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On the stability of solitary waves for classical scalar fields

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ABSTRACT. — We study the stability for the bound states of lowest action of certain nonlinear Klein-Gordon and Schrödinger equations by applying the Shatah-Strauss formalism. We extend the range of application of this formalism by using a recent existence theorem for minimum action solutions to a large class of equations including logarithmic Klein-Gordon and logarithmic Schrödinger equations and scalar field equations with fractional nonlinearities.

Furthermore we discuss the relation between different stability criteria considered in the literature.

RÉSUMÉ. — On étudie la stabilité des états liés d'action minimale de certaines équations de Klein-Gordon et de Schrödinger non linéaires en appliquant le formalisme de Shatah et Strauss. On étend le domaine de validité de ce formalisme par l'utilisation d'un théorème récent d'existence de solutions d'action minimale pour une classe étendue d'équations, comprenant les équations de Klein-Gordon et de Schrödinger logarithmiques et les équations de champs scalaires avec des non linéarités sous-linéaires. En outre, on discute les relations entre divers critères de stabilité considérés dans la littérature.

0. INTRODUCTION

In the study of solitary waves we have to consider three main steps: the existence, the stability and the general evolution problem.

In the last ten years interesting results have been obtained for the three problems, which will be detailed below. Concerning the existence problem one of us recently proved the existence of solitary waves having lowest energy for scalar fields with logarithmic and fractional nonlinearities [1].

Now our purpose is to study the stability of such solitary waves of the nonlinear Klein-Gordon equation

$$\text{(NLKG)} \qquad \phi_{tt} - \Delta\phi - g(\phi) = 0$$

and the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad - i\phi_t + \Delta\phi + f(\phi) = 0$$

by applying the Shatah-Strauss formalism [2] [3].

By a solitary wave we mean a solution of the form

$$\phi(x, t) = e^{i\omega t}u_\omega(x)$$

with ω real and $u_\omega(x)$ being in a suitable function space, i. e. ϕ is a standing wave with frequency ω. Stability has to be understood in the following sense: Let $\phi(0, x) = \phi_0(x)$ an initial value close to the standing wave solution with respect to a certain function space metric then $\phi(x, t)$ remains close to $u_\omega(x)e^{i\omega t}$ for all $t > 0$ with respect to this metric. Otherwise we call $u_\omega(x)$ unstable.

Our paper is organized as follows:

In Section I we summarize the existence results and introduce the notations we need.

In Section II we consider the stability and instability criteria. We recall the results of Shatah and Strauss which naturally also hold in the cases we investigate.

In Section III we derive general classifications for stability and instability from the general formalism and present a few examples, e. g. fractional nonlinearities, the logarithmic Klein-Gordon equation and logarithmic Schrödinger equation which are treated for the first time within this general framework.

Finally, in Section IV, we discuss the relations between different stability criteria. Especially we put the linearized operator into the context of Shatah-Strauss formalism.

1. EXISTENCE OF NONTRIVIAL STANDING WAVES

Consider the nonlinear Klein-Gordon equation

(NLKG) $$\phi_{tt} - \Delta\phi = g(\phi) \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 3.$$

This equation has nontrivial standing wave solutions $\phi(x, t) = e^{i\omega t}u(x)$ provided that

(*) $$- \Delta u = g_\omega(u)$$

where $g_\omega(u) \equiv g(u) + \omega^2 u$, has a nontrivial solution. Especially we are interested in finding solutions having least energy among all possible (nontrivial) solutions of (*). The existence of such a « ground state » was proven by Strauss [4], generalized by Berestycki and Lions [5] and extended to more general nonlinearities recently by Stubbe [1]. We require $g_\omega(u)$ to satisfy the hypotheses of [1].

We assume that $g: \mathbb{R} \to \mathbb{R}$ is odd and that $g \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$. Furthermore g and g_ω satisfy the following conditions:

(1.1) $$\lim_{u\to 0^+} \frac{g_\omega(u)}{u} < 0$$

(1.2) $$\lim_{u\to\infty} \frac{g(u)}{u^1} \leq 0, \qquad 1 = \frac{N+2}{N-2}$$

(1.3) There exists $\xi \equiv \xi(\omega)$ such that

$$G_\omega(\xi) = \int_0^\xi g_\omega(\sigma)d\sigma > 0$$

(1.4) $$\lim_{u\to 0^+} \frac{g(2u)}{g(u)} > 1.$$

We emphasize that the limit in (1.1) is allowed to be infinite which yields the difference to the existence conditions in the work of Berestycki and Lions [4] who required the nonlinearity to have a finite derivative at the origin. Therefore in our case the functional

$$V_\omega(u) = \int_{\mathbb{R}^N} G_\omega(u)dx$$

does not have to be well-defined on $H^1(\mathbb{R}^N)$. In [1] it was shown that $V_\omega(u)$ is well-defined and of class C^1 on the subspace

(1.5) $$W = \{ u \in H^1(\mathbb{R}^N) \mid G_\omega(u) \in L^1(\mathbb{R}^N) \}.$$

Indeed, W is a reflexive Banach space.

With the help of this framework one solves the following constrained minimization problem

$$\text{Minimize } \{ \mathrm{T}(u) \mid u \in \mathrm{W},\ \mathrm{V}_\omega(u) = 1 \}$$

where
$$\mathrm{T}(u) = \int_{\mathbb{R}^N} (\nabla u)^2 dx$$

from which one derives the existence of a minimum action solution of the field equation (*), i. e. a solution which minimizes the action

$$\mathrm{S}_\omega(u) = \frac{1}{2}\mathrm{T}(u) - \mathrm{V}_\omega(u) \tag{1.6}$$

among all possible solutions of the field equation.

In this section we want to put this existence result into the context of the nonlinear Klein-Gordon equation and of the Shatah-Strauss formalism [2] [3].

For this purpose we need the following functionals and sets:

There are three important physical quantities for solutions of the NLKG

$$\mathrm{E}(u, u_t) = \frac{1}{2}\int_{\mathbb{R}^N} |u_t|^2 dx + \frac{1}{2}\mathrm{T}(u) - \mathrm{V}_0(u) \qquad \text{(energy)} \tag{1.7}$$

$$\mathrm{Q}(u, u_t) = \mathrm{Im}\int_{\mathbb{R}^N} \bar{u} u_t dx \qquad \text{(charge)} \tag{1.8}$$

$$\mathrm{L}(u, u_t) = -\frac{1}{2}\int_{\mathbb{R}^N} |u_t|^2 dx + \frac{1}{2}\mathrm{T}(u) - \mathrm{V}_0(u) \qquad \text{(action)} \tag{1.9}$$

with
$$u \in \mathrm{W} \quad \text{and} \quad u_t \in \mathrm{L}^2(\mathbb{R}^N).$$

For standing waves $e^{i\omega t}u(x)$ we have

$$\begin{aligned}\mathrm{L}(u, i\omega u) &= \mathrm{E}(u, i\omega u) - \omega \mathrm{Q}(u, i\omega u)\\ &= \mathrm{S}_\omega(u).\end{aligned}$$

Then we will use the abbreviations

$$\begin{aligned}\mathrm{E}_\omega(u) &\equiv \mathrm{E}(u, i\omega u)\\ \mathrm{Q}_\omega(u) &\equiv \mathrm{Q}(u, i\omega u)\end{aligned}$$

Furthermore we need the functional

$$\mathrm{K}_\omega(u) = \frac{\mathrm{N}-2}{2}\mathrm{T}(u) - \mathrm{N}\mathrm{V}_\omega(u) \tag{1.10}$$

and the set

$$\mathrm{M}_\omega = \{ u \in \mathrm{W}_r,\ \mathrm{K}_\omega(u) = 0,\ u \not\equiv 0 \} \tag{1.11}$$

where W_r denotes the space of radial functions in W.

It is easy to see that M_ω is a C^1 hypersurface in W_r bounded away from zero (easy extension of Lemma 1.1 in [*2*]).

