ANNALES DE L'I. H. P., SECTION A

ERIK BALSLEV

Bounds on many-body resonances

Annales de l'I. H. P., section A, tome 47, nº 2 (1987), p. 185-198
<http://www.numdam.org/item?id=AIHPA_1987__47_2_185_0>

Ann. Inst. Henri Poincaré,
Vol. 47, n° 2, 1987, p. 185-198. *Physique théorique*

Bounds on many-body resonances

by

Erik BALSLEV (*)
University of Aarhus, Denmark
and
Institute for Advanced Study, Princeton

ABSTRACT. — For multiplicative, dilation-analytic pair potentials converging to zero at infinity we prove that the set of resonances of the Schrödinger operator forms a bounded set in any angular region of the associated Riemann surface bounded away from the set of accumulation points of resonances and from the limiting directions of the cuts defining the angular region.

RÉSUMÉ. — On démontre que, pour des potentiels à deux corps multiplicatifs, analytiques par dilatation et tendant vers zéro à l'infini, l'ensemble des résonances de l'opérateur de Schrödinger forme un ensemble borné dans tout secteur angulaire de la surface de Riemann associée qui ne s'approche pas de l'ensemble des points d'accumulation des résonances et des directions limites des coupures qui définissent le secteur angulaire.

INTRODUCTION

Bounds on resonances of two- and three-body Schrödinger operators with dilation-analytic, short-range potentials were obtained in [*4*], based on estimates of the symmetrized Faddeev kernel. In the present paper these results are generalized to n-body operators with dilation-analytic, multiplicative pair potentials converging to zero at infinity.

(*) The author would like to thank the Institute for Advanced Study for its hospitality and the National Science Foundation for financial support under Grant No. DMS-8610730 (1) (1.6).

The proof is based on estimates of the connected kernel $I(\varphi, \zeta)$ of the Weinberg-van Winter equation (cf. [2]), where $\varphi \in (-a, a)$ is the dilation angle $\left(0 < a \leqq \frac{\pi}{2}\right)$ and ζ is in the complement of the union of the set of half-lines $\{ \lambda + e^{-2i\varphi}\mathbb{R}^+ \mid \lambda \in \mathcal{T}_\varphi \}$, $\mathcal{T}_\varphi$ being the set of thresholds of the complex-dilated operator $H(\varphi)$. Adding a rather weak uniformity condition on the families of complex-dilated pair potentials $V_\alpha(\varphi)$, we obtain estimates of $I(\varphi, \zeta)$ on half-lines $\lambda + e^{-2i\varphi}\mathbb{R}^+$, uniformly in φ and λ as long as the distance of the half-lines $\lambda + e^{-2i\varphi}\mathbb{R}^+$ from $\mathcal{T}_\varphi$ is bounded below by some positive number. From these uniform estimates we prove boundedness of the set of resonances in certain regions of the Riemann surface associated with the given operator. If $\| I(\varphi, \lambda + e^{-2i\varphi}t) \| < 1$, then $\zeta = \lambda + e^{-2i\varphi}t$ is not a resonance on the particular sheet of the Riemann surface defined by λ and φ. The results may be described in more detail as follows.

The Riemann surface associated with an n-body problem may be very complicated due to the fact that the real and complex thresholds, which are the branchpoints of the surface, may accumulate at both real and complex threshold, their accumulation points may accumulate, etc., the process terminating after a finite number of steps with the zero-threshold.

Under our very general conditions on the potentials the estimates of the norm of $I(\varphi, \zeta)$ depend on the distance of ζ from the cuts with direction $e^{-2i\varphi}$. As the branch points accumulate, the distances between adjacent cuts go to zero and the estimating constants go to infinity. Therefore it is possible that the resonances go to infinity as the branch points accumulate, which define the sheets of the Riemann surface on which the resonances lie. We therefore consider a region bounded by an interval (λ', λ'') between consecutive real thresholds λ' and λ'' and the limiting half-lines $\lambda' + e^{-2ia}\mathbb{R}^+$, $\lambda'' + \mathbb{R}^+$. Within this region we restrict the attention to those sheets of the Riemann surface which are defined by the finite number of branch points with a minimum distance ε from the set of accumulation points of the threshold set.

Even with this restriction resonances may go to infinity along the bounding half-lines $\lambda' + e^{-2ia}\mathbb{R}^+$ and $\lambda'' + \mathbb{R}^+$, since the estimates are not uniform in φ up to $\varphi = 0$ and $\varphi = a$. Also resonances may go to infinity along a finite number of half-lines $\lambda_i + e^{-2i\varphi_i}\mathbb{R}^+$, $\lambda' < \lambda_i < \lambda''$, $0 < \varphi_i < a$, with the property that they contain two complex thresholds. This is due to the non-uniformity of the estimates on half-lines $\lambda + e^{-2i\varphi}\mathbb{R}^+$, passing between the two thresholds, as $\lambda \to \lambda_i$, $\varphi \to \varphi_i$.

Taking these restrictions into account, our main result (Theorem 3.9) is that for every $\varepsilon > 0$ resonances form a bounded set in the region bounded by (λ', λ'') and the half-lines $\lambda' + e^{-2i(a-\varepsilon)}\mathbb{R}^+$, $\lambda'' + e^{-2i\varepsilon}\mathbb{R}^+$ with omission of the sectors bounded by the half-lines $\lambda_i + e^{-2i(\varphi_i \pm \varepsilon)}\mathbb{R}^+$ and on all sheets of the Riemann surface defined by thresholds of distance at least ε from the set of accumulation points of the threshold set.

In the 3-body case the above result generalizes those of [4] by allowing long-range potentials and by not assuming complex thresholds to be simple resonances. On the other hand, we require local singularities to be at most like $r^{-1+\varepsilon}$, whereas $r^{-2+\varepsilon}$ type singularities are allowed in [4]. Two complex thresholds on the same half-line do not cause any problem in [4], because the results of [4] are based on estimates uniform up to the cuts. For the same reason the results of [4] rule out that resonances can go infinity along $e^{-2ia}\mathbb{R}^+$, provided the potentials have short range boundary values as $\varphi \to a_-$. Finally, for potentials going to zero faster then $r^{-2-\varepsilon}$ the possibility that resonances go to infinity along $\mathbb{R}^+$ is ruled out by establishing estimates uniform in φ up to $\varphi = 0$. It seems likely that these results hold in the n-body case, but their generalization would require more refined techniques.

