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An explicit determination of the space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle. — Part II: Petrov type D space-times

by

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ABSTRACT. — It is shown that there exist no Petrov type D space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle. Some related results concerning Maxwell's equations and Weyl's neutrino equation are also given.

RÉSUMÉ. — On démontre qu'il n'existe aucun espace-temps de type D de Petrov sur lequel l'équation invariante conforme des ondes scalaires satisfait au principe d'Huygens. On donne aussi quelques résultats de nature analogue pour les équations de Maxwell et pour l'équation de neutrino de Weyl.

1. INTRODUCTION

This paper is the second of a series devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation,

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Maxwell's equations and Weyl's neutrino equation. These equations may be written respectively as

$$\square u + \frac{1}{6} \mathrm{R} u = 0\,, \tag{1.1}$$

$$d\omega = 0\,, \qquad \delta\omega = 0\,, \tag{1.2}$$

$$\nabla_{\dot{\mathrm{A}}}{}^{\mathrm{B}} \phi_{\mathrm{B}} = 0\,, \tag{1.3}$$

where $\square$ denotes the Laplace Beltrami operator corresponding to the metric g_{ab} of the space-time V_4, u the unknown scalar function, R the curvature scalar, d the exterior derivative, δ the exterior co-derivative, ω the Maxwell 2-form, $\nabla_{\dot{\mathrm{A}}}{}^{\mathrm{B}}$ the covariant derivative on 2-spinors, and ϕ_{A} a valence 1-spinor. Our conventions are those of McLenaghan [*17*]. All considerations in this paper are entirely local.

According to Hadamard [*13*] *Huygens' principle* (in the strict sense) is valid for equation (1.1) if and only if for every Cauchy initial value problem and every $x_0 \varepsilon \mathrm{V}_4$, the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of $\mathrm{S} \cap \mathrm{C}^-(x_0)$ where S denotes the initial surface and $\mathrm{C}^-(x_0)$ denotes the past null conoid from x_0. Analogous definitions of the validity of the principle for Maxwell's equations (1.2) and Weyl's equation (1.3) have been given by Günther [*11*] and Wünsch [*27*] respectively in terms of the appropriate formulations of the initial value problems for these equations. *Hadamard's problem* for the equations (1.1), (1.2) or (1.3), originally posed only for scalar equations, is that of determining *all* space-times for which Huygens' principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens' principle, the determination may only be effected up to an arbitrary conformal transformation of the metric on V_4

$$\tilde{g}_{ab} = e^{2\phi} g_{ab}\,, \tag{1.4}$$

where ϕ is an arbitrary function.

Huygens' principle is valid for (1.1), (1.2) and (1.3) on any conformally flat space-time and also on any space-time conformally related to the exact plane wave space time [*10*] [*14*] [*28*], the metric of which has the form

$$ds^2 = 2dv \, \{ \, du + [\mathrm{D}(v)z^2 + \overline{\mathrm{D}}(v)\bar{z}^2 + e(v)z\bar{z}]dv \, \} - 2dz d\bar{z}\,, \tag{1.5}$$

in a special co-ordinate system, where D and e are arbitrary functions. These are the only *known* space-times on which Huygens' principle is valid for these equations. Furthermore, it has been shown [*15*] [*12*] [*28*] that these are the only conformally empty space-times on which Huygens' principle is valid. In the non-conformally empty case some further results have been obtained under various additional hypotheses, a review of which is given by one of us [*18*].

More recently, the authors have outlined a program [*2*] for the solution of Hadamard's problem based on the conformally invariant Petrov classification [22] [8], of the Weyl conformal curvature tensor. This involves the consideration of five disjoint cases which exhaust all the possibilities for non-conformally flat space-times. As a first stage in the implementation of this program, the case of Petrov type N (the most degenerate) was considered, where we have proved the following theorem: *Every Petrov type* N *space-time on which the conformally invariant scalar wave equation (1.1) satisfies Huygens' principle is conformally related to an exact plane wave space-time (1.5)*, [3] [4] (denoted by CM in the sequel). This result together with Günther's [10] solves Hadamard's problem in this case.

The proof of the above theorem was obtained by first solving the following sequence of necessary conditions for the validity of Huygens' principle for the equations (1.1), (1.2) and (1.3) [9] [24] [17] [16] [15] [27]:

III′ $$S_{abk;}{}^{k} - \frac{1}{2} C^{k}{}_{ab}{}^{l} L_{kl} = 0\,, \tag{1.6}$$

V′ $$\begin{aligned} \mathrm{TS}\big[&k_1 C^{k}{}_{ab}{}^{l}{}_{;}{}^{m} C_{kcdl;m} + 2k_2 C^{k}{}_{ab}{}^{l}{}_{;c} S_{kld} + 2(8k_1 - k_2) S_{ab}{}^{k} S_{cdk} \\ &- 2k_2 C^{k}{}_{ab}{}^{l} S_{klc;d} - 8k_1 C^{k}{}_{ab}{}^{l} S_{cdk;l} + k_2 C^{k}{}_{ab}{}^{l} C_{l}{}^{m}{}_{ck} L_{dm} + 4k_1 C^{k}{}_{ab}{}^{l} C^{m}{}_{cdl} L_{km}\big] = 0\,, \end{aligned} \tag{1.7}$$

where $$C_{abcd} := R_{abcd} - 2g_{[a[d}L_{b]c]}\,, \tag{1.8}$$

$$L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab} \tag{1.9}$$

$$S_{abc} := L_{a[b;c]}\,. \tag{1.10}$$

In the above C_{abcd} denotes the Weyl tensor, R_{ab} the Ricci tensor and TS [ ] the operator which takes the trace free symmetric part of the enclosed tensor. The quantities k_1 and k_2 appearing in Eq. (1.7) are constants whose values are given in the following table:

TABLE 1.

