invariant under general dyad transformations (2.17) and conformal transformations (2.22), imply that the repeated principal null congruence of  $C_{abcd}$  defined by the principal null vector field  $l^a$ , is *geodesic* and *shear-free*.

Before proceeding with the derivation of further dyad equations from III's and V's we employ the conformal transformation (2.22) to set (dropping tildes)

$$\rho = -\bar{\rho}. \tag{3.10}$$

This condition may always be imposed since the differential equation

$$D\phi = \frac{1}{2}(\rho + \bar{\rho}), \tag{3.11}$$

always has a solution.

The next contractions to consider are  $o^{AB} \iota^{CD} \bar{o}^{\dot{A}\dot{B}} \bar{\iota}^{\dot{C}\dot{D}}$  with V's and  $o^A \iota^B \bar{\iota}^{\dot{A}\dot{B}}$  with III's which yield respectively the equations

$$(k_2 - 4k_1) \left[ D(\varepsilon - \rho) + (\varepsilon - \rho)(\varepsilon - \bar{\varepsilon} + 2\rho) + \frac{1}{2} \Phi_{00} \right] + c. c. - 8(k_2 - 5k_1)\rho^2 - 32k_1 \varepsilon \bar{\varepsilon} + 4(8k_1 - k_2)\rho(\bar{\varepsilon} - \varepsilon) = 0, \tag{3.12}$$

$$D(\varepsilon - \rho) + (\varepsilon - \rho)(\varepsilon - \bar{\varepsilon} - 2\rho) + \frac{1}{2} \Phi_{00} = 0. \tag{3.13}$$

Eliminating the quantity  $D(\varepsilon - \rho)$  between these equations, we obtain an equation that may be written in the following form, by completing the square:

$$c\rho^2 + d \left[ \rho + \frac{1}{2}(e/d)(\bar{\varepsilon} - \varepsilon) \right]^2 - \frac{1}{4}(e^2/d)(\bar{\varepsilon} + \varepsilon)^2 + (e^2/b + f)\varepsilon\bar{\varepsilon} = 0, \tag{3.14}$$

where

$$\begin{aligned} c + d &:= 9k_1 - 2k_2, \\ e &:= 6k_1 - k_2, \\ f &:= -4k_1. \end{aligned} \tag{3.15}$$

The expression on the left-hand side of (3.14) will be negative definite provided that

$$c > 0, \quad d > 0, \quad e^2/d + f > 0. \tag{3.16}$$

These inequalities will hold if  $d$  satisfies the inequality

$$\frac{1}{4}(6k_1 - k_2)^2/k_1 < d < 9k_1 - 2k_2. \tag{3.17}$$

A number  $d$  exists for which (3.17) holds since the inequality

$$(6k_1 - k_2)^2 < 4k_1(9k_1 - 2k_2), \tag{3.18}$$

holds for each of the three possible pairs of values of  $k_1$  and  $k_2$  given in Table 1. Thus Eq. (3.14) implies

$$\rho = 0, \tag{3.19}$$

$$\varepsilon = 0, \tag{3.20}$$

for each of the three cases. We observe that the condition (3.19) is invariant under a general tetrad transformation (2.17) but not under a general conformal transformation (2.22). However, it is clear from the transformations (2.23) that the condition

$$\rho = \bar{\rho}, \tag{3.21}$$

is invariant under (2.22) and that this is the form that (3.19) must take in an arbitrary conformal gauge. Turning our attention to (3.20), we note that it is not invariant under (2.17) since

$$\varepsilon' = e^a \left( \varepsilon + \frac{1}{2} D w \right). \tag{3.22}$$

From the transformation formula

$$D' \Psi'_3 + 2\varepsilon' \Psi'_3 = e^{-2ib} (D \Psi_3 + 2\varepsilon \Psi_3), \tag{3.23}$$

which is a consequence of (2.19), (2.21) and (3.22), it follows that the form of the condition (3.20) invariant under (2.17) is

$$D \Psi_3 + 2\varepsilon \Psi_3 = 0. \tag{3.24}$$

This form of the condition is not invariant under a general conformal transformation (2.22).