1. NOTATIONS AND BACKGROUND

Let $S = \{1, \ldots, n\}$ be a system of n particles with masses $m_1, \ldots, m_n$. A cluster C is a subset of S containing at least two particles. Let $C_P = \{1, \ldots, p\}$, let $\bar{r}_{C_{j-1},j}$ be the position vector from the center-of-mass of C_{j-1} to j and $r_{C_{j-1},j} = |\bar{r}_{C_{j-1},j}|$, $r^2_{C_p} = r^2_{12} + r^2_{C_2,3} + \ldots + r^2_{C_{p-1},p}$. A decomposition D is a partition of S into a set of disjoint clusters and single particles. We write $D = D_k$, if D contains m clusters $C^1, \ldots, C^m$ and $k - m$ particles. Set $r^2_{D_k} = r^2_{C^1} + \ldots + r^2_{C^k}$. We write $D_l \subset D_k$ if $l > k$ and every cluster of D_l is contained in some cluster of D_k. Pairs are denoted by α, β, etc. A pair α is identified with the decomposition D_{n-1} containing α.

DEFINITION 1.1. — Let D and D_k be decompositions with clusters $G^1, \ldots, G^m$ and $C^1, \ldots, C^k$ respectively. Let $\tilde{G}^i = G^i \setminus \bigcup_{j=1}^{k} C^j$ and let $\tilde{r}_{ij}$ be the distance between the centers-of-mass of $\tilde{G}^i$ and C^j if $G^i \cap C^j \neq \emptyset$. Let D^k be the decomposition obtained by connecting all particles which are connected in either D or D_k. Let

$$r^2_{D,D_k} = r^2_{\tilde{G}^1} + \ldots + r^2_{\tilde{G}^m} + \sum_{k,j} \tilde{r}^2_{ij}.$$

LEMME 1.2. — *i*) Let C^1 and C^2 be clusters of the decomposition D_{k+1}, suppose that D_k is obtained from D_{k+1} by joining C^1 and C^2 and let $i \in C^1$, $j \in C^2$. Then

$$1 + r^2_{D_k} \leqq (1 + r^2_{D_{k+1}})(1 + r^2_{ij}),$$

ii) Let D, D_k, D^k and r_{D,D_k} be given as in Definition 1.1. Then there exists $C > 0$ such that

$$1 + r_{D,D_k}^2 \leqq C(1 + r_{D_k}^2).$$

Proof. — *i*) is proved in [2] Lemma 1.1. To prove *ii*), consider one term $\tilde{r}_{\tilde{G}^i}^2 + \tilde{r}_{ij}^2$. Then $G^i \cap C^j \neq \emptyset$, for some particle $p \in C^j$, $\tilde{r}_{ij} \leqq r_{\tilde{G}^i,p}$, hence

$$\tilde{r}_{\tilde{G}^i}^2 + \tilde{r}_{ij}^2 \leqq r_{G^i \cup C^j}^2 \leqq r_{D_k}^2$$

and *ii*) follows.

We shall make use of the spaces $L^p = L^p(\mathbb{R}^3)$ and the weighted L^∞-spaces $L_s^\infty = L_s^\infty(\mathbb{R}^3)$ defined for $s > 0$ by

$$L_s^\infty = \{ f \mid \| f \|_{L_s^\infty} = \| (1 + r)^s f \|_{L^\infty} < \infty \}.$$

Let $0 < a \leqq \frac{\pi}{2}$ and $S_a = \{ \rho e^{i\varphi} \mid \rho > 0, -a < \varphi < a \}$. Let $U(\rho)$ be the unitary group of dilations on $L^2(\mathbb{R}^{3(n-1)})$ defined by

$$(U(\rho)f)(\bar{r}) = \rho^{\frac{3(n-1)}{2}} f(\rho\bar{r}).$$

The symmetric operator V in $L^2(\mathbb{R}^3)$ is S_a-dilation-analytic, if V is Δ-compact and the $\mathscr{C}(\mathscr{D}_\Delta, L^2(\mathbb{R}^3))$-valued function $V(\rho) = U(\rho)VU(\rho)^{-1}$ on $\mathbb{R}^+$ has an analytic extension $V(z)$ to S_a. We set $V(\varphi) = V(e^{i\varphi})$. Note that if V is multiplicative, then $V(z)$ is multiplicative for all $z \in S_a$ (cf. [1]).

We assume that the interaction V is a sum of pair potentials $V_\alpha = V_\alpha(\bar{r}_\alpha)$ satisfying the following conditions, formulated for a real-valued, measurable function v on $\mathbb{R}^3$:

(A1) The operator of multiplication by v is S_a-analytic.

(A2) For some $q < 3 < p$, $s > 0$ and all $\varphi \in (-a, a)$, there exist decompositions

$$v(\varphi) = u(\varphi) + w(\varphi),\ u(\varphi) \in L^p \cap L^q,\ w(\varphi) \in L_s^\infty$$

such that the maps $\varphi \to u(\varphi)$, $\varphi \to w(\varphi)$ are continuous from $(-a, a)$ to $L^p \cap L^q$ and L_s^∞, respectively. Moreover, there exists for every $\varepsilon > 0$ a function $g_\varepsilon \in L^p \cap L^q$ such that

$$| [u(\varphi)](\bar{r}) | \leqq g_\varepsilon(\bar{r}) \quad \text{for} \quad |\varphi| \leqq a - \varepsilon,\ \bar{r} \in \mathbb{R}^3.$$

Under the assumption (A1) a selfadjoint, analytic family $H(z)$ is defined for $z \in S_a$ by

$$H(z) = z^{-2}H_0 + \sum_\alpha v_\alpha(z),$$

such that for fixed $\varphi \in (-a, a)$

$$H(\rho e^{i\varphi}) = U(\rho)H(e^{i\varphi})U(\rho)^{-1}. \tag{1.2}$$

For any cluster C and any decomposition D we define $H^C(z)$ and $H_D(z)$ by

$$H^C(z) = z^{-2}H_0^C + \sum_{\alpha \subseteq C} v_\alpha(z)$$

$$H_D(z) = z^{-2}H_0 + \sum_{\alpha \subseteq D} v_\alpha(z).$$

The resolvents are denoted by $R(z, \zeta)$, $R^C(z, \zeta)$, $R_D(z, \zeta)$, etc. We set $H(e^{i\varphi}) = H(\varphi)$, $R(e^{i\varphi}, \zeta) = R(\varphi, \zeta)$, etc. The spectrum, essential spectrum, discrete spectrum, point spectrum and resolvent set, which depend on φ only by (1.2), are denoted by $\sigma(\varphi)$, $\sigma_e(\varphi)$, $\sigma_d(\varphi)$ and $\rho(\varphi)$, respectively.

