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STEPHEN J. SUMMERS
EYVIND H. WICHMANN

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Concerning the condition of additivity in quantum field theory

by

Stephen J. SUMMERS
Department of Mathematics,
University of Rochester,
Rochester, N. Y. 14627,
USA

and

Eyvind H. WICHMANN
Department of Physics,
University of California,
Berkeley, CA 94720,
USA

ABSTRACT. — The condition of additivity for local von Neumann algebras is discussed within the framework of local quantum field theory. It is shown that this condition holds for algebras of observables associated with wedge-shaped regions in Minkowski space-time if the system of local algebras is associated with a local quantum field in a weak sense. And under somewhat stronger conditions additivity is shown to hold for arbitrary regions for the algebras of a certain minimal net generated by the quantum field.

RÉSUMÉ. — La condition d'additivité pour les algèbres de von Neumann locales est discutée dans le cadre de la théorie quantique locale des champs. On montre que cette condition est satisfaite pour les algèbres d'observables associées aux régions de l'espace-temps minkowskien en forme de coin si le réseau d'algèbres locales est associé dans un sens faible à un champ quantique local. Et sous des conditions un peu plus fortes, l'additivité pour des régions arbitraires est démontrée pour les algèbres d'un certain réseau minimal qui est engendré par les champs quantiques.

I. INTRODUCTION

In algebraic quantum field theory [*1*]-[*5*], an interesting problem is whether the condition of additivity holds for the net of local (observable or field) algebras of bounded operators that is the basic object of the theory. The reader is referred to [6] for a background discussion of additivity. We consider the case when the local net is a system of von Neumann algebras on a Hilbert space $\mathscr{H}$, i. e. to every element $\mathcal{O}$ of a suitable family $\mathscr{R}$ of subsets of Minkowski space-time $\mathscr{M}$ there corresponds a von Neumann algebra $\mathscr{A}(\mathcal{O})$ on $\mathscr{H}$. The mapping $\mathcal{O} \rightarrow \mathscr{A}(\mathcal{O})$ is subject to a number of standard conditions, such as Poincaré covariance, locality and isotony [2] [4].

Somewhat loosely stated, the condition of additivity holds if for any $\mathcal{O} \in \mathscr{R}$ and any collection $\{ \mathcal{O}_i \}_{i \in I} \subset \mathscr{R}$ such that $\cup \mathcal{O}_i \supset \mathcal{O}$, then

$$\{ \mathscr{A}(\mathcal{O}_i) \mid i \in I \}'' \supset \mathscr{A}(\mathcal{O}) . \tag{1.1}$$

(A prime on a set of operators denotes its commutant.) If (1.1) holds and $\cup \mathcal{O}_i = \mathcal{O}$, then by isotony one would have equality in (1.1). Recalling the physical interpretation of the algebra $\mathscr{A}(\mathcal{O})$ as the algebra generated by all observables localized in $\mathcal{O}$, additivity may be interpreted as the condition that any physical observable in $\mathcal{O}$ can be « constructed from » the observables in the sets $\mathcal{O}_i$, for any covering of $\mathcal{O}$.

In this generality the condition (1.1) does not hold for an arbitrary local net: restrictions on both the collection of regions $\mathscr{R}$ and on the algebras in the net are necessary. We shall thus need to narrow down the problem, and we do this by discussing the issue within a framework given in [7], which paper will hereafter be referred to as DSW. We shall assume the existence of a quantum field φ satisfying the standard axioms [8] with a cyclic vacuum vector Ω in $\mathscr{H}$, and this field will be assumed to be associated to the net of local algebras in a specific manner. Two different situations will be considered here.

In the more restrictive of these cases (see Chapter 3), the field φ will have an intrinsically local operator and will satisfy a generalized H-bound (these properties were identified in DSW and will be briefly reviewed in Chapter 3). Under these conditions it follows from DSW that the field can generate a minimal net $\{ \mathscr{E}(\mathcal{O}) \}_{\mathcal{O} \in \mathscr{R}_0}$ (where $\mathscr{R}_0$ is the collection of all open sets in Minkowski space-time) that satisfies the standard conditions [2] [4] and more. We shall show that this net satisfies additivity in the form (1.1) for any open covering $\{ \mathcal{O}_i \}_{i \in I} \subset \mathscr{R}_0$ of any $\mathcal{O} \in \mathscr{R}_0$. However, this net may not satisfy duality, i. e. $\mathscr{E}(\mathcal{O}) = \mathscr{E}(\mathcal{O}^c)'$ may not even hold for all double cones in $\mathscr{R}_0$, where $\mathcal{O}^c$ is the causal complement of $\mathcal{O}$; thus it may not be maximal [9]. Under the stated conditions the maximal local net was identified in DSW—it is the unique « AB-system » $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ associated with the field φ.

