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The quantum stability problem for time-periodic perturbations of the harmonic oscillator

by

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ABSTRACT. — In this paper we study the quantum stability problem for a one-dimensional harmonic oscillator perturbed by time-periodic operators. The perturbations we consider need not have exponentially decaying matrix elements in the basis of the unperturbed quasi-energy states. We can prove a perturbative stability result of the following form: given any $\varepsilon > 0$ and any perturbation P with $\| P \| \leqslant C\varepsilon^2$ for suitable norm $\| \, . \, \|$ and constant C, there exists a Borel set I of « resonant values » of the frequency Ω, whose Lebesgue measure $| I |$ is smaller than ε, such that for any $\Omega \notin I$, the pure-point nature of the spectrum of the quasi-energy operator is preserved under the perturbation P.

RÉSUMÉ. — Dans cet article on étudie le problème de la stabilité quantique pour l'oscillateur harmonique unidimensionnel perturbé par des opérateurs périodiques en temps. Dans la base des états propres de l'opérateur de quasi-énergie non perturbé, les éléments de matrice des perturbations que l'on considère ne sont pas nécessairement à décroissance exponentielle. On montre un résultat perturbatif de stabilité de la forme suivante : pour tout $\varepsilon > 0$ et toute perturbation P satisfaisant $\| P \| \leqslant C\varepsilon^2$ pour une norme $\| \, . \, \|$ et une constante C convenables, il existe un borélien I de « valeurs résonantes » de la fréquence Ω dont la mesure de Lebesgue $| I |$ est inférieure à ε, tel que pour tout $\Omega \notin I$, la nature purement ponctuelle du spectre de l'opérateur de quasi-énergie est préservée par la perturbation P.

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1. INTRODUCTION

A large variety of situations can occur when self-adjoint operators with dense point spectra are perturbed, even by bounded self-adjoint operators with small norms. The perturbed operator can, for instance, develop a continuous component in its spectrum [*4*] [*8*] [*13*] [*25*]. Examples of that type may be constructed with a perturbation of the form μB where $\mu \neq 0$ and B is a rank one perturbation [*25*]. This situation is connected to some « resonance » or « close to resonance » condition on the unperturbed operator. On the other hand, if the unperturbed operator is suitably « far from resonances », stability results for the pure-point nature of the spectrum can be established [*5*] [*6*] [*7*] [*12*] [*22*] [*24*], at least for sufficiently small perturbations. The stability result in this case is perturbative, and one may ask the question of a possible transition from point spectrum to continuous spectrum as the coupling increases. Although a transition of this type is known to occur in the solid state physics of incommensurate crystals [*7*] and presumably in disordered systems [*1*], it is an open question for the quasi-energy spectrum of general quantum systems with time-periodic forces: the exactly solvable case of time-periodic perturbations that are quadratic in the space coordinates actually exhibits a transition between pure-point and continuous spectrum of the quasi-energy operator as the coupling increases [*9*] [*11*]. But on the other hand there is a variety of cases with time-periodic *bounded* perturbations where this transition does not occur [*15*].

The usual perturbation theory of self-adjoint operators with dense point spectra exhibits « small divisors problems », and a natural way out is to use an « accelerated convergence method », as in the celebrated Kolmogorov-Arnol'd-Moser (KAM) result on perturbations of classical integrable systems [*16*] [*3*] [*17*]. The similarity between both approaches is not surprising, due to the following remark [*25*]: KAM's result on the preservation (under small perturbations of a smooth integrable non-resonant system) of most invariant tori can be expressed as the preservation of the pure-point spectrum of some unitary operator (the unitary implementation of the classical flow).

Quantum dynamics is also commonly described by the spectral property of its unitary evolution operator $U(t, t')$: then the Ruelle-Amrein-Georgescu theorem, if it holds, claims that $U(t, t')$ has pure point spectrum, if and only if any quantum state will stay essentially confined *uniformly in time* in the following sense [*2*] [*23*]:

$\forall \varphi \in L^2(\mathbb{R}^n)$ and $\forall \varepsilon > 0$, there exists $R > 0$ such that

$$\operatorname*{Sup}_{t} \int_{|x|>R} d^n x \, |(U(t,0)\varphi)(x)|^2 < \varepsilon . \quad (1.1)$$

On the contrary if $U(t, t')$ has some continuous part in its spectrum, there will be some quantum states that « escape » to infinity in configuration space. Thus we may ask the following question to which we will refer later as « quantum stability »:

if one perturbs slightly a quantum system whose unitary evolution operator has pure-point spectrum, will the resulting evolution operator still have pure-point spectrum ?

However, although there are plenty of classical integrable systems and plenty of perturbations for which the progressive destruction of invariant tori can be described [*16*] [*3*] [*17*] [*18*], there are relatively few quantum systems for which « quantum stability » has been studied, up to now, by these KAM methods. Among them are discrete Schrödinger hamiltonians with limit periodic potentials [*6*] [*7*] [*12*] [*22*] [*25*], and a time-periodic system called the « pulsed rotator model » by Bellissard [*5*]. A particularly important example of a quantum mechanical system with time-periodic interaction whose unitary evolution operator has a dense pure-point spectrum is that of a charged particle inside a quadrupole radio-frequency trap [*9*] [*11*]. It is intensively used, in the physics of ions, as a starting point for the building of « atomic clocks », and appears to exhibit remarkable stability properties [*20*]. Furthermore the stability of its classical dynamics is well understood, together with a semi-classical approach of the possible « destabilisation effect » of the quantum fluctuations [*10*]. However such a semi-classical approach cannot predict quantum stability for the very long term, and a much finer approach is required to predict the exact persistence of « trapping », at the quantum level, as time goes to infinity.

It is the purpose of this work to start such an approach: we show that the quasi-energy operator for a charged particle inside the radio-frequency trap is unitarily equivalent to the quasi-energy operator for time-periodic perturbations of a 3-dimensional harmonic oscillator. Under the simplifying assumption that these time-periodic perturbations decouple along the three coordinates, we are just led to study time-periodic perturbations of a one-dimensional harmonic oscillator. Then we proceed towards a perturbative stability result away from resonant frequencies, analogous to that of ref. [*5*]. However we encounter some difficulties connected with the lack of exponential decay of the perturbation *in the space of the quantum numbers of the unperturbed quasi-energy operator*. Therefore the usual KAM procedure which relies on this exponential decay property does not apply directly. A way out is inspired by the Nash-Moser ideas for going from the analytical to the differentiable case in the classical KAM theorem [*17*]. In this paper we present the approximation procedure that allows us to go from exponential decays to power law decays in the perturbative treatment of the quantum stability problem. We stress that it is much easier and simpler than the proof of the corresponding results in classical mechanics which relies on hard analysis [*14*] [*19*] [*27*] [*28*]. However this approximation procedure is shown to converge only for sufficiently high power law decays, and the allowed power laws are too strong to cover the physical case of time-periodic perturbations that are *well localized in configuration space.*

Thus we fail at the moment to provide a stability result for the true physical systems of radio-frequency ions' traps. But the present paper is a first step towards such a result, as indicated by the following remarks:

1) going from one to n space dimensions does not seem to raise major difficulties, and is presently under study;

2) in the same way as for the original Nash-Moser argument that has received several improvements towards the optimal result [*14*] [*21*] [*27*] [*28*], possible improvements in our approximation procedure can be searched to reach an optimal power law decay.