Equation	k_1	k_2
Scalar	3	4
Maxwell	5	16
Weyl	8	13

The final step in the proof involved the imposition of a further necessary Condition VII, valid for the scalar case, derived by Rinke and Wünsch [23]. However, it should be noted that Hadamard's problem still remains open for Maxwell's equations and Weyl's equation. The general solutions of Conditions III′ and V′ have been obtained for these equations [4], but it is not known whether Huygens' principle is actually satisfied on the resulting space-times other than the conformally plane wave space-times. The derivation of the analogue of Condition VII for these equations might settle the question, as it did in the scalar case.

Our analysis has now been extended to include the case of Petrov type D space-times. We recall that such space-times are characterized by the existence of pointwise linearly independent null vector fields l and n satisfying the following equations [8]:

$$C_{abc[d}l_{e]}l^b l^c = C_{abc[d}n_{e]}n^b n^c = 0\,. \tag{1.11}$$

The main results of this paper are contained in the following theorems:

THEOREM 1. — *The validity of Huygens' principle for the conformally invariant scalar wave equation (1.1), or Maxwell's equation (1.2), or Weyl's neutrino equation (1.3) on a Petrov type* D *space-time implies that both principal null congruences of the Weyl tensor are geodesic and shear-free, that is*

$$l_{a;b}l^b = f\,l_a, \qquad n_{a;b}n^b = g n_a\,, \tag{1.12}$$

$$l_{(a;b)}l^{a}{}_{;}{}^{b} = \frac{1}{2}(l_{a;}{}^{a})^2, \qquad n_{(a;b)}n^{a}{}_{;}{}^{b} = \frac{1}{2}(n_{a;}{}^{a})^2\,. \tag{1.13}$$

THEOREM 2. — *The validity of Huygens' principle for the conformally invariant scalar wave equation (1.1), or Maxwell's equations (1.2), or Weyl's equation (1.3) on a Petrov type* D *space-time satisfying*

$$C_{abcd} * C^{abcd} = 0\,, \tag{1.14}$$

implies that both principal null congruences of the Weyl tensor are hypersurface orthogonal, that is

$$l_{[a;b}l_{c]} = 0\,, \qquad n_{[a;b}n_{c]} = 0\,, \tag{1.15}$$

THEOREM 3. — *The validity of Huygens' principle for the conformally invariant scalar wave equation (1.1) on a Petrov type* D *space-time implies that both principal null congruences of the Weyl tensor are hypersurface orthogonal.*

THEOREM 4. — *There are no space-times of Petrov* D *where both principal null congruences of the Weyl tensor are hypersurface orthogonal, on which the conformally invariant scalar wave equation (1.1), or Maxwell's equations (1.2), or Weyl's equation (1.3) satisfy Huygen's principle.*

As a consequence of Theorems 1, 2, and 4, we obtain the following theorem, which solves Hadamard's problem for the equations (1.1), (1.2) and (1.3) on type D space-times satisfying (1.14):

THEOREM 5. — *There exist no Petrov type* D *space-times satisfying (1.14) on which the conformally invariant scalar wave equation (1.1) or Max-*

well's equations (1.2) or Weyl's equation (1.3) satisfies Huygens' principle.

In the case of the conformally invariant scalar wave equation (1.1), Theorems 1, 3 and 4 imply the stronger result, stated without proof in [*5*], which solves Hadamard's problem for this equation on type D space-times.

THEOREM 6. — *There exist no Petrov type* D *space-times on which the conformally invariant scalar wave equation (1.1) satisfies Huygens' principle.*

It is worth noting that Conditions III′ and V′ were sufficient to establish Theorems 1, 2, 4, and 5. However, the proofs of Theorems 3 and 5 depend also on Condition VII. A deeper analysis of Conditions III′ and V′ might permit the removal of the restriction (1.14) thereby completing the solution of Hadamard's problem for the Eqs. (1.2) and (1.3) in Petrov type D. Alternatively additional necessary conditions for (1.2) and (1.3) analogous to Condition VII for the scalar equation may be required, as they apparently are in the case of Petrov type N.

The results obtained thus far for the Petrov type N and type D cases lend weight to the conjecture that every space-time on which the conformally invariant scalar wave equation satisfies Huygens' principle is conformally related to the plane wave space-time (1.5) or is conformally flat [*2*] [*3*] [*4*]. The above theorems include Wünsch's result [*28*] that the validity of Huygens' principle for any one of the Eqs. (1.1), (1.2) or (1.3) on a 2×2-decomposable space-time implies that the space-time is conformally flat, since any such space-time is necessarily complex recurrent of Petrov type D or conformally flat [*16*].

The plan of the remainder of the paper is as follows. In Section 2, the formalisms used are briefly described. The proofs of the theorems are given in Section 3.

2. FORMALISMS

We use the two-component spinor formalism of Penrose [*20*] [*22*] and the spin coefficient formalism of Newman and Penrose (NP) [*19*] whose conventions we follow. In the spinor formalism, tensor and spinor indices are related by the complex connection quantities $\sigma_a{}^{A\dot{A}}(a = 1, \ldots, 4 ; A = 0, 1)$ which are Hermitian in the spinor indices $A\dot{A}$. Spinor indices are lowered by the skew symmetric spinors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ defined by $\varepsilon_{01} = \varepsilon_{\dot{0}\dot{1}} = 1$, according to the convention

$$\xi_A = \xi^B \varepsilon_{BA}, \tag{2.1}$$

where ξ_A is an arbitrary1-spinor. Spinor indices are raised by the respective inverses of these spinors denoted by ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$. The spinor equivalents of the Weyl tensor (1.8) and the tensor L_{ab} defined by (1.9) are given respectively by

$$C_{abcd}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}}\sigma^c{}_{C\dot{C}}\sigma^d{}_{D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\varepsilon_{AB}\varepsilon_{DC}, \tag{2.2}$$

$$L_{ab}\sigma^a{}_{A\dot{A}}\sigma^b{}_{B\dot{B}} = 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}), \tag{2.3}$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$ denotes the Weyl spinor, where $\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})}$ denotes the Hermitian trace-free Ricci spinor and where