An important consequence of the conditions (3.8), (3.9) and (3.19) is

$$\Phi_{00} = \Phi_{01} = 0, \quad (3.25)$$

which follow from NP Eqs. (4.2).

We are now able to use some of the remaining conformal freedom (2.22) preserving (3.19) and (3.20) to set

$$\tau = 0. \quad (3.26)$$

This is possible since there exists a solution of the following system of partial differential equations

$$D\phi = 0, \quad \delta\phi = \tau, \quad \bar{\delta}\phi = \bar{\tau}. \quad (3.27)$$

The proof that the integrability conditions for the above system are satisfied is identical to that given in CM1 and will be omitted here.

We complete the proof of (1.16) by noting that (3.8), (3.9), (3.19) and (3.26) are equivalent to

$$\Pi_{AA} = 0, \quad (3.28)$$

which by (2.20) is clearly invariant under a general tetrad transformation (2.18). The Eq. (1.16) now follows from (2.9). The conditions (1.18) are a direct consequence of (3.28) and NP Eqs. (4.2).

In order to establish (1.17), we express the covariant derivative of  $\Psi_{ABCD}$  with respect to any dyad  $\{o_A, \iota_A\}$  where  $o_A$  is any repeated principal spinor satisfying (1.13) and (1.14). Using (2.14) with (3.28) we obtain

$$\begin{aligned} \Psi_{ABCD;E\dot{E}} = & -4o_{(ABC}\iota_{D)}[(\Delta\Psi_3 + 2\gamma\Psi_3)o_E\bar{o}_{\dot{E}} - (\bar{\delta}\Psi_3 + 2\alpha\Psi_3)o_E\bar{\iota}_{\dot{E}} \\ & - (\delta\Psi_3 + 2\beta\Psi_3)\iota_E\bar{o}_{\dot{E}} + (D\Psi_3 + 2\varepsilon\Psi_3)\iota_E\bar{\iota}_{\dot{E}}] \\ & + o_{ABCD}[(\Delta\Psi_4 - 4\nu\Psi_3 + 4\gamma\Psi_4)o_E\bar{o}_{\dot{E}} - (\bar{\delta}\Psi_4 - 4\lambda\Psi_3 + 2\alpha\Psi_4)o_E\bar{\iota}_{\dot{E}} \\ & - (\delta\Psi_4 - 4\mu\Psi_3 + 4\beta\Psi_4)\iota_E\bar{o}_{\dot{E}} + (D\Psi_4 - 4\pi\Psi_3 + 4\varepsilon\Psi_4)\iota_E\bar{\iota}_{\dot{E}}]. \end{aligned} \quad (3.29)$$

The Eq. (1.17) follows from the above and (3.24). This completes the proof of Theorem 1.

#### 4. PROOF OF THEOREM 2

The results established in Sec. 3 may be summarized as follows: Conditions III's and V's imply that there exists a dyad  $\{o_a, \iota_a\}$  and a conformal transformation  $\phi$  such that

$$\kappa = \sigma = \rho = \tau = \varepsilon = 0, \quad (4.1)$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \quad \Psi_3 = -1, \quad (4.2)$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0, \quad (4.3)$$

$$D\alpha = D\beta = 0. \quad (4.4)$$

We recall that the conditions (4.2) uniquely determine the dyad. In contrast, the conditions (4.1) are preserved by any conformal transformation satisfying

$$D\phi = \delta\phi = 0, \quad (4.5)$$

which does not completely determine the functional dependence of  $\phi$  on the co-ordinates. This remaining conformal freedom is used to set

$$\Phi_{11} = c, \quad (4.6)$$

where  $c$  is constant. This is achieved by the use of the following transformation law for  $\Phi_{11}$ , induced by (2.22) subject to (4.5):

$$\tilde{\Phi}_{11} = e^{-2\phi}\Phi_{11}. \quad (4.7)$$

The existence of a function  $\phi$  for which (4.6) holds is guaranteed by the equations

$$D\Phi_{11} = \delta\Phi_{11} = 0, \quad (4.8)$$

which arise from the Bianchi identities [23].