3) a perturbative treatment *in a neighborhood of a selected quasi-energy* should allow to overcome the encountered difficulty, because if the perturbation is well localized in configuration space, then any given quasi-energy level is exponentially weakly coupled to all other levels. This alternative approach, suggested by Bellissard is presently under study.

This paper is organized as follows: in section 2 we study perturbations of infinite diagonal matrices by matrices whose matrix elements are exponentially decaying. In section 3 we give a further approximation argument which extends this perturbative treatment to the case of perturbations whose matrix elements have only power law decays. In both cases, the unperturbed diagonal elements are, in a suitable sense, close to $i_1\Omega + i_2$ ($i_1, i_2 \in \mathbb{Z}$, $\Omega \in \mathbb{R}$). In section 4 we give applications of these results, in particular to time-periodic perturbations of the one-dimensional harmonic oscillator. We also establish the connection with perturbations of quadrupole radio-frequency traps.

2. PERTURBATIONS OF INFINITE DIAGONAL MATRICES BY EXPONENTIALLY DECAYING MATRIX ELEMENTS

In this section, we shall consider infinite matrices in $\mathbb{Z}^2$ that depend on some real parameter Ω. The question is: starting from a given diagonal matrix D, and perturbing it by some perturbation P, both being hermitian, is it possible to diagonalize the perturbed D + P ? For which values of Ω will it be possible?

The answer will strongly depend on the Ω dependence of the eigenvalues of D, and in particular on their degeneracy or « near degeneracy » for some values of Ω that are called « resonant ». If these eigenvalues have the simple form $i_1\Omega + i_2$, for $i = (i_1, i_2) \in \mathbb{Z}^2$, then there exists a simple number theoretic classification of real values of Ω into « resonant » or « non-resonant » ones [5]:

. the rational numbers *and* the Liouville numbers are resonant; they are everywhere dense on $\mathbb{R}$ but however very rare, because the Lebesgue measure of the union of Liouville and rational numbers is zero;

. the remaining numbers, called diophantine numbers, whose set has full Lebesgue measure, are non-resonant.

In this section and the next one we shall consider diagonal matrices whose matrix elements $d_i(\Omega)$ are close to $i_1\Omega + i_2$ in a suitable sense, so that a classification similar to the above one into « resonant » and « non-resonant » values of Ω can be performed. This classification allows one to control the « small divisors problems » that occur in the perturbation of D.

As far as the perturbation P is concerned, we want its matrix elements P_{ij} to decay suitably when $|i-j|$ becomes large, so that part of this decay can compensate for the small denominators that occur in perturbation theory. Then an iterative procedure coupled with an accelerated convergence method inspired from Kolmogorov's work [16] will enable us to perform the perturbation theory and the diagonalization for small coupling. So far, similar ideas have been developed for perturbations having exponentially decaying matrix elements $|P_{ij}| \leqslant Ce^{-r|i-j|}$ [5] [22]. In this section, we will adapt this approach for exponentially decaying matrix elements to our class of unperturbed diagonal matrices $D(\Omega)$, in such a way that an extension to power law decays of $|P_{ij}|$ will be possible in the next section.

In order to state the results of this section in a convenient way, it is suitable to introduce algebras in the space of sequences and of infinite matrices.

DEFINITION 2.1. — B being any closed Borel set in $\mathbb{R}$, let $\mathcal{M}_B$ be the algebra (for pointwise addition and multiplication of sequences) of sequences $a = (a_i)_{i\in\mathbb{Z}^2}$ of functions from B to $\mathbb{R}$ with the norm

$$\| a \|_{\mathcal{M}_B} = \operatorname*{Sup}_{\substack{i\in\mathbb{Z}^2 \\ \Omega,\Omega'\in B}} |a_i(\Omega)| + \left| \frac{a_i(\Omega) - a_i(\Omega')}{\Omega - \Omega'} \right| . \tag{2.1}$$

Given any $k \in \mathbb{Z}^2$, T_k is the operator of translations by k:

$$(T_k a)_i = a_{i+k} . \tag{2.2}$$

DEFINITION 2.2. — Given $A = (a_{ij})_{i,j\in\mathbb{Z}^2}$ an infinite matrix, and given $k \in \mathbb{Z}^2$, we denote by A_k the sequence

$$A_k = (a_{i\,i+k})_{i\in\mathbb{Z}^2}$$

and by diag A the diagonal matrix whose diagonal sequence is A_0.

DEFINITION 2.3. — Given $r > 0$ and B any closed Borel set in $\mathbb{R}$, we define $M_r(B)$ to be the set of functions A from B to the space of infinite matrices such that $A_k \in \mathcal{M}_B$ $\forall k \in \mathbb{Z}^2$, and that the norm $\| A \|_{r,B}$ is finite, where:

$$\| A \|_{r,B} = \sum_{k \in \mathbb{Z}^2} e^{|k|r} \| A_k \|_{\mathcal{M}_B}$$

$$k = (k_1, k_2) \in \mathbb{Z}^2 , \quad |k| = |k_1| + |k_2| .$$

REMARK 2.1. — It is clear that if $A \in M_\infty(B)$, $A(\Omega)$ is a bounded diagonal matrix for any $\Omega \in B$. Furthermore the sum over k has been introduced so that in the limit $r = 0$, any $X \in M_0(B)$ be a function from B to the space of bounded matrices in $l^2(\mathbb{Z}^2)$. Namely if $a \in l^2(\mathbb{Z}^2)$, $Xa = \sum_k X_k . T_k a$ where the dot denotes the pointwise multiplication of sequences. Therefore $|(Xa)_i| \leqslant \sum_k \| X_k \|_{\mathcal{M}_B} |a_{i+k}|$ and belongs to $l^2(\mathbb{Z}^2)$ because convolution by l^1 is bounded in l^2.

Our result is as follows ($|B|$ denotes the Lebesgue measure for B a Borel set).