An AB-system is a system of von Neumann algebras $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ satisfying the following conditions. Let $\mathscr{K}$ be the set of all closed double cones with non-empty interiors, $\mathscr{K}^c$ the set of causal complements K^c of all $K \in \mathscr{K}$, and $\mathscr{W}$ the set of all wedge-shaped regions bounded by two characteristic planes in $\mathcal{M}$ (see DSW). To every $W \in \mathcal{W}$ corresponds a von Neumann algebra $\mathcal{A}(W)$ such that $W_1, W_2 \in \mathcal{W}$ and $W_1 \subset W_2$ imply that $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$, and to every $K \in \mathcal{K}$ correspond two von Neumann algebras $\mathcal{B}(K)$ and $\mathcal{A}(K^c)$ defined by

$$\mathcal{B}(K) = \cap \{ \mathcal{A}(W) \mid W \in \mathcal{W} \text{ and } W \supset K \}, \tag{1.2}$$

$$\mathcal{A}(K^c) = \{ \mathcal{A}(W) \mid W \in \mathcal{W} \text{ and } W \subset K^c \}''. \tag{1.3}$$

For a detailed discussion of such systems of algebras, we refer to DSW and also to [*10*].

If such an AB-system is associated with a quantum field φ in a certain weak sense (see DSW and Chapter 2), then it satisfies duality

$$\mathcal{A}(W) = \mathcal{A}(\overline{W}^c)', \ W \in \mathcal{W}; \ \mathcal{B}(K) = \mathcal{A}(K^c)', \ K \in \mathcal{K}, \tag{1.4}$$

and is thus maximal. (It is possible to extend the AB-system to other (open) space-time regions $\mathcal{O} \in \mathcal{R}_0$ by defining $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated by $\{ \mathcal{B}(K) \mid K \in \mathcal{K}, K \subset \mathcal{O} \}$.) If this field satisfies a generalized H-bound, then from DSW it is known that it has many intrinsically local operators and a minimal net $\{ \mathcal{E}(\mathcal{O}) \}_{\mathcal{O} \in \mathcal{R}_0}$ can be generated. From DSW it follows that $\mathcal{E}(W) \subset \mathcal{A}(W)$, $\mathcal{E}(K^0) \subset \mathcal{B}(K)$, $\mathcal{E}(K^c) \subset \mathcal{A}(K^c)$, for all $W \in \mathcal{W}$ and $K \in \mathcal{K}$, where K^0 is the interior of K. In fact, it will be proven below that $\mathcal{E}(W) = \mathcal{A}(W)$ for all $W \in \mathcal{W}$, but that there are examples for which $\mathcal{E}(K^0) \neq \mathcal{B}(K)$ (and the point is *not* that K^0 is open and K is closed).

For AB-systems one natural form of the condition of additivity for double cones is the following. For any $K \in \mathcal{K}$ and any $\{ K_\alpha \}_{\alpha \in I} \subset \mathcal{K}$ such that K is covered by the interiors of K_α, $\alpha \in I$,

$$\mathcal{B}(K) \subset \{ \mathcal{B}(K_\alpha) \mid \alpha \in I \}''. \tag{1.5}$$

It is not yet known whether such a condition holds for an *arbitrary* AB-system satisfying the necessary condition that $\mathcal{B}(K)\Omega$ is dense in $\mathcal{H}$ for all $K \in \mathcal{K}$. But we shall show (without assuming that the field satisfies a generalized H-bound) that for any $W \in \mathcal{W}$, *any* collection $\{ K_\alpha \}_{\alpha \in I} \subset \mathcal{K}$ the interiors of whose elements cover W, and for any AB-system $\{ \mathcal{A}(W), \mathcal{B}(K), \mathcal{A}(K^c) \}$ weakly associated with a quantum field φ, one has

$$\mathcal{A}(W) \subset \{ \mathcal{B}(K_i) \mid i \in I \}''. \tag{1.6}$$

A number of further results of this type will be proven in Chapter 2. Since $\mathcal{E}(W) = \mathcal{A}(W)$ for all $W \in \mathcal{W}$, (1.6) is a satisfying result for any wedge algebra and is stronger than the result shown for wedge algebras in Chapter 3. The proof relies on the knowledge of the geometrical significance of the modular automorphism group of any wedge algebra, knowledge that is completely lacking for algebras associated to other space-time regions.

We emphasize that the assumptions made in this paper are not, after all, very restrictive and refer the reader to DSW for a discussion of this point. In particular, the assumptions made in Chapter 2 are absolutely minimal, while those required in Chapter 3 obtain in all known quantum field models with a positive mass gap.