The connected parts $I^r_{D_k}(\varphi, \zeta)$ and $I^l_{D_k}(\varphi, \zeta)$ and the disconnected part $D_{D_k}(\varphi, \zeta)$ of the resolvent $R_{D_k}(\varphi, \zeta)$ are given by

$$I^r_{D_k}(\varphi, \zeta) = \sum_{D_{n-1} \subset \ldots \subset D_k} \sum_{\{\alpha_1 \ldots \alpha_{n-k}\}} v_{\alpha_{n-k}}(\varphi) R_{D_{k+1}}(\varphi, \zeta)$$

$$\ldots v_{\alpha_{n-i+1}}(\varphi) R_{D_i}(\varphi, \zeta) \ldots v_{\alpha_1}(\varphi) R_0(\varphi, \zeta) \quad (1.3)$$

$$I^l_{D_k}(\varphi, \zeta) = I^*_r(-\varphi, \bar{\zeta}) \quad (1.4)$$

where $\sum_{D_{n-1} \subset \ldots \subset D_k}$ is over all sequences of decompositions and $\sum_{\{\alpha_1 \ldots \alpha_{n-k}\}}$ is over all sequences of pairs such that α_{n-i+1} connects two clusters from D_i to form a cluster from D_{i-1};

$$I^l(\varphi, \zeta) = I^l_{D_1}(\varphi, \zeta); \; I^r(\varphi, \zeta) = I^r_{D_1}(\varphi, \zeta);$$

$$D_{D_k}(\varphi, \zeta) = \sum_{D_m \subset D_k} (-1)^m (m-1)! \, R_{D_m}(\varphi, \zeta). \quad (1.5)$$

The Weinberg-van Winter equation will be used in the forms

$$R_{D_k}(\varphi, \zeta) = D_{D_k}(\varphi, \zeta) + I^l_{D_k}(\varphi, \zeta) R_{D_k}(\varphi, \zeta) \quad (1.6)$$

$$R_{D_k}(\varphi, \zeta) = D_{D_k}(\varphi, \zeta) + R_{D_k}(\varphi, \zeta) I^r_{D_k}(\varphi, \zeta). \quad (1.7)$$

A threshold λ of $H(\varphi)$ is a sum of discrete eigenvalues μ_i of operators $H^{C_i}(\varphi)$ of disjoint clusters C_i. Let $\mathcal{T}(\varphi)$ denote the set of thresholds of $H(\varphi)$.

The spectrum $\sigma(\varphi)$ of $H(\varphi)$ was characterized in [3] as follows:

The spectrum $\sigma(\varphi)$ of $H(\varphi)$ consists of

i) $\sigma_e(\varphi) = \{ \lambda + e^{-2i\varphi}\mathbb{R}^+ \mid \lambda \in \mathcal{T}(\varphi) \}$

ii) $\sigma^r_d = \sigma_p(H) \backslash \mathcal{T}(\varphi)$

iii) $\sigma^c_d(\varphi)$, contained in the angle bounded by $\lambda_e + \mathbb{R}^+$ and $\lambda_e + e^{-2i\varphi}\mathbb{R}^+$, where $\lambda_e = \min \{ \lambda \in \mathcal{T}(\varphi) \cap \mathbb{R} \}$ and otherwise independent of φ unless a point gets absorbed by $\sigma_e(\varphi)$.

DEFINITION. — The set $\mathscr{R}$ of resonances of H is defined by $\mathscr{R} = \bigcup_{0<\varphi<a} \sigma_d^c(\varphi)$.

Let $\lambda' < \lambda''$ be two consecutive real thresholds of H, and let $I = (\lambda'\lambda'')$. Let $\mathcal{O}_I$ be the domain bounded by I and the half-lines $\lambda' + e^{-2ia}\mathbb{R}^+$, $\lambda'' + \mathbb{R}^+$. For $\varepsilon > 0$, let $\mathcal{O}_I^\varepsilon$ be the domain bounded by I and the half-lines $e^{-2i(a-\varepsilon)}\mathbb{R}^+$, $\lambda'' + e^{-2i\varepsilon}\mathbb{R}^+$. The set of thresholds in $\mathcal{O}_I$ and $\mathcal{O}_I^\varepsilon$ are denoted by $\mathscr{T}_I$ and $\mathscr{T}_I^\varepsilon$ respectively. The set of resonances in $\mathcal{O}_I$ and $\mathcal{O}_I^\varepsilon$ are denoted by $\mathscr{R}_I$ and $\mathscr{R}_I^\varepsilon$ respectively. For any cluster C and consecutive real thresholds (λ', λ'') of H^C, we denote by $\mathscr{R}_I^C$ and $\mathscr{R}_I^{C,\varepsilon}$ the resonances of H^C in $\mathcal{O}_I$ and $\mathcal{O}_I^\varepsilon$ respectively. The distance between two sets A_1 and A_2 is denoted by $d(A_1, A_2)$. For $\delta > 0$, let $I_\delta = (\lambda' + \delta, \lambda'' - \delta)$ and $I_{\delta,\varepsilon} = I_{\delta(\sin 2\varepsilon)^{-1}}$. For $\varepsilon > 0$, $0 < \delta < \dfrac{\lambda'' - \lambda'}{2}\sin 2\varepsilon$, $\lambda \in I_{\delta,\varepsilon}$ and any cluster C, set

$$J_{\varepsilon,\delta,\lambda}^C = \{ \varphi \in [\varepsilon, a - \varepsilon], d(\lambda + e^{-2i\varphi}\mathbb{R}^+, \sigma(H^C(\varphi))) \geq \delta \} ;$$

for any decomposition D, set

$$J_{\varepsilon,\delta,\lambda}^D = \{ \varphi \in [\varepsilon, a - \varepsilon] \mid d(\lambda + e^{-2i\varphi}\mathbb{R}^+, \sigma(H_D(\varphi))) \geqq \delta \}$$

and let

$$M_{\varepsilon,\delta,\lambda} = \bigcap_D J_{\varepsilon,\delta,\lambda}^D = \{ \varphi \in [\varepsilon, a - \varepsilon] \mid d(\lambda + e^{-2i\varphi}\mathbb{R}^+, \sigma_e(H(\varphi))) \geqq \delta \}.$$

2. BASIC ESTIMATES

LEMMA 2.1. — Given $\varkappa, \varepsilon > 0$, there exists $K = K(\varkappa, \varepsilon) < \infty$ and decompositions

$$v_\alpha(\varphi) = u_\alpha(\varphi) + w_\alpha(\varphi)$$

such that for $\varepsilon \leqq \varphi \leqq a - \varepsilon$ and all α

$$\| u_\alpha(\varphi) \|_p < \varkappa, \quad \| u_\alpha(\varphi) \|_q < \varkappa, \quad \| w_\alpha(\varphi) \|_{L_S^\infty} < K.$$

Proof. — We refer to the proof of [*4*] Lemma 4.1.

LEMMA 2.2. — For every $p \in \mathbb{N}$ there exists $C = C(p)$ such that for every decomposition D and all $A \in \mathscr{B}(\mathscr{H})$,

$$\| (1 + r_D^2)^{-s}A \| \leqq \| (1 + r_D^2)^{-2^p s}A \|^{2^{-p}} \| A \|.$$

Proof. — $\|(1+r_D^2)^{-s}A\| = \|A^*(1+r_D^2)^{-2s}A\|^{1/2} \leqq \|(1+r_D^2)^{-2s}A\|^{1/2}\|A\|^{1/2}$. Repeating this estimate p times, we obtain the Lemma.

