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« Geometry » of spin 3 gauge theories

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ABSTRACT. — Some « geometrical » aspects of spin 3 gauge theories are developed in arbitrary dimension. The spin 3 analogs of the (linearized) Riemann and Weyl tensors are introduced and studied. « Curvature », whose vanishing implies that the field is pure gauge, is shown to differ from « Riemann » here, but the « Weyl » tensor remains (when $D \geqq 4$) the criterion for « conformal flatness ». In $D = 3$, where a natural rank-preserving curl exists, the « topologically massive » theory is defined and analyzed.

RÉSUMÉ. — Quelques aspects « géométriques » des théories de jauge de spin 3 sont développés en dimension arbitraire. Les analogues, en spin 3, des tenseurs (linéarisés) de Riemann et de Weyl sont introduits et étudiés. La « courbure » (dont l'annulation implique que le champ est une pure jauge) s'avère ne plus coincider avec « Riemann », en revanche « Weyl » reste (quand $D \geqq 4$) le critère pour savoir si un champ est « conforme » à une pure jauge. En dimension $D = 3$, où existe un rotationnel naturel qui préserve l'ordre des tenseurs, on définit et analyse la théorie « topologiquement massive ».

§ 1. INTRODUCTION

Higher integer spin gauge theories differ profoundly from the spin 2 case because there is no geometrical unification between the background Minkowski tensor $\eta_{\mu\nu}$ and the dynamical field ϕ into a single object of the form $\eta_{\mu\nu} + \phi_{\mu\nu} =: g_{\mu\nu}$. This absence of an invertible « metric » field implies that both self-interactions and couplings to other systems are strongly restricted. Nevertheless, it is possible to treat the free fields in a « geometrical » fashion, much like that of linearized gravity in flat space. This treatment, first elaborated by de Wit and Freedman [*1*], is further developed here. We hope that the detailed study of these « linearized geometrical structures » will be useful for the study of possible corresponding non-linear geometric structures, thereby providing us with an efficient tool to attack the difficult problem of interactions (for recent work on the latter problem see for example [*2*] [*3*]). We will also analyze the special case of dimension D = 3, where the analog of a Chern-Simons term exists and can also be put in geometrical form. We will deal primarily with spin 3 (in arbitrary dimension) to avoid excessive index proliferation, but most of our considerations apply for higher spins. We leave a detailed general treatment of arbitrary, integer and half-integer, spin gauge fields to further work.

This paper is organized as follows: we end the introduction by explicating our notation and conventions; in section 2 we define and study various spin 3 analogs of well-known spin 2 geometrical objects: the field (§ 2.1), the « affinities » (§ 2.2), the « Ricci » and « Einstein » tensors (§ 2.3), the « Riemann » tensor (§ 2.4), and finally the « Weyl » tensor (§ 2.5). We discuss the concepts of spin 3 « curvature » (different from « Riemann » here) and of « conformal curvature » in sections 3 and 4 respectively. Section 5 introduces a generalized « curl operator », defined in D = 3; this operator is used in section 6 which studies the spin 3 analog of « topologically massive » gauge theories. Some technical details are relegated to the appendices.

NOTATION

Our metric, in a space-time of dimension D, is $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, \ldots, +1)$, greek indices taking the values $0, 1, \ldots, D-1$. The (totally antisymmetric) Levi-Civita tensor is normalized to $\varepsilon_{012\ldots} = +1 = -\varepsilon^{012\ldots}$. The symbol $:=$ in $A := B$ means that A is *defined* as being B, while $A =: B$ means that B is defined as being A. We use the symbol $\equiv$ only for (algebraic or differential) identities. As a rule, our notation is chosen so as to simplify the appearance of formulas. For example, we shall sometimes take the liberty of putting the dummy indices of a contraction in a « wrong » position if it increases the legibility of the expression: e. g. $\partial^\alpha \varphi^\alpha{}_{\mu\nu}$ denotes $\partial^\alpha \phi_{\alpha\mu\nu} \equiv \eta^{\alpha\beta}\partial_\alpha \phi_{\beta\mu\nu}$.

Parentheses denote *unnormalized* symmetrization, and mean the *sum* of the *minimal* number of terms required to achieve symmetry, taking into account any manifest symmetries of the constituents [4]. For example, if $B_{\mu\nu}$ is known to be symmetric (sometimes indicated as $B_{\underline{\mu\nu}}$), one has

$$A_{\alpha(\mu}B_{\nu\lambda)} := A_{\alpha\mu}B_{\nu\lambda} + A_{\alpha\nu}B_{\lambda\mu} + A_{\alpha\lambda}B_{\mu\nu}\,. \tag{1.1}$$

Then, if $A_{\mu\nu}$ is also known to be symmetric, $A_{(\mu\nu}B_{\rho\sigma)}$ denotes a sum of 6 terms, except when $A_{\mu\nu} \equiv B_{\mu\nu}$ in which case $A_{(\mu\nu}A_{\rho\sigma)}$ denotes a sum of only 3 terms. Any dummy indices are ignored in the process of symmetrization, and are sometimes indicated in the « wrong » position to simplify parentheses. For example, if $B_{\mu\nu\lambda}$ is symmetric,

$$A_{(\mu\alpha}B^{\alpha}{}_{\nu\lambda)} \equiv A_{(\mu}{}^{\alpha}B_{\alpha\nu\lambda)} \equiv A_{(\mu}{}^{\alpha}B^{\alpha}{}_{\nu\lambda)} = A_{\mu\alpha}B^{\alpha}{}_{\nu\lambda} + A_{\nu\alpha}B^{\alpha}{}_{\lambda\mu} + A_{\lambda\alpha}B^{\alpha}{}_{\mu\nu}\,. \tag{1.2}$$

Antisymmetrization, likewise not normalized, is indicated by square brackets around (or, for two indices, a hook below) the relevant indices. For example,

$$\partial_{[\alpha}\phi_{\mu]} \equiv \partial_{\underline{\alpha}}\phi_{\underline{\mu}} := \partial_\alpha\phi_\mu - \partial_\mu\phi_\alpha \tag{1.3}$$

and, if $F_{\mu\nu}$ is already antisymmetric (sometimes indicated as $F_{\overset{\vee}{\mu\nu}}$)

$$\partial_{[\lambda}F_{\mu\nu]} := \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,. \tag{1.4}$$

When needed, *normalized* symmetrization (resp. antisymmetrization) will be denoted by a parenthesis (resp. square bracket) qualified by the index 1 (= « effective » number of terms involved). For example, when B (resp. F) is symmetric (resp. antisymmetric):

$$A_{\alpha(\mu}B_{\nu\lambda)_1} := \frac{1}{3}\,A_{\alpha(\mu}B_{\nu\lambda)} \tag{1.5}$$

$$\phi_{[\mu}F_{\nu\lambda]_1} := \frac{1}{3}\,\phi_{[\mu}F_{\nu\lambda]}\,. \tag{1.6}$$

When possible and convenient we will use an index-free notation: T_n denoting a n-tensor $T_{\mu_1\ldots\mu_n}$.

Trace-free tensors will often bear a tilde. In the following ϕ_3 (the « field »), and ξ_2 (« generalized gauge parameter ») will always denote *symmetric* tensors, while ϕ_1 and ξ_0 will denote their traces, e. g.

$$\phi_\lambda := \phi^{\alpha\alpha}{}_{\lambda}\,. \tag{1.7}$$

Similarly the symmetric and trace-free « gauge parameter » $\tilde{\xi}_2$ will have $\tilde{\xi}_0 \equiv 0$. In index-free notation the dot means maximal contraction on (usually symmetric) neighbouring indices, for example, $\phi_3 \cdot G_3$ (with G_3 symmetric) denotes $\phi_{\mu\nu\lambda}G^{\mu\nu\lambda}$, and $\partial\cdot\phi_3$ denotes $\partial^\alpha\phi^\alpha{}_{\mu\nu}$.

When using the apparatus of (linearized) affinities, it will be convenient to drop the 1/2 factors from the Christoffel symbols, so that our $\Gamma^{\alpha}{}_{\mu\nu}$ for spin 2 is twice the usual one. Note that our Γ's have the opposite sign of those in [*1*].

It has also been found convenient to choose a sign convention for the « Ricci » and « Riemann » tensors which is opposite to the one of [*5*] for spin 2, so that our $R_{\mu\nu}$ and $R_{\alpha\mu\beta\nu}$ are minus two times theirs.

§ 2. SPIN 3 « GEOMETRY »

2.1 The field.

The spin 3 is here described by a symmetric 3-tensor ϕ_3, although a non-symmetric « vielbein » formulation is also possible [*4*]. Its action is invariant under « gauge transformations », $\delta_{\tilde{\xi}}\phi_3$, induced by an arbitrary symmetric *trace-free* $\tilde{\xi}_2$:

$$\delta_{\tilde{\xi}}\phi_{\mu\nu\lambda} := \partial_{(\mu}\tilde{\xi}_{\nu\lambda)}, \qquad \tilde{\xi}_{\alpha}{}^{\alpha} \equiv 0 . \tag{2.1 a}$$

This defines our choice of ϕ_3 as against possible field redefinitions of the form $\phi_3 + c\eta_{(2}\phi_{1)}$. The « generalized gauge transformations »,

$$\delta_{\xi}\phi_3 := \partial_{(1}\xi_{2)}, \qquad \xi_0 \not\equiv 0 , \tag{2.1 b}$$

induced by an arbitrary symmetric ξ_2 will also be considered.

In the following we shall mean by « spin 3 geometry » the study of the equivalence classes of spin 3 fields modulo the $(\tilde{\xi})$ gauge transformations.

2.2 The hierarchy of affinities.

« Affinities » or « generalized Christoffel symbols » in the terminology of [*1*], are introduced, as in spin 2, to transform (as closely as possible) like (single term) gradients of $\tilde{\xi}_2$. This leads to a natural definition of gauge invariants by taking traces and/or curls. For spin 2 (linearized general relativity) $\delta_{\xi}\phi_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$ and $\Gamma^{\alpha}{}_{\mu\nu} := \partial_{(\mu}\phi_{\nu)}{}^{\alpha} - \partial^{\alpha}\phi_{\mu\nu}$, so that

$$\delta_{\xi}\Gamma^{\alpha}{}_{\mu\nu} = 2\partial_{\mu\nu}\xi^{\alpha} . \tag{2.2}$$

Here the nearest one can come to this requirement is through

$$\overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu\lambda} := \partial_{(\mu}\phi_{\nu\lambda)}{}^{\alpha} - \partial^{\alpha}\phi_{\mu\nu\lambda} , \tag{2.3}$$

which transforms as:

$$\delta_{\xi}\overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu\lambda} = 2\partial_{(\mu\nu}\xi_{\lambda)}{}^{\alpha} . \tag{2.4}$$

A second affinity, defined by

$$\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda} := \partial_{(\mu}\overset{(1)}{\Gamma}{}^{\alpha}{}_{\nu\lambda)\beta} - 2\partial_{\beta}\overset{(1)}{\Gamma}{}^{\alpha}{}_{\mu\nu\lambda} \tag{2.5}$$

is separately symmetric in $(\alpha\beta)$ and $(\mu\nu\lambda)$, as shown by its explicit expression in terms of ϕ_3:

$$\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda} = 2\partial_{(\mu\nu}\phi^{\alpha\beta}_{\lambda)} - \partial^{(\alpha}_{(\mu}\phi^{\beta)}_{\nu\lambda)} + 2\partial^{\alpha\beta}\phi_{\mu\nu\lambda}\,. \tag{2.6}$$