THEOREM 2.1. — Given B a closed Borel set, and r a positive constant, let P be an infinite matrix belonging to $M_r(B)$, and D a diagonal matrix in $M_\infty(B)$ whose diagonal sequence is of the form $d_i(\Omega) = i_1\Omega + i_2 + a_i(\Omega)$ with

$$\| a \|_{\mathcal{M}_B} + 3/2 \| P \|_{r,B} \leqslant 1/4 . \tag{2.3}$$

Assume moreover that there exist $\gamma > 0$, $\rho : 0 < \rho < r$, $\sigma > 1$ and $d > 0$ independent of γ, r and ρ such that

$$\| P \|_{r,B} \leqslant d\gamma^2 \rho^{2\sigma+1} . \tag{2.4}$$

Then there exists a closed Borel set $B' \subset B$ satisfying

$$| B \backslash B' | < \gamma \tag{2.5}$$

and an invertible matrix $V \in M_{r-\rho}(B')$ with

$$\| V - 1 \|_{r-\rho,B'} \text{ and } \| V^{-1} - 1 \|_{r-\rho,B'} \leqslant \frac{\| P \|_{r,B}}{d\gamma^2\rho^{2\sigma+1}} \tag{2.6}$$

such that

$$V^{-1}(D + P)V = \Delta \tag{2.7}$$

where Δ is a diagonal matrix belonging to $M_\infty(B')$.

Furthermore its diagonal sequence $\Delta_0 = (\delta_i)_{i \in \mathbb{Z}^2}$ satisfies

$$\| \delta - d \|_{\mathcal{M}_{B'}} \leqslant 3/2 \| P \|_{r,B} . \tag{2.8}$$

COROLLARY. — If in addition D and P are hermitian, the same result holds with V unitary.

REMARK 2.2. — (2.3), (2.4) and (2.8) imply that the resulting diagonal sequence Δ_0 is also close to $i_1\Omega + i_2$ in the sense that $b_i(\Omega) = \delta_i(\Omega) - i_1\Omega - i_2$ satisfies

$$\| b \|_{\mathcal{M}_{B'}} \leqslant 1/4 .$$

This important « stability property » will enable us to take Δ again as an unperturbed diagonal matrix in the next section. In the next lemma, we show how this property leads to a useful control of the non-resonant condition.

LEMMA 2.2. — Let $a \in \mathcal{M}_B$ be such that $\| a \|_{\mathcal{M}_B} \leqslant 1/4$ and, denoting $\omega = (\Omega, 1)$, $i = (m, n)$, let $b \in \mathcal{M}_B$ be the sequence

$$i \rightarrow b_i(\Omega) = i \, . \, \omega + a_i(\Omega)$$

then

i) if $\mathrm{I} = \{ \Omega \in \mathrm{B} / \operatorname*{Sup}_i | (b - \mathrm{T}_k b)_i | < \gamma | k |^{-\sigma} \quad \forall k \in \mathbb{Z}^2 \backslash \{ 0 \} \}$

some $\gamma > 0$, $\sigma > 1$, there exists a positive constant $C(\sigma)$ independent of γ such that the Lebesgue measure of I is smaller than $C(\sigma)\gamma$.

ii) if furthermore γ is chosen to be smaller than 1/2, then $\forall \Omega \in \mathrm{B} \backslash \mathrm{I}$ we have

$$\| (b - \mathrm{T}_k b)^{-1} \|_{\mathcal{M}_{B \backslash I}} \leqslant 2\gamma^{-2} | k |^{2\sigma + 1} . \tag{2.9}$$

REMARK 2.3. — $b_i(\Omega)$ is a small perturbation of the sequence $i.\omega$ that satisfies a « non-resonance condition » similar to (2.9) provided Ω is diophantine. Here, contrarily to other works [*22*], we do not expect an exact stability result of the non-resonance condition under small perturbations. Instead of it, we have an approximate stability result by excluding a set of small Lebesgue measure of the resonant parameter Ω. The important feature is that the variation with respect to Ω of the difference $b_i(\Omega) - i.\omega$ is small.

Proof of Lemma 2.2. — Given b the above sequence, and η any positive number $< 1/2$, we define

$$\mathrm{I}_k(\eta) = \{ \Omega \in \mathrm{B} / \operatorname*{Sup}_i | (b - \mathrm{T}_k b)_i | < \eta \} .$$

In order that $\mathrm{I}_k(\eta)$ be non-empty for $k \neq 0$, it is clear that the first component k_1 of k must be non-zero because

$$(b - \mathrm{T}_k b)_i = k_1\Omega + k_2 + a_i(\Omega) - a_{i+k}(\Omega)$$

would be $> 1/2$. But if Ω_1 and Ω_2 both belong to $I_k(\eta)$, we have using (2.1) and (2.3):

$$|\Omega_1 - \Omega_2| \leqslant \frac{2\eta}{|k_1| - 2\|a_i\|_{\mathscr{M}_B}} \leqslant \frac{2\eta}{|k_1| - 1/2}$$

which implies that the Lebesgue measure of $I_k(\eta)$ is smaller than $2\eta/(|k_1| - 1/2)$ so that

$$\sum_{k:|k|=N} |I_k(\eta)| \leqslant 4\eta \operatorname{Log} 2N.$$

But $I = \bigcap_{k\in\mathbb{Z}^2\setminus\{0\}} I_k(\gamma|k|^{-\sigma})$, and therefore letting $\eta = \gamma N^{-\sigma}$ we obtain part *i)* of lemma 2.2, with

$$C(\sigma) = 4\sum_{1}^{\infty} N^{-\sigma} \operatorname{Log} 2N < \infty \qquad (\sigma > 1). \tag{2.10}$$

Now using definition 2.1 of $\|\,.\,\|_{\mathscr{M}_B}$ we easily get *ii)*.

For the proof of theorem 2.1 we shall use an iterative procedure based on the accelerated convergence method of Kolmogorov, Arnol'd and Moser [*16*] [*3*] [*17*]. Our proof is inspired from similar results obtained by Bellissard on the « pulsed rotator model » [*5*], and from ideas borrowed to Pöschel [*22*]. We first need the following lemmas:

LEMMA 2.3. — $M_r(B)$ is an algebra for the multiplication of matrices:

$$\|XY\|_{r,B} \leqslant \|X\|_{r,B}\|Y\|_{r,B}.$$

LEMMA 2.4. — Let $X \in M_r(B)$ be such that $\|X - 1\|_{r,B} < 1$ (1 : identity matrix). Then X is invertible in $M_r(B)$ and we have

$$\|X^{-1} - 1\|_{r,B} \leqslant \|X - 1\|_{r,B}(1 - \|X - 1\|_{r,B})^{-1}.$$

Proofs. — Let $Z = XY$. Then with T_l defined by (2.2) we have

$$Z_k = \sum_{l\in\mathbb{Z}^2} X_l \,.\, T_l Y_{k-l}$$

where the . means the pointwise multiplication of sequences. But the norm in $\mathscr{M}_B$ is invariant under translations, so

$$\|Z_k\|_{\mathscr{M}_B} \leqslant \sum_{l\in\mathbb{Z}^2} \|X_l\|_{\mathscr{M}_B}\|Y_{k-l}\|_{\mathscr{M}_B}$$

$$e^{|k|r}\|Z_k\|_{\mathscr{M}_B} \leqslant \sum_{l\in\mathbb{Z}^2} e^{|l|r}\|X_l\|_{\mathscr{M}_B} e^{|k-l|r}\|Y_{k-l}\|_{\mathscr{M}_B}.$$

Now lemma 2.3 follows because convolution by l^1 is bounded in l^1. Then taking a Neumann series for $A^{-1} - 1$:

$$A^{-1} - 1 = \sum_{n=1}^{\infty} (1 - A)^n$$

and using lemma 2.3 yields lemma 2.4.