Furthermore by the virial theorem or Pohozaev identity (see [*4*] [*6*] for a proof) each solution u_ω of the field equation satisfies:

$$K_\omega(u_\omega) = 0 \tag{1.12}$$

$$S_\omega(u_\omega) = \frac{1}{N} T(u_\omega). \tag{1.13}$$

Now we state our existence theorem in analogy to Theorem 1.1 of Shatah [*2*]:

THEOREM 1.1.

$$d(\omega) \equiv \inf_{u \in M_\omega} S_\omega(u)$$

is achieved for some $u_\omega \not\equiv 0$ and

$$d(\omega) = \inf \left\{ \frac{1}{N} T(u) \mid K_\omega(u) \leq 0, u \not\equiv 0 \right\}.$$

Furthermore, if $\tilde{d}(\omega) \equiv \inf \{ T(u) \mid u \in W, V_\omega(u) = 1 \}$ then $\tilde{d}(\omega)$ is achieved for some $\tilde{u}_\omega \not\equiv 0$ and we have the relations

$$d(\omega) = \frac{1}{N} \left(\frac{N-2}{2N} \right)^{\frac{N-2}{2}} \tilde{d}(\omega)^{N/2}$$

$$u_\omega(x) = \tilde{u}_\omega(x/\sqrt{\theta(\omega)}) \qquad \text{where} \qquad \theta(\omega) = \frac{N-2}{2N} T(\tilde{u}_\omega).$$

In addition u_ω satisfies the field equation

$$-\Delta u_\omega = g_\omega(u_\omega). \tag{*}$$

Proof. — First we show the equivalence of the first two minimization problems.

Let $v_1 \in W_r$ such that $K_\omega(v_1) > 0$. $v_\sigma = v_1(x/\sigma)$ satisfies

$$K_\omega(v_\sigma) = \sigma^{N-2} \left[\frac{N-2}{2} T(v_1) - \sigma^2 N V_\omega(v_1) \right].$$

Thus, if $\sigma = \sigma_0 \equiv \left(\frac{N-2}{2N} \frac{T(v_1)}{V_\omega(v_1)} \right)^{1/2}$, we have $K_\omega(v_{\sigma_0}) = 0$ and since $0 < \sigma_0 < 1$ the inequality

$$\frac{1}{N} T(v_{\sigma_0}) = \frac{1}{N} \sigma_0^{N-2} T(v_1) < \frac{1}{N} T(v_1)$$

holds. Since $S_\omega(v) = \frac{1}{N} T(v) + \frac{1}{N} K_\omega(v)$, then

$$\begin{aligned} d(\omega) &\equiv \inf_{v \in M_\omega} S_\omega(v) \\ &= \inf \left\{ \frac{1}{N} T(v),\ v \in W_r,\ K_\omega(v) = 0,\ v \not\equiv 0 \right\} \\ &= \inf \left\{ \frac{1}{N} T(v),\ v \in W_r,\ K_\omega(v) \leq 0,\ v \not\equiv 0 \right\}. \end{aligned}$$

Now we prove the relation between $d(\omega)$ and $\tilde{d}(\omega)$.

Let $v \in W_r$ such that $K_\omega(v) = 0$. For $\sigma = \left(\frac{N-2}{2N}\right)^{-\frac{1}{N}} T(v)^{-\frac{1}{N}} v_\sigma$ satisfies $V_\omega(v_\sigma) = 1$. Thus

$$\tilde{d}(\omega) \leq T(v_\sigma) = \left(\frac{N-2}{2N}\right)^{-\frac{N-2}{N}} T(v)^{2/N}$$

which implies

$$d(\omega) \geq \frac{1}{N} \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \tilde{d}(\omega)^{N/2}.$$

On the other hand if $v \in W_r$ such that $V_\omega(v) = 1$ we choose

$$\sigma = \sigma_0 = \left(\frac{N-2}{2N} T(u)\right)^{1/2} \quad \text{and obtain} \quad K_\omega(v_{\sigma_0}) = 0.$$

But

$$d(\omega) \leq \frac{1}{N} T(v\sigma_0) = \frac{1}{N} \left(\frac{N-2}{2N}\right)^{\frac{N-2}{N}} T(v)^{N/2}$$

which gives

$$d(\omega) \leq \frac{1}{N} \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \tilde{d}(\omega)^{N/2}.$$

Applying the existence result of [1] there is an $\tilde{u}_\omega \in W_r$ such that $T(\tilde{u}_\omega) = \tilde{d}(\omega)$ and $V(\tilde{u}_\omega) = 1$. Furthermore in [1] it is shown that $u_\omega(x) = \tilde{u}_\omega(x/\sqrt{\sigma(\omega)})$ has least action among all functions satisfying $K_\omega(u) = 0$. Thus $\frac{1}{N} T(u_\omega) = d(\omega)$ and in addition u_ω satisfies the field equation.

REMARK 1.2. — Alternatively one can solve directly the minimization problem for $d(\omega)$ as Shatah [2] did. The extension to nonlinearities satisfying condition (1.1) is an easy application of the results presented in [1].

REMARK 1.3. — In addition it can be shown that

$$d(\omega) = \inf \left\{ S_\omega(u), u \in W_\mu, \frac{1}{N} \int |\nabla u|^2 = d(\omega) \right\}$$

i. e. the solution set of $d(\omega) = \inf_{u \in M_\omega} S_\omega(u)$ is also the solution set of

$$\inf S_\omega(u) = d(\omega) \qquad \text{and} \qquad \frac{1}{N} T(u) = d(\omega).$$

REMARK 1.4. — Uniqueness of the minimum action solution u_ω is not known in general. Nevertheless we assume that there is a choice u_ω achieving the infimum in Theorem 1.1 such that

$$\omega \mapsto u_\omega$$

is of class C^2. For a proof of this fact under the assumption of uniqueness and for $g \in C^2$ see [3]. $u_\omega \in C^1$ is sufficient, but our assumption simplifies the presentation. At least it is valid for the examples considered in Section 3. The search for standing waves of the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad - i\phi_t + \Delta\phi + f(\phi) = 0 \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 3$$

leads to

$$(**) \qquad - \Delta u = f_\omega(u)$$

where $f_\omega(u) = f(u) + \omega u$. We require $f_\omega(u)$ to satisfy the same hypotheses as $g_\omega(u)$. Then also Theorem 1.1 holds.

The relevant quantities for (NLS) are

$$(1.10) \qquad E(u) = \frac{1}{2} T(u) - V_0(u) \qquad \text{(energy)}$$

$$(1.11) \qquad Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \qquad \text{(charge)}$$

$$(1.12) \qquad L(u, u_t) = \frac{1}{2} \operatorname{Im} \int_{\mathbb{R}^N} u\bar{u}_t + E(u) \quad \text{(action)}$$

and therefore we have

$$\begin{aligned} L(u, i\omega u) &\equiv S_\omega(u) = E(u) - \omega Q(u) \\ &\equiv \frac{1}{2} T(u) - V_\omega(u) \end{aligned}$$

2. STABILITY AND INSTABILITY CRITERIA

The main purpose of this section is to recall the results of Shatah and Strauss [3].

Since their formalism essentially depends on the minimum property of $d(\omega)$ resp. $u_\omega(x)$ in a suitable function space and on the definiteness of certain functionals, most of the work to implement their formalism has abready been done in Section I.

Furthermore we restrict ourselves to the NLKG unless otherwise stated.

2.1. **Properties of** $d(\omega)$.

We recall the properties of $d(\omega)$ which will be useful for the stability analysis.

LEMMA 2.1. — Assume that $\omega > 0$ and g_ω satisfies the conditions which garantee the existence of a ground state solution. Then $d(\omega)$ is a positive decreasing function of ω and its derivative is given by

$$d'(\omega) = -\,Q_\omega(u_\omega).$$

Proof. — $d(\omega) = S_\omega(u_\omega) = E_\omega(u_\omega) - \omega Q_\omega(u_\omega)$. Therefore

$$\begin{aligned} d'(\omega) &= \left\langle DE_\omega(u_\omega) - \omega DQ_\omega(u_\omega), \frac{du_\omega}{d\omega} \right\rangle - Q_\omega(u_\omega) \\ &= -\,Q_\omega(u_\omega) \end{aligned}$$

THEOREM 2.2. — $d(\omega)$ is convex at ω_0 if and only if $E(u, v)$ restricted to the manifold

$$M_0 \equiv \{ (u, v) \in W_r \oplus L_r^2 \mid Q(u, v) = Q_{\omega_0}(u_{\omega_0}) \}$$

has a local minimum at u_{ω_0}.