Under a generalized gauge transformation $\overset{(2)}{\Gamma}$ transforms as a (multi-) gradient:

$$\delta_{\xi}\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda} = 6\partial_{\mu\nu\lambda}\xi^{\alpha\beta}\,. \tag{2.7}$$

2.3 The « Ricci » and « Einstein » tensors.

It appears from eq. (2.7) that the $(\alpha\beta)$-trace of $\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda}$ is $\tilde{\xi}$-gauge invariant. This leads to the definition of a « Ricci » tensor,

$$R_{\mu\nu\lambda} := \frac{1}{2}\overset{(2)}{\Gamma}{}^{\alpha\alpha}{}_{\mu\nu\lambda} = \Box\phi_{\mu\nu\lambda} - \partial^{\alpha}_{(\mu}\phi^{\alpha}{}_{\nu\lambda)} + \partial_{(\mu\nu}\phi_{\lambda)} \tag{2.8}$$

(ϕ_1 denoting the trace of ϕ_3, see eq. (1.7)). « Ricci » transforms as:

$$\delta_{\xi}R_{\mu\nu\lambda} = 3\partial_{\mu\nu\lambda}\xi^{\alpha\alpha}\,; \qquad \delta_{\tilde{\xi}}R_{\mu\nu\lambda} = 0\,. \tag{2.9}$$

As in the spin 2 case, it is convenient to introduce along with « Ricci » and its trace, R_1,

$$R_{\lambda} := R^{\alpha\alpha}_{\ \ \lambda} = 2\left\{\Box\phi_{\lambda} - \partial^{\alpha\beta}\phi^{\alpha\beta}_{\lambda} + \frac{1}{2}\partial^{\alpha}_{\lambda}\phi^{\alpha}\right\}, \tag{2.11}$$

an « Einstein » tensor, G_3, defined, in any dimension D, as

$$G_{\mu\nu\lambda} := R_{\mu\nu\lambda} - \frac{1}{2}\eta_{(\mu\nu}R_{\lambda)}, \qquad G_{\lambda} = -\frac{D}{2}R_{\lambda}\,. \tag{2.12}$$

G_3 satisfies a « conservation identity » up to a trace:

$$\partial^{\lambda}G_{\mu\nu\lambda} - D^{-1}\eta_{\mu\nu}\partial^{\lambda}G_{\lambda} \equiv 0\,, \tag{2.13 a}$$

appropriate to the $\tilde{\xi}$ nature of the gauge invariance.

When written in terms of « Ricci », the differential identity (2.13 *a*) reads:

$$\partial^{\lambda}R_{\mu\nu\lambda} - \frac{1}{2}\partial_{(\mu}R_{\nu)} \equiv 0\,. \tag{2.13 b}$$

In the spin 2 case, the analog of the differential identity (2.13) (whose physical importance was first understood by Einstein) is often called the « contracted Bianchi identity », because it can be obtained by contracting the full (5-indexed) differential identities (first derived by Bianchi) satisfied by the Riemann tensor. As we shall see below the situation is very different in spin 3 where the identity (2.13) can be derived, by contraction, from the *algebraic* symmetries of « Riemann », R_6, and not from the differential (Bianchi) identities of R_6. In the following, we will refer to (2.13) as the « Einstein identities ».

One checks also that the operator: $\phi_3 \rightarrow G_3(\phi_3)$ is self-adjoint in the sense that for any pair of symmetric 3-tensors the scalar $\psi_3 \cdot G_3(\phi_3) - G_3(\psi_3) \cdot \phi_3$ is a total divergence. This property is the basis for obtaining the free field equations $G_3(\phi_3) = 0$ from the Lagrangian $\phi_3 \cdot G_3(\phi_3)$. Note that (unlike for spin 2) $D = 2$ does not have any special Euler characteristic here; in particular $\phi_3 \cdot G_3$ is not a total divergence.

We conclude our discussion of the « Ricci » tensor with the definition of *harmonic gauges*, introduced in [*1*]. Consider the combination

$$\tilde{H}_{\mu\nu} := \partial_\alpha \phi^\alpha{}_{\mu\nu} - \frac{1}{2} \partial_{(\mu} \phi_{\nu)} . \tag{2.14}$$

$\tilde{H}_2$ is clearly traceless, and transforms as

$$\delta_{\tilde{\xi}} \tilde{H}_{\mu\nu} = \Box \tilde{\xi}_{\mu\nu} . \tag{2.15}$$

Therefore $\tilde{H}_2$ may be gauged away; $\tilde{H}_2 = 0$ defines the class of harmonic gauges. Now the identity,

$$R_{\mu\nu\lambda} \equiv \Box \phi_{\mu\nu\lambda} - \partial_{(\mu} \tilde{H}_{\nu\lambda)} , \tag{2.16}$$

tells us that, in harmonic gauges, $R_3^H = \Box \phi_3$. If we further define

$$\psi_{\mu\nu\lambda} := \phi_{\mu\nu\lambda} - \frac{1}{2} \eta_{(\mu\nu} \phi_{\lambda)} ; \tag{2.17}$$

then the field equation, $G_3 = S_3$ (with some given source S_3), reads, in harmonic gauges

$$\Box \psi_{\mu\nu\lambda} = S_{\mu\nu\lambda} , \tag{2.18 a}$$

$$\partial_\alpha \psi^\alpha{}_{\mu\nu} - \frac{1}{D} \eta_{\mu\nu} \partial_\alpha \psi^\alpha = 0 . \tag{2.18 b}$$

2.4 The « Riemann » tensor.

The first major difference from spin 2 occurs in the definition of the « Riemann » tensor, which is of higher derivative order than « Ricci » (for spin 1, on the contrary, « Riemann » ($F_{\mu\nu}$) is of lower order than « Ricci » ($\partial_\mu F^{\mu\nu}$)).

There are three natural definitions of a « curvature tensor » which can be proven to lead to algebraically equivalent, though different, tensors (see below). We define « Riemann », R_6, to be the curl on each index of ϕ_3:

$$R_{\alpha\mu\beta\nu\gamma\lambda} := \partial_{\alpha\beta\gamma}\phi_{\mu\nu\lambda} \tag{2.19 a}$$

which, in explicit form, is a sum of eight terms:

$$\begin{aligned} R_{\alpha\mu\beta\nu\gamma\lambda} := {} & \partial_{\alpha\beta\gamma}\phi_{\mu\nu\lambda} - \partial_{\mu\beta\gamma}\phi_{\alpha\nu\lambda} \\ & - \partial_{\alpha\nu\gamma}\phi_{\mu\beta\lambda} + \partial_{\mu\nu\gamma}\phi_{\alpha\beta\lambda} \\ & - \partial_{\alpha\beta\lambda}\phi_{\mu\nu\gamma} + \partial_{\mu\beta\lambda}\phi_{\alpha\nu\gamma} \\ & + \partial_{\alpha\nu\lambda}\phi_{\mu\beta\gamma} - \partial_{\mu\nu\lambda}\phi_{\alpha\beta\gamma} . \end{aligned} \tag{2.19 b}$$

« Riemann », so defined, is a ξ-gauge (and not merely $\tilde{\xi}$-gauge) invariant tensor whose algebraic, and differential, symmetries are direct generalizations of the spin 2 $R_{\alpha\mu\beta\nu} = \partial_{\alpha\beta}\phi_{\mu\nu}$.

Indeed, $R_{\alpha\mu\beta\nu\gamma\lambda}$ is antisymmetric in each pair $(\alpha\mu)$, $(\beta\nu)$ and $(\gamma\lambda)$, symmetric under pair exchange, obeys the cyclic identity on any three indices, e. g.

$$R_{\alpha\mu\beta[\nu\gamma\lambda]} \equiv 0 , \tag{2.20}$$

and a cyclic differential (« Bianchi ») identity with respect to any pair, e. g.

$$R_{\alpha\mu\beta\nu[\gamma\lambda,\rho]} \equiv 0 . \tag{2.21}$$

We have phrased the algebraic symmetries of R_6 in the usual, explicit, way; however the combination of the manifest pair antisymmetries and of the further symmetry expressed in (2.20) is sufficient [*6*] to conclude that R_6 has, in fact, the symmetry of the following (GL(D)) Young tableau:

$$R_{\alpha\mu\beta\nu\gamma\lambda} = \begin{array}{|c|c|c|} \hline \mu & \nu & \lambda \\ \hline \alpha & \beta & \gamma \\ \hline \end{array} . \tag{2.22}$$

By equation (2.22) we mean that the symmetry-type of R_6 corresponds, starting from an arbitrary 6-tensor, to, first, separately symmetrizing over the row indices, and, then, separately antisymmetrizing over the column indices (once realized, this is clear from the definition (2.19 *a*) of R_6 because $\partial_{\alpha\beta\gamma}\phi_{\mu\nu\lambda}$ is already separately symmetric in $(\alpha\beta\gamma)$ and $(\mu\nu\lambda)$). (Young tableau symmetries are compactly and clearly reviewed in [*6*], see also e. g. [*7*] [*8*]). Now, the so-called « hook formula » [*9*] [*6*]-[*8*], gives very simply the dimension of the representation of GL(D) corresponding to the Young tableau (2.22). As R_6 clearly spans this representation, this gives the number of algebraically independent components of R_6:

$$N(R_6) = \frac{(D-1)D^2(D+1)^2(D+2)}{144} . \tag{2.23}$$

In particular, when D = 1, 2, 3 and 4, R_6 has respectively 0, 1, 10 and 50 independent components.

Before studying the algebraic decomposition of R_6 into parts irreducible under the orthogonal group $0(D-1,1)$ let us discuss the other two natural definitions of a spin 3 curvature tensor alluded to above. De Wit and Freedman [1] have introduced the following ξ-gauge invariant third order affinity:

$$\overset{(3)}{\Gamma}{}^{\alpha\beta\gamma}{}_{\mu\nu\lambda} := \partial_{(\mu}\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\nu\lambda)\gamma} - 3\partial_{\gamma}\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda}\,. \tag{2.24}$$

This tensor is algebraically equivalent to R_6 as shown by the relations:

$$\overset{(3)}{\Gamma}{}^{\alpha\beta\gamma}{}_{\mu\nu\lambda} \equiv - R^{\alpha}{}_{.(\mu}{}^{\beta}{}_{.\nu}{}^{\gamma}{}_{.\lambda)} \equiv - 6R^{(\alpha}{}_{.(\mu}{}^{\beta}{}_{.\nu}{}^{\gamma)_1}{}_{.\lambda)_1}\,, \tag{2.25 a}$$

$$R_{\alpha\mu\beta\nu\gamma\lambda} \equiv -\frac{1}{24}\overset{(3)}{\Gamma}{}_{\alpha\beta\gamma\,\mu\nu\lambda}\,. \tag{2.25 b}$$

Equation (2.25 a) shows that $\overset{(3)}{\Gamma}{}_6$ is the spin 3 analog of the Jacobi tensor in gravity (see e. g. [5])

$$J^{\alpha\beta}{}_{\mu\nu} := \frac{1}{2}R^{\alpha}{}_{.(\mu}{}^{\beta}{}_{.\nu)} \tag{2.26}$$

(note that, in spin 2, R_4 has the following Young tableau symmetry:

$$R_{\alpha\mu\beta\nu} = \left(\begin{array}{|c|c|}\hline \mu & \nu \\ \hline \alpha & \beta \\ \hline\end{array}\right).$$

In these « Jacobi » tensors the symmetries among $(\alpha\beta\ldots)$ and $(\mu\nu\ldots)$ are manifest while the antisymmetries are hidden. The latter imply however that any further symmetrization, e. g. over $(\alpha\beta\gamma\mu)$, gives zero. These (symmetric) cyclic identities then imply a further exchange (anti-)symmetry,

$$J_{\alpha_1\ldots\alpha_S\mu_1\ldots\mu_S} \equiv (-)^S J_{\mu_1\ldots\mu_S\alpha_1\ldots\alpha_S}\,.$$

These Jacobi-type symmetries correspond to using « transposed » Young symmetrizers, the columns, e. g. in eq. (2.22), being antisymmetrized first, before symmetrizing the rows.