$$\Lambda = (1/24)R. \tag{2.4}$$

The covariant derivative of spinors is denoted by « ; » and satisfies

$$\sigma_a{}^{A\dot{A}}{}_{;b} = \varepsilon_{AB;b} = 0. \tag{2.5}$$

It will be necessary in the sequel to express spinor equations in terms of a spinor dyad $\{o_A, \iota_A\}$ satisfying the completeness relation

$$o_A\iota^A = 1. \tag{2.6}$$

Associated to the spinor dyad is a null tetrad $\{l, n, m, \overline{m}\}$ defined by

$$l^a = \sigma^a{}_{A\dot{A}}o^A\bar{o}^{\dot{A}}, \qquad n^a = \sigma^a{}_{A\dot{A}}\iota^A\bar{\iota}^{\dot{A}}, \qquad m^a = \sigma^a{}_{A\dot{A}}o^A\bar{\iota}^{\dot{A}}, \tag{2.7}$$

whose only non-zero inner products are

$$l_a n^a = -m_a\overline{m}^a = 1. \tag{2.8}$$

The metric tensor may be expressed in terms of the null tetrad by

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\overline{m}_{b)}. \tag{2.9}$$

The NP spin coefficients associated with the dyad are defined by the equations

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}} + \iota_A II_{B\dot{B}}, \tag{2.10}$$

$$\iota_{A;B\dot{B}} = o_A III_{B\dot{B}} - \iota_A I_{B\dot{B}}, \tag{2.11}$$

where

$$I_{B\dot{B}} := \gamma o_B\bar{o}_{\dot{B}} - \alpha o_B\bar{\iota}_{\dot{B}} - \beta\iota_B\bar{o}_{\dot{B}} + \varepsilon\iota_B\bar{\iota}_{\dot{B}}, \tag{2.12}$$

$$II_{B\dot{B}} := -\tau o_B\bar{o}_{\dot{B}} + \rho o_B\bar{\iota}_{\dot{B}} + \sigma\iota_B\bar{o}_{\dot{B}} - \kappa\iota_B\bar{\iota}_{\dot{B}}, \tag{2.13}$$

$$III_{B\dot{B}} := \nu o_B\bar{o}_{\dot{B}} - \lambda o_B\bar{\iota}_{\dot{B}} - \mu\iota_B\bar{o}_{\dot{B}} + \pi\iota_B\bar{\iota}_{\dot{B}}. \tag{2.14}$$

The NP components of the Weyl spinor and trace-free Ricci spinor are defined respectively by

$$\begin{aligned} &\Psi_0 := \Psi_{ABCD}o^{ABCD}, && \Psi_1 := \Psi_{ABCD}o^{ABC}\iota^D, \\ &\Psi_2 := \Psi_{ABCD}o^{AB}\iota^{CD}, && \Psi_3 := \Psi_{ABCD}o^A\iota^{BCD}, \\ &\Psi_4 := \Psi_{ABCD}\iota^{ABCD}, && \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\Phi_{00} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{o}^{\dot{A}\dot{B}}, && \Phi_{01} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}, \\ &\Phi_{02} := \Phi_{AB\dot{A}\dot{B}}o^{AB}\bar{\iota}^{\dot{A}\dot{B}}, && \Phi_{11} := \Phi_{AB\dot{A}\dot{B}}o^A\iota^B\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}, \\ &\Phi_{12} := \Phi_{AB\dot{A}\dot{B}}o^A\iota^B\bar{\iota}^{\dot{A}\dot{B}}, && \Phi_{22} := \Phi_{AB\dot{A}\dot{B}}\iota^{AB}\bar{\iota}^{\dot{A}\dot{B}}, \end{aligned} \tag{2.16}$$

where the notation $o_{A_1 \ldots A_p} := o_{A_1} \ldots o_{A_p}$, etc. has been used. The NP differential operators are defined by

$$D := l^a \frac{\partial}{\partial x^a}, \qquad \Delta := n^a \frac{\partial}{\partial x^a}, \qquad \delta := m^a \frac{\partial}{\partial x^a}. \tag{2.17}$$

The equations relating the curvature components to the spin coefficients, and the commutation relations satisfied by the above differential operators may be found in NP.

The subgroup of the proper orthochronous Lorentz group $L_+^\uparrow$ preserving the directions of the vectors l and n is given by

$$l' = e^a l, \qquad n' = e^{-a} n, \qquad m' = e^{ib} m, \tag{2.18}$$

where a and b are real-valued functions. The corresponding transformation of the spinor dyad is given by

$$o' = e^{w/2} o, \qquad \iota' = e^{-w/2} \iota, \tag{2.19}$$

where $w = a + ib$. These transformations induce transformations of the spin coefficients and curvature components which will be needed later.

The following discrete transformation of the dyad preserving (2.6) is also important

$$\hat{o} = -\iota, \qquad \hat{\iota} = o. \tag{2.20}$$

This transformation induces the following transformation of the NP operators, spin coefficients and curvature components

$$\hat{D} = \Delta, \qquad \hat{\Delta} = D, \qquad \hat{\delta} = -\bar{\delta}, \tag{2.21}$$

$$\hat{\gamma} = -\varepsilon, \qquad \hat{\alpha} = \beta, \qquad \hat{\beta} = \alpha, \qquad \hat{\varepsilon} = -\gamma, \tag{2.22}$$

$$\left.\begin{array}{llll} \hat{\tau} = \pi, & \hat{\rho} = -\mu, & \hat{\sigma} = -\lambda, & \hat{\kappa} = \nu, \\ \hat{\pi} = \tau, & \hat{\mu} = -\rho, & \hat{\lambda} = -\sigma, & \hat{\nu} = \kappa, \end{array}\right\} \tag{2.23}$$

$$\hat{\Psi}_0 = \Psi_4, \quad \hat{\Psi}_1 = -\Psi_3, \quad \hat{\Psi}_2 = \Psi_2, \quad \hat{\Psi}_3 = -\Psi_1, \quad \hat{\Psi}_4 = \Psi_0. \tag{2.24}$$

$$\left.\begin{array}{lll} \hat{\Phi}_{00} = \Phi_{22}, & \hat{\Phi}_{01} = -\Phi_{21}, & \hat{\Phi}_{02} = \Phi_{20}, \\ \hat{\Phi}_{11} = \Phi_{11}, & \hat{\Phi}_{12} = -\Phi_{10}, & \hat{\Phi}_{22} = \Phi_{00}. \end{array}\right\} \tag{2.25}$$

