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Semi-classical estimates for resolvents and asymptotics for total scattering cross-sections

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ABSTRACT. — We study the semi-classical asymptotic behavior as $h \to 0$ of total scattering cross-sections for Schrödinger operators $-(1/2)h^2\Delta + V$, when energies are fixed in non-trapping energy ranges. The proof is based on the semi-classical estimates for resolvents and the argument applies to the asymptotics for forward scattering amplitudes.

RÉSUMÉ. — On étudie le comportement asymptotique semi-classique quand $h \to 0$ de la section efficace totale pour l'opérateur de Schrödinger $-(1/2)h^2\Delta + V$, pour des énergies dans des régions sans trajectoires piégées. Les démonstrations sont basées sur des estimations semi-classiques de résolvantes, et les arguments s'appliquent au comportement asymptotique de l'amplitude de diffusion vers l'avant.

§ 0. INTRODUCTION

In the present paper we study the semi-classical asymptotics for total scattering cross-sections of Schrödinger operators $H(h) = -(1/2)h^2\Delta + V$, $0 < h < 1$, in the n-dimensional space R^n_x, $n \geqq 2$. As is well known (e. g. [*10*]), if a real-valued potential $V(x)$ satisfies $|V(x)| \leqq (1+|x|)^{-\rho}$ with $\rho > (n+1)/2$, then the scattering matrix $S(\lambda, h)$ with energy $\lambda > 0$ can be defined as a unitary operator acting on $L^2(S^{n-1})$, S^{n-1} being the $(n-1)$-dimensional unit sphere and also $S(\lambda, h) - Id$ is an integral operator of Hilbert-Schmidt class. The kernel of this operator is represented in terms of the scattering amplitude $f(\omega \to \theta; \lambda, h)$, $(\omega, \theta) \in S^{n-1} \times S^{n-1}$, with the incoming direction ω and the outgoing one θ. (The precise definition of $f(\omega \to \theta; \lambda, h)$ is given in section 1.) The total scattering cross-section $\sigma(\omega; \lambda, h)$ is defined by

$$\sigma(\omega; \lambda, h) = \int_{S^{n-1}} |f(\omega \to \theta; \lambda, h)|^2 d\theta$$

and also the averaged total cross-section $\sigma_a(\lambda, h)$ is defined by averaging $\sigma(\omega; \lambda, h)$ over ω. The aim of the present paper is to study the asymptotic behavior of $\sigma(\omega; \lambda, h)$ in the semi-classical limit $h \to 0$.

We shall first formulate the main theorem precisely together with some assumptions and then make several comments on recent related results.

We make the following assumption on $V(x)$.

ASSUMPTION $(V)_\rho$. — $V(x)$ is a real C^∞-smooth function and satisfies

$$|\partial_x^\alpha V(x)| \leqq C_\alpha \langle x \rangle^{-\rho-|\alpha|}$$

for some $\rho > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

We further assume that the energy $\lambda > 0$ under consideration is non-trapping in the following sense.

Non-trapping condition. Let $\{x(t; y, \eta), \xi(t; y, \eta)\}$ be the solution to the Hamilton system $\dot{x} = \xi$, $\dot{\xi} = -\nabla_x V$ with initial state (y, η). We say that energy $\lambda > 0$ is non-trapping, if for any $R \gg 1$ large enough, there exists $T = T(R)$ such that $|x(t; y, \eta)| > R$ for $|t| > T$, when $|y| < R$ and $\lambda = |\eta|^2/2 + V(y)$.

We require some notations to formulate the main theorem. We fix the incoming direction $\omega \in S^{n-1}$ and denote by Π_ω the hyperplane orthogonal to ω. We write $x = y + \omega s$, $s \in R^1$, with $y \in \Pi_\omega$. Under the above assumptions and notations, the main theorem is stated as follows.

THEOREM 1. — *Assume* $(V)_\rho$ *with* $\rho > (n+1)/2$, $n \geqq 2$, *and that the energy* $\lambda > 0$ *is fixed in a non-trapping energy range. Then the total scattering cross-section* $\sigma(\omega; \lambda, h)$ *obeys the following asymptotic formula as* $h \to 0$:

$$\sigma(\omega; \lambda, h) = 4 \int_{\Pi_\omega} \sin^2 \left[2^{-1}(2\lambda)^{-1/2} h^{-1} \int_{-\infty}^{\infty} V(y + s\omega) ds \right] dy + o(h^{-\nu})$$

with $\nu = (n-1)/(\rho-1)$.