Finally, in view of the multi-gradient transformation law (2.7) of $\overset{(2)}{\Gamma}$ one could consider, in naive analogy with the spin 2 case, the ξ-gauge invariant curl of $\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda}$ on, say, λ:

$$\overline{R}{}^{\alpha\beta}{}_{\mu\nu\kappa\lambda} := \partial_{\kappa}\overset{(2)}{\Gamma}{}^{\alpha\beta}{}_{\mu\nu\lambda}\,. \tag{2.27}$$

Apart from its manifest symmetries (in $(\alpha\beta)$ and $(\mu\nu)$) and antisymmetries (in $(\kappa\lambda)$) this tensor also has several hidden symmetries relating the indices in the various pairs. As the resulting symmetry type is not mathematically canonical we shall not pause to explicate it, but content ourselves by quoting the relations which prove that $\overline{R}_6$ is algebraically equivalent to R_6:

$$\overline{R}^{\alpha\beta}{}_{\mu\nu\kappa\lambda} \equiv R^{\alpha}{}_{.(\mu.}{}^{\beta}{}_{\nu)\kappa\lambda}\,, \tag{2.28 a}$$

$$R_{\alpha\beta\mu\nu\kappa\lambda} \equiv \frac{1}{3}\overline{R}_{\alpha\underline{\mu\beta}\nu\kappa\lambda}. \tag{2.28 b}$$

2.5 The traces of « Riemann », and the « Weyl » tensor.

The « Riemann » tensor spans the symmetry class (2.22) which is an irreducible representation of GL(D). To complete our study of R_6 we must discuss its algebraic decomposition with respect to irreducible representations of the Lorentz group $0\,(D-1,1)$. Let us first consider the traces of R_6. It is clearly seen that there is only *one* independent (single) trace of R_6, say R_4,

$$R_{\alpha\mu\nu\lambda} := R_{\alpha\mu\sigma\nu\sigma\lambda}\,, \tag{2.29 a}$$

which is easily found to be the following curl of « Ricci »:

$$R_{\alpha\mu\nu\lambda} \equiv \partial_{\underline{\alpha}} R_{\underline{\mu}\nu\lambda}\,. \tag{2.29 b}$$

The symmetries of R_4 are: antisymmetry with respect to the pair $(\alpha\mu)$, symmetry w. r. t. $(\nu\lambda)$, plus the following cyclic symmetry:

$$R_{[\alpha\mu\nu]\lambda} \equiv 0\,. \tag{2.30}$$

In terms of symmetry classes this means that R_4 belongs to the following Young tableau:

$$R_{\alpha\mu\nu\lambda} \subset \begin{array}{|c|c|c|} \hline \mu & \nu & \lambda \\ \hline \alpha \\ \cline{1-1} \end{array} \tag{2.31}$$

(as is clear from (2.29 *b*); we replace the equal sign of (2.22) by an inclusion symbol because we shall see that R_4 spans only a subset of its symmetry class).

Let us consider a general T_4 spanning the full symmetry class (2.31) (i. e. (2.30)). It possesses, in general, two independent (single) traces: an antisymmetric one,

$$\check{T}_2 := \check{T}r\,(T_4) := T_{\alpha\mu\sigma\sigma}\,, \tag{2.32 a}$$

and a symmetric-traceless one,

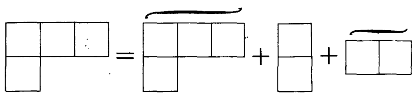
$$\tilde{T}_2 := \tilde{T}r\,(T_4) := T_{\alpha\sigma\mu\sigma} - \frac{1}{2}T_{\alpha\mu\sigma\sigma}\,, \tag{2.32 b}$$

and therefore all its double traces vanish. Then we have the following 0(D)-irreducible decomposition of T_4,

$$T_{\alpha\mu\nu\lambda} = \tilde{T}_{\alpha\mu\nu\lambda} + \frac{1}{2(D+2)} \underset{\sqcup}{\check{T}_{\alpha(\mu}\eta_{\nu\lambda)}} + \frac{1}{D} \underset{\sqcup}{\tilde{T}_{\alpha(\mu}\eta_{\nu\lambda)}} \tag{2.33 a}$$

(the antisymmetrization being effected last), where $\tilde{T}_4$ (defined by (2.33 *a*) with (2.32)) is completely tracefree.

In diagrammatic notation (2.33 *a*) reads



(2.33 *b*)

where we use the (unconventional) notation of « tilded » Young tableaux to represent the irreducible representation of 0(D) spanned by the set of completely tracefree tensors having the corresponding Young tableau symmetry (e. g. [*7*] [*8*]).

Applying the previous decomposition to (2.31) one discovers first that the fact that $R_4 = \mathrm{Tr}\,(R_6)$, with (2.22), implies the following *algebraic* identity for the traces of R_4:

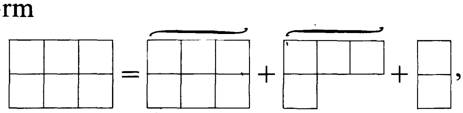
$$\tilde{R}_2 := \tilde{\mathrm{T}}\mathrm{r}\,(R_4) \equiv 0\,. \tag{2.34 a}$$

Now, written out explicitly in terms of « Ricci » $\equiv R_3$, $\tilde{R}_2$ reads

$$\tilde{R}_{\alpha\mu} \equiv -\left\{ \partial^{\lambda} R_{\alpha\mu\lambda} - \frac{1}{2}\, \partial_{(\alpha} R_{\mu)} \right\}. \tag{2.34 b}$$

Therefore we see, as announced in § 2.3, that, in spin 3, the « Einstein identities » (2.13) derive, by contraction, from the algebraic symmetries of « Riemann », and not from its differential (Bianchi) identities (2.21).

In summary, we have proven that the 0(D)-irreducible decomposition of R_6 is of the form



(2.35 *a*)

which an explicit calculation shows to be obtainable from

$$R_{\alpha\mu\beta\nu\gamma\lambda} = C_{\alpha\mu\beta\nu\gamma\lambda} + \frac{1}{D}\left\{ \underset{\sqcup\!\sqcup}{S_{\alpha\mu\nu\lambda}\eta_{\beta\gamma}} + \underset{\sqcup\!\sqcup}{S_{\beta\nu\lambda\mu}\eta_{\gamma\alpha}} + \underset{\sqcup\!\sqcup}{S_{\gamma\lambda\mu\nu}\eta_{\alpha\beta}} \right\}, \tag{2.35 b}$$

with

$$S_{\alpha\mu\nu\lambda} = \tilde{R}_{\alpha\mu\nu\lambda} + \frac{D}{4(D+1)(D+2)} \underset{\sqcup}{\check{R}_{\alpha(\mu}\eta_{\nu\lambda)}}, \tag{2.36}$$

$\check{R}_2$ being the only independent double trace of R_6,

$$\check{R}_{\alpha\mu} := \check{\mathrm{T}}\mathrm{r}\,(R_4) := R_{\alpha\mu\sigma\sigma} \equiv R_{\alpha\mu\rho\sigma\rho\sigma}, \tag{2.37 a}$$

i. e. in terms of the trace of Ricci, $R_1 = R_{\mu\sigma\sigma}$,

$$\check{R}_{\alpha\mu} = \partial_\alpha R_\mu \, . \tag{2.37 b}$$

The trace-free part of R_4, $\tilde{R}_4$, appearing in (2.36) is defined by eq. (2.33 *a*) (with $\tilde{R}_2 \equiv 0$), i. e. by

$$\tilde{R}_{\alpha\mu\nu\lambda} := R_{\alpha\mu\nu\lambda} - \frac{1}{2(D+2)} \check{R}_{\alpha(\mu}\eta_{\nu\lambda)} \tag{2.38}$$

(beware of the fact that $\tilde{R}_4$ is *not* the $\alpha\mu$ curl of the tracefree part, $\tilde{R}_3$, of Ricci).

Equation (2.35 *b*) in fact *defines* the « Weyl » tensor C_6, i. e. the completely tracefree part of « Riemann » with the same symmetries. From this point of view the tensor S_4, (or $D^{-1}S_4$), (though reducible according to (2.36)) plays a special role. It is the spin 3 analog of the gravity tensor,

$$S_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2(D-1)} R g_{\alpha\beta} \, , \tag{2.39}$$

which appears in the Weyl decomposition

$$R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} + \frac{1}{D-2} S_{\alpha\beta} g_{\mu\nu} \, . \tag{2.40}$$

$S_{\alpha\beta}$ (or $(D-2)^{-1}S_{\alpha\beta}$) is known to play an important role in the description of the conformal geometry of space-times (for example through the definition of the normal conformal Cartan connection, see e. g. [*10*]). Its spin 3 counterpart S_4 will also be important in our investigation of the « conformal curvature » in spin 3. For completeness, let us give its direct expression (obtained from (2.36), (2.38)), in terms of the traces of R_6 (analogously to (2.39)):

$$S_{\alpha\mu\nu\lambda} = R_{\alpha\mu\nu\lambda} - \frac{1}{4(D+1)} \check{R}_{\alpha(\mu}\eta_{\nu\lambda)} \, . \tag{2.41}$$

From the irreducible decompositions (2.33 *b*), (2.35 *a*) it is easy to compute the number of independent components of the various algebraic parts of R_6. We find

$$N(C_6) = \frac{(D-3)(D-2)(D-1)(D+2)(D+3)(D+4)}{144} \, , \tag{2.42 a}$$

$$N(\tilde{R}_4) = \frac{(D-2)(D-1)(D+1)(D+4)}{8} \, , \tag{2.42 b}$$

$$N(\check{R}_2) = \frac{(D-1)D}{2} \, , \tag{2.42 c}$$

which add up to (2.23). It is to be noticed that the polynomials in D appearing in eqs. (2.42) give the correct counting in any dimension $D \geqq 1$.

This contrasts with the spin 2 case where the corresponding polynomial formulas are correct only if $D \geqq 3$, $D \leqq 2$ necessitating a special counting.

From eq. (2.42 *a*) we see immediately that the « Weyl » tensor vanishes identically if $D \leqq 3$, while it has 14 independent components in $D = 4$. More precisely in $D = 1$, R_6, and all its traces, vanishes identically, though the spin 3 geometry is still described by one function of one variable. This is the first indication that « Riemann » does not carry full information about the spin 3 « curvature ». In $D = 2$, C_6 and $\tilde{R}_4$ vanish identically and R_6 (through eq. (2.35 *b*)) can be completely expressed in terms of $\check{R}_2 = \partial_{[\alpha} R_{\mu]}$, which possesses only one independent component. However the spin 3 geometry is described by two functions of two variables (and is « conformally trivial » see appendix C). In $D = 3$, only C_6 vanishes identically, and eq. (2.35 *b*) then expresses R_6 in terms of its traces, i. e. in terms of the curl of « Ricci ». « Riemann » has then ten independent components, i. e. as many as « Ricci ». However, while $R_3 = 0$ implies $R_6 = 0$, the converse is not true because for instance,

$$\phi_{\mu\nu\lambda} = \eta_{(\mu\nu}\partial_{\lambda)}\chi \;\Rightarrow\; R_{\mu\nu\lambda} = 3D\partial_{\mu\nu\lambda}\chi \;\Rightarrow\; R_6 \equiv 0\,. \tag{2.43}$$

§ 3. SPIN 3 « CURVATURE »

Curvature is defined as the obstruction to triviality of a field, i. e. a quantity which vanishes if and only if ϕ is pure gauge. For spin 1 it is of course $F_{\mu\nu} = \partial_{[\mu}\phi_{\nu]}$, while for spin 2 it is the Riemann (or Jacobi) tensor. For higher spins things are more complicated as was already indicated by the counting of the degrees of freedom given above in $D \leqq 3$ and by the fact that « Riemann » is invariant under the wider class of ξ—(rather than $\tilde{\xi}$—) gauge transformations. We shall here prove the following two theorems that completely describe the concept of curvature for spin 3 (in dimension $D > 2$).