We also shall need the following transformation of the null tetrad

$$\tilde{l}_a = e^{\Phi} l_a, \qquad \tilde{n}_a = e^{\Phi} n_a, \qquad \tilde{m}_a = e^{\Phi} m_a, \tag{2.26}$$

which induces via (2.9), the conformal transformation of the metric (1.4).

3. PROOF OF THEOREMS

Recall from CM that the spinor form of the conditions (1.6) and (1.7) are given by

$$\text{III's}\quad \Psi_{ABKL;}{}^{K\dot{A}L\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L};}{}^{\dot{K}A\dot{L}B} + \Psi_{AB}{}^{KL}\Phi_{KL\dot{A}\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\overline{\Phi}_{\dot{K}\dot{L}AB}=0, \tag{3.1}$$

$$\begin{aligned}\text{V's}\quad & k_1\Psi_{ABCD;K\dot{K}}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D};}{}^{\dot{K}K} + k_2\Psi^{K}{}_{(ABC;D)(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{L};}{}^{\dot{L}}{}_{K} \\ & + k_2\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})(A}\Psi_{BCD)L;}{}^{L}{}_{\dot{K}} - 2(8k_1-k_2)\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K};}{}^{\dot{K}}{}_{D)} \\ & - k_2\Psi^{K}{}_{(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{L}|;}{}^{\dot{L}}{}_{|K|D)\dot{D})} - k_2\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|L|;}{}^{L}{}_{|\dot{K}|D)\dot{D})} \\ & + 4k_1\Psi^{K}{}_{(ABC}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|;}{}^{\dot{L}}{}_{D)K\dot{D})} + 4k_1\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|L|;}{}^{L}{}_{\dot{D})D)\dot{K}} \\ & + 2(k_2-4k_1)\Psi^{K}{}_{(ABC}\Phi_{D)K\dot{K}(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})}{}^{\dot{K}} - 2(4k_1+k_2)\Lambda\Psi_{ABCD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}=0.\end{aligned} \tag{3.2}$$

We now make the hypothesis that the space-time is of Petrov type D. These space-times are characterised by the existence of null vectors l and n satisfying Eq. (1.11). In terms of spinors, this is equivalent to the existence of two 1-spinors o_A and ι_A satisfying Eq. (2.6) such that

$$\Psi_{ABCD} = 6\Psi\iota_{(AB}o_{CD)}\,. \tag{3.3}$$

Selecting $\{o_A, \iota_A\}$ as the spinor dyad, it follows from (2.15) and (3.3) that $\Psi_2 := \Psi$ is the only non-vanishing NP component of the Weyl spinor. It should be noted that Ψ_2 is invariant under the continuous transformation (2.19) and the discrete transformation (2.20). However, the conformal transformation (2.26) induces the transformation

$$\tilde{\Psi} = e^{-2\phi}\Psi\,. \tag{3.4}$$

We proceed by substituting for Ψ_{ABCD} in Eqs. (3.1) and (3.2) from (3.3). The covariant derivatives of o_A and ι_A that appear are eliminated using Eqs. (2.10) and (2.11) respectively. The dyad form of the resulting equations is obtained by contracting them with appropriate products of o^A and ι^A and their complex conjugates. In view of the conformal invariance of conditions III's and V's [*17*] [*26*], it follows that each dyad equation must be individually invariant under the conformal transformations (2.26). The first contraction to consider is $o^{ABCD}\bar{o}^{\dot{A}\dot{B}\dot{C}\dot{D}}$ with Condition V's which yields the equation

$$k_1\,|\,\kappa\Psi\,|^2 = 0\,. \tag{3.5}$$

This implies

$$\kappa = 0\,, \tag{3.6}$$

since $k_1\Psi \neq 0$, by assumption. The result of the $\iota^{ABCD}\bar{\iota}^{\dot{A}\dot{B}\dot{C}\dot{D}}$ contraction with Condition V's, may be obtained by the application of the discrete transformation (2.20) to equation (3.6), which yields

$$\nu = 0\,. \tag{3.7}$$

The conditions (3.6) and (3.7), which are invariant under the transformations (2.18) and (2.26), imply that the principal null congruences of C_{abcd} defined by the principal null vector fields l^a and n^a are *geodesic*, which is equivalent to the conditions (1.12).

The next contractions to consider are $o^{ABCD}\bar{o}^{\dot{A}\dot{B}}\bar{l}^{\dot{C}\dot{D}}$ and $o^{ABC}{}_{l}{}^{D}\bar{o}^{\dot{A}\dot{B}\dot{C}}\bar{l}^{\dot{D}}$ with V's which yield, respectively,

$$\sigma[2(k_2-8k_1)\mathrm{D}\overline{\mathrm{H}}+(52k_1+k_2)\bar{\rho}]-(k_2+4k_1)[\mathrm{D}\sigma+\sigma(3\rho+\bar{\varepsilon}-3\varepsilon)]=0, \quad (3.8)$$

$$\begin{aligned}&8(3k_2-20k_1)\rho\bar{\rho}-16k_1\mathrm{DHD}\overline{\mathrm{H}}+4(12k_1-k_2)(\bar{\rho}\mathrm{DH}+\rho\mathrm{D}\overline{\mathrm{H}})\\&\quad+(k_2-4k_1)[\mathrm{D}^2\mathrm{H}+\mathrm{DH}^2-(2\rho+\varepsilon+\bar{\varepsilon})\mathrm{DH}-3\mathrm{D}\rho-3\rho(\rho-\varepsilon-\bar{\varepsilon})+9\sigma\bar{\sigma}\\&\qquad+\Phi_{00}+\text{c. c.}]=0, \quad (3.9)\end{aligned}$$