We should note that if $C^{-1} \langle x \rangle^{-\rho} \leqq V(x) \leqq C \langle x \rangle^{-\rho}$, $C > 1$, then the leading term is comparable to the order $0(h^{-\nu})$.

The proof of Theorem 1 is based on semi-classical estimates for resolvents. We denote by the same notation $H(h)$ the unique self-adjoint realization in $L^2(R^n)$ and by $R(z; H(h))$, $\operatorname{Im} z \neq 0$, the resolvent of $H(h)$; $R(z; H(h)) = (H(h) - z)^{-1}$. Let $L^2_\alpha(R^n_x)$ be the weighted L^2 space defined by

$$L^2_\alpha(R^n_x) = \{ f(x) : \langle x \rangle^\alpha f(x) \in L^2(R^n_x) \}.$$

Then, by the principle of limiting absorption ([1], [3], etc.), there exist bounded operators $R(\lambda \pm i0; H(h)) : L^2_\alpha \to L^2_{-\alpha}$, $\alpha > 1/2$, defined by

$$R(\lambda \pm i0; H(h)) = \underset{\kappa \downarrow 0}{s\text{-}\lim}\, R(\lambda \pm i\kappa; H(h)), \quad \lambda > 0,$$

strongly in $L^2_{-\alpha}(R^n_x)$. The next theorem plays an essential role in proving the main theorem.

THEOREM 2 *(resolvent estimate).* — *Assume* $(V)_\rho$ *with* $\rho > 0$ *and that* $\lambda > 0$ *is non-trapping. Denote by* $\| \ \|$ *the operator norm when considered as an operator from* $L^2(R^n_x)$ *into itself. Then, for any* $\alpha > 1/2$,

$$\| \langle x \rangle^{-\alpha} R(\lambda \pm i0; H(h)) \langle x \rangle^{-\alpha} \| = 0(h^{-1})$$

as $h \to 0$. *Furthermore, if* λ *ranges over a compact interval in a non-trapping energy range, then the above bound is uniform in* λ.

Now, we shall make several comments on recent results related to the main theorem.

1) The bound

$$\int_a^b \sigma(\omega; \lambda, h) = 0(h^{-\nu}), \qquad 0 < a < b < \infty,$$

is proved by Enss-Simon [2] when $|V(x)| \leqq C \langle x \rangle^{-\rho}$, $\rho > (n+1)/2$. It should be noted that the above bound is obtained without assuming the non-trapping condition for $\lambda \in [a, b]$.

2) The weak bound for the averaged cross-section $\sigma_a(\lambda, h)$ without averaging over energy λ; $\sigma_a(\lambda, h) = 0(h^{-\nu})$ is proved by Sobolev-Yafaev [13] without assuming the non-trapping condition.

3) Recently, Yafaev [16] has obtained the sharp bound $\sigma(\omega; \lambda, h) = 0(h^{-\nu})$ in the high energy case

(0.1) $$\lambda > \inf_{0 < \beta \leqq 2} \sup_x (\beta^{-1} |x| (\partial/\partial |x|) V + V).$$

The proof is based on the sharp resolvent estimate

(0.2) $$\| (|x| + r)^{-\alpha} R(\lambda \pm i0; H(h)) (|x| + r)^{-\alpha} \| \leqq C h^{-1} r^{1-2\alpha}, \qquad \alpha > 1/2,$$

with C independent of h, $0 < h < 1$, and r, $r \geqq 1$. Under assumption $(V)_\rho$, the restriction (0.1) implies that λ is in a non-trapping energy range. Our resolvent estimate (Theorem 2) is a special case of (0.2) with $r = 1$.