Proof (see [3]). — Since $E(u, v) \geq S_\omega(u) + \omega Q(u, v)$ we have

$$\begin{aligned} E(u, v) &\geq d(\omega) + \omega Q_{\omega_0}(u_{\omega_0}) \\ &= d(\omega) - \omega d'(\omega_0) \end{aligned}$$

for any (u, v) nearby $(u_{\omega_0}, \omega_0 u_{\omega_0})$ and suitable ω.

If $d(\omega)$ is convex the above inequality implies

$$\begin{aligned} E(u, v) \geq d(\omega_0) - \omega_0 d'(\omega_0) &= E(u_{\omega_0}, i\omega u_{\omega_0}) \\ &= E_{\omega_0}(u_{\omega_0}). \end{aligned}$$

Conversely consider the family $v_\omega(x) = u_\omega(x_{\sigma(\omega)})$ with $\sigma(\omega) = \left(\dfrac{Q_{\omega_0}(u_{\omega_0})}{Q_\omega(u_\omega)}\right)^{1/N}$.
Then $Q_\omega(v_\omega) = Q_{\omega_0}(u_{\omega_0})$ and

$$\begin{aligned} E_\omega(v_\omega) &= \omega Q\omega(v_\omega) + S_\omega(v_\omega) \\ &= \omega Q\omega_0(u_{\omega_0}) + \frac{1}{2}\sigma(\omega)^{N-2}T(u_\omega) - \sigma^N(\omega)V_\omega(u_\omega) \\ &= -\,\omega d'(\omega_0) + \left(\frac{1}{2}\sigma^{N-2} - \frac{N-2}{2N}\sigma^N\right)T(u_\omega) \\ &\leq -\,\omega d'(\omega_0) + d(\omega) \qquad \text{since} \quad N \geq 3. \end{aligned}$$

But by assumption

$$E_\omega(v_\omega) \geq E_{\omega_0}(u_{\omega_0}) = d(\omega_0) - \omega_0 d'(\omega_0)$$

which implies the convexity of $d(\omega)$ at ω_0.

REMARK 2.3. — For the Schrödinger equation the inequality for $E(u, v)$ is replaced by the equality

$$\begin{aligned} E_\omega(u) &= L(u, i\omega u) + \omega Q(u) \\ &= S_\omega(u) + \omega Q(u). \end{aligned}$$

A simple consequence of Theorem 2.2 is the following

COROLLARY 2.4. — *a)* If $d(\omega)$ is strictly concave at ω_0 then E, subject to the constraint $Q = Q_{\omega_0}(u_{\omega_0})$ does not have a local minimum at u_{ω_0} and in particular $E_\omega(v_\omega) < E_{\omega_0}(u_{\omega_0})$ for ω nearby ω_0 and $\omega \neq \omega_0$.

b)
$$\frac{d^2}{d\omega^2} E_\omega(v_\omega)\,|_{\omega=\omega_0} \leq d''(\omega_0).$$

Finally we observe the following interesting property of $d(\omega)$:

COROLLARY 2.5. — $d(\omega)^{-\frac{2}{N-2}}$ is an increasing convex function of ω.

Proof. — Since $d(\omega)$ is decreasing the growth property is immediate. Now $\left(d^{-\frac{2}{N-2}}\right)''(\omega) = -\dfrac{2}{N-2} d^{-2\frac{N-1}{N-2}}\left(d''d - \dfrac{N}{N-2} d'^2\right)(\omega)$ and the proposition is trivial if $d''(\omega) \leq 0$.

Let $d''(\omega_0) > 0$ for some ω_0. Then by Theorem 2.2 the energy, subject to the constraint $Q = Q_{\omega_0}(u_{\omega_0})$, has a local minimum at u_{ω_0}. The curve v_ω used in the proof of theorem 2.2 satisfies

$$E_\omega(v_\omega) - \omega Q_\omega(v_\omega) = \mu(\omega) d(\omega)$$

with
$$\mu(\omega) = \frac{N}{2}\sigma(\omega)^{N-2} - \frac{N-2}{2}\sigma(\omega)^N.$$

At ω_0 we have

$$\mu(\omega_0) = 1$$

$$\mu'(\omega_0) = \left(\frac{N}{2}(N-2)\sigma(\omega_0)^{N-3} - \frac{N}{2}(N-2)\sigma(\omega_0)^{N-1}\right)\sigma'(\omega_0) = 0$$

$$\mu''(\omega_0) = -N(N-2)\sigma'^2(\omega_0) = -\frac{N-2}{N}\left(\frac{d''(\omega_0)}{d'(\omega_0)}\right)^2.$$

So, upon expanding $\mu(\omega)d(\omega)$ in power of $\Delta\omega = \omega - \omega_0$, we obtain

$$\begin{aligned} E_\omega(v_\omega) - \omega Q_\omega(v_\omega) &= \left[1 + \frac{1}{2}\mu''(\omega_0)(\Delta\omega)^2\right]\left[d(\omega_0) + d'(\omega_0)\Delta\omega + \frac{1}{2} d''(\omega_0)(\Delta\omega)^2\right] \\ &= d(\omega_0) + d'(\omega_0)\Delta\omega + \frac{1}{2}[\mu''(\omega_0)d(\omega_0) + d''(\omega_0)](\Delta\omega)^2 \end{aligned}$$

which implies

$$\begin{aligned}
E_\omega(v_\omega) - E_{\omega_0}(u_{\omega_0}) &= \frac{1}{2}(\mu''(\omega_0)d(\omega_0) + d''(\omega_0))(\Delta\omega)^2 \\
&= \frac{1}{2}\left(d''(\omega_0) - \frac{N-2}{N}d(\omega_0)\left(\frac{d''(\omega_0)}{d'(\omega_0)}\right)^2\right)(\Delta\omega)^2 \\
&= \frac{(N-2)^2}{2N}d''(\omega_0)\frac{d^{2\frac{N-1}{N-2}}(\omega_0)}{d'(\omega_0)^2}(d(\omega_0)^{-\frac{2}{N-2}})''\cdot(\Delta\omega)^2 .
\end{aligned}$$

This expression is positive. The proposition follows since $d''(\omega_0) > 0$, $d(\omega_0) > 0$, $d'(\omega_0) \neq 0$.

2.3. **The Evolution equations.**

In this subsection we summarize the stability/instability results obtained by Shatah and Strauss in [*3*].

We consider the Cauchy problems

$$\text{(NLKG)} \qquad \begin{gathered} \phi_{tt} - \Delta\phi - g(\phi) = 0 \\ \phi(0, x) = u_0(x) \qquad \phi_t(0, x) = v_0(x) \end{gathered}$$

with

$$(u_0, v_0) \in W_r \oplus L_r^2$$

and

$$\text{(NLS)} \qquad \begin{gathered} -i\psi_t + \Delta\psi + f(\psi) = 0 \\ \psi(0, x) = u_0(x) \end{gathered}$$

with

$$u_0 \in W_r .$$

There is a wide literature on both problems but restricted to nonlinearities g (resp. f) having a finite derivative at the origin. Under this restriction the NLKG was studied by Pecher [*7*], Strauss [*8*] and Ginibre and Velo (see e. g. the survey article [*9*] and references therein).

We know that for NLKG there exists $T > 0$ such that $\phi \in C([0, T), W_r)$, $\phi_t \in C([0, T), L_r^2)$ and ϕ is unique. Furthermore the charge and energy are conserved quantities, i. e.

$$\begin{aligned} Q(u, u_t) &= Q(u_0, v_0) \\ E(u, u_t) &= E(u_0, v_0) \end{aligned}$$

for all $t \in [0, T)$.