where

$$\mathrm{H} := \ln\Psi, \quad (3.10)$$

and « c. c. » denotes the complex conjugate of the preceeding terms. Now NP Eq. (4.2 *b*) gives with $\kappa = 0$,

$$\mathrm{D}\sigma = \sigma(\rho + \bar{\rho} + 3\varepsilon - \bar{\varepsilon}), \quad (3.11)$$

so that upon eliminating $\mathrm{D}\sigma$ from (3.8), we obtain

$$\sigma[(k_2 - 8k_1)\mathrm{D}\overline{\mathrm{H}} + 24k_1\bar{\rho} - 2(4k_1 + k_2)\rho] = 0. \quad (3.12)$$

If we assume $\sigma \neq 0$, Eq. (3.12) implies

$$(k_2 - 8k_1)\mathrm{D}\overline{\mathrm{H}} + 24k_1\bar{\rho} - 2(4k_1 + k_2)\rho = 0. \quad (3.13)$$

At this stage, it will prove convenient to use the conformal freedom to set

$$\Psi\overline{\Psi} = 1, \quad (3.14)$$

which by (3.10), is equivalent to

$$\mathrm{H} + \overline{\mathrm{H}} = 0. \quad (3.15)$$

The real part of Eq. (3.13) now becomes

$$(8k_1 - k_2)(\rho + \bar{\rho}) = 0, \quad (3.16)$$

which implies

$$\rho + \bar{\rho} = 0, \quad (3.17)$$

since from Table 1, $k_2 \neq 8k_1$. The above result implies that Eq. (3.13) may be written as

$$\mathrm{DH} = c\rho, \quad (3.18)$$

where

$$c := (32k_1 + 2k_2)/(8k_1 - k_2) > 0. \quad (3.19)$$

The relation (3.17) and the NP Eq. (4.2*a*) also imply that

$$\mathrm{D}\rho = \rho(\varepsilon + \bar{\varepsilon}), \quad (3.20)$$

$$\Phi_{00} = -\rho^2 - \sigma\bar{\sigma}. \quad (3.21)$$

Finally, by virtue of Eqs. (3.15), (3.17), (3.18), (3.20) and (3.21), the Eq. (3.9) may be written as

$$[12k_1-k_2+2(k_2-2k_1)(c+1)(c-3)+(8k_1-k_2)(c-3)^2]\rho^2 \\ +4(k_2-4k_1)\sigma\bar{\sigma}=0. \quad (3.22)$$

It follows from Table 1 that the coefficient of ρ^2 is positive in all cases while the coefficient of $\sigma\bar{\sigma}$ is correspondingly negative. Consequently Eq. (3.22) implies $\rho = \sigma = 0$, which contradicts the assumption $\sigma \neq 0$. We conclude that Eqs. (3.8) and (3.9) together with NP Eqs. (4.2a) and (4.2b), imply that

$$\sigma = 0. \quad (3.23)$$

It may be shown in an identical manner that the equations arising from the contractions $\iota^{ABCD}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$ and $o^{A}{}_{}\iota^{BCD}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ with V's which may be obtained by applying the transformation (2.20) to Eqs. (3.8) and (3.9), imply that

$$\lambda = 0. \quad (3.24)$$

The conditions (3.23) and (3.24), which are invariant under the transformations (2.18) and (2.26) imply that the principal null congruences defined by l^a and n^a, respectively, are *shear-free*. This is equivalent to the conditions (1.13). The proof of Theorem 1 is now complete.

We now proceed to prove Theorem 2 and 3, which requires the use of Eq. (3.9). After the elimination of $D\rho$ by means of NP Eq. (4.2a) it takes the following form in the scalar case:

$$D^2H+DH^2-(2\rho+\varepsilon+\bar{\varepsilon})DH-6\rho^2-2\Phi_{00}+24\rho\bar{\rho} \\ +3DHD\bar{H}-8(\bar{\rho}DH+\rho D\bar{H})+\text{c. c.}=0. \quad (3.25)$$

We also need the equation resulting from the $\iota^{AB}\bar{\iota}^{\dot{A}\dot{B}}$ contraction with Condition III, which after the elimination of $D\rho$ reads

$$D^2\Psi - (6\rho + \varepsilon + \bar{\varepsilon})D\Psi + 2(3\rho^2 - \Phi_{00})\Psi + \text{c. c.} = 0. \quad (3.26)$$

It should be noted that the form of Eqs. (3.25) and (3.26) are invariant under a general tetrad transformation (2.18) and a conformal transformation (2.26), which induce the following transformations on the spin coefficients :

$$\rho'=e^a\rho,\ \varepsilon'=e^a\left(\varepsilon+\frac{1}{2}Da+\frac{1}{2}iDb\right),\ \mu'=e^{-a}\mu,\ \tau'=e^{ib}\tau,\ \pi'=e^{-ib}\pi, \quad (3.27)$$

$$\tilde{\rho} = e^{-\phi}(\rho - D\phi), \qquad \tilde{\mu} = e^{-\phi}(\mu - \Delta\phi). \quad (3.28)$$