II. THE ADDITIVITY PROPERTY FOR WEDGE ALGEBRAS

Throughout this paper the notation will be as in DSW, and although we shall repeat here some definitions, the reader is referred to DSW for any further details. We consider an irreducible, local, hermitian scalar field $\varphi(x)$ satisfying the standard axioms [8]. For reasons of simplicity we restrict our discussion to the case of a single hermitian scalar field; the generalization to the case of an arbitrary number of finite-component quantum fields is straightforward and presents no new complications. Let $U(\mathscr{P}_+^\uparrow)$ be the representation of the Poincaré group associated with $\varphi(x)$.

The field is regarded as defined on the domain customarily denoted by D_0, which is the smallest linear manifold which contains the vacuum vector Ω and which is mapped into itself by any averaged field operator $\varphi[f]$, $f \in \mathscr{S}(\mathscr{M})$. $\mathscr{P}_0(\mathscr{M})$ (resp. $\mathscr{P}_0(\mathscr{O})$, $\mathscr{O} \subset \mathscr{M}$) signifies the smallest unital $*$-algebra containing all field operators $\varphi[f]$, $f \in \mathscr{S}(\mathscr{M})$ (resp. $f \in \mathscr{S}(\mathscr{M})$ with support contained in $\mathscr{O}$). Moreover, if $X \in \mathscr{P}_0(\mathscr{M})$, then $X^\dagger \equiv X^* \upharpoonright D_0$.

In this chapter we shall discuss additivity properties of wedge algebras, for which strong results can be proven, primarily because for wedge algebras the geometrical significance of the corresponding modular automorphism groups is known [7] [11] [12]. Throughout this paper we shall make the following assumptions. We assume that there exists an AB-system $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ such that the set $\cup \{ \mathscr{B}(K) \mid K \in \mathscr{K} \}$ is irreducible and such that for each $K \in \mathscr{K}$ every $A \in \mathscr{A}(K^c)$ *commutes weakly* on D_0 with every averaged field operator $\varphi[f]$ for which $\operatorname{supp}(f) \subset K$, i. e.

$$\langle \varphi[f]^\dagger \Phi, A\Psi \rangle = \langle A^*\Phi, \varphi[f]\Psi \rangle \tag{2.1}$$

for all $\Phi, \Psi \in D_0$. Under these circumstances, the AB-system was said in DSW to satisfy Scenario G. By Theorem 2.7 in DSW it follows that: *a*) the vacuum vector Ω is cyclic and separating for $\mathscr{B}(K)$ for all $K \in \mathscr{K}$; *b*) the AB-system is local and TCP- and Poincaré-covariant; *c*) the AB-system is locally generated in the sense that

$$\mathscr{A}(W) = \{ \mathscr{B}(K) \mid K \in \mathscr{K}, K \subset W \}'', \tag{2.2}$$

for all $W \in \mathscr{W}$, which condition, in view of (1.3), implies that

$$\mathscr{A}(K^c) = \{ \mathscr{B}(K_0) \mid K_0 \in \mathscr{K}, K_0 \subset K^c \}'', \text{ all } K \in \mathscr{K};$$

d) the AB-system satisfies the conditions of duality (1.4); in fact, it satisfies the *special condition of duality:* if $W_R \in \mathscr{W}$ is defined by

$$W_R \equiv \{ (x_1, x_2, x_3, x_4) \in \mathscr{M} \mid x_3 > | x_4 | \}, \tag{2.3}$$

where x_4 is the time coordinate, then the linear manifold $\mathscr{A}(W_R)\Omega$ is a core for the complex extension $V(i\pi)$ of the velocity transformation subgroup (in the 3-direction) of $U(\mathscr{P}_+^\uparrow)$, and for any $A \in \mathscr{A}(W_R)$ one has

$$JV(i\pi)A\Omega = A^*\Omega, \tag{2.4}$$

where J is the product of the TCP-operator θ associated with $\varphi(x)$ and the rotation by angle π about the 3-direction in $U(\mathscr{P}_+^\uparrow)$. The one-parameter group $\{ V(t) \}_{t \in \mathbb{R}} \subset U(\mathscr{P}_+^\uparrow)$ of the velocity transformations in the 3-direction is thus the modular automorphism group of the pair $(\mathscr{A}(W_R), \Omega)$ and J is the corresponding modular conjugation. This property is very important in all of the following.

Actually, much more can be said about an AB-system satisfying (2.1), but we refer to DSW for further information. The AB-system is uniquely determined by the premises stated above.