As the first step in the proof of Theorem 2, we establish the following:

**LEMMA.** — *The Conditions III' and V' imply that, with respect to the dyad defined by (4.2) and conformal gauge defined by (4.6), the vanishing of any one of the spin coefficients  $\alpha$ ,  $\beta$  or  $\pi$  implies the vanishing of the remaining two and  $\Phi_{11}$ .*

*Proof.* — We begin the proof by deriving the required equations from III' and V' in dyad form, together with the Bianchi identities and their integrability conditions. The first equation to consider is the  $\sigma^{AB}{}_{\dot{C}\dot{D}}\sigma^{\dot{C}\dot{D}}{}_{A\dot{B}}$  contraction with V's which yields

$$D\pi = 0. \quad (4.9)$$

This equation may also be obtained by the contraction  $\iota^{AB}{}_{\dot{C}\dot{D}}\iota^{\dot{C}\dot{D}}{}_{A\dot{B}}$  with III's. The remaining Bianchi identities may be written as

$$D\Phi_{12} = 2\pi\Phi_{11}, \quad (4.10)$$

$$\delta\Phi_{12} = 2(\bar{\alpha} + 2\bar{\pi}) + 2\bar{\lambda}\Phi_{11} - 2\bar{\alpha}\Phi_{12}, \quad (4.11)$$

$$\bar{\delta}\Phi_{12} = -2\beta + 2\bar{\mu}\Phi_{11} - 2\bar{\beta}\Phi_{12}, \quad (4.12)$$

$$D\Phi_{22} = -2(\beta + \bar{\beta}) + 2\pi\Phi_{12} + 2\bar{\pi}\Phi_{21}, \quad (4.13)$$

$$\bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 2(\gamma + 2\mu) - 2\nu\Phi_{11} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} - 2(\alpha + \bar{\beta})\Phi_{22}. \quad (4.14)$$

The integrability conditions for the above identities, obtained from the NP commutation relations, are

$$\delta(\alpha + 2\pi) + \bar{\delta}\bar{\beta} + (3\beta - \bar{\alpha})(\alpha + 2\pi) + \bar{\beta}(\alpha + \bar{\beta}) + \Phi_{11} = 0, \quad (4.15)$$

$$\delta(\gamma + 2\mu) + \Delta\beta + (3\beta + \bar{\alpha})(\gamma + 2\mu) - \bar{\lambda}(\alpha + 2\pi) + \beta(3\mu + \gamma + \bar{\gamma}) + \Phi_{12} - c.c. = 0. \quad (4.16)$$

The next step is to extract the remaining conditions from III's. The contractions  $o^A{}_I B_{\bar{I}}{}^{\bar{A}}{}^{\bar{B}}$  and  $l^{AB}{}_{\bar{I}}{}^{\bar{A}}{}^{\bar{B}}$  yield respectively

$$2\delta\beta - \delta\bar{\beta} - \bar{\delta}(\bar{\alpha} + 2\bar{\pi}) - D(\bar{\gamma} + 2\bar{\mu}) + (2\beta - \bar{\beta})(\bar{\alpha} + \beta) \\ + (\bar{\alpha} + 2\bar{\pi})(\alpha - 3\bar{\beta} + \pi) - 3\bar{\beta}\pi - 2\Phi_{11} = 0, \quad (4.17)$$

$$\delta(2\mu + \gamma) + \Delta\beta + (\bar{\alpha} + 3\beta)(\gamma + 2\mu) - \bar{\lambda}(\alpha + 2\pi) + \beta(3\mu + \gamma + \bar{\gamma}) + \Phi_{12} + c.c. = 0. \quad (4.18)$$

The first of these equations and Eq. (4.15) may be rewritten as

$$\delta\beta = -\beta(\bar{\alpha} + \beta), \quad (4.19)$$

$$\delta(\alpha + 2\pi) = (\bar{\alpha} - 3\beta)(\alpha + 2\pi) - \Phi_{11}, \quad (4.20)$$

where use has been made of NP Eqs. (4.2). Similarly, (4.16) and (4.18) may be combined to yield

$$\delta(\gamma + \mu) + (\bar{\alpha} + 3\beta)(\gamma + \mu) - \bar{\lambda}(\alpha + \pi) + \beta\mu = 0. \quad (4.21)$$