LEMMA 2.3. — Given $\varepsilon, \delta > 0$

$$\| (1 + r_D^2)^{-s}R_0(\varphi, \lambda + e^{-2i\varphi}t) \| \xrightarrow[t\to\infty]{} 0,$$

uniformly for $\varepsilon \leqq \varphi \leqq a - \varepsilon$, $\lambda \in \mathbb{R}$, $|\lambda| \geqq \delta$.

Proof. — We note that

$$R_0(\varphi, \lambda + e^{-2i\varphi}t) = e^{2i\varphi}R_0(\lambda e^{2i\varphi} + t). \tag{2.1}$$

Choose $p \in \mathbb{N}$ such that $2^p s > \frac{1}{4}$. By [2] Lemma 2.5, there exists $C = C(\delta, \varepsilon)$ such that for $\varepsilon \leqq \varphi \leqq a - \varepsilon$, $\lambda \in \mathbb{R}$, $|\lambda| \geqq \delta$, $t \geqq 1$

$$\| (1 + r_D^2)^{-2^p s} R_0(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq Ct^{-\frac{1}{4}}.$$

By Lemma 2.2 this implies

$$\| (1 + r_D^2)^{-s} R_0(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq Ct^{-\frac{1}{4}}(\delta \sin 2\varepsilon)^{-1}$$

and the lemma follows.

LEMMA 2.4 (*Iorio-O'Carroll*). — Given $\varepsilon, \delta > 0$, there exists $C = C(\varepsilon, \delta)$, such that for $\varepsilon \leqq \varphi \leqq a - \varepsilon$, $\lambda \in \mathbb{R}$, $|\lambda| \geqq \delta$, $t \in \mathbb{R}$ and all pairs α, β

i) $\| u_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) u_\beta(\varphi) \| \leqq$
$$\leqq C \max_{\varphi \in [\varepsilon, a-\varepsilon]} \{ (\| u_\alpha \|_p^{1/2} + \| u_\alpha \|_q^{1/2})(\| u_\beta |_p^{1/2} + \| u_\beta \|_q^{1/2}) \}$$

ii) $\| u_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq C \max_{\varphi \in [\varepsilon, a-\varepsilon]} \{ \| u_\alpha \|_p^{1/2} + \| u_\alpha \|_q^{1/2} \}$.

Proof. — *i)* is proved in [5], and *ii)* follows from *i)*, using (2.1) and the 1st resolvent equation.

LEMMA 2.5. — Given $\varepsilon, \delta > 0$, there exists $C = C(\varepsilon, \delta)$, such that for $\varepsilon \leqq \varphi \leqq a - \varepsilon$, $\lambda \in \mathbb{R}$, $|\lambda| \geqq \delta$, $t \in \mathbb{R}$ and all α, β

$$\| v_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq C$$
$$\| v_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) v_\beta(\varphi) \| \leqq C.$$

Proof. — This follows from Lemma 2.4 and (2.2), since $\| w_\alpha(\varphi) \|_{L^\infty_\varepsilon} \leqq C(\varepsilon)$ for $\varepsilon \leqq \varphi \leqq a - \varepsilon$.

3. THE MAIN RESULT

We make the following induction assumptions:

(B1) For every system of at most $n - 1$ particles, every pair of consecutive real thresholds (λ', λ'') of H^C and every $\varepsilon > 0$, the set $\mathscr{R}_I^{C,\varepsilon}$ is bounded.

(B2) For every ε, $\delta > 0$ there exists $C = C(\varepsilon, \delta)$ such that for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in J^C_{\varepsilon,\delta,\lambda}$ and $f \in \mathscr{H}$

$$\int_{-\infty}^{\infty} \| R^C(\varphi, \lambda + e^{-2i\varphi}t) f \|^2 dt \leqq C \| f \|^2.$$

LEMMA 3.1. — Given $\varepsilon, \delta > 0$, there exists $C = C(\varepsilon, \delta)$ such that for every decomposition D and for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in J^{D}_{\varepsilon,\delta,\lambda}$ and $f \in \mathscr{H}$,

$$i) \qquad \int_{-\infty}^{\infty} \| R_D(\varphi, \lambda + e^{-2i\varphi}t)f \|^2 dt \leqq C \| f \|^2$$

$$ii) \qquad \sup_{t \in \mathbb{R}} \| R_D(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq C .$$

Proof. — *i)* follows from (B1) and (B2) and the representation

$$R_D(\varphi, \zeta) = (R^{C_1} * \ldots * R^{C_k} * R^{D}_{rel})(\varphi, \zeta) \tag{3.1}$$

where $C^1 \ldots C^k$ are the clusters of D, $R^{D}_{rel}(\varphi, \zeta) = (H^{D}_{0,rel}(\varphi) - \zeta)^{-1}$, $H_{0,rel}$ the free Hamiltonian of the centers-of-mass of $C^1, \ldots, C^k$, and the convolution is defined by integration over suitable contours (cf. [*2*] and [*6*] for details). Using the uniformity in λ and φ assumed in (B1) and (B2) the contours can be chosen depending on φ and λ so as to obtain the uniform estimate *i)*. The estimate *ii)* is an easy consequence of *i)*, applying [*2*] Lemma 2.7.

LEMMA 3.2. — Given $\varepsilon, \delta > 0$, there exists $C = C(\varepsilon, \delta)$ such that for all D, α, $\lambda \in I_\delta$, $\varphi \in J^{D}_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$

i) $\| v_\alpha(\varphi) R_D(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq C$

ii) $\| u_\alpha(\varphi) R_D(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq C \sup_{\varphi \in [\varepsilon, a-\varepsilon]} \{ \| u_\alpha \|_p^{1/2} + \| u_\alpha \|_q^{1/2} \}$.

Proof.

$$v_\alpha(\varphi) R_D(\varphi, \lambda + e^{-2i\varphi}t) =$$
$$= v_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) + \sum_{\beta \subseteq D} v_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) v_\beta(\varphi) \cdot R_D(\varphi, \lambda + e^{-2i\varphi}t),$$

and *i)* follows from Lemmas 2.5 and 3.1.

Moreover,

$$u_\alpha(\varphi) R_D(\varphi, \lambda + e^{-2i\varphi}t) = u_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t) \left(1 + \sum_{\beta \subseteq D} v_\beta(\varphi) R_D(\varphi, \lambda + e^{-2i\varphi}t) \right),$$

and *ii)* follows from *i)* and Lemma 2.4 *ii)*.