Let the (open) wedge $W_L \in \mathscr{W}$ be defined as the causal complement $W_L = \overline{W_R}^c$ of the closure of the wedge defined in (2.3). From the above it readily follows that the weak commutation relation (2.1) holds for any $\Phi, \Psi \in D_0$, any $A \in \mathscr{A}(W_L)$ and any test function f with support in W_R. An equivalent statement of this is that $\varphi[f]$ has a closed extension X_e affiliated with $\mathscr{A}(W_R)$ such that $X_e^* \supset \varphi[f]^\dagger$ (DSW).

We can now state and prove a general additivity property for the wedge algebras.

THEOREM 2.1. — *Let the premises be as described above in this chapter, i. e. let* $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ *be an irreducible AB-system satisfying Scenario G in DSW. Let* $W \in \mathscr{W}$ *and let* $\{ K_i \mid i \in I \}$ *be a set of double cones in* $\mathscr{K}$ *whose interiors cover* W. *Then*

$$\mathscr{A}(W) \subset \{ \mathscr{B}(K_i) \mid i \in I \}'' \equiv \mathscr{A}. \tag{2.5}$$

Proof. 1. — It suffices to consider the case $W = W_R$, in view of the Poincaré covariance of the AB-system. Let $x \in W_R$, in which case x is in the interior K_i^0 of K_i for some $i \in I$. Since W_R is open, there exists a double cone $K(x) \in \mathscr{K}$ with x as its center and such that $K(x) \subset W_R \cap K_i$. One then has $\mathscr{B}(K(x)) \subset \mathscr{B}(K_i)$. If such a $K(x)$ is selected for each $x \in W_R$, then the set $\{ K(x) \mid x \in W_R \}$ also has the property that the interiors of its elements cover W_R, and it has the further property that $K(x) \subset W_R$ for all $x \in W_R$. It follows that $\mathscr{A} \supset \{ \mathscr{B}(K(x)) \mid x \in W_R \}''$. In view of this, it suffices to prove the assertion of the theorem for a covering with $K_i \subset W_R$ for all $i \in I$.

2. Let $A \in \mathscr{A}'$. Hence $A \in \mathscr{B}(K_i)' = \mathscr{A}(K_i^c)$ for all $i \in I$, and it follows

that for such an A (2.1) holds for all $\Phi, \Psi \in D_0$ and every f such that $\operatorname{supp}(f) \subset K_i$ for some $i \in I$. From this it readily follows that (2.1) also holds for every test function f with support in W_R, for any $A \in \mathscr{A}'$, since there is a partition of unity in the test function space corresponding to $\{ K_i \mid i \in I \}$ and since if $f_i \to f$ in the topology of $\mathscr{S}(\mathscr{M})$, then $\varphi[f_i] \to \varphi[f]$ strongly on D_0 [8]. Since D_0 is left invariant by the field operators, one can conclude that

$$\langle X^\dagger \Omega, A\Omega \rangle = \langle A^* \Omega, X\Omega \rangle \qquad (2.6)$$

for all $A \in \mathscr{A}'$ and all $X \in \mathscr{P}_0(W_R)$. It was shown in [11] that $\mathscr{P}_0(W_R)\Omega$ is a core for $V(i\pi)$ and that $JV(i\pi)X\Omega = X^\dagger\Omega$ for all $X \in \mathscr{P}_0(W_R)$, and that it then follows from (2.6) that $A\Omega$ is in the domain of $V(-i\pi)$ and

$$JV(-i\pi)A\Omega = A^*\Omega, \text{ all } A \in \mathscr{A}'. \qquad (2.7)$$

3. Since it may be assumed that $K_i \subset W_R$ for all $i \in I$, one has $\mathscr{A} \subset \mathscr{A}(W_R)$, and hence $\mathscr{A}' \supset \mathscr{A}(W_L) = \mathscr{A}(W_R)'$. From this and (2.5), it then follows by Lemma 2.5 in DSW that $\mathscr{A}' = \mathscr{A}(W_L)$, and hence that $\mathscr{A} = \mathscr{A}(W_R)$. ■

The reasoning in the proof depends in an essential manner on the special condition of duality, and specifically on the « geometrical » nature of the modular automorphism group for $(\mathscr{A}(W_R), \Omega)$. For the algebras $\mathscr{B}(K)$, $K \in \mathscr{K}$, it is an open question whether the modular automorphism group has any kind of geometrical interpretation at all (excepting the free, massless field [13]). From examples involving generalized free fields it is known that the operators $\varphi[f]$ with $\operatorname{supp}(f) \subset K$ need not generate the algebra $\mathscr{B}(K)$ for $K \in \mathscr{K}$ [9] [14].