We now turn our attention to Condition V. The contractions  $l^{ABCD}{}_{\bar{O}}{}^{\bar{A}}{}^{\bar{B}}{}^{\bar{C}}{}^{\bar{D}}$  and  $o^A{}_I BCD{}_{\bar{O}}{}^{\bar{A}}{}^{\bar{B}}{}^{\bar{C}}{}^{\bar{D}}$  give respectively,

$$\bar{\delta}(\alpha + 2\pi) = (\bar{\beta} - 3\alpha)(\alpha + 2\pi) - q_1\alpha\bar{\beta} - q_2\bar{\beta}\pi, \quad (4.22)$$

where

$$q_1 := 32k_1/(4k_1 - k_2), \quad (4.23)$$

$$q_2 := 4(16k_1 - k_2)/(4k_1 - k_2), \quad (4.24)$$

and  $\bar{\delta}\bar{\beta}$  has been removed with the use of (4.19), and

$$(k_2 + 12k_1)(\bar{\delta}\beta + \delta\bar{\beta} + \alpha\beta + \bar{\alpha}\bar{\beta}) - 4(k_2 - 4k_1)\left(\alpha\bar{\alpha} + \alpha\bar{\pi} + \bar{\alpha}\pi - \frac{1}{2}\Phi_{11}\right) \\ + 32k_1\pi\bar{\pi} - 2(5k_2 - 52k_1)\beta\bar{\beta} = 0, \quad (4.25)$$

where NP Eqs. (4.2) have been used. The following additional condition may be generated from the  $[\bar{\delta}, \delta]$  commutator applied to  $\alpha + 2\pi$ :

$$2(q_1\alpha + q_2\pi)\delta\bar{\beta} + (2q_1 - q_2)\bar{\beta}\delta\beta + \bar{\beta}[-2q_1\alpha\bar{\alpha} + (2q_1 - q_2)\beta\bar{\beta} + (4q_1 - q_2)\alpha\beta \\ - 2q_2\bar{\alpha}\pi + 2q_2\beta\pi] + 2[2\alpha + (q_1 - q_2)\bar{\beta} + 8\pi]\Phi_{11} = 0. \quad (4.26)$$

We proceed with the proof of the lemma by assuming

$$\alpha = 0. \quad (4.27)$$

Eliminating  $\bar{\delta}\beta$  and  $\delta\bar{\beta}$  from (4.26), using NP Eq. (4.21), we obtain

$$[2(q_2 - 8)\pi + q_2\bar{\beta}]\Phi_{11} = 0. \quad (4.28)$$

If  $\Phi_{11} = 0$ , Eq. (4.25) takes the form

$$(3k_2 - 20k_1)\beta\bar{\beta} - 8k_1\pi\bar{\pi} = 0. \quad (4.29)$$

This equation implies  $\beta = \pi = 0$ , for all three cases. On the other hand, if

$$\Phi_{11} \neq 0, \quad (4.30)$$

Eq. (4.28) implies

$$\beta = 2(8 - q_2)\bar{\pi}/q_2. \quad (4.31)$$

Comparison of (4.19) and (4.22) now yields  $(q_1 - 2)\pi^2 = 0$ , which implies  $\Phi_{11} = 0$ , by (4.20), contradicting (4.30).

We next assume

$$\beta = 0, \quad (4.32)$$

which by (4.26) implies

$$(\alpha + 4\pi)\Phi_{11} = 0. \quad (4.33)$$

If  $\Phi_{11} = 0$ , Eq. (4.25) may be written as

$$(k_2 - 4k_1)(\alpha + \pi)(\bar{\alpha} + \bar{\pi}) - (k_2 + 4k_1)\pi\bar{\pi} = 0, \quad (4.34)$$

which implies  $\alpha = \pi = 0$ , in each of the three cases. If  $\Phi_{11} \neq 0$ , Eq. (4.33) implies

$$\alpha = -4\pi. \quad (4.35)$$

Subtracting twice (4.20) from NP Eq. (4.21), we obtain  $\Phi_{11} = 0$ , which is a contradiction.