THEOREM 1. — The vanishing of « Riemann », R_6, is necessary and sufficient for ϕ_3 to be (at least locally) pure *generalized* gauge:

$$(R_{\alpha\mu\beta\nu\gamma\lambda}(\phi_3) = 0) \Leftrightarrow (\exists\, \xi_{\mu\nu}\,;\, \phi_{\mu\nu\lambda} = \partial_{(\mu}\xi_{\nu\lambda)})\,. \tag{3.1}$$

THEOREM 2. — In dimension $D > 2$, ϕ_3 is (locally) pure ($\tilde{\xi}$) gauge if and only if both « Riemann » *and* « Ricci » (or both « Weyl » *and* « Ricci ») vanish:

$$(\exists\, \tilde{\xi}_2\,;\, \tilde{\xi}_0 \equiv 0 \text{ and } \phi_3 = \partial_{(1}\tilde{\xi}_{2)}) \Leftrightarrow (R_6(\phi_3) = 0 \text{ and } R_3(\phi_3) = 0)\,. \tag{3.2}$$

Therefore, in $D > 2$, the spin 3 « curvature » is the union of « Riemann » and « Ricci », or, if one prefers to use independent tensors, of « Weyl » and « Ricci ». This results holds true in D $\leqq$ 2 if one imposes some suitable boundary conditions.

3.1 **Proof of theorem 1.**

The generalized gauge invariance of R_6 means that $R_6 = 0$ is a necessary criterion of ξ-gauge triviality. It remains to prove that it is sufficient. We start by noticing that the spin 3 « Riemann » (2.19 *a*) is linked to the usual spin 2 Riemann by

$$R^{(S=3)}_{\sigma\alpha\tau\beta\check{\mu}\check{\nu}}(\phi_3) \equiv \partial_{\sigma\tau\mu}\phi_{\alpha\beta\nu} \equiv \partial_{\sigma\tau}(\partial^{\mu}\phi^{\nu}_{\alpha\beta}) \equiv R^{(S=2)}_{\sigma\alpha\tau\beta}(\overset{\mu\nu}{h_2}), \tag{3.3 a}$$

where the index pair $(\mu\nu)$ is consider as fixed (or « inert ») and where

$$\overset{\check{\mu}\check{\nu}}{h_2} \equiv h^{\check{\mu}\check{\nu}}_{\alpha\beta} := \partial^{\mu}\phi^{\nu}_{\alpha\beta}\,. \tag{3.3 b}$$

Therefore the vanishing of $R_6(\phi_3)$ implies the vanishing of $R_4(\overset{\check{\mu}\check{\nu}}{h_2})$, and hence, by the usual curvature theorem in spin 2, that $\overset{\check{\mu}\check{\nu}}{h_2}$ is (at least locally) pure (spin 2) gauge:

$$R_6(\phi_3) = 0 \Rightarrow \exists \xi_{\alpha\check{\mu}\check{\nu}}\,; \qquad \partial_{[\mu}\phi_{\nu]\alpha\beta} = \partial_{(\alpha}\xi_{\beta)\check{\mu}\check{\nu}}\,. \tag{3.4}$$

Now it is proven in appendix A (lemma 3) that the second equality in (3.4) implies the following structure for $\xi_{\alpha\check{\mu}\check{\nu}}$:

$$\xi_{\alpha\check{\mu}\check{\nu}} = a_{\check{\alpha}\check{\mu}\check{\nu}} + b_{\check{\alpha}\check{\beta}\check{\mu}\check{\nu}}x^{\beta} + \partial_{[\mu}\psi_{\nu]\alpha}, \tag{3.5}$$

where a_3 and b_4 are constant tensors (antisymmetric in the caretted indices), and ψ_2 some *symmetric* 2-tensor field. Inserting the information (3.5) into (3.4) leads to

$$\exists \psi_{\mu\nu}\,; \qquad \partial_{[\mu}\phi_{\nu]\alpha\beta} = \partial_{\mu}\{\partial_{(\alpha}\psi_{\beta)\nu}\}\,. \tag{3.6}$$

Introducing now the fully symmetric tensor

$$\omega_{\nu\alpha\beta} := \phi_{\nu\alpha\beta} - \partial_{(\nu}\psi_{\alpha\beta)}\,, \tag{3.7}$$

eq. (3.6) can be rewritten as

$$\partial_{[\mu}\omega_{\nu]\alpha\beta} = 0\,. \tag{3.8}$$

Now by Poincaré's lemma (see appendix A, lemma 1), eq. (3.8) implies that, for fixed $\alpha\beta$, $\omega_{\nu\alpha\beta}$ is some ν-gradient

$$\exists \omega_{\alpha\beta}\,; \qquad \omega_{\nu\alpha\beta} = \partial_{\nu}\omega_{\alpha\beta}\,, \tag{3.8}$$

ω_2 being symmetric. Then the full symmetry of ω_3 (and Poincaré's lemma again) implies that $\omega_{\alpha\beta}$ is also an α-gradient. Finally the symmetry of ω_2 gives simply

$$\exists \omega ; \qquad \omega_{\nu\alpha\beta} = \partial_{\nu\alpha\beta}\omega . \tag{3.9}$$

Inserting this information into eq. (3.7) and defining a symmetric ξ_2 by

$$\xi_{\mu\nu} := \psi_{\mu\nu} + \frac{1}{3}\,\partial_{\mu\nu}\omega , \tag{3.10}$$

leads finally to

$$\phi_{\mu\nu\lambda} = \partial_\mu \xi_{\nu\lambda} + \partial_\nu \xi_{\lambda\mu} + \partial_\lambda \xi_{\mu\nu} , \tag{3.11}$$

which completes the proof of theorem 1. Note that this proof is local and, in principle, fully constructive (thanks to the tools of appendix A). Other proofs are possible based, for instance, on decomposing ϕ_3 in a transverse-traceless part, a transverse-trace part and a generalized gauge.

3.2 Proof of theorem 2.

From the results of § 2 the necessary character, for ($\tilde{\xi}$) gauge triviality, of the double criterion $R_6 = 0$, $R_3 = 0$, is clear, as well as its equivalence to $C_6 = 0$, $R_3 = 0$. Let us prove that $R_6 = 0 = R_3$ is also sufficient. We first use theorem 1 which tells us that $R_6 = 0$ implies the form (3.11) for ϕ_3. Replacing this information into $R_3 = 0$ gives from eq. (2.9).

$$\partial_{\mu\nu\lambda}\xi^{\alpha\alpha} = 0 , \tag{3.12}$$

which implies that

$$\xi^{\alpha\alpha} = a + b_\lambda x^\lambda + \frac{1}{2}\, c_{\underline{\lambda\rho}} x^\lambda x^\rho , \tag{3.13}$$

where a, b_1 and c_2 are some constant tensors ($c_{\underline{\lambda\rho}} = c_{\underline{\rho\lambda}}$). Now, if we can find some constant tensors, $a_{\underline{\mu\nu}}$, $b_{\underline{\mu\nu}\lambda}$, $c_{\underline{\mu\nu}\,\underline{\lambda\rho}}$ (symmetric in the indicated pairs) such that

$$a = a_{\alpha\alpha}, \qquad b_\lambda = b_{\alpha\alpha\lambda}, \qquad c_{\lambda\rho} = c_{\alpha\alpha\lambda\rho} \tag{3.14}$$

and

$$b_{(\mu\nu\lambda)} = 0, \qquad c_{(\mu\nu\lambda)\rho} = 0, \tag{3.15}$$

then the quantity

$$\zeta_{\mu\nu}(x) := a_{\mu\nu} + b_{\mu\nu\lambda}x^\lambda + \frac{1}{2}\, c_{\mu\nu\lambda\rho}x^\lambda x^\rho , \tag{3.16}$$

will satisfy

$$\zeta_{\alpha\alpha}(x) = \xi_{\alpha\alpha}(x), \qquad \text{and} \qquad \partial_{(\lambda}\zeta_{\mu\nu)} = 0 . \tag{3.17}$$

Then, if we introduce the gauge parameter

$$\tilde{\xi}'_{\mu\nu} := \xi_{\mu\nu} - \zeta_{\mu\nu}, \qquad \tilde{\xi}'_{\alpha\alpha} = 0 , \tag{3.18}$$

we will have

$$\phi_{\mu\nu\lambda} = \partial_{(\mu}\tilde{\xi}'_{\nu\lambda)}, \tag{3.19}$$

as was to be proven. Finally, the existence of the constant tensors a_2, b_3, c_4 with the above symmetries (plus (3.15)) is proven by direct construction: for instance, using two vectors to make up b_3, and $\eta_{\mu\nu}$ and a symmetric d_2 to make up c_4, namely

$$c_{\mu\nu\lambda\rho} = 2\eta_{\mu\nu}d_{\lambda\rho} + 2\eta_{\lambda\rho}d_{\mu\nu} - \eta^{(\lambda}_{(\mu}d^{\rho)}_{\nu)} . \tag{3.20}$$

The first construction breaks down only in $D = 1$ (where anyway b_1 has more independent components than b_3), while the second breaks down in $D \leqq 2$ (where $N(c_2) > N(c_4)$). And indeed, in the case $D = 2$, it can be proven that a term in $\xi^{\alpha\alpha}$, eq. (3.13), of the type $\frac{1}{2}\,\tilde{c}_{\lambda\rho}x^\lambda x^\rho$ *cannot* be eliminated by a $\tilde{\xi}$-gauge transformation. Therefore R_6 and R_3 fully describe the curvature in $D \leqq 2$ only if one imposes some further conditions (e. g. boundary, or fall-off, conditions) that eliminate such dangerous terms in the solutions of eq. (3.12).

3.3 **Applications of the curvature theorem.**

As an application of theorem 2 we conclude that in $D = 3$ the free field equations $G_3 = 0$ imply « flat space », i. e. that there are no excitations and ϕ_3 is pure ($\tilde{\xi}$) gauge, just as for spin 2. Indeed this follows immediately from the fact (proven in § 2.5) that in $D = 3$ the Weyl tensor vanishes identically, or, in other words, that one can algebraically express R_6 in terms of the curl of R_3 (see eq. (2.35 *b*) [Note that the latter result can also be directly obtained by taking multiple duals of R_6 over the antisymmetric pairs, and then by expanding the products of ε's into antisymmetrized products of δ's]. By contrast, a direct proof of this absence of free spin 3 dynamics in $D = 3$ from the field equations is quite tedious. We shall see below (§ 6) that with suitably modified field equations, one can define and study an interesting topologically massive spin 3 dynamics in $D = 3$. The previous « flat space » result holds also in $D \leqq 2$ if some suitable boundary conditions are imposed.