The following result was established in [11], without any assumption about a possible association of a quantum field with the AB-system in question. Let $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ be a Poincaré-covariant AB-system which satisfies the special condition of duality. Suppose $\mathscr{K} \ni K \subset W_R$ with $\mathscr{B}(K)\Omega$ dense. Then

$$\mathscr{A}(W_R) = \{ V(t)\mathscr{B}(K)V(t)^{-1} \mid t \in \mathbb{R} \}''. \qquad (2.8)$$

At first sight this might seem rather miraculous since the set of double cones obtained by velocity transformations of the double cone K by no means covers W_R. This is, however, again a consequence of the fact that these velocity transformations form the modular automorphism group for $(\mathscr{A}(W_R), \Omega)$.

It might also seem as if the quoted result settles the question of additivity for the wedge algebras more generally than Theorem 2.1, i.e. without the assumption that there is a quantum field locally associated with the AB-system. But the conclusion of Theorem 2.1 does not follow from (2.8) unless the given covering $\{ K_i \}_{i \in I}$ has the special property that there exists a double cone $K \subset W_R$ such that every image of K under a velocity transformation in the 3-direction is contained in at least one of the K_i, $i \in I$.

It is clear that an arbitrary covering by double cones will not have this property. We have, however, the following result which bears some kinship to the result quoted from [*11*].

PROPOSITION 2.2. — *Let* $\{ \mathcal{A}(W), \mathcal{B}(K), \mathcal{A}(K^c) \}$ *be an irreducible AB-system satisfying Scenario G in DSW, and let* Λ *be the « hyperbolic arc »*

$$\Lambda = \{ c(\tau) \equiv (0, 0, q \cdot \cosh(\tau), q \cdot \sinh(\tau)) \mid \tau \in \mathbb{R} \},$$

for some $q > 0$. *Let* $\{ K_i \mid i \in I \}$ *be a set of double cones such that* $K_i \subset W_R$ *and such that the interiors of the* K_i *cover* Λ. *Then*

$$\mathcal{A}(W_R) = \{ \mathcal{B}(K_i) \mid i \in I \}'' \equiv \mathcal{A}.$$

Proof. — 1. With $x_0 \in W_R$ it is easily seen that there exists $\tau_0 > 0$ such that $c(\tau_0) - x_0$ is forward timelike and $x_0 - c(-\tau_0)$ is backward timelike, i. e. x_0 is contained in the (open) double cone with $c(\tau_0)$ and $c(-\tau_0)$ as apices. Since the arc $\Lambda(\tau_0) = \{ c(\tau) \mid |\tau| \leq \tau_0 \}$ is compact, there exists a finite subset $I(\tau_0)$ of I such that the interiors of the double cones in the set $\{ K_i \mid i \in I(\tau_0) \}$ cover $\Lambda(\tau_0)$. Let $\rho > 0$ and denote by $K_\rho(y)$ the double cone

$$K_\rho(y) = \{ x \in \mathcal{M} \mid |x_4 - y_4| + |\vec{x} - \vec{y}| \leq \rho \}$$

obtained by a translation by y of the double cone $K_\rho(0)$ centered at the origin. It is possible to choose $\rho > 0$ sufficiently small so that for any τ with $|\tau| \leq \tau_0$ and for any $y \in \mathcal{M}$ with $|y_4| + |\vec{y}| < \rho$ the double cone $K_\rho(c(\tau) + y)$ is contained in the interior of at least one of the K_i, $i \in I$. Hence,

$$\mathcal{A}(\tau_0) \equiv \{ \mathcal{B}(K_\rho(c(\tau) + y)) \mid |\tau| \leq \tau_0, |y_4| + |\vec{y}| < \rho \}'' \subset \mathcal{A}.$$

2. Let $A \in \mathcal{A}'$, so that $A \in \mathcal{A}(\tau_0)'$, and let $B \in \mathcal{B}(K_\rho(0))$. Then $[A, T(c(\tau) + y)BT(c(\tau) + y)^{-1}] = 0$ for all τ such that $|\tau| \leq \tau_0$ and all y such that $|y_4| + |\vec{y}| < \rho$. Here $T(x)$ denotes the unitary operator representing a translation by x in $U(\mathcal{P}_+^\uparrow)$. Since the arc $\Lambda(\tau_0)$ has an everywhere timelike tangent vector, it readily follows from the work of Araki [*15*] that $[A, T(x)BT(x)^{-1}] = 0$ for all x in the double cone with $c(\tau_0)$ and $c(-\tau_0)$ as apices, and in particular, $[A, T(x_0)BT(x_0)^{-1}] = 0$. This implies that $\mathcal{B}(K_\rho(x_0)) \subset \mathcal{A}(\tau_0) \subset \mathcal{A}$. It has thus been shown that for each $x_0 \in W_R$ there exists a double cone $K(x_0)$ centered at x_0 such that $\mathcal{B}(K(x_0)) \subset \mathcal{A}$. The set $\{ K(x_0) \mid x_0 \in W_R \}$ satisfies the premises of the covering set in Theorem 2.1, so it follows that $\mathcal{A} \supset \mathcal{A}(W_R)$. Since obviously $\mathcal{A} \subset \mathcal{A}(W_R)$, one has $\mathcal{A} = \mathcal{A}(W_R)$. ■