LEMMA 2.5. — Let D be a diagonal matrix whose diagonal sequence d satisfies:

$$\| (d - T_k d)^{-1} \|_{\mathcal{M}_B} \leqslant F(|k|) \tag{2.11}$$

for some function $F : \mathbb{R}^+ \to \mathbb{R}^+$. Then given any $P \in M_r(B)$ with diag $P=0$, and any $\rho : 0 < \rho < r$, there exists a unique $W \in M_{r-\rho}(B)$ with diag $W = 0$ solution of

$$[D, W] + P = 0 . \tag{2.12}$$

Furthermore we have

$$\| W \|_{r-\rho,B} \leqslant \varphi(\rho) \| P \|_{r,B} \tag{2.13}$$

where

$$\varphi(\rho) = \operatorname*{Sup}_{r \geqslant 0} F(r) e^{-r\rho} . \tag{2.14}$$

Proof. — From (2.12) we get immediately that

$$W_k = (d - T_k d)^{-1} . P_k$$

(recall that $P_0 = 0$). Then

$$\| W_k \|_{\mathcal{M}_B} \leqslant F(|k|) \| P_k \|_{\mathcal{M}_B}$$

whence

$$\| W \|_{r-\rho,B} \leqslant \| P \|_{r,B} \operatorname*{Sup}_{k \in \mathbb{Z}^2} F(|k|) e^{-\rho|k|}$$

which completes the proof of lemma 2.5.

Proof of Theorem 2.1. — Let $(\rho_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ be sequences of positive numbers such that

$$\sum_0^{\infty} \rho_n = \rho < r \tag{2.15}$$

$$\sum_0^{\infty} \gamma_n = \gamma < 1/2 . \tag{2.16}$$

We define a sequence $(\theta_n)_{n\in\mathbb{N}}$ as follows:

$$\theta_n = 2^{1-2^{-n}} \| \mathrm{P} \|_{r,\mathrm{B}} \sum_{\nu=0}^{n-1} (\gamma_\nu{}^{-2}\varphi(\rho_\nu))^{2^{-\nu-1}} \tag{2.17}$$

where φ is defined by (2.14).

Assume inductively, starting from $\mathrm{D}_0 = \mathrm{D}$, $\mathrm{P}_0 = \mathrm{P}$, $\mathrm{V}_0 = 1$, $\mathrm{B}_0 = \mathrm{B}$ that we can find for $n = 0, 1, \ldots, \mathrm{N}$.

i) a sequence of Borel sets

$$\mathrm{B}_{n+1} \subset \mathrm{B}_n \subset \mathrm{B}_{n-1} \subset \ldots \subset \mathrm{B}$$

ii) a sequence of diagonal matrices

$\mathrm{D}_n \in \mathrm{M}_\infty(\mathrm{B}_{n-1})$ whose diagonal sequences d_n satisfy

$$\| (d_n - \mathrm{T}_k d_n)^{-1} \|_{\mathscr{M}_{\mathrm{B}_n}} \leqslant \gamma_{n-1}{}^{-2}\mathrm{F}(|k|) \tag{2.18}$$

iii) a sequence of invertible matrices $\mathrm{V}_n \in \mathrm{M}_{r_n}(\mathrm{B}_n)$ where

$$r_n = r - \sum_0^n \rho_\nu \tag{2.19}$$

satisfying

$$\| \mathrm{V}_n - \mathrm{V}_{n-1} \|_{r_n,\mathrm{B}_n} \leqslant 2\gamma_{n-1}{}^{-2}\varphi(\rho_{n-1})\theta_{n-1}{}^{2^{n-1}} \tag{2.20}$$

iv) a sequence of matrices $\mathrm{P}_n \in \mathrm{M}_{r_n}(\mathrm{B}_n)$ s. t.

$$\| \mathrm{P}_n \|_{r_n,\mathrm{B}_n} \leqslant \theta_n{}^{2^n} \tag{2.21}$$

such that for any $\Omega \in \mathrm{B}_n$:

$$\mathrm{V}_n{}^{-1}(\mathrm{D} + \mathrm{P})\mathrm{V}_n = \mathrm{D}_n + \mathrm{P}_n . \tag{2.22}$$

Then we show that for $n = \mathrm{N} + 1$ we can construct B_{n+1}, D_n, V_n and P_n satisfying (2.19)-(2.22).

Using lemma 2.5, let $\mathrm{W}_n \in \mathrm{M}_{r_{n+1}}(\mathrm{B}_{n+1})$ be solution of

$$[\mathrm{D}_n + \operatorname{diag} \mathrm{P}_n, \mathrm{W}_n] + \mathrm{P}_n - \operatorname{diag} \mathrm{P}_n = 0 \tag{2.23}$$

with

$$\operatorname{diag} (\mathrm{W}_n - 1) = 0 \tag{2.24}$$

It satisfies

$$\| \mathrm{W}_n - 1 \|_{r_{n+1},\mathrm{B}_{n+1}} \leqslant \gamma_n{}^{-2}\varphi(\rho_n)\theta_n{}^{2^n} . \tag{2.25}$$

Thus letting

$$\begin{aligned} \mathrm{V}_{n+1} &= \mathrm{V}_n\mathrm{W}_n \\ \mathrm{D}_{n+1} &= \mathrm{D}_n + \operatorname{diag} \mathrm{P}_n \\ \mathrm{P}_{n+1} &= \mathrm{W}_n{}^{-1}(\mathrm{P}_n - \operatorname{diag} \mathrm{P}_n)(\mathrm{W}_n - 1) \end{aligned} \quad . \tag{2.26}$$

we have, using (2.23) and (2.26):

$$V_{n+1}^{-1}(D + P)V_{n+1} = W_n^{-1}(D_n + P_n)W_n$$
$$= D_{n+1} + P_{n+1}. \tag{2.27}$$

Now using lemma 2.2, we prove that by restricting B_{N+1} to B_{N+2}, we obtain the diophantine condition (2.18) for $n = N + 1$ with

$$F(r) = 2C(\sigma)^2 r^{2\sigma+1} \tag{2.28}$$
$$| B_{N+1} \backslash B_{N+2} | < \gamma_{N+1} \tag{2.29}$$

provided $d_{N+1} = (d_{N+1}(i))_{i \in \mathbb{Z}^2}$ is close enough to $i.\omega$, which is ensured by condition (2.3) because

$$\| d_{N+1} - d \|_{\mathscr{M}_{B_{N+1}}} \leqslant \sum_{0}^{N} \| P_n \|_{r_n, B_n} \leqslant \frac{3}{2} \| P \|_{r,B}$$

as we shall see below (see (2.39)).