Let us also quickly discuss the consequences of the curvature theorem in $D = 4$. In any dimension, the curvature of a freely propagating spin 3 excitation is fully described by the independent components of C_6, since $R_3 = 0$. In $D = 4$, it is convenient to describe the 14 real components of C_6 in spinorial form, using van der Waerden 2-spinors (for reviews see e. g. [*6*] [*11*]). One finds that C_6 is equivalent to a totally symmetric 6-spinor ψ_{ABCDEF} (with 7 independent complex components). In « vacuum » ($R_3 = 0$) the Bianchi identities (2.21) imply the following equation

$$\partial^{AA'}\psi_{ABCDEF} = 0 , \tag{3.21}$$

which is the usual propagation equation for a free massless field described by a symmetric spinor [6]. The spinor form ψ_6 of C_6 is convenient for classifying the algebraic structure of the curvature (« Petrov-Penrose classification », [6] [11]). One can indeed always decompose ψ_6 in the symmetrized product of six 2-spinors:

$$\psi_{ABCDEF} = \alpha_{(A}\beta_B\gamma_C\delta_D\varepsilon_E\zeta_{F)}, \tag{3.22}$$

which defines the 6 « principal (real) null directions » of C_6 (each $\alpha_A \leftrightarrow$ null vector $a_\mu = \alpha_A\bar{\alpha}_{A'}$). Then, at each point, C_6 can be classified according to the coincidence scheme of the principal null directions, from the algebraically general type (6 different directions) to the « null » type (6 coincident directions). For instance the (monochromatic) plane wave solutions are found to be of the « null » type with κ_A ($\leftrightarrow$ k_μ = wave vector) as repeated null direction.

Finally let us add that one can also define a spin 3 « Bel-Robinson » tensor

$$T_{\alpha\beta\gamma\delta\varepsilon\zeta} := \psi_{ABCDEF}\bar{\psi}_{A'B'C'D'E'F'}, \tag{3.23}$$

which can (with more work) directly be expressed as a sum of contracted products of C_6 and its duals. As for the spin 2 T_4 [12] (see also [5], p. 382), our T_6 is easily checked to be symmetric, traceless and divergence-free on all indices.

§ 4. SPIN 3 « CONFORMAL CURVATURE »

Let us define a « conformal transformation » of the field ϕ_3 by

$$\delta_\omega \phi_{\mu\nu\lambda} := \omega_{(\mu}\eta_{\nu\lambda)}, \tag{4.1}$$

where ω_1 is an arbitrary vector field. This definition is a natural generalization of the spin 2 case ($\delta_\omega \phi_2 = \omega_0 \eta_2$) and gives rise, even at the « linearized » level here considered, to an interesting « geometrical » structure which closely parallels the Riemannian conformal geometry. « *Conformal curvature* » of the field ϕ_3 is defined as the obstruction to conformal triviality of ϕ_3, i. e. as a quantity which vanishes if and only if there exist (at least locally) a vectorial conformal weight ω_1 and a tracefree gauge parameter $\tilde{\xi}_2$ such that:

$$\phi_{\mu\nu\lambda} = \omega_{(\mu}\eta_{\nu\lambda)} + \partial_{(\mu}\tilde{\xi}_{\nu\lambda)}. \tag{4.2}$$

For spin 2, conformal curvature is fully described in dimension $D \geqq 4$ by the Weyl tensor C_4 (see e. g. [13]). In $D = 3$, C_4 vanishes identically, and the Riemannian conformal curvature is described by Cotton's (symmetric) 2-tensor [14] (see also [5], p. 541)

$$D_{\mu\nu} := \varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu}, \tag{4.3 a}$$

where

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} \tag{4.3 b}$$

is the tensor (2.39) (for D = 3) which appears in the Weyl decomposition (2.40). In D = 2 (resp. 1). the concept of conformal curvature is empty because all Riemannian geometries are conformally flat (resp. flat). We shall here prove the following analogous results that completely specify the concept of conformal curvature for spin 3:

THEOREM 3. — In dimension $D \geqq 4$, the vanishing of « Weyl », $C_6(\phi_3)$, is necessary and sufficient for ϕ_3 to be (at least locally) conformally trivial.

THEOREM 4. — In dimension D = 3 (where $C_6(\phi_3)$ vanishes identically), a necessary and sufficient criterion for conformal triviality of ϕ_3 is the vanishing of the symmetric 3-tensor $D_3(\phi_3)$ defined in appendix B (eq. (B.5)) (D_3 is like D_2 in (4.3 *a*) the dual of a curl, but it is of *fifth* derivative order in ϕ_3 !).

THEOREM 5. — In dimensions $D \leqq 2$, all spin 3 fields, ϕ_3, are conformally trivial.

Therefore in the usual case $D \geqq 4$, the Weyl tensor $C_6(\phi_3)$ embodies the full spin 3 conformal curvature. This is to be contrasted with the more complicated geometrical description of spin 3 curvature (§ 3). One should keep in mind that this simplicity may be due to our « naive » definition of a spin 3 conformal transformation. In particular, the special role played here (and also in eq. (2.43)) by the « purely longitudinal » conformal transformations, $\delta_{\chi_0}\phi_3 = \eta_{(2}\partial_{1)}\chi_0$, deserves some further study (see end of appendix B).

Proofs. — Proofs of theorems 4 and 5 are given respectively in appendices B and C; let us consider here in detail the case $D \geqq 4$.

The necessity of the vanishing of C_6 is clear from $\delta_\omega R_6 \sim \partial_3\omega_1\eta_2$ which can only belong to the last two Young tableaux of (2.35 *a*). Let us now prove that the condition

$$C_{\alpha\mu\beta\nu\gamma\lambda}(\phi_3) = 0 \tag{4.4}$$

is also sufficient for the existence of ω_1 and $\tilde{\xi}_2$ in eq. (4.2). One first notices that a gradient term, $\partial_\lambda\zeta$, in ω_λ is equivalent in eq. (4.2) to adding a trace to $\tilde{\xi}_2 \to \xi_2 = \tilde{\xi}_2 + \zeta\eta_2$. It is then sufficient to prove that ϕ_3 is conformally trivial modulo a *generalized* gauge transformation. By theorem 1 (§ 3), one needs only to prove the existence of ω_1 such that $R_6(\phi_3) = R_6(\omega_{(1}\eta_{2)})$. The decomposition (2.35) then gives (4.4), plus the condition:

$$S_4(\phi_3) = S_4(\omega_{(1}\eta_{2)}) . \tag{4.5}$$

Computing the right-hand side of (4.5) from the equation (2.41) one gets the explicit condition:

$$D\partial_{\nu\lambda}\omega_{\alpha\mu} + \frac{1}{2}\{\partial_\alpha J_{(\nu}\eta_{\lambda)\mu} + \eta_{\mu(\nu}\partial_{\lambda)}J_\alpha\} = S_{\alpha\mu\nu\lambda}(\phi_3), \tag{4.6}$$

where we have put

$$\omega_{\alpha\mu} := \partial_\alpha\omega_\mu, \tag{4.7}$$

$$J_\mu := \partial^\alpha\omega_{\alpha\mu} = \Box\,\omega_\mu - \partial_\mu(\partial^\alpha\omega_\alpha). \tag{4.8}$$

The equations (4.6)-(4.8) constitute a third order partial differential system in ω_1, with a given « source » S_4 (ϕ_3). Our problem is to find its integrability conditions. All the components of $\omega_2 = \partial_1 \times \omega_1$ are coupled in this system; it is then convenient to decouple them by the following procedure. First, taking the α-divergence of eq. (4.6) gives a simpler system for J_1 (eq. (4.8)). After some algebraic manipulation one can transform it to the simple form

$$\partial_{\nu\lambda}J_\mu = \sigma_{\mu\nu\lambda}, \tag{4.9}$$

where

$$\sigma_{\mu\nu\lambda} := \frac{1}{2}\frac{1}{(D-2)(2D-1)}\{(2D-3)S'_{\mu\nu\lambda} + S'_{\nu\lambda\mu} + S'_{\lambda\mu\nu}\}, \tag{4.10}$$

$$S'_{\mu\nu\lambda} := 2\partial^\alpha S_{\alpha\mu\nu\lambda} - \frac{1}{D}\partial^\alpha S_{\alpha(\nu..}{}^{\sigma\sigma}\eta_{\lambda)\mu}. \tag{4.11}$$

Now, replacing (4.9) in the ρ-gradient of eq. (4.6) yields a decoupled third order equation for ω_2:

$$\partial_{\nu\lambda\rho}\omega_{\alpha\mu} = \Omega_{\alpha\mu\nu\lambda\rho}, \tag{4.12}$$

where

$$\Omega_{\alpha\mu\nu\lambda\rho} := \frac{1}{2D}\{2\partial_\rho S_{\alpha\mu\nu\lambda} - \Sigma_{\alpha\nu\rho}\eta_{\lambda\mu} - \Sigma_{\alpha\lambda\rho}\eta_{\nu\mu}\}, \tag{4.13}$$

$$\Sigma_{\mu\nu\lambda} := \sigma_{(\mu\nu)\lambda}. \tag{4.14}$$

Frobenius's theorem easily gives the necessary and sufficient conditions for the (complete) integrability of eq. (4.12) (considered as an equation for ω_2)

$$\Omega_{\alpha\mu\nu\lambda\rho} = 0, \tag{4.15 a}$$

$$\partial_\sigma\Omega_{\alpha\mu\nu\lambda\rho} = 0. \tag{4.15 b}$$

Now a long (and not so straightforward) calculation, using the Bianchi identities (2.21), as well as the Einstein identities (2.13), allows one to prove the identity

$$\Omega_{\alpha\mu\nu\lambda\gamma} \equiv \frac{1}{D-2}\partial^\beta C_{\alpha\mu\beta\nu\gamma\lambda}. \tag{4.16}$$

As for the second condition (4.15 *b*) one first transforms it into the following simpler equivalent condition:

$$\partial_{\sigma}\Sigma_{\alpha\nu\rho} = 0\,. \tag{4.15 b'}$$

Finally, another intricate calculation (using « Bianchi » and « Einstein ») allows one to prove the second identity:

$$\partial_{\gamma}\Sigma_{\mu\nu\lambda} \equiv \frac{2\mathrm{D}}{(\mathrm{D}-2)(\mathrm{D}-3)}\,\partial^{\alpha\beta}\mathrm{C}_{\alpha\mu\beta\nu\gamma\lambda}\,. \tag{4.17}$$

The two identities (4.16), (4.17) prove (when $\mathrm{D} \geqq 4$) that the necessary condition (4.4) is *sufficient* to ensure *complete* integrability of the eq. (4.12) in ω_2. The last step is to prove the existence of solutions in ω_1 satisfying the original system (4.6)-(4.8). Some algebraic transformations prove that eq. (4.12), when satisfied, implies that the ρ-gradient of eq. (4.6) is satisfied. Then it suffices to impose (4.6) at one point, say $x = x_0$ (this is always possible). One must also impose as further initial conditions on ω_2:

$$\partial_{[\nu}\omega_{\alpha\mu]}(x_0) = 0\,, \tag{4.18 a}$$

$$\partial_{\nu[\lambda}\omega_{\alpha\mu]}(x_0) = 0\,. \tag{4.18 b}$$

As eq. (4.6) preserves the constraint (4.18 *a*), the solution of eq. (4.12) satisfying the above initial conditions yields a solution ω_1 of the original system (4.6)-(4.8). This completes the proof of theorem 3. It is clear from the appearance of the denominators (D − 2) and (D − 3) in (eq. (4.10) and) the identities (4.16), (4.17) that the cases D = 2 or 3 must be treated separately (see appendices B and C).