This proposition might be regarded *prima facie* as somewhat of a curiosity in view of the special nature of the hyperbolic arc. However, we note that the twice iterated causal complement Λ^{cc} of Λ is equal to W_R. By similar reasoning one can also prove the proposition with Λ replaced by any continuous curve Γ which has a timelike tangent vector at each of its points, and such that $\Gamma^{cc} = W_R$. The algebra $\mathcal{A}(W_R)$ can thus be generated by sets of double cone algebras such that the corresponding double cones do not cover W_R. Results of this nature depend strongly on the spectrum condition for the translation subgroup $\{ T(x) \mid x \in \mathscr{M} \}$ and on the special condition of duality. We note here that for our proof to go through, it is essential that the algebra generated by the « covering algebras » contains a subalgebra which is invariant under conjugations by all velocity transformations, and which has the vacuum vector as a cyclic vector. We depended on the relationship between the field $\varphi(x)$ and the algebras of the AB-system to arrive at this conclusion.

For an AB-system the condition of additivity for the algebras $\mathscr{A}(W)$, $W \in \mathscr{W}$, trivially implies a condition of additivity for the algebras $\mathscr{A}(K^c)$, $K \in \mathscr{K}$. This follows from (1.3) and the fact that any open covering of K^c, $K \in \mathscr{K}$, is also an open covering of every $W \subset K^c$. In particular, we have the following.

PROPOSITION 2.3. — *Let $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ be an irreducible AB-system satisfying Scenario G in DSW. Let $K \in \mathscr{K}$ and let $\{ K_i \mid i \in I \} \subset \mathscr{K}$ be such that the interiors of the K_i cover K^c. Then $\mathscr{A}(K^c) \subset \{ \mathscr{B}(K_i) \mid i \in I \}''$.*

III. THE ADDITIVITY PROPERTY FOR THE MINIMAL NET

In this chapter we must strengthen our assumptions to obtain results for local algebras associated with regions other than wedges. We do not know if these assumptions are necessary; our intention is to prove that they are sufficient. It is widely felt (see, e. g. [*6*] [*16*]) that a net of local algebras « generated » by quantum fields should inherit the additivity property from the existence of a partition of unity in the test function space of the quantum field. But this has been verified only for the free field.

We therefore feel it is of interest to show that the following assumptions, known to obtain in all quantum field models with a positive mass gap that have been constructed, are sufficient to verify this intuition, with certain limits. The remarks made previously about extending results to the case of an arbitrary number of finite-component fields are valid here, as well.

We recall two definitions from DSW.

DEFINITION 3.1. — Let $K_s \in \mathscr{K}$ and let $X_s = X_s^\dagger \in \mathscr{P}_0(K_s)$. For any $R \subset \mathscr{M}$ denote by $\mathscr{P}_{0s}(R)$ the smallest unital $*$-algebra which contains all Poincaré transforms $U(\lambda)X_sU(\lambda)^{-1}$ of X_s with λ any element of the Poincaré group such that $K_{s,\lambda} \subset R$, where $K_{s,\lambda}$ is the image of K_s under the natural action of λ on $\mathscr{M}$. The operator X_s is said to be *intrinsically local* (and locally associated with K_s) if and only if the following two conditions hold:

a) The linear manifold $D_{0s} \equiv \mathscr{P}_{0s}(\mathscr{M})\Omega$ is dense in $\mathscr{H}$.

b) The von Neumann algebra $a_s \equiv a((X_s \upharpoonright D_{0s})^{**})$ generated by the closure of the restriction of X_s to D_{0s} is locally associated with K_s in the sense that $U(\lambda)a_sU(\lambda)^{-1}$ commutes with a_s whenever $K_{s,\lambda}$ is spacelike relative to K_s.

DEFINITION 3.2. — For any $\alpha > 0$, let $\omega_\alpha(s) \equiv (1+s^2)^{\alpha/2}$, and let $\omega_\alpha \equiv \omega_\alpha(H)$, where H is the generator of the time-translation subgroup of $U(\mathscr{P}_+^\uparrow)$. The field $\varphi(x)$ is said to satisfy a generalized H-bound if and only if there exists a constant α, with $1 > \alpha > 0$, such that the following conditions hold:

a) For any test function f the domain $D(\overline{\varphi[f]})$ of the closure of $\varphi[f]$ on D_0 contains $\exp(-\omega_\alpha)\mathscr{H}$.

b) For any test function f the operator $\overline{\varphi[f]}\exp(-\omega_\alpha)$ is bounded.