Using (2.25) and lemma 2.4 we see that W_N is invertible and satisfies

$$\| W_N^{-1} - 1 \|_{r_{N+1}, B_{N+1}} \leqslant 2 \| W_N - 1 \|_{r_{N+1}, B_{N+1}} < 1$$

provided

$$\gamma_N^{-2} \varphi(\rho_N) \theta_N^{2^N} < 1/2 \tag{2.30}$$

that we shall verify below. Using (2.26), (2.17), (2.18) and lemma 2.3 we get

$$\| P_{N+1} \|_{r_{N+1}, B_{N+1}} \leqslant 2\gamma_N^{-2} \varphi(\rho_N) \theta_N^{2^{N+1}} = \theta_{N+1}^{2^{N+1}}$$
$$\| V_{N+1} - V_N \|_{r_{N+1}, B_{N+1}} \leqslant \| W_N - 1 \|_{r_{N+1}, B_{N+1}} \| V_N \|_{r_N, B_N}$$
$$\leqslant 2 \| W_N - 1 \|_{r_{N+1}, B_{N+1}}$$

which implies (2.20) provided

$$2 \sum_{n=0}^{\infty} \gamma_n^{-2} \varphi(\rho_n) \theta_n^{2^n} \leqslant 1 \tag{2.31}$$

that we shall verify below.

It remains to be shown that the above iterative procedure converges as $n \to \infty$ under some condition that will also imply (2.29)-(2.31).

Let $B' = \lim_{n \to \infty} B_n$. It is a closed Borel set, and the Lebesgue measure of its complement in B is the sum of the Lebesgue measure of the disjoint Borel sets $B_n \backslash B_{n-1}$. Therefore using (2.16) and (2.28) we get

$$| B \backslash B' | < \gamma .$$

Taking $\rho_n = \rho 2^{-n-1}$, $\gamma_n = \gamma 2^{-n-1}$ and F given by (2.28) we see that θ_n converges as $n \to \infty$ to

$$\theta_\infty = (3d)^{-1}\gamma^{-2}\rho^{-2\sigma-1}\,\|\, \mathrm{P} \,\|_{r,\mathrm{B}} \tag{2.32}$$

for some constant d only depending on σ. Furthermore it is easy to see, using Lagrange multipliers that (2.32) is the supremum of the limiting values of θ_n for sequences ρ_n and γ_n satisfying (2.15) (2.16).

Moreover, since $\gamma_n^{-2}\varphi(\rho_n)$ increases with n, we have

$$2\gamma_n^{-2}\varphi(\rho_n)\theta_n^{2^n} \leqslant \theta_\infty^{2^n} \tag{2.33}$$

and we easily see by induction, using (2.33), that

$$\theta_n^{2^n} \leqslant \theta_0\theta_\infty^{2^n-1} \tag{2.34}$$

Therefore due to (2.20) we have

$$\|\, \mathrm{V}_n - \mathrm{V}_{n-1} \,\|_{r_n,\mathrm{B}_n} \leqslant \theta_\infty^{2^{n-1}}$$

which implies that V_n converges to $\mathrm{V} \in \mathrm{M}_{r-\rho}(\mathrm{B}')$ as $n \to \infty$ with

$$\|\, \mathrm{V} - 1 \,\|_{r-\rho,\mathrm{B}'} \leqslant \sum_{n=0}^{\infty} \theta_\infty^{2^n} < \frac{\theta_\infty}{(1-\theta_\infty)} \tag{2.35}$$

provided $\theta_\infty < 1$. Furthermore V is invertible in $\mathrm{M}_{r-\rho}(\mathrm{B}')$ and

$$\|\, \mathrm{V}^{-1} - 1 \,\|_{r-\rho,\mathrm{B}'} \leqslant \frac{2\theta_\infty}{1-\theta_\infty}. \tag{2.36}$$

Therefore (2.30) and (2.31) also hold, together with

$$\|\, \mathrm{V} \,\|_{r-\rho,\mathrm{B}'} \text{ and } \|\, \mathrm{V}^{-1} \,\|_{r-\rho,\mathrm{B}'} \leqslant 2 \tag{2.37}$$

provided $\theta_\infty < 1/3$ which reads

$$\|\, \mathrm{P} \,\|_{r,\mathrm{B}} \leqslant d\gamma^2\rho^{2\sigma+1}. \tag{2.38}$$

Furthermore $\mathrm{D}_n - \mathrm{D} = \sum_{\nu=0}^{n} \operatorname{diag} \mathrm{P}_n$ converges to $\Delta - \mathrm{D}$ in $\mathrm{M}_\infty(\mathrm{B}')$ whose sequence $\delta - d$ of diagonal elements satisfy (2.8) (using (2.34)):

$$\|\, \delta - d \,\|_{\mathscr{M}_{\mathrm{B}'}} \leqslant \sum_{n=0}^{\infty} \theta_n^{2^n} \leqslant \theta_0\Big(\sum_{n=1}^{\infty} \theta_\infty^{2^n-1} + 1\Big) \leqslant \frac{3}{2}\|\, \mathrm{P} \,\|_{r,\mathrm{B}}. \tag{2.39}$$

This completes the proof of theorem 2.1.

Proof of Corollary. — Up to now we have « diagonalized » D + P by means of an invertible operator V which is not unitary. However if

$\| a \|_{\mathcal{M}_B} + 3/2 \| P \|_{r,B} \leqslant 1/4$, then for $\Omega \in B'$ the sequence δ itself is close to $i_1\Omega + i_2$ in the sense that

$$\| \delta_i - i_1\Omega - i_2 \|_{\mathcal{M}_{B'}} \leqslant 1/4 .$$

Thus we can again use lemma 2.3 and find $B'' \subset B'$ with $| B' \backslash B'' | < \gamma'$ such that for $\Omega \in B''$

$$\operatorname*{Sup}_i | (\delta - T_k\delta)_i^{-1} | \leqslant C(\sigma)\gamma'^{-1} | k |^{\sigma} .$$

This implies that all eigenvalues $\delta_i(\Omega)$ of $D + P$ are simple for $\Omega \in B''$. Therefore when $D + P$ is hermitian, all its eigenstates $V\Psi_i$ are orthogonal to each other, which implies that the matrix V^*V is diagonal in the basis $(\Psi_i)_{i\in\mathbb{Z}^2}$ of the eigenstates of D. Thus by replacing B' by B'', V by $V(V^*V)^{-1/2}$ which is unitary, and Δ by $(V^*V)^{1/2}\Delta(V^*V)^{-1/2}$ which is diagonal, we get (2.7) with V unitary.