§ 5. THE (SYMMETRIZED) CURL IN D = 3

The curl is a tensor rank-preserving operator in D = 3, and may therefore be used to formulate (parity-violating) action terms there. Before doing so, we develop the properties of the *symmetrized* curl when acting on symmetric tensors. It is a mapping C, from symmetric tensors, ϕ_{S}, to similar ones, $\psi_{\mathrm{S}} = \mathrm{C}(\phi_{\mathrm{S}})$, according to

$$[\mathrm{C}(\phi_{\mathrm{S}})]_{\mu_1\ldots\mu_{\mathrm{S}}} := \varepsilon_{\alpha\beta(\mu_1}\partial^{\alpha}\phi^{\beta}{}_{\mu_2\ldots\mu_{\mathrm{S}})}\,. \tag{5.1}$$

Note that, in keeping with our general notation, C is defined as an unnormalized sum of *s* terms (the introduction of an s^{-1} normalization factor would very much complicate the nice, rank independent, properties of C). The C operator is self-adjoint in that

$$\chi_{\mathrm{S}}\cdot\mathrm{C}(\phi_{\mathrm{S}}) - \phi_{\mathrm{S}}\cdot\mathrm{C}(\chi_{\mathrm{S}}) = \text{total divergence}\,. \tag{5.2}$$

We also define the « symmetric exterior derivative », d, and the « divergence », δ, (without the Hodge minus sign) by

$$\phi_S \xrightarrow{d} \psi_{S+1} : (d\phi)_{\mu_1 \ldots \mu_{S+1}} := \partial_{(\mu_1} \phi_{\mu_2 \ldots \mu_{S+1})}, \tag{5.3}$$

$$\phi_S \xrightarrow{\delta} \psi_{S-1} : (\delta\phi)_{\mu_1 \ldots \mu_{S-1}} := \partial^\alpha \phi_{\alpha\mu_1 \ldots \mu_{S-1}}, \tag{5.4}$$

as well as the corresponding symmetric exterior product with η_2 and the trace:

$$\phi_S \xrightarrow{\eta} \psi_{S+2} : (\eta\phi)_{\mu_1 \ldots \mu_{S+2}} := \eta_{(\mu_1\mu_2} \phi_{\mu_3 \ldots \mu_{S+2})}, \tag{5.5}$$

$$\phi_S \xrightarrow{\mathrm{Tr}} \psi_{S-2} : (\mathrm{Tr}\,\phi)_{\mu_1 \ldots \mu_{S-2}} := \eta^{\alpha\beta} \phi_{\alpha\beta\mu_1 \ldots \mu_{S-2}}. \tag{5.6}$$

Then it is easily checked that C commutes with d, δ, η, Tr, and hence with their compositions, including

$$\Box \equiv \delta d - d\delta. \tag{5.7}$$

As a corollary, C commutes with the Ricci, and Einstein, operators (R_3 = Ricci (ϕ_3), ...) since

$$\mathrm{Ricci} = \Box - d\delta + \frac{1}{2} d^2\,\mathrm{Tr}, \tag{5.8 a}$$

$$\mathrm{Einstein} = \mathrm{Ricci} - \frac{1}{2}\eta\,\mathrm{Tr}\,\mathrm{Ricci} \tag{5.8 b}$$

(note that our symmetric version of the exterior derivative is not nilpotent, in fact $d^2(\phi_S) = 2\,\partial_{(2}\phi_{S)}$; for completeness let us also quote: $2\delta = \mathrm{Tr}\,d - d\,\mathrm{Tr}$).

The iteration $C^2(\phi) := C[C(\phi_S)]$ depends on the rank of $\phi_S : C^2(\phi_1)$, for example, is the Maxwell operator

$$[C^2(\phi_1)]_\mu = \Box\phi_\mu - \partial^\alpha_\mu \phi_\alpha. \tag{5.9 a}$$

For spin 3,

$$[C^2(\phi_3)]_{\mu\nu\lambda} = 9\,\Box\,\phi_{\mu\nu\lambda} - 5\partial^\alpha_{(\mu}\phi^\alpha_{\nu\lambda)} + 2\partial_{(\mu\nu}\phi^{\alpha\alpha}_{\lambda)} - 2\eta_{(\mu\nu}(\Box\phi^{\alpha\alpha}_{\lambda)} - \partial^{\alpha\beta}\phi^{\alpha\beta}_{\lambda)}). \tag{5.9 b}$$

Note that the overall sign in eqs (5.9) is signature dependent (our choice $(-++)$ implies $\varepsilon^{\mu\nu\lambda}\varepsilon_{\alpha\beta\gamma} = -\,\delta^{\mu\nu\lambda}_{\alpha\beta\gamma}$ and the + signs in (5.9).)

In the next section we shall construct a « Chern-Simons »-like term (6.1). It is therefore amusing to note that there is also a Pontryagin-like invariant in $D = 4$ which is a total divergence, $\partial_\nu J^\nu = \partial_0 J^0 + \ldots + \partial_3 J^3$, such that J^3 is the Chern-Simons density. Consider

$$P_4 = -\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\overset{(1)}{\Gamma}{}^\alpha{}_{\beta\rho\sigma}[R_{\mu\nu\rho\tau\sigma\tau} + \partial_\nu R_{(\mu}\eta_{\rho\sigma)}] = \varepsilon^{\alpha\beta\mu\nu}\overset{(1)}{\Gamma}{}^\alpha{}_{\beta\rho\sigma}\partial_\nu G_{\mu\rho\sigma}. \tag{5.10}$$

Owing to the symmetries of $\overset{(1)}{\Gamma}$, its $\varepsilon^{\alpha\beta\nu\cdot}\partial_\nu\overset{(1)}{\Gamma}{}^\alpha{}_{\beta..}$ curl vanishes identically, so that

$$P_4 \equiv \partial_\nu J^\nu, \qquad J^\nu := \varepsilon^{\alpha\beta\mu\nu}\overset{(1)}{\Gamma}{}^\alpha{}_{\beta\rho\sigma}G_{\mu\rho\sigma}, \tag{5.11}$$

and the last component, J^3, is indeed $(-2/3$ times) the Chern-Simons term (6.1).

§ 6. TOPOLOGICALLY MASSIVE THEORY (IN D = 3)

For $s = 1$ in D = 3 topological mass terms have been defined for the free abelian theory [15] [16], and generalized to the (much deeper) Chern-Simons characteristics in the non-abelian case [16] [18]. This can be extended also to gravity [18]. Here we obtain their analog for $s = 3$ (at the abelian level, of course). There is precisely one gauge invariant action constructed in terms of $\varepsilon_{\alpha\beta\gamma}$, $\partial_\mu\phi_3$ and R_3 or G_3, namely

$$\int d^3x C_3(\phi_3)\cdot G_3(\phi_3) = 3\int d^3x \varepsilon^{\alpha\beta}{}_{\mu}\partial_\alpha\phi_{\beta\nu\lambda}G^{\mu\nu\lambda} = -\frac{3}{2}\int d^3x \varepsilon_{\alpha}{}^{\beta}{}_{.\mu}\overset{(1)}{\Gamma}{}^{\alpha}{}_{\beta\nu\lambda}G^{\mu\nu\lambda}. \tag{6.1}$$

We now consider the action

$$I = -\frac{1}{2}\int d^3x\phi_3\cdot G_3(\phi_3) + \frac{1}{6m}\int d^3x C_3(\phi_3)\cdot G_3(\phi_3), \tag{6.2}$$

where the sign of the gauge action is taken ghost-like relative to the usual one to ensure that the final excitation be non-ghost; the coefficient 1/6 (actually $1/2s$ in general) ensures that the effective mass is $|m|$, and the sign of the « Chern-Simons »-like term has been chosen so as to correlate $m > 0$ with positive helicity. These remarks apply also to the lower spins.

Since the (symmetrized) curl, C, and, Einstein, G, operators commute (§ 5) and are both self-adjoint, the ϕ-variation of the action is

$$-\int G(\phi)\cdot\delta\phi + (3m)^{-1}\int C[G(\phi)]\cdot\delta\phi$$

and the resulting field equation becomes $C[G_3(\phi_3)] = 3mG_3(\phi_3)$, i. e. explicitly

$$\varepsilon_{\alpha\beta(\mu}\partial^\alpha G^\beta{}_{\nu\lambda)} = 3mG_{\mu\nu\lambda}. \tag{6.3}$$

Taking the divergence and trace of (6.3), one finds (with the help of the Einstein identities) that (with $G_1 := \mathrm{Tr}\, G_3$)

$$\delta G_3 = 0, \tag{6.4}$$

$$C(G_1) = 3mG_1, \tag{6.5}$$

which implies

$$\delta G_1 = 0. \tag{6.6}$$

Therefore there are no spin 2 or 0 excitations. Iterating the equation (6.3), using the C^2 formulas (5.9), we find that

$$(\square - m^2)G_3 + (\text{spin 1 terms}) = 0, \tag{6.7 a}$$

$$(\square - (3m)^2)G_1 = 0. \tag{6.7 b}$$

Thus (as for the $s \leqq 2$ cases) the m^2 dependence is non-tachyonic although we had no control on the sign of m^2; the spin 3 part has mass $|m|$, but there is also a spin 1 part with mass $3|m|$, which is in fact a ghost (see eq. (6.24) below). This problem is characteristic of higher-spin massive theories of the usual type in arbitrary dimension, and requires introduction of auxiliary fields, a complication which can be understood in terms of a Kaluza-Klein compactification from a (D + 1)-dimensional massless theory [*4*].

A straightforward ghost removal (as in normally massive theory) consists in introducing an auxiliary vector field χ_1, coupled gauge invariantly to ϕ_3. The interaction Lagrangian is uniquely

$$L_{\text{int}} = -\frac{1}{3}\chi_1 \cdot G_1(\phi_3) \tag{6.8}$$

whose normalization is arbitrary, and chosen here for later convenience. The most general auxiliary Lagrangian is

$$L_{\text{aux}} = \frac{1}{2}\left[a\chi_1 \cdot \Box\chi_1 - b(\partial\cdot\chi_1)^2 + cm^2(\chi_1)^2 + dm\chi_1\cdot C(\chi_1)\right]. \tag{6.9}$$

We must determine values of these four parameters (if possible) to eliminate the lower spin, G_1, and χ_1 itself in the extended action $I + I_{\text{int}} + I_{\text{aux}}$. First, one notes that $\partial\cdot G_1$ will vanish (on shell) if $\partial\cdot\chi_1$ does; the divergence of the χ equation then requires $a + b = 1/2$ (and $c \neq 0$) for $\partial\cdot\chi_1$ to necessarily vanish. Next, the χ equation is recast into a form in which $aC(\Box\chi_1)$ equals lower derivatives of χ_1; to avoid third derivatives on χ_1 (and transverse χ_1 excitations), we impose $a = 0$. The resulting equation is of the form

$$(c - 3d)mC(\chi_1) + \left(d + \frac{4}{3}\right)\Box\chi_1 = 3cm^2\chi_1, \tag{6.10}$$

and provides us with a means of getting $\chi_1 = 0$, which simultaneously ensures vanishing of G_1. This determines all the parameters:

$$a = 0, \qquad b = \frac{1}{2}, \qquad c = -4, \qquad d = -\frac{4}{3}, \tag{6.11}$$

and implies (on shell)

$$\chi_1 = 0, \qquad G_1 = 0, \qquad (\Box - m^2)G_3 = 0. \tag{6.12}$$

That the helicity of the remaining excitation is purely ± 3 (if $m \gtrless 0$) will be seen in detail shortly, but it is already clear that the vector mode is gone. For simplicity, we omit the auxiliary field part of the action, since its effect is just to remove the lower helicity, we will do so directly.