We now assume that the principal null congruence defined by the vector field l is *not* hypersurface orthogonal. This is equivalent to the inequality

$$\rho \neq \bar{\rho}. \quad (3.29)$$

We use the tetrad transformation (2.18) and conformal transformation (2.26) to set

$$\rho = i. \tag{3.30}$$

It follows from NP Eq. (4.2*a*) that

$$\varepsilon + \bar{\varepsilon} = 0, \tag{3.31}$$

$$\Phi_{00} = 1. \tag{3.32}$$

We use some of the remaining freedom in (2.18) to set

$$\varepsilon = 0. \tag{3.33}$$

It follows from the above that the Eqs. (3.26) and (3.25) take the form

$$D^2(\Psi + \bar{\Psi}) - 6iD(\Psi - \bar{\Psi}) - 8(\Psi + \bar{\Psi}) = 0, \tag{3.34}$$

$$\frac{D^2\Psi}{\Psi} + \frac{D^2\bar{\Psi}}{\bar{\Psi}} + 6\frac{D\Psi}{\Psi}\frac{D\bar{\Psi}}{\bar{\Psi}} - 14i\left(\frac{D\Psi}{\Psi} - \frac{D\bar{\Psi}}{\bar{\Psi}}\right) + 56 = 0. \tag{3.35}$$

A first consequence of these equations is the inequality

$$\Psi^2 \neq \bar{\Psi}^2. \tag{3.36}$$

If this inequality does not hold, Eqs. (3.34) and (3.35) imply that

$$(D\Psi/\Psi)^2 = -12, \qquad \text{or} \quad D\Psi = 0, \tag{3.37}$$

both of which are impossible. We note this result also holds for the Eqs. (1.2) and (1.3). It may be shown similarly by the application of the discrete transformation that $\mu \neq \bar{\mu}$ implies the inequality (3.36). Since $\Psi^2 = \bar{\Psi}^2$, is equivalent to Eq. (1.14) the proof of Theorem 2 is complete.

We proceed with the proof of Theorem 3 by solving Eqs. (3.34) and (3.35) obtaining

$$D^2\Psi = 2(\Psi - \bar{\Psi})^{-1}[3D\Psi D\bar{\Psi} - 10i\Psi D\bar{\Psi} - i(30\Psi - 7\bar{\Psi})D\Psi + 4\Psi^2 - 32\Psi\bar{\Psi}]. \tag{3.38}$$

Applying the D operator to this equation and using it to eliminate the second derivatives from the result, we obtain the following expression for the third derivative of Ψ

$$\begin{aligned} D^3\Psi = 2(\Psi - \bar{\Psi})^{-2}[&21D\Psi^2 D\bar{\Psi} - 21D\Psi D\bar{\Psi}^2 + 86i\Psi D\Psi D\bar{\Psi} \\ &+ 281i\bar{\Psi}D\Psi D\bar{\Psi} + 70i\bar{\Psi}D\Psi^2 + 37i\Psi D\bar{\Psi}^2 - 652\Psi\bar{\Psi}D\Psi - 464\bar{\Psi}^2 D\Psi \\ &+ 824\Psi\bar{\Psi}D\bar{\Psi} - 90\Psi^2 D\Psi - 1796\bar{\Psi}^2 D\bar{\Psi} + 240i\bar{\Psi}^3 - 1336i\Psi\bar{\Psi}^2 + 368i\Psi^2\bar{\Psi}]. \end{aligned} \tag{3.39}$$

The next step is to invoke the necessary condition VII [*23*] for the validity of Huygens' principle for (1.1). This condition in the form we require is given by CM Eq. (1.15) and will not be repeated here. We only need the condition that results by contracting the spinor equivalent of this equation with $o^{A\dot{B}CD}\iota^{E}\bar{o}^{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}}\bar{\iota}^{\dot{E}}$. In the gauge where Eqs. (3.30) and (3.32) hold, this equation has the following form:

$$5i(3\Psi+\overline{\Psi})D^3\Psi+5(3\Psi-26\overline{\Psi})D^2\Psi-17D^2\Psi D^2\overline{\Psi}+56D\Psi D\overline{\Psi}$$
$$+i(75\Psi+47\overline{\Psi})D\Psi-45\Psi^2-636\Psi\overline{\Psi}+\text{c. c.}=0. \quad (3.40)$$

Eliminating the $D^2\Psi$ and $D^3\Psi$ from the above using Eqs. (3.38) and (3.39), we obtain the following simplified equation:

$$7050\overline{\Psi}^2D\Psi^2-612D\Psi^2D\overline{\Psi}^2+63030\Psi\overline{\Psi}^3-79947\Psi^2\overline{\Psi}^2-7275\Psi^4$$
$$+54705i\Psi^3D\Psi+18995i\Psi^3D\overline{\Psi}+16656\Psi^2D\Psi D\overline{\Psi}+73313i\Psi\overline{\Psi}^2D\Psi$$
$$-160687i\Psi^2\overline{\Psi}D\Psi+13080i\Psi D\Psi^2D\overline{\Psi}+6096i\Psi D\Psi D\overline{\Psi}^2$$
$$-62684\Psi\overline{\Psi}D\Psi D\overline{\Psi}-42610\Psi\overline{\Psi}D\Psi^2+\text{c. c.}=0. \quad (3.41)$$

Integrability conditions for the above equation may be obtained by repeated application of the D operator. The higher order derivatives in these conditions may be removed using Eqs. (3.38) and (3.39). We shall need (3.41) and the first two of these integrability conditions. They may be written in a manifestly real form by the substitutions

$$\Psi = y(X+i), \qquad D\Psi = y(U+iV), \quad (3.42)$$

where X, $y \neq 0$, U and V are real quantities, as follows:

$$f_1 := 6048X^4-12916X^3V+20397X^2U^2+2617X^2V^2+1746XU^2V$$
$$+1746XV^3+153U^4+306U^2V^2+153V^4-3225X^2U-7050XUV$$
$$+4794U^3+4794UV^2+29061X^2-48626XV+32250U^2+7420V^2$$
$$-76925U+37563=0, \quad (3.43)$$

$$f_2 := 361648X^5-314460X^4V+1258566X^3U^2-117370X^3V^2$$
$$+136500X^2U^2V+66660X^2V^3+25434XU^4+49104XU^2V^2$$
$$+23670XV^4+1836U^4V+3672U^2V^3+1836V^5-866621X^3U$$
$$-1434167X^2UV+489828XU^3+410518XUV^2+34464U^3V$$
$$+34464UV^3+1852304X^3-1150729X^2V+1479810XU^2+107724XV^2$$
$$+23360U^2V-121790V^3-5144269XU-1109787UV+2365936X$$
$$+1392483V=0, \quad (3.44)$$