4) Furthermore, under the assumption (0.1), Yafaev [*16*] has obtained the asymptotic formula as $h \to 0$ of $\sigma(\omega; \lambda, h)$ for a certain class of potentials such as $V(x)$ behaving like

$$V(x) = \Phi(x/|x|)|x|^{-\rho} + 0(|x|^{-\rho}), \qquad |x| \to \infty,$$

with $\Phi \in C(S^{n-1})$;

$$\sigma(\omega; \lambda, h) = 4\sigma_0(\omega)(2\lambda)^{-\nu/2}h^{-\nu} + o(h^{-\nu}), \tag{0.3}$$

where

$$\sigma_0(\omega) = \int_0^\infty r^{n-2}\sin^2(r^{1-\rho}/2)dr \times \int_{S^{n-2}} \left| \int_0^\pi \Phi(\psi \sin\theta + \omega\cos\theta)\sin^{\rho-2}\theta d\theta \right|^\nu d\psi.$$

Formula (0.3) follows from Theorem 1 by taking spherical coordinates in Π_ω. The proof for (0.3) in [*16*] seems to rely on the homogeneous property of $V(x)$ at infinity. Theorem 1 will be considered as an extension to the non-homogeneous case as well as to the case of non-trapping energies, although the restrictive regularity condition $(V)_\rho$ is assumed.

5) Now assume that $V(x) \in C_0^\infty(R_x^n)$. Then we can easily guess from Theorem 1 that $\sigma(\omega; \lambda, h) = 2\sigma_{cl}(\omega) + o(1)$ as $h \to 0$, where $\sigma_{cl}(\omega)$ denotes a classical scattering cross-section for the incoming direction ω (i. e. cross-section of the support of $V(x)$ along the direction ω). We will prove this result in section 4, assuming a certain condition on the corresponding classical systems in addition to the non-trapping condition. This problem has been conjectured in [*2*] and Yajima [*17*] has proved this convergence when averaged over λ without assuming the non-trapping condition.

§ 1. TOTAL SCATTERING CROSS-SECTIONS

Throughout this section we assume $V(x)$ to satisfy $(V)_\rho$ with $\rho > (n+1)/2$. We collect some basic facts about the stationary scattering theory for Schrodinger operators $H(h) = -(1/2)h^2\Delta + V$. For details, see, for example, the book [*10*].

Let $H_0(h) = -(1/2)h^2\Delta$ and denote by $\phi_0(x; \lambda, \omega, h)$, $(\lambda, \omega) \in (0, \infty) \times S^{n-1}$, the generalized eigenfunction associated with $H_0(h)$;

$$\phi_0(x; \lambda, \omega, h) = \exp(ih^{-1}\sqrt{2\lambda}\langle x, \omega\rangle),$$

where $\langle , \rangle$ denotes the scalar product in R^n. We further define $\psi_0(x; \lambda, \omega, h)$ by

$$\psi_0(x; \lambda, \omega, h) = c_0(\lambda, h)\phi_0(x; \lambda, \omega, h)$$

with the normalized constant

$$c_0(\lambda, h) = (2\pi h)^{-n/2}(2\lambda)^{(n-2)/4}.$$

Then the unitary operator $F_0(h): L^2(R^n_x) \to L^2((0,\infty); L^2(S^{n-1}))$ defined by

$$(F_0(h)f)(\lambda, \omega) = \int_{R^n} \overline{\psi}_0(x; \lambda, \omega, h) f(x) dx$$

gives the spectral representation for $H_0(h)$ in the sense that $H_0(h)$ is transformed into the multiplication by λ in the space $L^2((0,\infty); L^2(S^{n-1}))$.

The generalized eigenfunction $\phi_\pm(x; \lambda, \omega, h)$ of $H(h)$ is given by

$$\phi_\pm = \phi_0 - R(\lambda \pm i0; H(h)) V \phi_0$$

and we define $\psi_\pm(x; \lambda, \omega, h)$ as in the same way as ψ_0; $\psi_\pm = c_0(\lambda, h)\phi_\pm$. Let $S(\lambda, h): L^2(S^{n-1}) \to L^2(S^{n-1})$ be the scattering matrix defined for the pair $(H_0(h), H(h))$. As is well known, $S(\lambda, h)$ is unitary and takes the form

$$S(\lambda, h) = \mathrm{Id} - (2\pi i) T(\lambda, h).$$

Here $T(\lambda, h)$ is an integral operator of Hilbert-Schmidt class with the kernel

$$T(\theta, \omega; \lambda, h) = (V\psi_+(.\,; \lambda, \omega, h), \psi_0(.\,; \lambda, \theta, h)),$$

where (,) denotes the L^2 scalar product.