3. PERTURBATION OF INFINITE DIAGONAL MATRICES BY OPERATORS WITH POWER LAW DECAYING MATRIX ELEMENTS

The class of unperturbed diagonal matrices is the same as in the previous section, but the class of perturbations P we shall consider is that of infinite matrices where matrix elements satisfy (the diagonal part of P has been incorporated in D so that diag P = 0)

$$| P_{ij} | \leqslant C | i - j |^{-r} \qquad i \neq j$$

C and r being independent of i and j, and $r > 9$. Our result is as follows:

THEOREM 3.1. — Let D be a diagonal matrix whose diagonal sequence $d_i(\Omega)$ is close to $i_1\Omega + i_2$ $(i = (i_1, i_2) \in \mathbb{Z}^2)$ in the following sense: $a_i = d_i - i_1\Omega - i_2$ satisfies for some closed Borel set B:

$$\| a \|_{\mathcal{M}_B} \leqslant \varepsilon < 1/4 . \tag{3.1}$$

Let P be an infinite matrix belonging to $M_0(B)$ with diag P = 0 and satisfying for some $\gamma : 0 < \gamma < 1/2$, $r > 9$

$$\| P_k \|_{\mathcal{M}_B} \leqslant C\gamma^2 | k |^{-r} \tag{3.2}$$

where

$$C < \left(\frac{1}{4} - \varepsilon\right)e^{-3} \operatorname{Inf}\left(\gamma^{-2}\left(\frac{3}{2}\zeta(r)\right)\right)^{-1}, d\zeta(\sigma)^{-2} \Big/ \sum_1^{\infty} \frac{(n+1)^{2\sigma+1}}{n^{r-4\sigma-1}} \tag{3.3}$$

d is as in theorem 2.1, σ is any number between 1 and $\dfrac{r-3}{6}$ and

$$\zeta(\alpha) = \sum_{1}^{\infty} n^{-\alpha} \qquad \text{for} \quad \alpha > 1 . \tag{3.4}$$

Then there exists a closed Borel set $B' \subset B$ satisfying

$$| B \backslash B' | < \gamma \tag{3.5}$$

and a unitary matrix $\Phi \in M_0(B')$ such that Δ defined by

$$\Phi^{-1}(D + P)\Phi = \Delta \tag{3.6}$$

is a diagonal matrix belonging to $M_\infty(B')$.

Proof. — Define for any $N \in \mathbb{N}$ an infinite matrix $P^{(N)}$:

$$P_{ij}^{(N)} = \begin{cases} 0 & \text{if} \quad |i-j| \neq N \\ P_{ij} & \text{if} \quad |i-j| = N . \end{cases} \tag{3.7}$$

Then it is clear that $P = \sum_{1}^{\infty} P^{(N)}$ and, using (3.2) that

$$\| P^{(N)} \|_{\frac{1}{N},B} \leqslant e \| P_N \|_{\mathcal{M}_B} \leqslant eC\gamma^2 N^{-r} \qquad N \geqslant 1 . \tag{3.8}$$

Therefore D and $P^{(1)}$ satisfy respectively assumptions (2.3) and (2.4) of theorem 2.1.

Thus there exists a Borel set I_1 of Lebesgue measure smaller than $\gamma_1 = \gamma/\zeta(\sigma)$ where

$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma} \tag{3.9}$$

such that there exist $U^{(1)} \in M_{\frac{1}{2}}(B\backslash I_1)$ and $\Delta_1 \in M_\infty(B\backslash I_1)$ satisfying for any $\Omega \in B\backslash I_1$

$$U^{(1)^{-1}}(D + P^{(1)})U^{(1)} = \Delta_1 . \tag{3.10}$$

Now the diagonal sequence $\delta^{(1)}$ of Δ_1 satisfies, by denoting $B_1 = B\backslash I_1$ and $b_i = \delta_i^{(1)} - i_1\Omega - i_2$

$$\| b \|_{\mathcal{M}_{B_1}} \leqslant \| a \|_{\mathcal{M}_B} + 3/2 \, \| P^{(1)} \|_{1,B} \leqslant 1/4 .$$

This allows one to consider Δ_1 as the unperturbed diagonal matrix, and to perturb it by $U^{(1)^{-1}}P^{(2)}U^{(1)}$ in order to get one step further. Namely

$$U^{(1)^{-1}}(D + P^{(1)} + P^{(2)})U^{(1)} = \Delta_1 + U^{(1)^{-1}}P^{(2)}U^{(1)} \tag{3.11}$$

so that one may diagonalize (3.11) by another use of theorem 2.1.

Assume that this diagonalization procedure has been performed inductively up to order N. This means that we have found a Borel set $B_N \subset B$, a unitary matrix $\phi_N \in M_{\frac{1}{N+1}}(B_N)$ and a diagonal matrix $\Delta_N \in M_\infty(B_N)$ such that the following holds

$$\Phi_N^{-1}\left(D + \sum_{n=1}^{N} P^{(n)}\right)\Phi_N = \Delta_N \qquad \forall \Omega \in B_N \tag{3.12}$$

$$|B \backslash B_N| < \gamma \sum_{1}^{N} n^{-\sigma}/\zeta(\sigma) \tag{3.13}$$

$$\operatorname{Log} \| \Phi_N^{\pm 1} \|_{\frac{1}{N+1}, B_N} \leqslant \sum_{1}^{N} \varepsilon_n \tag{3.14}$$

with

$$\varepsilon_n = \frac{Ce^3\zeta(\sigma)^2}{d} \frac{(n+1)^{2\sigma+1}}{n^{r-1-4\sigma}} \tag{3.15}$$

$$\| \delta^{(N)} - d \|_{\mathscr{M}_{B_N}} \leqslant \frac{3}{2} e^3 C\gamma^2 \sum_{1}^{N} n^{-r} \tag{3.16}$$

$\delta^{(N)}$ being the diagonal sequence of Δ_N. But due to (3.16), (3.3) and (3.1), $\delta_i^{(N)}$ is close to $i_1\Omega + i_2$ in the sense that $b_i^{(N)} = \delta_i^{(N)} - i_1\Omega - i_2$ satisfies

$$\| b_i^{(N)} \|_{\mathscr{M}_{B_N}} \leqslant 1/4 .$$

Furthermore $\Phi_N^{-1}P^{(N+1)}\Phi_N$ belongs to $M_{\frac{1}{N+1}}(B_N)$ and satisfies

$$\| \Phi_N^{-1}P^{(N+1)}\Phi_N \|_{\frac{1}{N+1}, B_N} \leqslant Ce^3\gamma^2(N+1)^{-r} \tag{3.17}$$

because, using (3.3), the series $(\varepsilon_n)_{n\in\mathbb{N}}$ converges with $\sum_{1}^{\infty} \varepsilon_n \leqslant 1$, so that $\| \Phi_N \|_{\frac{1}{N}, B_N}$ and $\| \Phi_N^{-1} \|_{\frac{1}{N+1}, B_N}$ are bounded above by e.