For the NLS the situation is similar if the nonlinearities have a finite derivative at the origin. This problem was mainly studied by Ginibre and Velo [*9*] [*10*].

Again we know (at least for reasonable nonlinearities) the existence of a solution ψ which is continuous in H_r^1 and satisfies

$$E(\psi) = E(u_0)$$
$$Q(\psi) = Q(u_0)$$

for all $t \in [0, T)$.

If the nonlinearity has an infinite derivative at the origin the treatment of NLKG and NLS is much more difficult and we have only partial results: The Cauchy problem for the logarithmic NLKG and logarithmic NLS was solved by Cazenave and Haraux [*11*] [*12*] by « regularizing » the evolution equation. In general we cannot expect to have strong solutions of the Cauchy problems but we assume that there is always a weak solution satisfying the energy inequality

$$E(\phi, \phi_t) \leq E(u_0, v_0)$$

resp.

$$E(\psi) \leq E(u_0).$$

See e. g. Strauss [*8*], where this is proven if $ug(u) \leqslant 0$. Strauss and Shatah studied the behaviour of the set $K = \{ e^{i\theta}(u_{\omega_0}, i\omega_0 u_{\omega_0}) \}$ in $H_r^1 \oplus L_r^2$ under the flow governed by NLKG (resp. $K = \{ e^{i\theta}u_{\omega_0} \}$ in H_r^1 for NLS). Their analysis also applies in our case for the space $W_r \oplus L_r^2$ (resp. W_r).

The behaviour of K depends on $d(\omega)$ in the following way:

THEOREM 2.7.

a) If $d''(\omega_0) < 0$ then K is unstable

b) If $d''(\omega_0) > 0$ then K is stable

For a proof we refer to the papers of Shatah and Strauss [*2*] [*3*].

In the next section we derive general classifications for stability and instability and apply theorem 2.7 to different models in physics.

3. APPLICATIONS

At the beginning of this section we will present general classifications for stability/instability regions in relation to the existence conditions for standing waves. We restrict ourselves to NLKG but similar results also hold for NLS.

Let $\hat{\omega} > \omega^* \geq 0$ and $(\omega^{*2}, \hat{\omega}^2)$ be the maximal interval of frequencies ω for which g_ω satisfies the conditions (1.1)-(1.4). Note that if $\omega^* = 0$ the left hand side of this interval may be closed. This case will be treated later.

If the interval is open then g_{ω^*} does not satisfy (1.3) and $g_{\hat{\omega}}$ does not satisfy (1.1).

We consider the standing wave equation

$$- \Delta u_\omega = g_\omega(u_\omega) \qquad \omega \in (\omega^*, \hat{\omega}) .$$

Then we have the following

THEOREM 3.1. — There exists $\omega_n \to \omega^*$ such that $d''(\omega_n) > 0$.

Proof. — Theorem 3.1 is an easy consequence of

PROPOSITION 3.2.

$$d(\omega) \to \infty \quad \text{as} \quad \omega \to \omega^* .$$

Proof of Prop. 3.2. — We prove the proposition by contradiction. Assume that $d(\omega)$ remains bounded as $\omega \to \omega^*$ then $\| u_\omega \|_W$ is bounded where $\| \cdot \|_W$ denotes the norm of W introduced in [*1*].

Since W is reflexive there exists a sequence $\omega_K \to \omega^*$ and $v \in W_r$ such that $u_{\omega_K} \to v$ weakly in W_r.

Now we have

$$\begin{aligned} K_{\omega^*}(v) &= \frac{N-2}{2} T(v) - NV_{\omega^*}(v) \\ &\leq \lim_{K \to \infty} K_{\omega_K}(u_{\omega_K}) = 0 \end{aligned}$$

On the other hand $K_{\omega^*}(u) > 0$ for all $u \in W_r$, $u \equiv 0$ and therefore $v = 0$ which implies $u_{\omega_K} \to 0$ strongly in W_r. Since $d(\omega)$ is monotone decreasing we have

$$0 = \lim_{k \to \infty} d(\omega_K) > d(\omega_1) > 0$$

which yields the desired contradiction.

As a consequence there exist stable standing waves for ω close to ω^*.

The next theorem is related to the case where a zero frequency solution exists. This result was also proven by Shatah and Strauss [*3*].

THEOREM 3.3. — If $\omega^* = 0$ and g_0 satisfies condition (1.3), then $d''(\omega) < 0$ for ω close to zero.

Proof. — Since $d'(\omega) = - Q_\omega(u_\omega) = - \omega \int (u_\omega)^2 dx$ we see

$$\begin{aligned} d''(0) &= \lim_{\omega \to 0^+} d''(\omega) \\ &= \lim_{\omega \to 0^+} - \int | u_\omega |^2 dx - \omega \frac{d}{d\omega} \int | u_\omega |^2 dx \\ &= - \int | u_0 |^2 dx < 0 . \end{aligned}$$

Thus the zero frequency solution of NLKG is always unstable.

For ω close to $\hat{\omega}$ the situation is much more difficult. The only result we have is

THEOREM 3.4. — Let $\hat{\omega}$ be infinite; then there exists $\omega_n \to \infty$ such that $d''(\omega_n) > 0$.

Proof. — First let us remark that $\hat{\omega}$ is infinite if the derivative of g_ω at zero is infinite.

Now we have $\lim_{\omega \to \infty} d(\omega) = 0$ by Proposition 2.5.

Since $d(\omega) > 0$, $d'(\omega) \leq 0$ the Theorem follows. (d cannot be concave everywhere).

If $\hat{\omega}$ is finite a lot of possibilities can occur which depend on the special structure of the nonlinearity $g_\omega(u)$ as will be seen in the examples.

Finally let us remark that in many applications there exists an ω_c such that $d''(\omega_c) = 0$ and $d''(\omega) < 0$ for ω arbitrarily close to ω_c. Then ω_c must be an unstable frequency since the set of stable frequencies is open.

3.1. Examples: Nonlinear Klein-Gordon equations.

We start with an example which was already discussed by Shatah and Strauss [*2*] [*3*] but we repeat it for the sake of completeness.

PROPOSITION 3.5. — The equation

$$\phi_{tt} - \Delta\phi = -\phi + |\phi|^{p-1}\phi \qquad 1 < p < \frac{N+2}{N-2}$$

has both stable and unstable standing waves for $1 < p < 1 + \frac{4}{N}$. If $p \geq 1 + \frac{4}{N}$ all standing waves are unstable.

Proof. — u_ω satisfies

$$-\Delta u_\omega = -(1-\omega^2)u_\omega + |u_\omega|^{p-1}u_\omega$$

with $\omega^2 \in (0, 1)$. Now we define

$$v(x) = \lambda^{-2/p-1}u_\omega(x/\lambda), \qquad \lambda = (1-\omega^2)^{1/2}.$$

Then v satisfies

$$-\Delta v = -v + |v|^{p-1}v$$

and thus

$$d(\omega) = (1-\omega^2)^\alpha d(0)$$

where $\alpha = \frac{2}{p-1} - \frac{N-2}{2} > 0$. Calculating the second derivative of $d(\omega)$ yields

$$d''(\omega) = 2\alpha((2\alpha-1)\omega^2 - 1)(1-\omega^2)^{\alpha-2}d(0).$$

CASE 1. — If $p \geq 1 + \frac{4}{N}$, then $\alpha \leq 1$, $d(\omega)$ is concave and u_ω is always unstable.

CASE 2. — If $p < 1 + \frac{4}{N}$, $d''(\omega)$ changes sign at $\omega_c = (2\alpha - 1)^{-1/2}$. u_ω is stable for $\omega_c < |\omega| < 1$ and unstable for $|\omega| \leq \omega_c$.

If $p < 1 + \frac{4}{N}$ we see that for given charge $Q < Q(\omega_c)$ there exist two solitary waves, a stable and a unstable one.