Finally, we assume

$$\pi = 0. \quad (4.36)$$

Solving (4.20) and NP Eq. (4.21) for  $\bar{\delta}\beta$ , we obtain

$$\bar{\delta}\beta = -\alpha\beta - \beta\bar{\beta} - 2\Phi_{11}. \quad (4.37)$$

The  $[\bar{\delta}, \delta]$  commutator of  $\beta$  now yields

$$\beta\Phi_{11} = 0. \quad (4.38)$$

The case  $\beta = 0$  has already been considered, so we assume  $\Phi_{11} = 0$ . It then follows that (4.25) may be written as

$$(k_2 - 4k_1)\alpha\bar{\alpha} + (3k_2 - 20k_1)\beta\bar{\beta} = 0. \quad (4.39)$$

This equation implies  $\alpha = \beta = 0$ , for all three cases and thus we conclude that  $\Phi_{11} = 0$ , and the proof of the lemma is complete.

We now proceed to the proof of Theorem 2. We first observe that by (3.29) the conditions (1.20), (1.21) and (1.22) may be expressed, respectively, as follows:

$$\bar{\delta}\Psi_3 + 2\alpha\Psi_3 = 0, \quad (4.40)$$

$$\delta\Psi_3 + 2\beta\Psi_3 = 0, \quad (4.41)$$

$$D\Psi_4 - 4\pi\Psi_3 + 4\epsilon\Psi_4 = 0, \quad (4.42)$$

in any dyad for which  $o_A$  satisfies (1.13) and (1.14). The above conditions are form invariant under a general dyad transformation (2.18) provided

that  $\Pi_{AA} = 0$ , and (3.24) is satisfied. In the canonical dyad for which (4.2) holds, the conditions (4.40) to (4.42) reduce to

$$\alpha = 0, \quad (4.43)$$

$$\beta = 0, \quad (4.44)$$

$$\pi = 0. \quad (4.45)$$

Invoking the Lemma, we conclude that the hypothesis of Theorem 2 implies

$$\alpha = \beta = \pi = 0. \quad (4.46)$$

It follows from NP Eqs. (4.2) that the D derivative of all the remaining spin coefficients except  $\nu$  vanish while the  $\delta$  and  $\bar{\delta}$  derivatives are given by

$$\delta\gamma = \Phi_{12}, \quad \bar{\delta}\gamma = -1 \quad (4.47)$$

$$\delta\lambda - \bar{\delta}\mu = 1 + \Phi_{21}. \quad (4.48)$$

From Eq. (4.21), we find

$$\delta\mu = -\Phi_{12}. \quad (4.49)$$

The  $o^A l^B \bar{c}^C \bar{d}^D \bar{c}^E \bar{d}^F$  contraction with V's yields

$$\bar{\delta}\mu = \frac{1}{2} + q_3 \Phi_{21}, \quad (4.50)$$

where

$$q_3 := \frac{3}{2}(k_2 - 4k_1)/(k_2 + 12k_1). \quad (4.51)$$

It then follows from (4.48) that

$$\delta\lambda = \frac{3}{2} + q_4 \Phi_{21}, \quad (4.52)$$

where

$$q_4 := \frac{1}{2}(5k_2 + 12k_1)/(k_2 + 12k_1). \quad (4.53)$$

It also follows from (4.46) that the Bianchi identities reduce to

$$D\Phi_{12} = \delta\Phi_{12} = \bar{\delta}\Phi_{12} = D\Phi_{22} = 0, \quad (4.54)$$

$$\bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 2(\gamma + 2\mu) + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21}. \quad (4.55)$$

The last condition from V's is given by

$$(k_2 - 4k_1) \left[ \Delta(\gamma + 2\mu) + (3\gamma + \bar{\gamma})(\gamma + 2\mu) + \frac{1}{2}\Phi_{22} \right] - 2k_2\bar{\gamma}(\gamma + 2\mu) - 2(8k_1 - k_2)(\gamma + 2\mu)(\bar{\gamma} + 2\bar{\mu}) - 4k_1(\lambda\bar{\lambda} + \mu\bar{\mu}) + c.c. = 0. \quad (4.56)$$