But according to (3.3), it is clear that (3.17) implies

$$\| \Phi_N^{-1}P^{(N+1)}\Phi_N \|_{\frac{1}{N+1}, B_N} \leqslant d\gamma_{N+1}^{\ 2}\left(\frac{1}{N+1} - \frac{1}{N+2}\right)^{2\sigma+1} \tag{3.18}$$

with

$$\gamma_n = \gamma n^{-\sigma}/\zeta(\sigma) . \tag{3.19}$$

Therefore all assumptions of theorem 2.1 are satisfied for $D = \Delta_N$ and $P = \Phi_N^{-1} P^{(N+1)} \Phi_N$ so that there exist a Borel set $B_{N+1} \subset B_N$ with

$$| B_N \backslash B_{N+1} | < \gamma_{N+1} \tag{3.20}$$

a unitary matrix $U^{(N+1)} \in M_{\frac{1}{N+2}}(B_{N+1})$ and a diagonal matrix $\Delta_{N+1} \in M_\infty(B_{N+1})$ such that

$$U^{(N)^{-1}}(\Delta_N + \Phi^{(N)^{-1}} P^{(N+1)} \Phi^{(N)}) U^{(N)} = \Delta_{N+1} \tag{3.21}$$

for any $\Omega \in B_{N+1}$. Thus taking $\Phi^{(N+1)} = \Phi^{(N)} U^{(N)}$, (3.12)-(3.16) are satisfied with $N + 1$ instead of N. We now prove the convergence of this inductive procedure. Let $B' = \lim_{N \to \infty} B_N$. By (3.9) and (3.13) we obtain immediately (3.5). $\Phi^{(N)} = U^{(1)} U^{(2)} \ldots U^{(n)}$ converges in $M_0(B')$ to some unitary matrix Φ satisfying $\| \Phi \|_{0,B'} \leqslant e$ and Δ_N converges in $M_\infty(B')$ to a diagonal matrix Δ whose diagonal sequence satisfies, using (3.16) and (3.3):

$$\| \delta - d \|_{\mathcal{M}_{B'}} \leqslant -\varepsilon + 1/4 .$$

This completes the proof of Theorem 3.1.

REMARK 3.1. — It is clear that the farther D is from the diagonal matrix $d_i = i_1 \Omega + i_2$ $(i = (i_1, i_2) \in \mathbb{Z}^2)$, the smaller the perturbation P has to be. On the other hand, as in section 2, the smaller we want the resonant set to be, the smaller the perturbation has to be.

4. APPLICATIONS. PERTURBATIONS OF THE QUANTUM MECHANICAL QUADRUPOLE RADIO-FREQUENCY TRAP

The application we have in mind is that of time-periodic perturbations of the one-dimensional quantum harmonic oscillator. In the first part of this section, we indicate which class of time-periodic perturbations is covered by the perturbative analysis of sections 2 and 3. Finally in the second part of this section we show the connection between this problem and that of perturbations of the quadrupole radio-frequency trap. We also stress the progress we have made towards a perturbative proof of the quantum stability for these physical systems, but also the limitations of the present result.

It has been known for a long time [*26*] [*15*] that a convenient way to study quantum mechanical systems subject to time-periodic forces is to analyse the spectral properties of the Floquet operator, or equivalently of the quasi-energy operator. The latter is the self-adjoint extension in $\mathcal{K} = \mathcal{H} \otimes L^2(\mathbb{T}_\Omega)$ (where $\mathbb{T}_\Omega$ is the one-dimensional torus $[0, 2\pi/\Omega]$

and $\mathscr{H}$ the Hilbert space of square integrable functions in configuration space) of

$$- i\hbar\frac{\partial}{\partial t} + \mathrm{H}(t) = \mathrm{K} \tag{4.1}$$

with periodic boundary conditions in t, $\mathrm{H}(t)$ being the time-periodic hamiltonian assumed to be self-adjoint in $\mathscr{H}$ for all t. Now assume we consider time-periodic perturbations $\mathrm{V}(x, t)$ of the one-dimensional harmonic oscillator. Then the quasi-energy operator can be written as

$$\mathrm{K} = \mathrm{D} + \mathrm{P} \tag{4.2}$$

where (setting $\hbar = 1$) P is the time-periodic perturbation, and

$$\mathrm{D} = - i\frac{\partial}{\partial t} + \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2\right) \tag{4.3}$$

has the complete set of eigenfunctions

$$\Psi_{m,n}(x, t) = e^{im\Omega t} \otimes \varphi_n(x) \qquad m \in \mathbb{Z},\ n \in \mathbb{N} \tag{4.4}$$

φ_n being the normalized Hermite function of order n. The spectrum of D is therefore pure-point, so that in the basis $(\Psi_{m,n})_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{N}}}$, D is the diagonal matrix whose diagonal sequence is $d_i = i.\omega$

$$\omega = (\Omega, 1)$$
$$i = (i_1, i_2) \in \mathbb{Z} \times \mathbb{N}\,.$$

It can be continued to a diagonal matrix in $\mathbb{Z}^2$ by taking $d_i = 0$ if $i_2 < 0$. Assume P is a bounded operator in $\mathscr{H}$ whose matrix elements P_{ij} in the basis $(\Psi_i)_{i=(m,n)\in\mathbb{Z}\times\mathbb{N}}$ are smooth functions of Ω satisfying (P_k being the sequence $\mathrm{P}_k = (\mathrm{P}_{i\,i+k})_{i\in\mathbb{Z}\times\mathbb{N}}$) one of the two following assumptions:

$$\exists\gamma < 1/2 \text{ and } r > 0: \quad \sum_{k\in\mathbb{Z}^2} \| \mathrm{P}_k \|_{\mathscr{M}_\mathrm{B}} e^{|k|r} < \mathrm{Inf}\left(\frac{1}{6}, d\gamma^2 r^{2\sigma+1}\right) \tag{$\mathscr{A}$}$$

(d being the constant found in theorem 2.1)

$$\exists\gamma < 1/2,\ r > 9, \text{ and } \sigma : 1 < \sigma < \frac{r-3}{6} \text{ such that } \quad \| \mathrm{P}_k \|_{\mathscr{M}_\mathrm{B}} \leqslant \mathrm{C}\gamma^2 |k|^{-r} \tag{$\mathscr{C}$}$$

with
$$\mathrm{C} < \frac{1}{4e^3}\,\mathrm{Inf}\left(\frac{2}{3\gamma^2\zeta(r)}, \frac{d}{\zeta(\sigma)^2 \displaystyle\sum_{1}^{\infty} \frac{(n+1)^{2\sigma+1}}{n^{r-4\sigma-1}}}\right).$$

An immediate application of theorems (2.1) and (3.1) yields:

THEOREM 4.1. — Let D be given by (4.3), and P satisfy one of the assumptions ($\mathscr{A}$) or ($\mathscr{C}$). Then D + P has pure-point spectrum for Ω away from a resonant set I of Lebesgue measure smaller than γ.