There are many known examples of quantum fields satisfying a generalized H-bound and having an intrinsically local operator. For information on when these two conditions are known to hold and when they are expected to hold, see DSW. There it is also shown that these two properties imply that condition *b*) in Definition 3.1, where K_s is replaced by any $K \in \mathscr{K}$ containing the support of f, is satisfied by every field operator $\varphi[f]$. One can therefore generate a minimal local net $\{\mathscr{E}(\mathscr{O})\}_{\mathscr{O}\in\mathscr{R}_0}$ associated with the field $\varphi(x)$ as follows:

$$\mathscr{E}(K^0) = \{\, a(\varphi[f]^{**}) \mid \mathrm{supp}(f) \subset K^0 \,\}'' . \tag{3.1}$$

for any open double cone K^0, and

$$\mathscr{E}(\mathscr{O}) = \{\, \mathscr{E}(K^0) \mid K^0 \subset \mathscr{O}, K \in \mathscr{K} \,\}'' \tag{3.2}$$

for any open subset $\mathscr{O}$ of $\mathscr{M}$. It follows from the results in DSW that this net satisfies the standard conditions of isotony, locality and Poincaré-covariance, and more.

One can also generate a maximal net $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ associated with the field as follows:

$$\mathscr{A}(W) = \{\, U(\lambda)a_sU(\lambda)^{-1} \mid K_{s,\lambda} \subset W \,\}'', \; W \in \mathscr{W} , \tag{3.3}$$

and the algebras $\mathscr{B}(K)$, $\mathscr{A}(K^c)$ are then given by (1.2), (1.3). This net also satisfies the standard axioms, duality *and* the special condition of duality. Moreover, it follows from DSW that

$$\mathscr{E}(W) \subset \mathscr{A}(W), \mathscr{E}(K^0) \subset \mathscr{B}(K), \mathscr{E}(K^c) \subset \mathscr{A}(K^c) , \tag{3.4}$$

for all $W \in \mathscr{W}$, $K \in \mathscr{K}$.

THEOREM 3.1. — *Let $\varphi(x)$ be as described, satisfy a generalized H-bound, and have an intrinsically local field operator $\varphi[f_s]$. Then with the nets $\{\mathscr{E}(\mathscr{O})\}_{\mathscr{O}\in\mathscr{R}_0}$*

and $\{\mathcal{A}(W), \mathcal{B}(K), \mathcal{A}(K^c)\}$ *constructed as above,* $\mathcal{E}(\mathcal{O}) \subset \{\mathcal{E}(\mathcal{O}_i) \mid i \in I\}''$ *for any open set* $\mathcal{O} \subset \mathcal{M}$ *and any collection* $\{\mathcal{O}_i\}_{i \in I}$ *of open sets satisfying* $\cup_{i \in I} \mathcal{O}_i \supset \mathcal{O}$. *Moreover, for any* $W \in \mathcal{W}$, $\mathcal{E}(W) = \mathcal{A}(W)$, *and for any* $K \in \mathcal{K}$, $\mathcal{E}(K^c) = \mathcal{A}(K^c)$.

Proof. — 1. By Theorem 5.6 in DSW it follows, as already noted above, that the AB-system defined by (3.3) satisfies the special condition of duality, and it furthermore follows that $\mathcal{E}(W) = \mathcal{A}(W)$, in view of the definition (3.2). From this definition it also follows that $\mathcal{E}(K^c) = \mathcal{A}(K^c)$ for all $K \in \mathcal{K}$.

2. Let $\mathcal{O}$ be an open subset of $\mathcal{M}$, and let $\{\mathcal{O}_i\}_{i \in I}$ be an open covering of $\mathcal{O}$. Since every $\mathcal{O}_i$ is a union of open double cones, and since the algebras $\mathcal{E}(\mathcal{O}_i)$ satisfy the condition of isotony, one may readily deduce that the conclusion in the theorem for an arbitrary open covering follows if the conclusion holds in the special case of any covering by open double cones. Without loss of generality it is thus assumed that the sets $\mathcal{O}_i$, $i \in I$, are all open double cones. For such a covering, define the algebra $\mathcal{A}$ by