LEMMA 3.3. — Let ε, δ, $\tau > 0$. There exists $C = C(\varepsilon, \delta, \tau)$ and $w_{\alpha_1}(\varphi), \ldots, w_{\varphi_{n-1}}(\alpha) \in L^\infty_s$ such that for $\lambda \in I_{\delta,\varepsilon}$, $M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$ and $k = 1, \ldots, n-1$ $\| w_{\alpha_k}(\varphi) \|_{L^\infty_s} \leqq C$ and any term of (1.3) satisfies the estimate

$$\| v_{\alpha_{n-k}}(\varphi) R_{D_{k+1}}(\varphi, \zeta) \ldots v_{\alpha_1}(\varphi) R_0(\varphi, \zeta) \| \leqq$$
$$\leqq \tau + \| w_{\alpha_{n-k}}(\varphi) R_{D_{k+1}}(\varphi, \zeta) \ldots w_{\alpha_1}(\varphi) R_0(\varphi, \zeta) \| . \tag{3.2}$$

Proof. — Consider the case $k = 1$, the proof for $k > 1$ is similar. By definition of $M_{\varepsilon,\delta,\lambda}$, $d(\lambda + e^{-2i\varphi}t, \sigma(H_D(\varphi))) \geqq \delta$ for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$ and for all D. By Lemmas 3.1 and 3.2 there exists $C = C(\varepsilon, \delta)$ such that *ii*) of Lemma 3.1 and *ii*) of Lemma 3.2 hold. Let $\varkappa_1 = (\tau/2nC^n)^2$. By Lemma 2.1, with $\varkappa = \varkappa_1$, there exist $K_1 = K_1(\varkappa_1, \varepsilon) = K_1(\varepsilon, \delta, \tau) < \infty$ and decompositions $v_{\alpha_{n-1}}(\varphi) = u_{\alpha_{n-1}}(\varphi) + w_{\alpha_{n-1}}(\varphi)$ such that

$$\sup_{\varphi\in[\varepsilon,a-\varepsilon]} \{ \| u_{\alpha_{n-1}}(\varphi) \|_p^{1/2} + \| u_{\alpha_{n-1}}(\varphi) \|_q^{1/2} \} \leqq \frac{\tau}{nC^n}$$

and

$$\sup_{\varphi\in[\varepsilon,a-\varepsilon]} \| w_{\alpha_{n-1}}(\varphi) \|_{L_s^\infty} = K_1 .$$

This implies by Lemma 3.2

$$\| v_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta)v_{\alpha_{n-2}}(\varphi)R_{D_3}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \| \leqq$$
$$\leqq \frac{\tau}{n} + \| w_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta)v_{\alpha_{n-2}}(\varphi)R_{D_3}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \| .$$

Let $\varkappa_2 = \left(\dfrac{\tau}{2nC^nK_1}\right)^2$. By Lemma 2.1, with $\varkappa = \varkappa_2$, there exists $K_2 = K_2(\varkappa_2, \varepsilon) = K_2(\varepsilon, \delta, \tau) < \infty$ and decompositions $v_{\alpha_{n-2}}(\varphi) = u_{\alpha_{n-2}}(\varphi) + w_{\alpha_{n-2}}(\varphi)$ such that

$$\sup_{\varphi\in[\varepsilon,a-\varepsilon]} \{ \| u_{\alpha_{n-2}}(\varphi) \|_p^{1/2} + \| u_{\alpha_{n-2}}(\varphi) \|_q^{1/2} \} \leqq \frac{\tau}{nC^kK_1}$$
$$\sup_{\varphi\in[\varepsilon,a-\varepsilon]} \| w_{\alpha_{n-2}}(\varphi) \|_{L_s^\infty} = K_2$$

By Lemma 3.1 *ii*) and Lemma 3.2 *i*) and *ii*) this implies

$$\| w_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \| \leqq$$
$$\leqq \frac{\tau}{n} + \| w_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta)w_{\alpha_{n-2}}(\varphi)R_{D_3}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \| .$$

Repeating this procedure n times, we arrive at

$$\| v_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta)v_{\alpha_{n-2}}(\varphi)R_{D_3}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \leqq$$
$$\leqq \tau + \| w_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta)w_{\alpha_{n-2}}(\varphi)R_{D_3}(\varphi, \zeta) \dots w_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \|$$

where

$$\sup_{\varphi\in[\varepsilon,a-\varepsilon]} \| w_{\alpha_{n-i}}(\varphi) \|_{L_s^\infty} = K = K_n(\varepsilon, \delta, \tau),$$

and the lemma is proved.

LEMMA 3.4. — Let $\varepsilon, \delta > 0$, $\beta < \dfrac{1}{2} < \alpha$, $\alpha + \beta \leqq 1$ be given. Then there exists $C = C(\varepsilon, \delta, \alpha, \beta)$ such that for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$ and all D and D_k, with r_{D,D_k} given by Definition 1.1,

$$\| (1 + r_{D,D_k}^2)^{-\frac{\alpha}{2}} R_{D_k}(\varphi, \lambda + e^{-2i\varphi}t)(1 + r_{D,D_k}^2)^{\frac{\beta}{2}} \| \leqq C. \tag{3.3}$$

Proof. — In the special case when $G^i \cap C^j = \emptyset$ for all clusters G^i of D and C^j of D_k, (3.3) with r_{D,D_k} replaced by r_D was proved in [2] Lemma 5.1 for $\lambda \in I_{\delta,\varepsilon}$, $t \in \mathbb{R}$ and φ fixed in $M_{\varepsilon,\delta,\lambda}$. The proof was based on the convolution representation (3.1) of $R_{D_k}(\varphi, \zeta)$ and (3.3) with R_{D_k} replaced by $R_0^{G^0}$, G^0 being the union of all clusters of D.

It is easy to check that the proof is valid also for r_{D,D_k} utilizing the validity of (3.3) with R_{D_k} replaced by the resolvent of the free Hamiltonian of the particles in $\bigcup_{i=1}^{m} \tilde{G}^i$ and the centers-of-mass of $C_1, \ldots, C_k$ (cf. Def. 1.1). Using Lemma 3.1 one also verifies easily that C can in fact be chosen independent of $\varphi \in M_{\varepsilon,\delta,\lambda}$.

LEMMA 3.5. — Let $\beta < \frac{1}{2} < \alpha$, $\alpha + \beta \leqq 1$, be fixed. Let $\varepsilon, \delta, \tau > 0$, let $A(\varphi, \lambda + e^{-2i\varphi}t) \in \mathcal{B}(\mathcal{H})$ and assume that for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$

$$\| A(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq K.$$

Then there exist $1 \geqq \eta > 0$, $p > 0$ and $C = C(\varepsilon, \delta, \tau, K)$ such that for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$ and all D and D_k

$$\| (1 + r_D^2)^{-\frac{\alpha}{2}} R_{D_k}(\varphi, \lambda + e^{-2i\varphi}t) A(\varphi, \lambda + e^{-2i\varphi}t) \| \leqq$$
$$\leqq C \| (1 + r_D^2)^{-\frac{\beta}{2}} A(\varphi, \lambda + e^{-2i\varphi}t) \|^{\eta} + p\tau^{\eta}. \tag{3.4}$$

Proof. — For $k = n$, (3.4) follows from Lemma 3.4. We now make the following assumption:

The estimate (3.4) holds for all D and D_m with $m \geqq k + 1$. $\quad (3.4)_k$

We shall then prove (3.4) for $m = k$, and the lemma will follow. Let $\eta > 0$, $p > 0$ and $C = C(\varepsilon, \delta, \tau, K)$ denote different constants, fixed in each estimate.