Examples. — Let P be of the form $f(t)\mathrm{V}$ where f is a $\frac{2\pi}{\Omega}$-periodic real function with exponentially decaying Fourier coefficients, and V is an operator acting on $L^2(\mathbb{R})$. Let p be the self-adjoint realization of $-i\frac{d}{dx}$ on $L^2(\mathbb{R})$, and take V to be one of the following trace class operators:

$$\mathrm{V} = \lambda e^{-x^2} e^{-p^2/2} e^{-x^2/2} \tag{4.5}$$

$$\mathrm{V} = \lambda e^{-x^2/2}(1 + p^2)^{-2r} e^{-x^2/2} \qquad r > 9\,. \tag{4.6}$$

Then for λ sufficiently small, (4.5) belongs to the class ($\mathscr{A}$) with $r = \operatorname{Log} 2/2$, and (4.6) belongs to the class ($\mathscr{C}$). This easily follows from explicit calculations on Hermite functions.

We now recall a result on the three-dimensional quadrupole radio-frequency trap that makes a connection with the one-dimensional harmonic oscillator [*9*] [*11*]. If $\mathrm{V}(t) = \mathrm{V}_1 + \mathrm{V}_2 \cos \Omega t$ is the electric potential between the electrodes of the quadrupolar trap, the time-periodic hamiltonian for a particle of mass m and of charge e inside the cavity is

$$\mathrm{H}(t) = \frac{p^2}{2m} + \frac{e\mathrm{V}(t)}{z_0{}^2}\left(z^2 - \frac{x^2 + y^2}{2}\right) \tag{4.7}$$

where $p = (p_x, p_y, p_z)$ is the three-dimensional momentum.

THEOREM 4.2. — Assume that the time-periodic hamiltonian $\mathrm{H}(t)$ given by (4.7) has stable three-dimensional classical orbits (in particular this holds if $|\mathrm{V}_1/\mathrm{V}_2| <$ some α, and Ω is sufficiently large). Then the quasi-energy operator (4.1) is unitarily equivalent in to the operator

$$-i\hbar\frac{\partial}{\partial t} + \lambda_0 \mathrm{H}_z + \lambda_1(\mathrm{H}_x + \mathrm{H}_y) \tag{4.8}$$

$\left(\text{with } \frac{2\pi}{\Omega}\text{-periodic boundary conditions in } t\right)$, where H_z is the one-dimensional hamiltonian $-\frac{1}{2}\frac{d^2}{dz^2} + \frac{z^2}{2}$ (and similarly for H_x and H_y) and where λ_0/Ω and λ_1/Ω are the Floquet exponents of the z and x classical motions respectively.

REMARK 4.1. — It is clear that the dynamics for each coordinate are decoupled, and therefore we only have to deal with one-dimensional problems.

REMARK 4.2. — If the parameters belong to the first stability area (namely $|V_1/V_2| < \alpha$ and Ω sufficiently large) and if we increase the coupling constant ($V_1 \to \lambda V_1, V_2 \to \lambda V_2$ with $\lambda > 1$) the motion becomes instable; furthermore the straight line V_1/V_2 = constant can by chance hit another stability region in the plane (V_1, V_2) and we can (theoretically) have several transitions stability $\to$ instability $\to$ stability. They have not been observed experimentally.

Actually, real radio-frequency traps used in practice never reduce exactly to the ideal quadrupole case described by the hamiltonian (4.7) but are perturbed versions of it. The realistic perturbations P of the quadrupole r. f. hamiltonian $H(t)$ are local, due to the presence of holes in the electrodes, to finite size effects, or to distorsion of the hyperboloid shape of the cavity. By the unitary equivalence stated above, there is a time-periodic operator $P(t)$ in $L^2(\mathbb{R}^3)$ such that the sum $K + P$ is unitarily equivalent to

$$- i\hbar\frac{\partial}{\partial t} + \lambda_0 H_z + \lambda_1(H_x + H_y) + P(t). \tag{4.9}$$

Therefore we can ask the question whether the pure-point character of the spectrum of (4.8) is preserved under small perturbations $P(t)$. If the answer is yes, then we will have a result of quantum stability in the form expressed by (1.1). In general $P(t)$ acts on all variables x, y and z, so the quantum system described by the quasi-energy operator (4.9) is a true three-dimensional system. However we make the severe simplification that it decouples along the three coordinates x, y and z, and that we can restrict ourselves to one-dimensional problems of the form

$$- i\frac{\partial}{\partial t} + H + f(t)V \tag{4.10}$$

where H is the one-dimensional harmonic oscillator hamiltonian, f is $\frac{2\pi}{\Omega}$-periodic, and V is an operator acting on $L^2(\mathbb{R})$. Now we ask the following question: if we start with a « realistic » local potential $V(x)$, what kind of decrease of the matrix elements $V(n, n')$ of V in the basis of the normalized Hermite functions can be expected? We take for instance $V(x) = e^{-x^2}$ as a prototype of well localized potentials; in this case explicit calculations can be performed which yield:

$$|V(n, n')| = \frac{(n + n')!}{2^{n+n'}\left(\frac{n + n'}{2}\right)!\sqrt{n!n'!}}. \tag{4.11}$$

It is convenient to rewrite (4.11) in terms of $N = \frac{n + n'}{2}$ which is the coordinate along the principal diagonal of the matrix $V(n, n')$, and of $k = \frac{n - n'}{2}$ which is the coordinate in the direction orthogonal to this principal diagonal. When N is large, we get:

$$| V(n, n') | \sim \frac{N!}{(N(N + k)!(N - k)!)^{1/2}}.$$

If $|k| > N^{\alpha}$, $|V(n, n')|$ which is for fixed N a decreasing function of k is smaller than

$$N^{-1/2} e^{-N^{2\alpha - 1}} \leqslant N^{-1/2} e^{-k^{2\alpha - 1}}$$

thus we get an exponential decay provided $\alpha > 1/2$. However in the range $|k| \leqslant N^{1/2}$ we cannot obtain a better decay than $|k|^{-1}$. But the power law decay that we have been able to deal with perturbatively in section 3 is much stronger than $|k|^{-1}$. Thus the perturbative treatment of section 3 ia a first progress towards a quantum stability result for the quadrupole r. f. trap, but it fails to reach the physical case of small power law decays. Nevertheless we hope that a refinement of the method can be performed that takes into account the support property of the part of the interaction having this slow power law decay, namely a slowly enlarging region around the principal diagonal.

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Note added in proof. — By an improved version of the KAM procedure, using analyticity arguments closer to Arnold's original one, the author is now able to largely extend the class of allowed perturbations. More precisely, condition (2.4) in theorem 2.1 has been replaced by $\| P \|_{r,B} \leqslant d\gamma\rho^{\sigma+1/2}$ for a different, but related norm $\| \circ \|_{r,B}$ of the perturbation.

This allows in theorems 3.1 and 4.1 to weaken condition (3.2) into $\| P_k \|_{\mathcal{M}_B} \leqslant C\gamma \, | k |^{-r}$ with $r > 4.5$.

This result will appear in a subsequent publication.