Next we consider a nonlinear Klein-Gordon equation with a fractional nonlinearity, i. e.

$$\phi_{tt} - \Delta\phi = - |\phi|^{p-1}\phi - m^2\phi, \qquad 0 < p < 1.$$

Such models were proposed as an example for a classical field theory with « spontaneously bounded domains » [*13*]. Indeed, the state of lowest energy with frequency ω, which exists if $\omega^2 > m^2$, is spherically symmetric and confined to a finite volume of $\mathbb{R}^N$ [*1*].

We find

PROPOSITION 3.6. — All standing waves of

$$\phi_{tt} - \Delta\phi = - |\phi|^{p-1}\phi - m^2\phi \qquad 0 < p < 1$$

are stable.

Proof.

$$v(x) = \lambda^{-\frac{2}{p-1}} u_\omega(x/\lambda)$$

satisfies

$$-\Delta v = - |v|^{p-1}v + v$$

where $\lambda = (\omega^2 - m^2)^{1/2}$. Therefore

$$d(\omega) = (\omega^2 - m^2)^\alpha \frac{1}{N} T(v)$$

where $\alpha = -\frac{N-2}{2} + \frac{1}{p-1} < 0$ which implies that $d(\omega)$ is always strictly convex.

An important model is the logarithmic Klein-Gordon equation

$$\phi_{tt} - \Delta\phi = k\phi \log |\phi|^2, \qquad k > 0$$

which was often investigated [*14*] [*15*] [*16*].

Standing waves exist for all $\omega \in \mathbb{R}$. The states of lowest energy obtained by theorem 1.1 are explicitly given by

$$\begin{aligned} u_\omega(x) &= \exp\left(-\frac{\omega^2}{2k} + \frac{N}{2} - \frac{k}{2}x^2\right) \\ &= \exp\left(-\frac{\omega^2}{2k}\right) u_0(x) \end{aligned}$$

and they are unique [*17*].

We state the following:

PROPOSITION 3.7. — The logarithmic Klein-Gordon equation admits both stable and unstable standing waves, more precisely

a) If $\omega^2 > k/2$ then u_ω is stable

b) If $\omega^2 \leq k/2$ then u_ω is unstable

Proof. — Obviously $d(\omega) = \exp\left(-\frac{\omega^2}{k}\right)d(0)$.

Differentiating twice yields

$$d''(\omega) = \frac{4}{k^2}\left[\omega^2 - \frac{k}{2}\right]\exp\left(-\frac{\omega^2}{k}\right)d(0)$$

which proves the proposition.

For given charge $Q < Q(u\sqrt{k/2})$ there exist two solitary waves with different energy. The state of lower energy is stable, while the other is unstable.

The same result was obtained by Marques and Ventura investigating the linearized stability equation [*18*]. They also gave an interpretation of the unstable states considering them as resonances.

Finally we consider an example illustrating how to get informations at the right hand side of the existence range for standing waves:

$$\phi_{tt} - \Delta\phi = -\phi + \varepsilon|\phi|^{p-1}\phi - |\phi|^{q-1}\phi$$

where $\varepsilon > 0$, $1 < p < q < \frac{N+2}{N-2}$. For ε large enough the interval $(\omega^*, 1)$ is nonempty.

By Theorem 3.1 there exist stable standing waves for ω close to ω^*.

PROPOSITION 3.8.

a) If $p < 1 + \frac{4}{N}$ there are stable standing waves for ω close to 1.

b) If $p \geq 1 + \frac{4}{N}$ the standing waves are unstable for ω close to 1.

Proof. — u_ω satisfies $-\Delta u_\omega = -(1-\omega^2)u_\omega + \varepsilon u_\omega^p - u_\omega^q$ therefore $v(x) = \lambda^{-\frac{2}{p-1}}v_\omega(x/\lambda)$, $\lambda = (1-\omega^2)^{\frac{1}{2}}$ satisfies $-\Delta v = -v + \varepsilon v^p - (1-\omega^2)^{\frac{q-p}{q-1}}v^q$ which implies that $d(\omega) \sim (1-\omega^2)^\alpha$, $\alpha = \frac{2}{p-1} - \frac{N-2}{2}$ for ω close to 1.

3.2. Examples: Nonlinear Schrödinger equations.

Again we start with an example considered by Strauss and Shatah [3].

PROPOSITION 3.9. — For the equation

$$i\phi_t - \Delta\phi = |\phi|^{p-1}\phi \qquad 1 < p < \frac{N+2}{N-2}$$

all standing waves u_ω are stable if $p < 1 + \frac{4}{N}$ and all standing waves are unstable if $p > 1 + \frac{4}{N}$.

Proof. — Let $\phi_\omega(x, t) = e^{+i\omega t}u_\omega(x)$, $\omega < 0$.
Again we calculate $d(\omega)$ explicitly:

$$d(\omega) = (-\omega)^\alpha d(0), \qquad \alpha = \frac{2}{p-1} - \frac{N-2}{2}$$

which implies

$$d''(\omega) = \alpha(\alpha - 1)(-\omega)^{\alpha-2}d(0)$$

and we conclude using Theorem 2.7.

For the limiting case $p = 1 + \frac{4}{N}$ we obtain $d(\omega) = (-\omega)d(0)$ and Theorem 2.7 does not apply. But these solutions are unstable as proved in [19]. The instability result when $p > 1 + \frac{4}{N}$ was first obtained in [20] while the stability result was also proven by a compactness method [21]. Weinstein [22] proved the above stability result by a spectral analysis of the linearized (about u_ω) Schrödinger operators in dimension $N = 3$.

The logarithmic Schrödinger equation

$$i\phi_t - \Delta\phi = k\phi \log |\phi|^2$$

has standing waves of least energy

$$u_\omega(x) = e^{-\frac{\omega}{2k}}u_0(x)$$

for all $\omega \in \mathbb{R}$. The following proposition is easily proven by using the above representation of u_ω:

PROPOSITION 3.10. — u_ω is stable for all ω.

This result was also obtained by Cazenave using compactness methods [23].

Finally we consider an example which is not related directly to the framework presented in this paper, but which allows also an explicit calculation of $d(\omega)$, namely the so called Pekard-Choquard (or Hartree) equation

$$i\phi_t + \Delta\phi + \left(\int_{\mathbb{R}^3} |x-y|^{-1} |\phi(y)|^2 dy\right)\phi = 0 .$$

Standing waves $e^{i\omega t}u_\omega(x)$ satisfy the stationary equation ($\omega > 0$):

$$\Delta u - \omega u + \left(\int_{\mathbb{R}^3} |x-y|^{-1} |u(y)|^2 dy\right)u = 0 .$$

The minimization problem was solved by Lieb [*24*]. For each $\omega > 0$ there exists $u_\omega \in \mathrm{H}^1(\mathbb{R}^3)$ which satisfies the above equation and has least energy among all possible solutions.

The Cauchy problem was studied by Ginibre and Velo [*25*]: for each $\phi_0 \in \mathrm{H}^1(\mathbb{R}^3)$ there exists a unique solution $\phi \in \mathrm{C}(\mathbb{R}_0^+, \mathrm{H}^1(\mathbb{R}^3))$. In addition, charge and energy are conserved quantities for this solution.

One can verify that the Sahath-Strauss formalism also applies to this problem and we obtain

PROPOSITION 3.11. — $d''(\omega) > 0$ for each $\omega > 0$ and therefore all standing waves u_ω are stable.

Proof. — Since $v(x) = \omega^{-1}u_\omega(\omega^{-1/2}x)$ satisfies

$$\Delta v - v + \left(\int_{\mathbb{R}^3} |x-y|^{-1} |v(y)|^2 dy\right)v = 0$$

we see

$$d(\omega) = \omega^{+3/2}d(1)$$

which yields the proposition.

The same stability result was obtained by Cazenave and Lions [*21*] using the concentration compactness principle.