By (1.6) and $(3.4)_k$ it suffices to prove (3.4) with R_{D_k} replaced by $I_{D_k}^l R_{D_k}$. We estimate a typical term as follows, using Lemma 3.3 and then repeatedly $(3.4)_k$, Lemmas 2.2 and 1.1 *i*), obtaining for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$

$$\|(1+r_D^2)^{-\frac{\alpha}{2}} R_0(\varphi, \zeta) v_{\alpha_1}(\varphi) R_{D_{n-1}}(\varphi, \zeta) v_{\alpha_2}(\varphi) \ldots v_{\alpha_{n-k}}(\varphi) R_{D_k}(\varphi, \zeta) A(\varphi, \zeta)\| \leqq$$
$$\leqq C\|(1+r_{D^k}^2)^{-\frac{\alpha}{2}} R_{D_k}(\varphi, \zeta) A(\varphi, \zeta)\|^{\eta} + p\tau^{\eta} \tag{3.5}$$

where D^k is obtained by connecting all particles which are connected in either D or D_k. We have also used the inequality $(a + b)^\eta \leqq a^\eta + b^\eta$, valid for $0 < \eta \leqq 1$, $a, b > 0$.

By (1.7) and $(3.4)_k$ it suffices in order to estimate the r. h. s. of (3.5) to estimate the operator obtained by replacing R_{D_k} with $R_{D_k}I^r_{D_k}$. A typical term is estimated as follows, using Lemmas 1.1 *ii*), 3.4 and 2.2,

$$\|(1+r_{D^k}^2)^{-\frac{\alpha}{2}}R_{D_k}(\varphi, \zeta)v_{\alpha_{n-k}}(\varphi)R_{D_{k+1}}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta)A(\varphi, \zeta)\| \leqq$$
$$\leqq C\|(1+r_{D,D_k}^2)^{-\frac{\alpha}{2}}(1+r_{\alpha_{n-k}}^2)^{-\frac{\alpha}{2}}R_{D_{k+1}}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta)A(\varphi, \zeta)\|^\eta. \quad (3.6)$$

Using Lemma 3.3 and then repeatedly $(3.4)_k$, Lemmas 2.2 and 1.1 *i*), we obtain (3.4) for $m = k$, and the lemma is proved.

THEOREM 3.6. — Let ε, $\delta > 0$. Then

$$\| I^r(\varphi, \lambda + e^{-2i\varphi}t) \| \underset{t\to\infty}{\longrightarrow} 0,$$

uniformly for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$.

Proof. — A typical term of $I^r(\varphi, \lambda+e^{-2i\varphi}t)$ is given, setting $\zeta=\lambda+e^{-2i\varphi}t$, by

$$v_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta) \dots v_{\alpha_{n-i+1}}(\varphi)R_{D_i}(\varphi, \zeta) \dots v_{\alpha_1}(\varphi)R_0(\varphi, \zeta).$$

By Lemma 3.3 it suffices to consider the case where v_{α_j} is replaced by $w_{\alpha_j}, j = 1 \dots n-1$. Let $\tau > 0$ be given. Applying repeatedly Lemmas 3.1 *ii*) and 3.5, we obtain for some $p > 0$, $0 < \eta \leqq 1$, $C = C(\varepsilon, \delta, \tau) > 0$

$$\| w_{\alpha_{n-1}}(\varphi)R_{D_2}(\varphi, \zeta) \dots w_{\alpha_{n-i+1}}(\varphi)R_{D_i}(\varphi, \zeta) \dots w_{\alpha_1}(\varphi)R_0(\varphi, \zeta) \| \leqq$$
$$\leqq C \| (1 + r^2)^{-\frac{\alpha}{2}}R_0(\varphi, \zeta) \|^\eta + p\tau^\eta.$$

Note that η and p depend only on α and β, which are chosen fixed with $\beta < \frac{1}{2} < \alpha$, $\alpha + \beta \leqq 1$.

Thus, for any $\sigma > 0$ we can find $\tau > 0$ such that $p\tau^\eta < \frac{\sigma}{2}$ and then by Lemma 2.3 $T > 0$ such that for $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in M_{\varepsilon,\delta,\lambda}$ and $t \geqq T$

$$C(\varepsilon, \delta, \tau) \| (1 + r_S^2)^{-\frac{\alpha}{2}}R_0(\varphi, \lambda + e^{-2i\varphi}t) \|^\eta < \frac{\sigma}{2}.$$

The Theorem is proved.

DEFINITION 3.7. — Let $\lambda' < \lambda''$ be consecutive real thresholds with $I = (\lambda', \lambda'')$ and let $R_\varepsilon = \sup \{ |\mu| \mid \mu \in \mathcal{T}_I^\varepsilon \}$. By (B1), $R_\varepsilon < \infty$. Let $\mathcal{O}_I^{\varepsilon,R_\varepsilon} = \{ \zeta \in \mathcal{O}_I^\varepsilon \mid |\zeta - \lambda_0| \geq 2R_\varepsilon \}$, where $\lambda_0 = \dfrac{\lambda' + \lambda''}{2}$. Let $\mathcal{T}_I^{\varepsilon'}$ be the set of limit points of $\mathcal{T}_I^\varepsilon$, and let $\tilde{\mathcal{T}}_I^\varepsilon = \{ \mu \in \mathcal{T}_I^\varepsilon \mid d(\mu, \mathcal{T}_I^{\varepsilon'}) \geqq \varepsilon \}$. Let $\xi_1 \leqq \xi_2 \leqq \dots \leqq \xi_p$ be the set of points in $[\lambda', \lambda'']$ with the property that for $i = 1 \dots p$ there exists $\varphi_i \in [\varepsilon, a - \varepsilon]$ such that the half-line $\xi_i + e^{-2i\varphi_i}\mathbb{R}^+$ contains two points of the finite set $\tilde{\mathcal{T}}_{\mathrm{I}}^{\varepsilon}$. Let

$$\tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon} = \mathcal{O}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon} \Big\backslash \bigcup_{i=1}^{p} \bigcup_{|\eta|<\varepsilon} \{ \xi_i + e^{-2i(\varphi_i+\eta)} \},$$

$$\tilde{\mathcal{R}}_{\mathrm{I}}^{\varepsilon} = \mathcal{R}_{\mathrm{I}}^{\varepsilon} \Big\backslash \bigcup_{i=1}^{p} \bigcup_{|\eta|<\varepsilon} \{ \xi_i + e^{-2i(\varphi_i+\eta)} \}.$$