In harmonic gauge (§ 2.3), the field equation (6.3) reads

$$\Box\left[\varepsilon_{\alpha\beta(\mu}\partial^{\alpha}\psi^{\beta}{}_{\nu\lambda)_1} - m\psi_{\mu\nu\lambda}\right] = 0. \tag{6.13}$$

We will consider plane waves,

$$\psi_3(x) = \hat{\psi}_3(k)e^{ikx} + \hat{\psi}_3^*(k)e^{-ikx}. \tag{6.14}$$

We can remove the $\square$ in (6.13), because there are no massless solutions: indeed they would be Ricci-flat in D = 3 which means (§ 3.3) that they are pure gauge (alternatively, an explicit counting of the remaining freedom of the harmonic gauge when $k^2 = 0$ leads to the same conclusion). Then the positive frequency amplitude $\hat{\psi}_3$ obeys

$$i\varepsilon_{\alpha\beta(\mu}k^\alpha\hat{\psi}^\beta{}_{\nu\lambda)_1} = m\hat{\psi}_{\mu\nu\lambda}. \tag{6.15 a}$$

In addition, the harmonic gauge condition (2.18 *b*), and the trace of (6.15 *a*), imply that ψ_1 is divergence-free ($\delta\psi_1 = 0$), and hence also $\delta\psi_3 = 0$; therefore

$$k^\alpha\hat{\psi}_{\alpha\mu\nu} = 0. \tag{6.15 b}$$

The system (6.15) requires k^α to be time-like. We will define a positively oriented orthonormal triad $\vec{\varepsilon}_{(\alpha)}$, $\alpha = 0, 1, 2$, such that ($\varepsilon_{012} = +1$, $|\vec{k}| := (-\vec{k}^2)^{1/2}$)

$$\vec{\varepsilon}_{(\alpha)}\cdot\vec{\varepsilon}_{(\beta)} = \eta_{\alpha\beta}; \qquad \varepsilon_{\mu\nu\lambda}\varepsilon^\mu_{(0)}\varepsilon^\nu_{(1)}\varepsilon^\lambda_{(2)} = +1; \qquad k^\mu = |\vec{k}|\,\varepsilon^\mu_{(0)}. \tag{6.16 a}$$

In terms of the complex vectors

$$\vec{\varepsilon}_\pm := \frac{\vec{\varepsilon}_{(1)} \pm i\vec{\varepsilon}_{(2)}}{\sqrt{2}}, \tag{6.16 b}$$

we have

$$\vec{\varepsilon}^{\,2}_{(0)} = -1, \qquad \vec{\varepsilon}_+\cdot\vec{\varepsilon}_- = +1; \tag{6.16 c}$$

all other scalar products vanish. The $\vec{\varepsilon}_{(\alpha)}$ also obey $\vec{\varepsilon}_- = (\vec{\varepsilon}_+)^*$ and

$$i\varepsilon_{\alpha\beta\mu}\varepsilon^\alpha_{(0)}\varepsilon^\beta_\pm = \pm\,\varepsilon^\mu_\pm, \qquad i\varepsilon_{\mu\alpha\beta}\varepsilon^\alpha_+\varepsilon^\beta_- = -\,\varepsilon_{(0)\mu}. \tag{6.16 d}$$

Since $\hat{\psi}_3$ is $\vec{k}$-transverse, we may decompose it along tensor products of $\vec{\varepsilon}_\pm$

$$\hat{\psi}^{\mu\nu\lambda} = \psi_{+3}\varepsilon^\mu_+\varepsilon^\nu_+\varepsilon^\lambda_+ + \psi_{+1}\varepsilon^{(\mu}_+\varepsilon^\nu_+\varepsilon^{\lambda)}_- + \psi_{-1}\varepsilon^{(\mu}_+\varepsilon^\nu_-\varepsilon^{\lambda)}_- + \psi_{-3}\varepsilon^\mu_-\varepsilon^\nu_-\varepsilon^\lambda_-. \tag{6.17}$$

The full equations then read

$$\frac{3m}{|\vec{k}|}\psi_n = n\psi_n; \qquad n = \pm 1, \ \pm 3. \tag{6.18}$$

It follows that the only plane wave solutions are

$$m > 0 \Rightarrow \begin{cases} \hat{\psi}^{\mu\nu\lambda} = \psi_{+3}\varepsilon^\mu_+\varepsilon^\nu_+\varepsilon^\lambda_+, & k^2 + m^2 = 0 \\ \text{or } \hat{\psi}^{\mu\nu\lambda} = \psi_{+1}\varepsilon^{(\mu}_+\varepsilon^\nu_+\varepsilon^{\lambda)}_-, & k^2 + (3m)^2 = 0 \end{cases} \tag{6.19 a}$$

$$m < 0 \Rightarrow \begin{cases} \hat{\psi}^{\mu\nu\lambda} = \psi_{-3}\varepsilon^\mu_-\varepsilon^\nu_-\varepsilon^\lambda_-, & k^2 + m^2 = 0 \\ \text{or } \hat{\psi}^{\mu\nu\lambda} = \psi_{-1}\varepsilon^{(\mu}_+\varepsilon^\nu_-\varepsilon^{\lambda)}_-, & k^2 + (3m)^2 = 0 \end{cases} \tag{6.19 b}$$

That is the only modes are helicity ± 3 or ± 1, corresponding to m^2, $(3m)^2$ and the helicity sign is correlated with that of m. This is precisely as for $s = 1, 2$, except for the presence of helicity ± 1 (before introduction of the auxiliary field).

Let us now consider the stress tensor T^{ν}_{μ}. We will use the canonical (Noetherian) one because it is simplest to obtain, and suffices for discussion of the total momentum P_{μ}. Since the action (6.2) depends also on second derivatives, the formula for $T^{\nu}_{\mu}(\phi_3)$ is more complicated than the usual one:

$$T^{\nu}_{\mu} = L\delta^{\nu}_{\mu} - \phi_{,\mu}\left[\frac{\partial L}{\partial\phi_{,\nu}} - \partial_{\lambda}\left(\frac{\partial L}{\partial\phi_{,\nu\lambda}}\right)\right] - \phi_{,\mu\lambda}\frac{\partial L}{\partial\phi_{,\nu\lambda}} \tag{6.20}$$

(with contraction on the unwritten indices of $\phi = \phi_3$).

However we shall now see that $\partial L/\partial(\partial^2\phi)$ (and L itself) vanish on shell. The Lagrangian is

$$L = -\frac{1}{2}\,[\phi_3 - (3m)^{-1}C_3(\phi_3)]\cdot G_3(\phi_3). \tag{6.21}$$

We have seen above that, in harmonic gauge, $\psi_3 = (3m)^{-1}C_3(\psi_3)$, which implies $\phi_3 = (3m)^{-1}C_3(\phi_3)$. The vanishing of the bracketed quantity in (6.21) then implies the vanishing of both L and $\partial L/\partial(\partial^2\phi)$. It is then easy to get from eqs. (6.20)-(6.21) the value of T^{ν}_{μ} on shell, and in harmonic gauge:

$$T^{\nu}_{\mu} = -\frac{1}{2m}\,\varepsilon^{\nu}_{\cdot\alpha\beta}[\partial_{\mu}\psi^{\alpha\kappa\lambda}\Box\psi^{\beta}{}_{\kappa\lambda} - D^{-1}\partial_{\mu}\psi^{\alpha}\Box\psi^{\beta}], \tag{6.22}$$

and of course $T^{\nu}_{\mu,\nu} = 0$ on shell. Now the general $\psi^{\mu\nu\lambda}(x)$ decomposes in two pure helicity parts, say $\psi^{\mu\nu\lambda}_{+3}(x) + \psi^{\mu\nu\lambda}_{+1}(x)$ when $m > 0$, where

$$\psi^{\mu\nu\lambda}_{+3}(x) = \int_{H_m} \tilde{dk}[\psi_{+3}(k)\varepsilon^{\mu}_{+}\varepsilon^{\nu}_{+}\varepsilon^{\lambda}_{+}e^{ikx} + \text{c. c.}], \tag{6.23 a}$$

$$\psi^{\mu\nu\lambda}_{+1}(x) = \int_{H_{3m}} \tilde{dk}[\psi_{+1}(k)\varepsilon^{(\mu}_{+}\varepsilon^{\nu}_{+}\varepsilon^{\lambda)}_{-}e^{ikx} + \text{c. c.}], \tag{6.23 b}$$

$\tilde{dk} := (2\pi)^{1-D}d^{D-1}\vec{k}/k^0$ denoting the invariant volume element on (each of the two different) mass hyperboloids H_m and H_{3m} ($D = 3$ here). The total P_{μ} is then

$$P_{\mu} = \int T^0_{\mu}d^2x = \int_{H_m} \tilde{dk}\,|\,\psi_{+3}(k)\,|^2 k_{\mu} - \int_{H_{3m}} \tilde{dk}\,|\,\psi_{+1}(k)\,|^2 k_{\mu}. \tag{6.24}$$

The $m < 0$ case is obtained by replacing $(+3, +1)$ by $(-3, -1)$ in (6.23)-(6.24). Note that our choice of « ghost sign » for the kinetic term, $-\frac{1}{2}\phi_3\Box\phi_3$, has led to positive energy for the top helicity (as for $s = 1, 2$).

The helicity 1 part is ghost-like but introduction of the auxiliary action removes it, without otherwise altering T^{ν}_{μ}, since χ_1 vanishes on shell.

To confirm that what we have called « helicity + 3 » is really that requires in principle a detailed study of the angular momentum (and hence of the symmetric stress tensor), together with the delicate choice of nonsingular canonical variables (as dictated by the boosts), as was done for the lower spins in [*18*]. However, every step of the ingredients is so identical to the lower spin cases as to guarantee that the progression from 1 to 2 to 3 in the expression for the angular momentum will be valid; the polarization structure of $\psi_{\pm 3}$ as the coefficient of $\varepsilon^{\mu}_{\pm}\varepsilon^{\nu}_{\pm}\varepsilon^{\lambda}_{\pm}$ is in any case the standard criterion for the helicity value in terms of rotations properties.

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APPENDIX A

THREE LEMMAS

LEMMA 1 (Poincaré). — If $\partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha = 0$, then ξ_α is (locally) a gradient, more precisely $\xi_\beta = \partial_\beta \xi$, with

$$\xi(x) := \int_0^1 d\lambda x^\alpha \xi_\alpha(\lambda x). \tag{A.1}$$

LEMMA 2. — If an antisymmetric $\xi_{\mu\nu}$ satisfies $\partial_\alpha \xi_{\mu\nu} = \partial_{[\mu} \phi_{\nu]\alpha}$ for some *symmetric* ϕ_2, then $\xi_{\mu\nu}(x) = \xi_{\mu\nu}(0) + \partial_{[\mu}\eta_{\nu]}$ where

$$\eta_\nu(x) := \int_0^1 \frac{d\lambda}{\lambda} x^\alpha [\phi_{\nu\alpha}(\lambda x) - \phi_{\nu\alpha}(0)]. \tag{A.2}$$

LEMMA 3. — If $\xi_{\alpha\mu\nu}$ (antisymmetric in $(\mu\nu)$) satisfies $\partial_{(\alpha}\xi_{\beta)\mu\nu} = \partial_{[\mu}\phi_{\nu]\alpha\beta}$ for some *symmetric* ϕ_3, then

$$\xi_{\alpha\mu\nu} = a_{\alpha\mu\nu} + b_{\alpha\beta\mu\nu} x^\beta + \partial_{[\mu}\psi_{\nu]\alpha} \tag{A.3}$$

for some *symmetric* ψ_2, a_3 and b_4 being constant tensors having the indicated antisymmetries.

Proof of lemma 1. — It suffices to integrate over the scaling parameter λ, between 0 and 1, the easily verified consequence of $\partial_{[\alpha}\xi_{\beta]} = 0$:

$$\frac{d}{d\lambda}[\lambda \xi_\beta(\lambda x)] = \partial_\beta [x^\alpha \xi_\alpha(\lambda x)]. \tag{A.4}$$

We quote this method of proof of the well-known Poincaré lemma because it serves as a model for the other lemmas.