From this point on, it is more convenient to consider each case separately. We shall present the proof only in the scalar case. The proof in the two

remaining cases is essentially identical. The Eq. (4.56) with  $k_1 = 3$  and  $k_2 = 4$  reads

$$\Delta(\gamma + \bar{\gamma} + 2\mu + 2\bar{\mu}) + 3(\gamma^2 + \bar{\gamma}^2) + 14\gamma\bar{\gamma} + 24(\gamma\bar{\mu} + \bar{\gamma}\mu) + 6(\gamma\mu + \bar{\gamma}\bar{\mu}) + 43\mu\bar{\mu} + 3\lambda\bar{\lambda} + \Phi_{22} = 0. \quad (4.57)$$

By applying  $\delta$  to the above equation and subsequently employing the  $[\delta, \Delta]$  and  $[\bar{\delta}, \Delta]$  commutators of  $\gamma + 2\mu$  and the Eqs. (4.47) to (4.55), we obtain the following integrability condition:

$$30\lambda\delta\bar{\lambda} = 6\Delta\Phi_{12} + 20\gamma + 10\bar{\gamma} + 25\mu + 20\bar{\mu} - 45\bar{\lambda} + \Phi_{12}(56\gamma + 114\bar{\gamma} + 65\mu + 190\bar{\mu}) - 25\bar{\lambda}\Phi_{21}. \quad (4.58)$$

If  $\lambda = 0$ , Eq. (4.52) implies

$$\Phi_{12} = -15/7, \quad (4.59)$$

while (4.58) reduces to

$$70\gamma + 164\bar{\gamma} + 80\mu + 271\bar{\mu} = 0. \quad (4.60)$$

A  $\delta$  derivative of this equation yields an immediate contradiction.

We now consider the case  $\lambda \neq 0$ . This assumption allows the solution of (4.58) for  $\delta\bar{\lambda}$ . The next step is to compute the  $[\bar{\delta}, \delta]$  commutator of  $\lambda$ , and a further integrability condition obtained by taking  $\delta$  of this equation. Due to their length, these conditions are given in the Appendix as Eqs. (A.1) and (A.2). They were computed with the aid of the computer algebra system MAPLE [7] and the incorporated NP package [8]. When the quantities  $\Delta\Phi_{12}$  and  $\Delta\Phi_{21}$  are eliminated between (A.1) and (A.2), we obtain

$$(113\Phi_{12}\Phi_{21} + 55\Phi_{12} + 55\Phi_{21} + 75)(92447\Phi_{12}^2\Phi_{21}^2 + 160900\Phi_{12}^2\Phi_{21} + 160900\Phi_{12}\Phi_{21}^2 + 56925\Phi_{12}^2 + 56925\Phi_{21}^2 + 389100\Phi_{12}\Phi_{21} + 189000\Phi_{12} + 189000\Phi_{21} + 151875) = 0, \quad (4.61)$$

where it has been assumed that

$$\Phi_{12}(5 + 11\Phi_{12}) \neq 0, \quad (4.62)$$

essential for the elimination process. The case when (4.62) does not hold requires a separate treatment. The first factor of Eq. (4.61) which we shall denote by  $F$ , may be expressed as

$$F = (1/226^2) \{ 113 [113(\Phi_{12} + \Phi_{21}) + 110]^2 - 113^3(\Phi_{21} - \Phi_{12})^2 + 2463400 \}, \quad (4.63)$$

which is always greater than zero. The second factor,  $G$ , also has this property. To see this, we rewrite  $G$ , as follows, in terms of the real and imaginary parts of  $\Phi_{12}$ , denoted by  $x$  and  $y$  respectively:

$$G = 92447x^4 + 184894x^2y^2 + 92447y^4 + 321800x^3 + 321800xy^2 + 502950x^2 + 275250y^2 + 378000x + 151875. \quad (4.64)$$

Next, we determine the roots of the equation  $G = 0$ , for  $y^2$ . By an elementary analysis, it may be shown that these roots are both negative for all  $x$ . We conclude as a result of this contradiction that (4.62) is impossible. Thus, necessarily  $\Phi_{12}$  is constant.

Finally, we turn our attention to this case. The  $[\bar{\delta}, \delta]$  commutator applied to  $\bar{\lambda}$  yields, by virtue of (4.58), the equation

$$30\bar{\delta}\lambda\delta\bar{\lambda} + F = 0. \quad (4.65)$$

But this is also impossible since  $F$ , as shown in (4.63), is always greater than zero. This completes the proof of Theorem 2.