Let us remark that this example can also be understood as a special case of systems, since we can write the Pekard-Choquard equation as the following system of two equations:

$$\begin{aligned} i\phi_t + \Delta\phi + \mathrm{V}\phi &= 0 \\ \Delta\mathrm{V} + |\phi|^2 &= 0 \end{aligned}$$

A natural generalization of this problem is to study the stability of the standing waves solutions for Yukawa—coupled nucleon—and meson field under the time evolution

$$\begin{aligned} i\phi_t + \Delta\phi &= g\mathrm{V}\phi \\ \mathrm{V}_{tt} - \Delta\mathrm{V} + m^2\mathrm{V} &= -g|\phi|^2 . \end{aligned}$$

IV. SOME REMARKS ABOUT STABILITY CRITERIA

The Shatah-Strauss formalism gives sufficient conditions for stability/instability of a ground state solution u_ω for a scalar field in the sense of Liapunov as obtained in Theorem 2.7:

If the action $S_\omega(u_\omega)$, considered as a function of ω, is strictly convex/concave at ω_0 then u_{ω_0} is stable/unstable.

On the other hand in physics there are a lot of different stability criteria for localized solutions. Especially due to their simplicity the following criteria have been used by physicists.

1. Linear dynamical stability.

A localized solution is dynamically stable (in the sense of Liapunov) if small perturbations do not destroy it, i. e. one studies the behaviour of

$$u(x, t) = e^{i\omega t}u_\omega(x) + e^{i\omega t}\eta(x, t).$$

The first order approximations leads to a linear eigenvalue problem.

2. Energetic Stability.

The standing wave solutions are critical points of the Lagrangian (action) and critical points of the energy restricted to the manifold of constant charge. Now by physical intuition one expects that the standing waves are stable if they are local minima of the energy. Otherwise they should be unstable.

Now, what are the relations or contact points between these different stability criteria?

First we observe that the Shatah-Strauss formalism proves the validity of a part of the energetic stability. Indeed, by Theorem 2.2 the convexity of the action (as a functional of the frequency) is equivalent to the fact that the energy has a local minimum on the manifold of constant charge and the convexity of the action implies the stability of the considered standing wave.

On the other hand physicists mostly use the second part of the energy criterion. Their work can be described in the following way:

Find a suitable curve along which the energy does not have a local minimum in u_ω.

In order to illustrate we consider the usual *example* (prototype curve):

Let $u(\lambda) = \lambda u_\omega(\lambda^{2/N}x)$; then the charge is fixed along the curve $u(\lambda)$ and equal to $Q_\omega(u_\omega)$. Consider the function

$$E(\lambda) = E_\omega(u(\lambda)) = \frac{1}{2}T(u(\lambda)) - V_\omega(u(\lambda)) + \omega Q_\omega(u_\omega)$$
$$= \frac{1}{2}\lambda^{4/N}T(u_\omega) - \lambda^{-2}V_\omega(\lambda u_\omega) + \omega Q_\omega(u_\omega).$$

We compute

$$\frac{dE}{d\lambda} = \frac{2}{N}\lambda^{-\frac{4-N}{N}}T(u_\omega) + 2\lambda^{-3}V_\omega(\lambda u_\omega) - \lambda^{-2}\int_{\mathbb{R}^N} u_\omega g_\omega(\lambda u_\omega)dx.$$

Since u_ω is a critical point of the energy along $u(\lambda)$ we have

$$\left.\frac{dE}{d\lambda}\right|_{\lambda=1} = 0.$$

Calculating the second derivative at $\lambda = 1$ yields

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} = -\frac{4N+8}{N}\int_{\mathbb{R}^N} G_\omega(u_\omega)dx + \frac{3N+4}{N}\int_{\mathbb{R}^N} u_\omega g_\omega(u_\omega)dx$$
$$- \int_{\mathbb{R}^N} u_\omega^2 g'_\omega(u_\omega)dx.$$

If $\left.\dfrac{d^2E}{d\lambda^2}\right|_{\lambda=1} < 0$ the energy does not have a local minimum at u_ω.

Then the energy criterion states that u_ω is unstable.

Observe that all linear terms vanish in the above expression. Therefore we can write the instability criteria as follows:

If

$$(4.1) \qquad \int_{\mathbb{R}^\nu} - (4N+8)G(u_\omega) + (3N+4)u_\omega g(u_\omega) - Nu^2 g'(u_\omega)dx < 0$$

then u_ω is unstable.

If we apply (4.1) to $g(u) = |u|^{p-1}u - u, p > 1$ we see that (4.1) is satisfied if $p > 1 + \dfrac{4}{N}$ (!)

Although this result is very impressive it must not be true in general because we do not know whether a local minimum of energy is a *necessary* condition for stability.

We can only conclude $d''(\omega) \leq 0$ but it is not known what happens in inflection points of $d(\omega)$. We expect instability (see Section 3) but this still remains to be proven in general (see additional remark at the end of the paper).

Thus the Shatah-Strauss formalism and the energy criterion are « almost » equivalent. This relies on the fact that the convexity of $d(\omega)$ at ω_0 and the existence of a local minimum of the energy (with respect to fixed charge) are equivalent. Let us remark that this property does not hold for general field theories. For example, in [26] we prove that for classical nonlinear spinor fields the convexity of $d(\omega)$ and the local minimum property of the energy are no longer equivalent. As a consequence for such field theories thêse two stability criteria cannot coincide.

The linear dynamical stability is of course closely related to the energetic stability. The linearized operator is given by

$$\mathrm{H} = -\Delta - g'(u_\omega) - \omega^2 \tag{4.2}$$

where u_ω is the ground state of the nonlinear equation. As observed by Derrick [25] H acting on $L^2(\mathbb{R}^N)$ has at least one negative eigenvalue which is a consequence of dilation invariance : $w_\omega = r\dfrac{\delta u_\omega}{\delta r}$ satisfies

$$(w_\omega, \mathrm{H}w_\omega) = -(\mathrm{N}-2)\mathrm{T}(u_\omega) \tag{4.3}$$

where (., .) denotes the usual scalar product in $L^2(\mathbb{R}^N)$.

As a consequence of the spatial translation invariance of the nonlinear problem, the set $\left(\dfrac{\delta u_\omega}{\delta x_k}\right)_{1\le k\le \mathrm{N}}$ belongs to the kernel of H. Therefore there exists a least one spherically symmetric state having a lower energy.

Therefore one could expect instability which is not true in general. There are two reasons:

First most of the spectrum of the corresponding dynamical problem (Klein-Gordon-equation) lies on the imaginary axis so that one cannot conclude that linearized instability implies true stability.

Second the eigenfunction corresponding to the negative eigenvalue may not be in the tangent space of the manifold M_0 at $(u_{\omega_0}, i\omega_0 u_{\omega_0})$ and therefore the quadratic form defined by the constrained second variation of the energy can be positive definite (i. e. the energy can have a local minimum on M_0).

To overcome these difficulties Shatah and Strauss invented the « modified linearized operator » [3]:

$$\mathrm{T} = -\Delta - g'(u_{\omega_0}) - \omega_0^2 + \frac{4\omega_0^2}{(u_{\omega_0}, u_{\omega_0})}(u_{\omega_0}, .)u_{\omega_0}\,. \tag{4.4}$$

Then we have

$$\frac{d^2}{d\omega^2}\mathrm{E}_\omega(v_\omega)\bigg|_{\omega=\omega_0} = (\mathrm{T}y_0, y_0) \tag{4.5}$$

where
$$y_0 = \frac{\delta v_\omega}{\delta\omega}\bigg|_{\omega=\omega_0}$$

Using Corollary 2.4 we see

$$(4.6\,a) \qquad (\mathrm{T}y_0, y_0) > 0 \Rightarrow d''(\omega_0) > 0$$

$$(4.6\,b) \qquad d''(\omega_0) < 0 \Rightarrow (\mathrm{T}y_0, y_0) < 0$$

(4.6 *a*, *b*) gives the relation between the linear operator T and the Shatah-Strauss formalism resp. energy criterion.

In the following we are looking for similar relations by direct analysis of the linearized operators H and T.