$$\mathcal{A} = \{\mathcal{E}(\mathcal{O}_i) \mid i \in I\}''. \tag{3.5}$$

3. It now must be shown that $\mathcal{E}(\mathcal{O}) \subset \mathcal{A}$. In view of the definition of $\mathcal{E}(\mathcal{O})$ by (3.1) and (3.2) it suffices to show that if K is any *closed* double cone contained in $\mathcal{O}$, and if $f \in \mathcal{S}(\mathcal{M})$ with $\operatorname{supp}(f) \subset K$, then $a(\varphi[f]^{**}) \subset \mathcal{A}$. Or, equivalently stated, the closed operator $\overline{\varphi[f]}$ is affiliated with $\mathcal{A}$, which relationship is signified by $\overline{\varphi[f]}\, \hat{\varepsilon} \mathcal{A}$. Consider now a particular such K. The set $\{\mathcal{O}_i \mid i \in I\}$ is an open covering of K, and since K is compact there exists a finite sub-covering $\{\mathcal{O}_i \mid i \in I_f\}$ of K. For this sub-covering define, in analogy with (3.5),

$$\mathcal{A}_f = \{\mathcal{E}(\mathcal{O}_i) \mid i \in I_f\}''. \tag{3.6}$$

One has $\mathcal{A}_f \subset \mathcal{A}$, and it will now be shown that $\overline{\varphi[f]}$ is affiliated with $\mathcal{A}_f$. Let $\{\chi_i \mid i \in I_f, \chi_i \in \mathcal{S}(\mathcal{O}_i)\}$ be a partition of unity subordinate to the covering $\{\mathcal{O}_i \mid i \in I_f\}$. Hence $\overline{\varphi[f \cdot \chi_i]}\, \hat{\varepsilon} \mathcal{E}(\mathcal{O}_i)$ for each $i \in I_f$. Let $B \in \cap \{\mathcal{E}(\mathcal{O}_i)' \mid i \in I_f\} = \mathcal{A}'_f$. Since I_f is a *finite* set, and since $\operatorname{supp}(f \cdot \chi_i)$ is closed, it follows that there exists a $\delta > 0$ such that

$$U(t)BU(t)^{-1}\overline{\varphi[f \cdot \chi_i]} \subset \overline{\varphi[f \cdot \chi_i]}U(t)BU(t)^{-1}, \text{ all } -\delta < t < \delta, i \in I_f.$$

Thus, for all $\Phi, \Psi \in D_0$, one has

$$\left\langle U(t)BU(t)^{-1}\Phi, \varphi\left[\sum_{i \in I_f} f \cdot \chi_i\right]\Psi \right\rangle = \left\langle \varphi\left[\sum_{i \in I_f} f \cdot \chi_i\right]^{\dagger}\Phi, U(t)B^*U(t)^{-1}\Psi \right\rangle,$$

for all $-\delta < t < \delta$. Hence by Lemma 5.4 of DSW,

$$U(t)BU(t)\overline{\varphi\left[\sum_{i \in I_f} f \cdot \chi_i\right]} \subset \overline{\varphi\left[\sum_{i \in I_f} f \cdot \chi_i\right]}U(t)BU(t)^{-1}, \text{ all } -\delta < t < \delta. \tag{3.7}$$

The relation (3.7) also holds with B replaced by B*. Since

$$\sum_{i \in I_f} f \cdot \chi_i = f$$

we conclude from the above that $\overline{\varphi[f]}$ commutes in the strong sense of von Neumann with every $B \in \mathscr{A}'_f$, and hence $\overline{\varphi[f]} \,\hat{\varepsilon}\, \mathscr{A}_f \subset \mathscr{A}$. Since f is an arbitrary element of $\mathscr{S}(K)$, and K is an arbitrary closed double cone contained in $\mathcal{O}$, it follows that $\mathscr{E}(\mathcal{O}) \subset \mathscr{A}$. ■

We have thus demonstrated the feature of additivity for the particular « minimal net » $\mathcal{O} \rightarrow \mathscr{E}(\mathcal{O})$, $\mathcal{O} \in \mathscr{R}_0$, defined by (3.1) and (3.2). From the discussion in DSW it follows that the maximal net $\{ \mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c) \}$ defined through (3.3) is *unique* in the sense that it is the unique AB-system which satisfies the condition $\mathscr{E}(K^0) \subset \mathscr{B}(K)$ for all $K \in \mathscr{K}$. Since we have $\mathscr{E}(K^c) = \mathscr{A}(K^c)$ by Theorem 3.1, we see that if the minimal net satisfies the duality condition $\mathscr{E}(K^0)' = \mathscr{E}(K^c)$, then $\mathscr{E}(K^0) = \mathscr{B}(K)$, and additivity holds for the maximal net. This is not, however, a *necessary* condition for additivity to hold for the maximal net. An interesting question is, of course, whether additivity holds for the maximal net, and this question remains open.

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