For $\mathrm{P} \in \tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon}$ and $\lambda \in [\lambda', \lambda'']$, let $d_{\mathrm{P}}(\lambda) = d(\lambda\mathrm{P}, \mathcal{T})$. Clearly, $d_{\mathrm{P}}(\lambda)$ is a continuous function of λ and P. Let $\lambda_1(\mathrm{P}) < \dots < \lambda_r(\mathrm{P})$, $r = r(\mathrm{P})$ be the points in (λ', λ'') of intersection with $\mathbb{R}$ of lines through P and points of $\tilde{\mathcal{T}}_{\mathrm{I}}^{\varepsilon}$. Set $\lambda' = \lambda_0(\mathrm{P})$, $\lambda'' = \lambda_{r+1}(\mathrm{P})$ and let $\mathrm{D}(\mathrm{P}) = \min_{1 \leqq i \leqq r+1} \{ \max_{\lambda_{i-1}(\mathrm{P}) \leqq \lambda \leqq \lambda_i(\mathrm{P})} d_{\mathrm{P}}(\lambda) \}$. Clearly, $\mathrm{D}(\mathrm{P}) > 0$ for $\mathrm{P} \in \tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon}$.

LEMMA 3.8. — $\operatorname{Inf} \{ \mathrm{D}(\mathrm{P}) \mid \mathrm{P} \in \tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon} \} = \delta(\varepsilon) > 0$.

Proof. — Assume that $\inf \{ \mathrm{D}(\mathrm{P}) \mid \mathrm{P} \in \tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon} \} = 0$. Then there exists a sequence $\{ \mathrm{P}_n \} \subset \mathcal{O}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon}$ such that $\mathrm{D}(\mathrm{P}_n) \underset{n\to\infty}{\longrightarrow} 0$ and either

a) $\mathrm{P}_n \to \mathrm{P}_0$

or

b) $d(\mathrm{P}_n, \lambda_0) \underset{n\to\infty}{\longrightarrow} \infty$, $\operatorname{Arg}(\lambda_0 \mathrm{P}_n) \underset{n\to\infty}{\longrightarrow} -2\psi_0$, $\psi_0 \in [\varepsilon, a - \varepsilon]$.

In case *a)* we have for $i = 1, \dots, r(\mathrm{P}_0) + 1$ for some $\varkappa > 0$

$$\max \{ d_{\mathrm{P}_0}(\lambda) \mid \lambda \in [\lambda_{i-1}(\mathrm{P}_0) + \varkappa, \lambda_i(\mathrm{P}_0) - \varkappa] \} \geqq \frac{1}{2} \mathrm{D}(\mathrm{P}_0) > 0,$$

hence there exist $\tilde{\lambda}_i \in [\lambda_{i-1}(\mathrm{P}_0) + \varkappa, \lambda_i(\mathrm{P}_0) - \varkappa]$ such that

$$d_{\mathrm{P}_0}(\tilde{\lambda}_i) \geqq \frac{1}{2} \mathrm{D}(\mathrm{P}_0).$$

By the definition of $\tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon}$, $r(\mathrm{P}_n) = r(\mathrm{P}_0)$ for n large and $\lambda_i(\mathrm{P}_n) \to \lambda_i(\mathrm{P}_0)$ for $i = 1, \dots, r(\mathrm{P}_0) + 1$. Moreover, $d_{\mathrm{P}_n}(\tilde{\lambda}_i) \underset{n\to\infty}{\longrightarrow} d_{\mathrm{P}_0}(\tilde{\lambda}_i)$, so there exists N such that for $n > \mathrm{N}$ and $i = 1, \dots, r(\mathrm{P}_0) + 1$

$$d_{\mathrm{P}_n}(\tilde{\lambda}_i) \geqq \frac{1}{4} \mathrm{D}(\mathrm{P}_0)$$

and hence $\mathrm{D}(\mathrm{P}_n) \geqq \frac{1}{4} \mathrm{D}(\mathrm{P}_0)$, a contradiction.

In case *b)*, for every $\lambda \in \mathrm{I}$, $\operatorname{Arg}(\lambda \mathrm{P}_n) \to -2\psi_0$. Let $\lambda_1 < \dots < \lambda_r$ be the points in (λ', λ'') of intersection with $\mathbb{R}$ of lines with direction $e^{-2i\psi_0}$ containing a point of $\tilde{\mathcal{T}}_{\mathrm{I}}^{\varepsilon}$. By the definition of $\tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon,\mathrm{R}_\varepsilon}$ there is only one point of $\tilde{\mathcal{T}}_{\mathrm{I}}^{\varepsilon}$ on each such line, and $r(\mathrm{P}_n) = r(\mathrm{P}_0)$ for n large. Let

$$\mathrm{D} = \min_{i=1,\dots,r+1} \{ \max_{\lambda \in [\lambda_1, \lambda_{i+1}]} d(\lambda + e^{-2i\psi_0}\mathbb{R}^+, \mathcal{T}) \}$$

and choose $\varkappa > 0$, $\tilde{\lambda}_i \in [\lambda_{i-1} + \varkappa, \lambda_i - \varkappa]$ such that

$$d(\tilde{\lambda}_i + e^{-2i\psi_0}\mathbb{R}^+, \mathcal{T}) \geqq \frac{1}{2}\mathrm{D}.$$

Choose N such that for $n > \mathrm{N}$ and $i = 1, \ldots, r+1$, $d_{\mathrm{P}_n}(\tilde{\lambda}_i) \geqq \frac{1}{4}\mathrm{D}$ and $\tilde{\lambda}_i \in [\lambda_{i-1}(\mathrm{P}_n), \lambda_i(\mathrm{P}_n)]$. Then $\mathrm{D}(\mathrm{P}_n) \geqq \frac{1}{4}\mathrm{D}$, a contradiction.

THEOREM 3.9. — For every $\varepsilon > 0$ the set $\tilde{\mathcal{R}}_{\mathrm{I}}^{\varepsilon}$ is bounded.

Proof. — Let $\mathrm{R}_{\varepsilon/2}$ and $\delta(\varepsilon/2)$ be defined as in Definition 3.7 and Lemma 3.8, and set $\delta = \min\{\delta(\varepsilon/2), \delta(\sin\varepsilon)^{-1}\}$. Let P_1 and P_2 be the points of intersection of $\lambda' + e^{-2i(a-\varepsilon)}\mathbb{R}^+$ with $\lambda' + \delta + e^{-2i(a-\varepsilon/2)}$ and $\lambda'' + e^{-2i\varepsilon}\mathbb{R}^+$ with $\lambda'' - \delta + e^{-i\varepsilon}\mathbb{R}^+$, respectively.