Let $(u(\lambda); iv(\lambda))$ be a curve in $\mathrm{W}_r \oplus \mathrm{L}_r^2$ such that $u(0) = u_{\omega_0}$ and $v(0) = \omega_0 u(0) = \omega_0 u_{\omega_0}$. The charge should be fixed along this curve, i. e. $(u(\lambda), iv(\lambda))$ is a curve in M_0. It is enough to consider u and v as real valued functions. Then the following proposition holds:

PROPOSITION 4.1. — For $\mathrm{E} \equiv \mathrm{E}(u(\lambda), iv(\lambda))$ we have

$$(4.7) \qquad \left.\frac{d\mathrm{E}}{d\lambda}\right|_{\lambda=0} = 0$$

$$(4.8) \qquad \left.\frac{d^2\mathrm{E}}{d\lambda^2}\right|_{\lambda=0} = \int_{\mathbb{R}^N} (\dot{v}(0) - \omega_0(\dot{u}(0))^2 dx + \int_{\mathbb{R}^\nu} \dot{u}(0)\mathrm{H}\dot{u}(0)dx$$

and the tangent vectors $\dot{u}(0)$, $\dot{v}(0)$ satisfy

$$(4.9) \qquad \int_{\mathbb{R}^N} (\omega_0\dot{u}(0) + \dot{v}(0))u_{\omega_0}dx = 0\,.$$

Proof. — Since the charge $\mathrm{Q}(u, iv) = \int_{\mathbb{R}^N} u(\lambda)v(\lambda)dx$ is fixed we have

$$0 = \left.\frac{d\mathrm{Q}}{d\lambda}\right|_{\lambda=0} = \int_{\mathbb{R}^N} (\dot{u}(0)v(0) + \dot{v}(0)u(0))dx$$

which implies (4.9) and

$$0 = \left.\frac{d^2\mathrm{Q}}{d\lambda^2}\right|_{\lambda=0} = 2\int_{\mathbb{R}^N} \dot{v}(0)\dot{u}(0)dx + \int_{\mathbb{R}^N} (\omega\ddot{u}(0) + \ddot{v}(0))u_\omega dx\,.$$

Then we compute

$$\frac{d\mathrm{E}}{d\lambda} = \int_{\mathbb{R}^N} (\dot{v}v - \dot{u}\Delta u - \dot{u}g(u))dx$$

$$\frac{d^2\mathrm{E}}{d\lambda^2} = \int_{\mathbb{R}^N} (v\ddot{v} + \dot{v}^2 - \ddot{u}\Delta u + |\nabla\dot{u}|^2 - \ddot{u}g(u) - \dot{u}^2 g'(u))dx$$

from which we easily verify (4.7) and (4.8) using the relations for the derivations of the charge at $\lambda = 0$.

Defining

$$x \equiv \omega_0 \dot{u}(0) - \dot{v}(0) \in L_r^2$$
$$y = \dot{u}(0) \in W_r$$

we rewrite (4.8) and (4.9) as follows

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=0} = (x, x) + (y, Hy) \tag{4.10}$$

$$(x, u_{\omega_0}) = 2\omega_0(y, u_{\omega_0}) \tag{4.11}$$

We see that the second derivative in $(u_{\omega_0}, i\omega_0 u_{\omega_0})$ is given by a quadratic form in $L_r^2 \oplus L_r^2$. Now the energy has a local minimum in M_0 at $(u_{\omega_0}, i\omega_0 u_{\omega_0})$ iff the quadratic form given by (4.10) is positive under the constraint (4.11).

For a detailed analysis of this problem we assume that the linearized operator H satisfies the following condition:

(H) *All* zero modes of H are generated by spatial translation invariance of the non linear problem, i. e. the kernel of H is spanned by the set $\left(\frac{\delta u_{\omega_0}}{\delta x_k}\right)_{1 \le k \le N}$.

Furthermore—for technical simplicity—we restrict ourselves to non-linearities $g(u)$ which have finite derivatives at the origin.

Let us remark that the validity of condition (H) was proven by M. Weinstein in space dimensions $N = 1$ and $N = 3$ (at least for special cases) [*28*].

Using condition (H) one can prove the following

PROPOSITION 4.2. — If H satisfies condition (H) then it has exactly one negative eigenvalue.

A proof of proposition 4.2 was also given by M. Weinstein [*22*] [*28*].

Next we prove an estimate for the form given in (4.10):

PROPOSITION 4.3. — Let $[x, y] \in L^2 \oplus L^2$ satisfying (4.11).
Then the following inequality holds

$$(x, x) + (y, Hy) \ge (y, Ty) \tag{4.12}$$

with equality iff $x = \delta u_{\omega_0}$ for some $\delta \in \mathbb{R}$.

Proof. — Using the Cauchy-Schwartz inequality and (4.11) we obtain

$$(x, x) + (y, Hy) \ge \frac{(x, u_{\omega_0})^2}{(u_{\omega_0}, u_{\omega_0})} + (y, Hy) = 4\omega_0^2 \frac{(y, u_{\omega_0})^2}{(u_{\omega_0}, u_{\omega_0})} + (y, Hy)$$
$$\equiv (y, Ty). \qquad \square$$

In view of Proposition 4.3 it is enough to consider the infimum of the quadratic form defined by T. It is easy to see that

$$\alpha \equiv \inf_{\substack{(y,y)=1 \\ y\in L^2}} (y, Ty)$$

is nonpositive on L^2 since $\left(\frac{\delta u_{\omega_0}}{\delta x_k} T \frac{\delta u_{\omega_0}}{\delta x_k}\right) = 0$ for $1 \leq k \leq N$.

Especially we have

PROPOSITION 4.4:

a) $\alpha \equiv \inf_{(y,y)=1} (y, Ty)$ is attained for an $y^* \in L^2$.

b) $\alpha = 0$ iff $d''(\omega_0) \geq 0$.

Proof. — Let $\{ y_u \}$ be a minimizing sequence, i. e. $(y_u, y_u) = 1$ and $(y_u, Ty_u) \to \alpha$. Obviously $\{ y_u \}$ is bounded in H^1.

Thus there exist $y^* \in H^1$ and a subsequence of $\{ y_u \}$ (again denoted by $\{ y_u \}$) for which

$$y_u \to y^* \quad \text{weakly in } H^1 .$$

By weak convergence $(y_u, u_{\omega_0}) \to (y^*, u_{\omega_0})$.

Since $V \equiv g'_{\omega_0}(u_{\omega_0}) - g'_{\omega_0}(0) \in L^{N/2}$ we obtain by uniform decay of u_{ω_0} that $(y_u, Vy_u) \to (y^*, Vy^*)$.

For any $\varepsilon > 0$ we can choose n such that

$$(\nabla y_n, \nabla y_n) - g'_\omega(0)(y_n, y_n) - \alpha < (y_n, Vy_n) - 4\omega_0^2 \frac{(y_n, u_{\omega_0})^2}{(u_{\omega_0}, u_{\omega_0})} + \varepsilon .$$

Therefore $y^* \not\equiv 0$ since $g'_\omega(0) < 0$ and $\alpha \leq 0$ and ε is arbitrary.

By Fatou's lemma and weak convergence we have

$((y^*, y^*) \leq 1$ and $(\nabla y^*, \nabla y^*) \leq \liminf_n (\nabla y_n, \nabla y_n)$ and therefore

$$(y^*, Ty^*) \leq \liminf_u (y_u, Ty_u) = \alpha .$$

Suppose $(y^*, y^*) < 1$. Then we have

$$\alpha \leq \left(\frac{y^*}{\| y^* \|}, T\frac{y^*}{\| y^* \|}\right) = \frac{(y^*, Ty^*)}{(y^*, y^*)} \leq \frac{\alpha}{(y^*, y^*)} < \alpha$$

if $\alpha < 0$ which yields a contradiction. If $\alpha \geq 0$ we obtain equality. Thus we can take $(y^*, y^*) = 1$.

For *b)* we use the fact that y^* satisfies the Euler-Lagrange equation

$$Ty^* = \alpha y^* . \tag{4.13}$$

Let $d''(\omega_0) > 0$. It suffices to show $\alpha \geq 0$. Suppose $\alpha < 0$.

If $(y^*, u_{\omega_0}) = 0$ then α is an eigenvalue of H and therefore it must be the groundstate by proposition 4.2. Thus $y^* > 0$ which contradicts the orthogonality condition $(y^*, u_{\omega_0}) = 0$. Therefore $(y^*, u_{\omega_0}) \neq 0$.