## 5. DISCUSSION

To complete the proof of the conjecture for Petrov type III requires consideration of the case when none of  $\alpha$ ,  $\beta$ , and  $\pi$  vanish. Our analysis of this case has not yet led to a definite result. By a study of Conditions III' and V' and their integrability conditions we have been able to show that  $\alpha$ ,  $\beta$  and  $\pi$  satisfy a system of at least three complex polynomial equations *not involving* the remaining spin coefficients  $\gamma$ ,  $\mu$ ,  $\lambda$  and  $\nu$ . This suggests that either an inconsistency occurs or that the system has at most finitely many solutions for  $\alpha$ ,  $\beta$  and  $\pi$ . In the latter case, it may be shown that these spin coefficients must all vanish, in which case the proof of the conjecture is completed by applying Theorem 2.

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## APPENDIX

$$\begin{aligned}
& 18\Delta\Phi_{12}\Delta\Phi_{21} + 2[30\gamma - 135\lambda + 75\bar{\mu} + (342\gamma + 168\bar{\gamma} + 195\bar{\mu} + 570\mu)\Phi_{21} - 75\lambda\Phi_{12} \\
& \quad + 60\mu + 60\bar{\gamma}]\Delta\Phi_{12} - 25(56\gamma\lambda + 190\lambda\bar{\mu} + 65\lambda\mu + 114\gamma\lambda)\Phi_{12}^2 + \frac{1}{2}(8066\gamma\bar{\gamma} \\
& \quad + 25300\gamma\bar{\mu} + 18050\gamma\mu + 6384\gamma^2 + 12350\mu^2 + c.c. + 40325\bar{\mu}\bar{\mu} + 4015\lambda\bar{\lambda})\Phi_{12}\Phi_{21} \\
& \quad + 5(354\gamma\mu + 452\gamma\bar{\gamma} - 1076\bar{\gamma}\lambda + 660\gamma\bar{\mu} + 1330\bar{\gamma}\bar{\mu} + 112\gamma^2 + 456\bar{\gamma}^2 + 760\bar{\gamma}\mu \\
& \quad - 604\gamma\lambda + 555\lambda\bar{\lambda} - 710\lambda\mu - 1810\lambda\bar{\mu} + 260\mu^2 + 1085\bar{\mu}\bar{\mu} + 950\bar{\mu}^2)\Phi_{12} \\
& \quad + \left(100\gamma^2 + 325\gamma\mu + 350\gamma\bar{\mu} - 450\gamma\lambda - 325\gamma\bar{\lambda} + 125\gamma\bar{\gamma} + 250\mu^2 - \frac{1125}{2}\mu\lambda \right. \\
& \quad \left. - 450\mu\bar{\lambda} + c.c. + \frac{1025}{2}\mu\bar{\mu} + \frac{4275}{2}\lambda\bar{\lambda}\right) + c.c. = 0. \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
& (324 + 180\Phi_{21})\Delta\Phi_{12}\Delta\Phi_{21} + (540\gamma - 2430\lambda + 1350\bar{\mu} + 1080\bar{\gamma} + 1080\mu \\
& \quad + [4260\bar{\mu} + 10860\mu + 6456\gamma + 3424\bar{\gamma} - 1350\lambda]\Phi_{21} - 1350\lambda\Phi_{12} \\
& \quad + [1950\bar{\mu} + 3420\gamma + 1680\bar{\gamma} + 5700\mu]\Phi_{21}^2 - 750\lambda\Phi_{12}\Phi_{21})\Delta\Phi_{12} \\
& \quad + (1080\bar{\mu} + 1080\gamma - 2430\bar{\lambda} + 540\bar{\gamma} + 1350\mu + [3024\gamma + 6156\bar{\gamma} + 3510\mu \\