$$f_3 := 8804880X^6+2484740X^5V+78299976X^4U^2-9594440X^4V^2$$
$$+7329456X^3U^2V-755520X^3V^3+3715380X^2U^4+5218560X^2U^2V^2$$
$$+1573740X^2V^4+560844XU^4V+964224XU^2V^3+403380XV^5+5508U^6$$
$$+38556U^4V^2+60588U^2V^4+27540V^6-2516673384X^4U$$
$$-168015573X^3UV+45051393X^2U^3+20051758X^2UV^2+7370400XU^3V$$
$$+6140270XUV^3+164250U^5+649464U^3V^2+485214UV^4$$
$$+50473940X^4+18626114X^3V+11973335X^2U^2+40119540X^2V^2$$
$$-2733502XU^2V-19087092XV^3+1145964U^4+191996U^2V^2$$
$$-2113538V^4-170374964X^2U-133094677XUV-4763955U^3$$
$$-22479666UV^2+43590872X^2+148655966XV-43823231U^2$$
$$+41725086V^2+103033097U-50129388=0. \quad (3.45)$$

The essential feature of the system of polynomial equations (3.43) to (3.45) is that it possesses *finitely many* solutions. This conclusion follows from the application of Buchberger's Gröbner basis theory [*1*]. The reduced minimal Gröbner basis GB for the polynomial ideal generated by $F := \{ f_1, f_2, f_3 \}$ was computed using the Maple [*6*] Gröbner basis package of Czapor [*7*]. The basis GB contains twenty-one elements and does not contain the polynomial 1 implying that the system F has solutions possibly complex (Method 6.8). An examination of the elements in GB reveals that the power products X^4, U^8 and V^9 appear among the leading power products of the polynomials in GB. Thus by Buchberger's Method 6.9, we conclude that the system F has finitely many (possibly complex) solutions. Since our unknowns are real, the system in fact may possess no solutions which leads to a contradiction with the assumption $\rho \neq \bar{\rho}$ in Eq. (3.29).

We now assume that the system F possesses finitely many real solutions $[X_j, U_j, V_j]$, $j = 1, \ldots, r$ where $r \in \mathbb{P}$. It follows from (3.42) that for any j we have

$$\Psi = a_j \bar{\Psi}, \tag{3.46}$$

$$D\Psi = b_j \Psi, \tag{3.47}$$

where

$$a_j := (X_j + i)/(X_j - i), \tag{3.48}$$

$$b_j := (U_j + iV_j)/(X_j + i). \tag{3.49}$$

A consequence of these equations is that

$$b_j = \bar{b}_j. \tag{3.50}$$

The Eqs. (3.35), (3.47) and (3.50) together imply that

$$b_j^2 + 7 = 0, \tag{3.51}$$

which is impossible. We thus concluded that (3.29) does not hold and consequently we must habe

$$\rho = \bar{\rho}. \tag{3.52}$$

Since

$$1_{[a;b}l_{c]} = (\bar{\rho} - \rho) l_{[a} m_b \bar{m}_{c]} = 0, \tag{3.53}$$

it follows that the null congruence defined by l is hypersurface orthogonal. The application of the discrete transformation (2.20) to (3.52) yields

$$\mu = \bar{\mu} \tag{3.54}$$

which implies that

$$n_{[a;b}n_{c]} = 0. \tag{3.55}$$

This completes the proof of Theorem 3.

We now proceed with the proof of Theorem 4. The hypothesis of this theorem and Theorem 2 imply that

$$\kappa = \nu = \sigma = \lambda = 0, \tag{3.56}$$

$$\rho = \bar{\rho}, \tag{3.57}$$

$$\mu = \bar{\mu}. \tag{3.58}$$

We may use the same conformal transformation as that used to obtain (3.30), to set

$$\rho = 0 . \tag{3.59}$$

It immediately follows from the NP Eqs. (4.2) that

$$\Phi_{00} = \Phi_{01} = 0 . \tag{3.60}$$

From the Bianchi identities and Eqs. (3.59) and (3.60), we have

$$D\Psi = \frac{2}{3} D\Phi_{11} = - 2D\Lambda , \tag{3.61}$$

which implies that $D\Psi$ and $D^2\Psi$ are real. In view of the above, the Eq. (3.26) reduces to

$$D^2\Psi = (\varepsilon + \bar{\varepsilon})D\Psi . \tag{3.62}$$

It follows from Eqs. (3.59) to (3.62) and Eq. (3.9) that

$$D\Psi = 0 . \tag{3.63}$$

On account of the transformation laws (3.26), the condition (3.63) has the form

$$DH = 2\rho , \tag{3.64}$$

in an arbitrary conformal gauge. The application of the discrete transformation to Eq. (3.64) yields the analogous condition

$$\Delta H = - 2\mu . \tag{3.65}$$

We proceed with the proof by using the conformal transformation to achieve $\Psi\bar{\Psi} = 1$ or equivalently $H + \bar{H} = 0$. It follows immediately from this and Eqs. (3.57), (3.58), (3.64) and (3.65) that

$$\rho = \mu = 0 , \tag{3.66}$$

$$DH = \Delta H = 0 . \tag{3.67}$$

As a consequence of these equations and NP Eqs. (4.2) we also have

$$\Phi_{00} = \Phi_{01} = \Phi_{12} = \Phi_{22} = 0 . \tag{3.68}$$

We next contract Condition V's with $o^{A}{}_{\iota}{}^{BCD}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$ and $o^{ABC}{}_{\iota}{}^{D}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$, respectively, obtaining

$$\Delta\tau = \tau(\gamma - \bar{\gamma}) , \tag{3.69}$$

$$D\pi = \pi(\bar{\varepsilon} - \varepsilon) . \tag{3.70}$$

It is convenient at this stage to distinguish the cases $\tau + \bar{\pi} \neq 0$ and $\tau + \bar{\pi} = 0$

CASE $\tau + \bar{\pi} \neq 0$.