We now consider the function

$$g(\alpha) = (u_{\omega_0}, (1 + u_{\omega_0}^2(\mathrm{H} - \alpha)^{-1})u_{\omega_0}) \tag{4.14}$$

$g(\alpha)$ is well defined, smooth and increasing on $(\lambda_1, 0)$ where λ_1 denotes the lowest eigenvalue of H. By (4.13) we need $g(\alpha) = 0$.

Therefore a sufficient condition for g to have no zero on $(\lambda_1, 0)$ is

$$g(0) \equiv (u_{\omega_0}, (1 + 4\omega_0^2\mathrm{H}^{-1})u_{\omega_0}) < 0\,. \tag{4.15}$$

Since $\dfrac{\delta u_\omega}{\delta\omega}$ *satisfies the elliptic equation*

$$H\frac{\delta u_\omega}{\delta\omega} = 2\omega u_\omega \tag{4.16}$$

we obtain

$$g(0) = \left.\frac{d\,\mathrm{Q}_\omega(u_\omega\,;i\omega u_\omega)}{d\omega}\right|_{\omega=\omega_0} = -\,d''(\omega_0)\,. \tag{4.17}$$

On the other hand if $\alpha = 0$ then $g(\alpha) = 0$ cannot have a negative root.

Thus we proved proposition 4.4.

Now we are able to investigate what happens in the space of radial functions.

We state the following.

THEOREM 4.5. — There exists a positive constant $\mathrm{C}(\omega_0)$ such that for any $y \in \mathrm{L}_r^2$

$$(y, \mathrm{T}y) \geq \mathrm{C}(\omega_0)(y, y) \tag{4.18}$$

if and only if $d''(\omega_0) > 0$.

Proof. — Let $d''(\omega_0) > 0$. Then by Proposition 4.4 $(y, \mathrm{T}y)$ is nonnegative. We will show that $(y, \mathrm{T}y)$ is positive on $(\mathrm{Ker\,H})^\perp$, i. e. that (4.18) holds for all $y \in \mathrm{L}^2$ which satisfy

$$(y, \nabla u_{\omega_0}) = 0\,. \tag{4.19}$$

Then the first part of Theorem 4.5 is proven since all radial functions are in the orthogonal complement of Ker H.

Let

$$\beta = \inf_{\substack{y\in\mathrm{L}^2\\(y,y)=1\\(y,\nabla u_{\omega_0})=0}} (y, \mathrm{T}y)\,. \tag{4.20}$$

We will prove $\beta > 0$ by showing that the assumption $\beta = 0$ yields a contradiction.

If $\beta = 0$ one can show as in proposition 4.4 that the infimum is attained by a function y^* satisfying

$$(4.21)\qquad (y^*, y^*) = 1, \qquad (y^*, \nabla u_{\omega 0}) = 0$$
$$(y^*, \mathrm{T}y^*) = 0$$

Furthermore g satisfies the Euler Lagrange equation

$$(4.22)\qquad \mathrm{T}y^* = \beta y^* + \gamma \nabla u_{\omega 0}$$
$$= \gamma \nabla u_{\omega 0}$$

for some $\gamma \in \mathbb{R}^{\mathrm{N}}$.

Taking the linear product of (4.22) with $\nabla u_{\omega 0}$ we see $\gamma \equiv 0$.

Now, if $(y^*, u_{\omega 0}) = 0$ then $y^* = \nabla u_{\omega 0}$ which is impossible. Thus $(y^*, u_{\omega 0}) \neq 0$. But then we have $0 = g(0) = d''(\omega_0)$ which contradicts the assumption.

Therefore we conclude $\beta > 0$.

On the other hand let (4.18) hold. Then in particular

$$(4.23)\qquad (\mathrm{T}y_0, y_0) > 0$$

which implies $d''(\omega_0) > 0$ by (4.6 *a*).

REMARK 4.6. — In the same way one can show that the condition

$$(4.24)\qquad (u_{\omega 0}, (1 + 4\omega_0^2 \mathrm{H}^{-1}) u_{\omega 0}) > 0$$

implies $\alpha < 0$ (resp. $\beta < 0$).

Thus we put the linearized operator into the context of the Shatah-Strauss formalism resp. the energetic criterion.

REMARK 4.7. — Sometimes it is useful to take the representation in the orthonormal base $\{ e_j \}$ of eigenfunctions of H. Writing $u_{\omega 0} = \Sigma u_i e_i$ we finally obtain

$$(4.25)\qquad g(0) = \sum_j \left(1 + \frac{4\omega_0^2}{\lambda_j}\right) u_j^2$$

where λ_j denote the eigenvalues of H acting on L_r^2.

REMARK 4.8. — Suppose H possesses two negative eigenvalues. Then $g(\alpha) = 0$ has always a negative root for $\alpha \in (\lambda_1, \lambda_2)$ and therefore the energy cannot have a local minimum on M_0. Since this result does not despend on $g(0)$ (resp. the property of Q as a function of ω) it looks rather unnatural. Therefore we conjecture that H always possesses only one negative eigenvalue.

REMARK 4.9. — For the NLS a similar analysis was done in the already mentioned papers of M. Weinstein [*22*] [*28*] which yield the same result. Furthermore the linear evolution is investigated.

Finally we consider an example to illustrate the abstract framework (although not covered there):

EXAMPLE 4.10. — We consider the logarithmic Klein-Gordon equation

$$\phi_{tt} - \Delta\phi = K\phi \log |\phi|^2 .$$

The standing waves of lowest energy have the form

$$u_\omega(x) = \exp\left(-\frac{\omega^2}{2k} + \frac{N}{2} - \frac{k}{2}x^2\right).$$

Formally, the linearized operator is a shifted harmonic oscillator (note that it does not exist on W_r).

$$H = -\Delta + k^2x^2 - (N+2)k . \tag{4.25}$$

It is obvious that H satisfies condition (H) and there exists an orthonormal base of L_r^2 consisting of eigenfunctions of H.

The ground state of H is given by

$$e_1(x) = c_1(\omega)u_0(x)$$

with eigenvalue $\lambda_1 = -2k$. The first excited state on L_r^2 is given by

$$e_2(x) = c_2(\omega)\left(1 - \frac{2k}{N}r^2\right)u_0(x)$$

with eigenvalue $\lambda_2 = 2k$.

Now we have

$$\begin{aligned} g(0) &= \sum_j \left(1 + \frac{4\omega^2}{\lambda_j}\right)u_j^2 \\ &= c_1(\omega)^2\left[1 - \frac{2\omega^2}{k}\right] \end{aligned}$$

which yields the well-known result.

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Additional Remark. — When we finished this paper we got knowledge of a work concerned with the stability theory of solitary waves by Grillakis, Shatah and Strauss [*29*]. They investigated an abstract Hamiltonian system which is invariant under a one-parameter unitary group of operators (the gauge group for NLKG and NLS) and obtained the stability/instability result by an analysis of the linearized operator

$$\hat{\mathrm{H}} = \mathrm{D}^2\mathrm{E}(\vec{u}_\omega) - \omega \mathrm{D}^2\mathrm{Q}(\vec{u}_\omega)$$

where $\vec{u}_\omega = [u_\omega, i\omega u_\omega]$. This is a mathematical generalization of the (previous) Shatah-Strauss formalism [*2*] [*3*].

Especially they proved that if u_ω is stable then $d(\omega)$ must be convex resp. the energy has a local minimum on the manifold of constant charge. Therefore there is an equivalence between stability and the local minimum property of the energy as we claimed in Section 4.

Our mathematical results on the stability partially fit in the framework presented by Grillakis, Shatah and Strauss [*29*]. On the other hand we proved the applicability of the theory to a larger class of models. In particular for fractional or logarithmic nonlinearities $\mathrm{D}^2\mathrm{E}(\vec{u}_\omega)$ does not exist on W_r and therefore the theory of [*29*] does not apply.

In addition we presented for the first time a detailed rigorous study of the linearized operators which may be useful for future applications.

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