Proof of lemma 2. — Let us define for any field $f(x)$, $\bar{f}(x) := f(x) - f(0)$. Then using the same kind of scaling parameter as in lemma 1, one verifies that

$$\frac{d}{d\lambda}\bar{\xi}_{\mu\nu}(\lambda x) = \partial_\mu \left[\frac{1}{\lambda} x^\alpha \bar{\phi}_{\nu\alpha}(\lambda x)\right] \tag{A.5}$$

is a consequence of the hypotheses of lemma 2. The integration of (A.5) over λ, between 0 and 1, gives the result (A.2).

Proof of lemma 3. — We now introduce $\bar{\bar{f}}(x) := f(x) - f(0) - x^\alpha \partial_\alpha f(0)$. Then one verifies, under the hypotheses of lemma 3, the relation

$$\frac{d}{d\lambda}\left[\frac{1}{\lambda}\bar{\bar{\xi}}_{\alpha\mu\nu}(\lambda x)\right] + \partial_\alpha\left[\frac{1}{\lambda^2}x^\beta \bar{\bar{\xi}}_{\beta\mu\nu}(\lambda x)\right] = \partial_\mu\left[\frac{1}{\lambda^2}x^\beta \bar{\bar{\phi}}_{\nu\alpha\beta}(\lambda x)\right]. \tag{A.6}$$

Integrating (A.6) over λ $(0 \leqq \lambda \leqq 1)$ gives

$$\bar{\bar{\xi}}_{\alpha\mu\nu}(x) = -\partial_\alpha \zeta_{\mu\nu}(x) + \partial_\mu \theta_{\nu\alpha} \tag{A.7}$$

with an antisymmetric ζ_2 and a *symmetric* θ_2. Introducing the information (A.7) into $\partial_{(\alpha}\bar{\bar{\xi}}_{\beta)\mu\nu} = \partial_{[\mu}\bar{\bar{\phi}}_{\nu]\alpha\beta}$ gives

$$-2\partial_{\alpha\beta}\zeta_{\mu\nu} = \partial_{[\mu}\omega_{\nu]\alpha\beta}, \qquad \omega_{\nu\alpha\beta} := \bar{\bar{\phi}}_{\nu\alpha\beta} - \partial_{(\nu}\theta_{\alpha\beta)}. \tag{A.8}$$

Thanks to the complete symmetry of ω_3, one can now apply lemma 2 twice: first to get $\partial_\beta \zeta_{\mu\nu}$, and then $\zeta_{\mu\nu}$ itself, with a result of the form

$$-2\zeta_{\underset{\vee}{\mu\nu}} = \partial_{[\mu}\pi_{\nu]} . \tag{A.9}$$

Defining then

$$\overline{\psi}_{\underline{\mu\nu}} := \theta_{\underline{\mu\nu}} + \frac{1}{2}\partial_{(\mu}\pi_{\nu)} \tag{A.10}$$

one gets

$$\overline{\overline{\xi}}_{\alpha\mu\nu} \equiv \xi_{\alpha\mu\nu}(x) - \xi_{\alpha\mu\nu}(0) - x^\beta \partial_\beta \xi_{\alpha\mu\nu}(0) = \partial_{[\mu}\overline{\psi}_{\nu]\alpha} , \tag{A.11}$$

which is the desired result (A.3) except for a term proportional to $x^\beta \partial_{(\beta}\xi_{\alpha)\underset{\vee}{\mu\nu}}(0) = x^\beta \partial_{[\mu}\phi_{\nu]\alpha\beta}(0)$. Finally the latter term is eliminated if one modifies $\overline{\psi}_{\mu\nu}$ by adding a term proportional to $x^\alpha x^\beta \partial_\alpha \phi_{\beta\mu\nu}$.

Note that the proofs used above are all fully constructive.

APPENDIX B

THE SPIN 3 CONFORMAL CURVATURE TENSOR IN D = 3

We can follow the proof of the case $D \geqq 4$ in § 4 up to the integrability conditions (4.15 *a*) and (4.15 *b*) or, equivalently (4.15 *b'*). Now, we have seen in § 2 (eq. (2.42 *a*)), that the Weyl tensor $C_6(\phi_3)$ was identically zero in $D = 3$. This implies, thanks to the identity (4.16), that the first integrability condition (4.15 *a*) is automatically satisfied. And as the second identity (4.17) is vacuous in $D = 3$, we are led to introduce the tensor,

$$B_{\underset{\vee-}{\rho\sigma\nu\lambda}} := \frac{D-2}{2} \partial_\rho \Sigma_{\underbrace{\nu\lambda\sigma}} . \tag{B.1}$$

The vanishing of B_4 is the necessary and sufficient condition for the (complete) integrability of eq. (4.12) for ω_2. Using the same reasoning as in the proof for $D \geqq 4$, it is also necessary and sufficient for the (local) existence of an ω_1 satisfying the system (4.6)-(4.8). Therefore B_4 fully embodies the concept of conformal curvature in $D = 3$.

A (somewhat lengthy) calculation of B_4 using the definitions (B.1), (4.14), (4.11), (4.10) and (2.41) leads, taking into account the « Einstein identities » (2.13), to the following explicit expression of B_4 in terms of « Ricci »:

$$B_{\underset{\vee-}{\rho\sigma\nu\lambda}} = \partial_\rho B_{\underbrace{\sigma\nu\lambda}} , \tag{B.2}$$

$$B_{\underline{\mu\nu\lambda}} := \Box R_{\mu\nu\lambda} - \frac{1}{2(D+1)} S_{(\mu}\eta_{\nu\lambda)} - \frac{D}{2(D+1)} \partial_{(\mu\nu} R_{\lambda)} , \tag{B.3}$$

with

$$S_\mu := \partial^\alpha \check{R}_{\alpha\mu} = \Box R_\mu - \partial_\mu(\partial^\alpha R_\alpha), \qquad R_\mu = R_{\mu\sigma}{}^\sigma \tag{B.4}$$

(one should replace D by 3 in (B.3), but we have given the general result).

Now, in $D = 3$ we can replace B_4 by its (algebraically equivalent) ($\rho\sigma$)-dual:

$$D_{\mu\nu\lambda} := \frac{1}{2} \varepsilon_\mu{}^{;\rho\sigma} B_{\rho\sigma\nu\lambda} \equiv \varepsilon_\mu{}^{;\rho\sigma} \partial_\rho B_{\sigma\nu\lambda} . \tag{B.5}$$

The 3-tensor D_3 is the spin 3 analog of Cotton's tensor D_2 (in « dual » form) see eq. (4.3 *a*) and references quoted there. D_3 fully describes conformal curvature in $D = 3$ and has the following properties (besides being, naturally, conformally invariant, $\delta_\omega D_3 = 0$): D_3 is *symmetric, traceless and transverse,*

$$D_{\mu\nu\lambda} = D_{(\mu\nu\lambda)_1} , \tag{B.6}$$

$$D_{\mu\sigma}{}^\sigma = 0, \tag{B.7}$$

$$\partial^\lambda D_{\mu\nu\lambda} = 0 . \tag{B.8}$$

Note that the total symmetry of D_3 (which is not at all manifest in its definition (B.5), with B_3 totally symmetric) implies that $D_3 = 3^{-1} C(B_3)$ where C is the symmetrized curl operator of § 5. Also, let us remark that D_3 is of fifth ($2s-1$?) order in ϕ_3, possesses 7 algebraically independent components, but, because of (B.8), depends only on 2 functions of 3 variables.

Finally, let us note that in $D = 3$ the curl operator C might be a useful tool to study the

« purely longitudinal » conformal transformations (which are also « pure trace » generalized gauge, $\xi_2 = \eta_2 \chi$)

$$\delta_\chi \phi_{\mu\nu\lambda} := \eta_{(\mu\nu}\partial_{\lambda)}\chi = \eta d\chi \tag{B.9}$$

which have appeared in various parts of our geometrical investigations (eqs (2.43), proof of theorem 3). Indeed for a general conformal transformation $\delta_\omega \phi_3 = \omega_{(1}\eta_{2)}$ one finds (using for instance the general tools of § 5)

$$\frac{1}{3} \mathrm{C}[\mathrm{G}(\omega_{(1}\eta_{2)})] = [\partial_{(\mu\nu} - \square \eta_{(\mu\nu})f_{\lambda)} \tag{B.10}$$

where

$$f_\mu := \mathrm{C}(\omega_1) = \varepsilon^{\alpha\beta}{}_\mu \partial_\alpha \omega_\beta \,. \tag{B.11}$$

The structure (B.10) is clearly transverse on each index and vanishes iff (under suitable boundary conditions) $f_1 = 0$, i. e. iff δ_ω is of the longitudinal type (B.9).

APPENDIX C

CONFORMAL TRIVIALITY IN D = 2

The methods of proof used in the cases D $\geqq$ 4 (§ 4) and D = 3 (appendix B) do not work in the case D = 2. It is better to start afresh and to prove theorem 5 directly, i. e. the existence in D = 2, for all ϕ_3, of ω_1 and $\tilde{\xi}_2$ ($\mathrm{Tr}(\tilde{\xi}_2) = 0$) such that

$$\phi_{\mu\nu\lambda} = \omega_{(\mu}\eta_{\nu\lambda)} + \partial_{(\mu}\tilde{\xi}_{\nu\lambda)} . \tag{C.1}$$

Written out in full eq. (C.1) reads

$$\phi_{111} = 3\partial_1\tilde{\xi}_{11} + 3\omega_1 , \tag{C.2a}$$

$$\phi_{112} = 2\partial_1\tilde{\xi}_{12} + \partial_2\tilde{\xi}_{11} + \omega_2 , \tag{C.2b}$$

$$\phi_{122} = \partial_1\tilde{\xi}_{22} + 2\partial_2\tilde{\xi}_{12} + \omega_1 , \tag{C.2c}$$

$$\phi_{222} = 3\partial_2\tilde{\xi}_{22} + 3\omega_2 . \tag{C.2d}$$

For notational simplicity we have used in eqs (C.2) a positive metric $\eta_{\mu\nu} = \delta_{\mu\nu} = (+, +)$. Let us now define:

$$u_1 := \tilde{\xi}_{12} , \qquad u_2 := \tilde{\xi}_{11} = -\tilde{\xi}_{22} , \tag{C.3}$$

$$\mathrm{A} := \frac{1}{6}(3\phi_{112} - \phi_{222}) , \qquad \mathrm{B} := \frac{1}{6}(\phi_{111} - 3\phi_{122}) . \tag{C.4}$$

Then the system (C.2) is equivalent to eqs (C.2 *b*) and (C.2 *c*) (which give ω_1 and ω_2 in terms of $\tilde{\xi}_2$) together with the following differential system for $\tilde{\xi}_2 \sim (u_1, u_2)$:

$$\partial_1 u_1 + \partial_2 u_2 = \mathrm{A} , \tag{C.5a}$$

$$\partial_1 u_2 - \partial_2 u_1 = \mathrm{B} . \tag{C.5b}$$

The system (C.5) always admits solutions, for instance one can write ($a, b = 1, 2$; $\varepsilon_{ab} = -\varepsilon_{ba}$, $\varepsilon_{12} = +1$)

$$u_a = \partial_a\phi + \varepsilon_{ab}\partial_b\psi , \tag{C.6}$$

with ($\Delta = \partial_{11} + \partial_{22} = \eta^{\mu\nu}\partial_{\mu\nu}$)

$$\Delta\phi = \mathrm{A} , \tag{C.7a}$$

$$\Delta\psi = -\mathrm{B} . \tag{C.7b}$$

This completes the proof of theorem 5 (§ 4).

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