The dyad transformation (2.19) is used to set

$$\tau + \bar{\pi} = \bar{\tau} + \pi . \tag{3.71}$$

The operators D and Δ applied to Eq. (3.71) yield, on account of Eqs. (3.69), (3.70) and NP Eqs. (4.2),

$$\varepsilon = \bar{\varepsilon}, \tag{3.72}$$

$$\gamma = \bar{\gamma}. \tag{3.73}$$

This implies that

$$D\tau = \Delta\tau = D\pi = \Delta\pi = 0. \tag{3.74}$$

The $[\Delta, D]$ commutator applied to $\tau + \bar{\pi}$ then gives

$$(\delta + \bar{\delta})(\tau + \bar{\pi}) = 0, \tag{3.75}$$

which in conjunction with the NP Eqs. (4.2) implies

$$\tau\bar{\tau} = \pi\bar{\pi}, \tag{3.76}$$

$$\tau\pi - \bar{\tau}\bar{\pi} + (\tau + \bar{\pi})(\alpha - \bar{\beta} - \bar{\alpha} + \beta) + \Psi - \bar{\Psi} = 0. \tag{3.77}$$

The Eqs. (3.71) and (3.76) give the important relation

$$\tau = \pi. \tag{3.78}$$

When this is taken into account in NP Eqs. (4.2), we obtain

$$Q := \alpha - \bar{\beta} = -\bar{Q}, \tag{3.79}$$

$$\Phi_{02} = \Phi_{20}, \tag{3.80}$$

$$\Psi - \bar{\Psi} = (\tau + \bar{\tau})(\bar{\tau} - \tau - 2Q). \tag{3.81}$$

The operator δ applied to Eq. (3.77) yields

$$\delta(\Psi - \bar{\Psi}) = 3(\tau\Psi + \bar{\tau}\bar{\Psi}) - (\tau + \bar{\tau})(2\Phi_{11} + \Phi_{02}), \tag{3.82}$$

while the commutator $[\Delta, D]$ applied to H implies

$$(\delta + \bar{\delta})H = 0. \tag{3.83}$$

The next step is to obtain the equations resulting from the contractions $o^{AB}\bar{\iota}^{\dot{A}\dot{B}}$ and $o^{A}\iota^{B}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}$ with III's, which, with the help of NP Eqs. (4.2), may be written as

$$\Psi[\delta^2 H + \delta H^2 - (6\tau + Q)\delta H + 6\tau^2 - 2\Phi_{02}] + \text{c.c.} = 0, \tag{3.84}$$

$$\Psi[\delta^2 H + \delta H^2 - (5\tau + \bar{\tau} - Q)\delta H + 3\tau(\tau + \bar{\tau} - 2Q) - 3\Phi_{02} - 2\Phi_{11}] + \text{c.c.} = 0, \tag{3.85}$$

Subtraction of the first of these equations from the second yields

$$\Psi[(\bar{\tau} - \tau - 2Q)\delta H + 3\tau(\tau - \bar{\tau} + 2Q) + \Phi_{02} + 2\Phi_{11}] + \text{c.c.} = 0. \tag{3.86}$$

In view of Eqs. (3.10), (3.14), (3.81) and (3.82), this equation reduces to

$$2\delta H = 3(\tau + \bar{\tau}). \tag{3.87}$$

Combining Eqs. (3.84) and (3.87), one obtains

$$(\Psi + \bar{\Psi})(3\tau^2 - 18\tau\bar{\tau} + 3\bar{\tau}^2 - 8\Phi_{02}) = 0. \tag{3.88}$$

This implies

$$\Phi_{02} = \frac{3}{8}(\tau^2 + \bar{\tau}^2 - 6\tau\bar{\tau}), \tag{3.89}$$

since $\Psi + \bar{\Psi} = 0$, is impossible in this case. Finally we need the equation arising from the contraction $o^{ABC}\iota^{D}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ with V's, which after elimination of δH and Φ_{02}, may be written as

$$3(\tau + \bar{\tau})^2 = 16\tau\bar{\tau}. \tag{3.90}$$

However, this equation is incompatible with the inequality

$$(\tau + \bar{\tau})^2 \leq 4\tau\bar{\tau}. \tag{3.91}$$

From this contradiction we conclude that the assumption $\tau + \bar{\pi} \neq 0$ is untenable.

CASE $\tau + \bar{\pi} = 0$.

We begin by using the dyad transformation (2.19) to set

$$\tau = \bar{\tau}. \tag{3.92}$$

It follows immediately that

$$\pi = -\tau. \tag{3.93}$$

The NP Eqs. (4.2) imply that

$$\Psi = \bar{\Psi}, \tag{3.94}$$

which together with Eq. (3.14) yields

$$\Psi = \pm 1. \tag{3.95}$$

The above results in conjunction with the equation resulting from the contraction $o^{AB}\bar{\iota}^{\dot{A}\dot{B}}$ with III's, yield

$$\Phi_{20} = 3\tau^2. \tag{3.96}$$

We next impose the above conditions on the equation arising from the contraction $o^{ABC}\iota^{D}\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}\dot{C}\dot{D}}$ with V's thereby obtaining

$$\tau = 0. \tag{3.97}$$

Finally, we appeal to the last equation contained in V's which is obtained by the contraction $o^{AB}\iota^{CD}\bar{o}^{\dot{A}\dot{B}}\bar{\iota}^{\dot{C}\dot{D}}$. In view of the results already obtained it reads

$$3\Psi + 4\Lambda = 0. \tag{3.98}$$

However, this is incompatible with the equation

$$\Psi + 2\Lambda = 0, \tag{3.99}$$

which arises from NP Eq. (4.2*h*).

This completes the proof of Theorem 4.

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