By Theorem 3.6 we can choose R such that $\mathrm{R} > 2\mathrm{R}_{\varepsilon/2}$, $\mathrm{R} > \max\{d(\lambda_0, \mathrm{P}_1), d(\lambda_0, \mathrm{P}_2)\}$ and such that for $\lambda \in [\lambda' + \delta, \lambda'' - \delta]$, $\varphi \in \mathrm{M}_{\varepsilon/2,\delta,\lambda}$, $t \geqq \mathrm{R} - |\lambda'' - \lambda'|$

$$\| \mathrm{I}^r(\varphi, \lambda + e^{-2i\varphi}t) \| < 1.$$

Let $\mathrm{P} \in \tilde{\mathcal{O}}_{\mathrm{I}}^{\varepsilon} \cap \{\zeta \mid |\zeta - \lambda_0| \geqq \mathrm{R}\}$.

Then $\mathrm{D}(\mathrm{P}) \geqq \delta$, hence there exists for $i = 1, \ldots, r(\mathrm{P})+1$ $\tilde{\lambda}_i \in [\lambda_{i-1}(\mathrm{P}), \lambda_i(\mathrm{P})]$ such that $d(\tilde{\lambda}_i\mathrm{P}, \mathcal{T}_{\mathrm{I}}) \geqq \delta$. Let $\tilde{t}_i = d(\mathrm{P}, \tilde{\lambda}_i)$, $\tilde{\varphi}_i = \mathrm{Arg}(\tilde{\lambda}_i\mathrm{P})$. Then $\tilde{\varphi}_i \in \mathrm{J}_{\varepsilon/2,\delta,\tilde{\lambda}_i}$ and $\tilde{t} \geqq \mathrm{R} - |\lambda'' - \lambda'|$. It follows that for $i = 1, \ldots, r(\mathrm{P}) + 1$, $\| \mathrm{I}(\tilde{\varphi}_i, \tilde{\lambda}_i + e^{-2i\tilde{\varphi}_i}\tilde{t}_i) \| < 1$ and hence $\mathrm{P} \notin \tilde{\mathcal{R}}_{\mathrm{I}}^{\varepsilon}$. We have proved that $\tilde{\mathcal{R}}_{\mathrm{I}}^{\varepsilon} \subset \{\zeta \mid |\zeta - \lambda_0| < \mathrm{R}\}$.

Theorem 3.9 verifies the induction hypothesis (B1). We conclude by verifying (B2).

THEOREM 3.10. — For every ε, $\delta > 0$ there exists $\mathrm{C} = \mathrm{C}(\varepsilon, \delta)$ such that for $\lambda \in \mathrm{I}_{\delta,\varepsilon}$, $\varphi \in \mathrm{J}^{s}_{\varepsilon,\delta,\lambda}$

$$\int_{-\infty}^{\infty} \| \mathrm{R}(\varphi, \lambda + e^{-2i\varphi}t)f \|^2 dt \leqq \mathrm{C} \| f \|^2.$$

Proof. — We have

$$\mathrm{R}(\varphi, \zeta) = (1 - \mathrm{I}^l(\varphi, \zeta))^{-1}\mathrm{D}(\varphi, \zeta). \tag{3.7}$$

By Theorem 3.6, there exists $\mathrm{R}_0 > 0$ such that for $\lambda \in \mathrm{I}_{\delta,\varepsilon}$, $\varphi \in \mathrm{J}^{\mathrm{S}}_{\varepsilon,\delta,\lambda}$, $t \geqq \mathrm{R}_0$

$$\| \mathrm{I}^r(\varphi, \lambda + e^{-2i\varphi}t) \| = \| \mathrm{I}^l(-\varphi, \lambda + e^{2i\varphi}t) \| \leqq \frac{1}{2}$$

and hence

$$\| (1 - \mathrm{I}^l(\varphi, \lambda + e^{-2i\varphi}t))^{-1} \| \leqq 2. \tag{3.8}$$

Moreover, by Lemma 3.2, for any decomposition D and φ and λ as above, $r \in \mathbb{R}$, there exists $C = C(\varepsilon, \delta)$ such that

$$\| R_D(\varphi, \lambda + e^{-2i\varphi}t)f \| = \| (1 + R_D(\varphi, \lambda + e^{-2i\varphi}t) \sum_{\alpha \subset D} v_\alpha(\varphi) R_0(\varphi, \lambda + e^{-2i\varphi}t)f \|$$

$$\leqq C \| R_0(\varphi, \lambda + e^{-2i\varphi}t)f \|$$

and hence

$$\| D(\varphi, \lambda + e^{-2i\varphi}t)f \| \leqq C_1 \| R_0(\varphi, \lambda + e^{-2i\varphi})f \| . \tag{3.9}$$

From (3.7), (3.8) and (3.9) we obtain for some $C_2 = C_2(\varepsilon, \delta)$ and $\lambda \in I_{\delta,\varepsilon}$, $\varphi \in J^S_{\varepsilon,\delta,\lambda}$, $t \in \mathbb{R}$

$$\| R(\varphi, \lambda + e^{-2i\varphi}t)f \| \leqq C_2 \| R_0(\varphi, \lambda + e^{-2i\varphi}t)f \| \tag{3.10}$$

where we have used also the continuity of $R(\varphi, \lambda + e^{-2i\varphi}t)$ in λ, φ and t. The theorem follows from (3.10) and the fact that for $|\lambda| \geqq \delta > 0$, $\varepsilon \leqq \varphi \leqq a - \varepsilon$

$$\int_{-\infty}^{\infty} \| R_0(\varphi, \lambda + e^{-2i\varphi}t)f \|^2 dt \leqq C(\varepsilon, \delta) \| f \|^2 .$$

REFERENCES

[1] D. BABBITT and E. BALSLEV, A characterization of dilation-analytic potentials and vectors, *J. Funct. Anal.*, t. **18**, 1975, p. 1-14.
[2] E. BALSLEV, Decomposition of many-body Schrödinger operators, *Comm. Math. Phys.*, t. *52*, 1977, p. 127-146; Aarhus Univ. Preprint, 1976.
[3] E. BALSLEV and J. M. COMBES, Spectral properties of many-body Schrödinger operators with dilation-analytic interactions, *Comm. Math. Phys.*, t. **22**, 1971, p. 280-294.
[4] E. BALSLEV and E. SKIBSTED, Boundedness of two- and three-body resonances, *Ann. Inst. H. Poincaré*, t. **43**, 4, 1985, p. 369-397.
[5] R. J. IORIO and M. O'CARROLL, Asymptotic completeness for multiparticle Schrödinger Hamiltonians with weak potentials, *Comm. Math. Phys.*, t. **27**, 1972, p. 137-145.
[6] C. VAN WINTER, Invariant subspaces of analytic multiparticle Hamiltonians, *J. Math. Anal. Appl.*, t. **49**, 1975, p. 88-123.

(Manuscrit reçu le 25 février 1987)