& \quad + 10260\bar{\mu} + 1080\lambda]\Phi_{12} + [600\gamma + 300\bar{\gamma} + 750\mu + 600\bar{\mu} - 2700\bar{\lambda}]\Phi_{21} \\
& \quad + [1950\mu + 5700\bar{\mu} + 1680\gamma + 3420\bar{\gamma}]\Phi_{12}\Phi_{21} + 2376\lambda\Phi_{12}^2 - 750\bar{\lambda}\Phi_{21}^2)\Delta\Phi_{21} \\
& \quad - 9900\lambda^2\Phi_{12}^3 + (38144\gamma + 7926\bar{\gamma} + 67115\mu + 1990\bar{\mu})\lambda\Phi_{12}^2\Phi_{21} + (126500\gamma\bar{\mu} \\
& \quad + 90250\gamma\mu + 40330\gamma\bar{\gamma} + 61750\mu^2 + 31920\gamma^2 + c.c. - 11113\lambda\bar{\lambda} + 201625\mu\bar{\mu})\Phi_{12}\Phi_{21}^2 \\
& \quad - (14250\gamma + 7000\bar{\gamma} + 23750\mu + 8125\bar{\mu})\bar{\lambda}\Phi_{21}^3 - (8640\gamma + 17730\bar{\gamma} + 6705\mu \\
& \quad + 32850\bar{\mu} + 22320\lambda)\lambda\Phi_{12}^2 + (27125\mu\bar{\mu} + 16500\mu\bar{\gamma} + 17900\bar{\mu}\gamma - 52550\gamma\bar{\lambda} - 27700\gamma\bar{\lambda} \\
& \quad + 33250\mu\gamma - 88000\mu\bar{\lambda} - 32375\bar{\mu}\bar{\lambda} + 8850\bar{\mu}\bar{\gamma} - 1305\lambda\bar{\lambda} + 11300\gamma\bar{\gamma} + 23750\mu^2 \\
& \quad + 6500\bar{\mu}^2 + 11400\gamma^2 + 2800\bar{\gamma}^2)\Phi_{21}^2 + (16450\mu\lambda - 33550\bar{\mu}\lambda + 244200\gamma\bar{\mu} - 26190\lambda\bar{\lambda} \\
& \quad + 195700\gamma\bar{\mu} + 5420\gamma\lambda + 156488\gamma\bar{\gamma} + 171300\gamma\mu - 16820\bar{\gamma}\lambda + 245600\mu\bar{\gamma} + 390050\mu\bar{\mu} \\
& \quad + 60256\gamma^2 + 117650\mu^2 + 134900\bar{\mu}^2 + 68856\bar{\gamma}^2)\Phi_{12}\Phi_{21} + (29700\gamma\bar{\mu} - 76950\bar{\mu}\lambda \\
& \quad + 32220\mu\bar{\gamma} - 28350\mu\lambda + 15930\gamma\mu + 59850\bar{\gamma}\bar{\mu} - 44820\bar{\gamma}\lambda + 20340\gamma\bar{\gamma} - 4725\lambda\bar{\lambda} \\
& \quad - 25380\gamma\lambda + 48825\mu\bar{\mu} + 11700\mu^2 + 5040\gamma^2 + 20520\bar{\gamma}^2 + 42750\bar{\mu}^2 - 8100\lambda^2)\Phi_{12} \\
& \quad + (22840\gamma\bar{\gamma} - 4500\bar{\mu}\lambda - 5625\mu\lambda + 35720\gamma\bar{\mu} - 31680\bar{\gamma}\lambda - 4050\lambda\bar{\lambda} + 19180\gamma\bar{\mu} \\
& \quad - 85950\mu\bar{\lambda} - 4500\gamma\lambda + 63100\gamma\mu - 2250\bar{\gamma}\lambda + 33200\mu\bar{\gamma} - 37575\bar{\mu}\bar{\lambda} + 53950\mu\bar{\mu} \\
& \quad - 50670\gamma\bar{\lambda} + 21520\gamma^2 + 45250\mu^2 + 14200\bar{\mu}^2 + 6040\bar{\gamma}^2)\Phi_{21} + (5850\gamma\mu + 6300\bar{\gamma}\mu \\
& \quad - 4050\bar{\gamma}\lambda - 8100\gamma\lambda - 10125\mu\lambda - 8100\mu\bar{\lambda} + 4500\mu^2 + 1800\gamma^2 + c.c. \\
& \quad + 4500\gamma\bar{\gamma} + 9225\mu\bar{\mu} - 2025\lambda\bar{\lambda}) = 0. \tag{A.2}
\end{aligned}$$

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