We shall discuss the relation between the scattering amplitude $f(\omega \to \theta; \lambda, h)$ and the kernel $T(\theta, \omega; \lambda, h)$. Let $G_0(x-y; \lambda, h)$ be the Green function of $R(\lambda + i0; H_0(h))$. We know (p. 196, [*2*]) that $G_0(x; \lambda, h)$ behaves like

$$G_0 = c_1(\lambda, h) |x|^{-(n-1)/2} \exp(ih^{-1}\sqrt{2\lambda}|x|)(1 + o(1)) \quad \text{as} \quad |x| \to \infty,$$

where

$$c_1 = (2\pi)^{-(n-1)/2}(2\lambda)^{(n-3)/4} \exp(-i(n-3)(\pi/4)) h^{-(n+1)/2}.$$

We now write $x = |x|\theta$, $\theta \in S^{n-1}$. Then it follows from the Lippmann-Schwinger equation that $\phi_+(|x|\theta; \lambda, \omega, h)$ behaves like

$$\phi_+ = \phi_0 + f(\omega \to \theta; \lambda, h) |x|^{-(n-1)/2} \exp(ih^{-1}\sqrt{2\lambda}|x|)(1 + o(1))$$

as $|x| \to \infty$, where

$$f = -c_1(\lambda, h)(V\phi_+(.\,; \lambda, \omega, h), \phi_0(.\,; \lambda, \theta, h)).$$

The function $f(\omega \to \theta; \lambda, h)$ is called the scattering amplitude and it is related to the kernel $T(\theta, \omega; \lambda, h)$ through

$$f(\omega \to \theta; \lambda, h) = c_2(\lambda, h) T(\theta, \omega; \lambda, h),$$

where

$$c_2 (= -c_1 c_0^{-2}) = -(2\pi)^{(n+1)/2}(2\lambda)^{-(n-1)/4} \exp(-i(n-3)(\pi/4)) h^{(n-1)/2}.$$

Thus the total scattering cross-section $\sigma(\omega; \lambda, h)$ is represented as

$$\sigma(\omega; \lambda, h) = |c_2(\lambda, h)|^2 \int_{S^{n-1}} |T(\theta, \omega; \lambda, h)|^2 d\theta.$$

Let $K(\lambda; h) = T(\lambda, h) - T(\lambda, h)^*$. By unitarity of $S(\lambda, h)$, $T(\lambda, h)$ is normal and also

$$T(\lambda, h)^* T(\lambda, h) = i(2\pi)^{-1} K(\lambda, h).$$

Hence, $K(\lambda, h)$ is of trace class. We denote by $K(\theta, \omega; \lambda, h)$ the kernel of $K(\lambda, h)$. Then, $\sigma(\omega; \lambda, h)$ can be rewritten as

$$\sigma(\omega; \lambda, h) = ic_3(\lambda, h) K(\omega, \omega; \lambda, h), \tag{1.1}$$

where

$$c_3\,(= (2\pi)^{-1} \mid c_2 \mid^2) = (2\pi)^n (2\lambda)^{-(n-1)/2} h^{(n-1)}.$$

§ 2. BOUNDS FOR TOTAL CROSS-SECTIONS

As a preliminary step toward the proof of the main theorem, we shall first establish the semi-classical bounds for total cross-sections, accepting Theorem 2 (resolvent estimate) as proved. The proof of Theorem 2 will be given in sections 5 and 6.

THEOREM 2.1. — *Under the same assumptions as in Theorem 1, we have*

$$\sigma(\omega; \lambda, h) = O(h^{-\nu}), \qquad \nu = (n-1)/(\rho - 1),$$

as $h \to 0$.

Proof. — The proof is rather long and is divided into several steps.

STEP (0). — We begin by fixing several notations. We denote by $|\ |_0$ and $(\,,)$ the L^2 norm and scalar product, respectively. We fix the two positive constants γ and β as

$$\gamma = 1/(\rho - 1), \qquad \beta = (1 + \delta_0)\gamma > \gamma, \tag{2.1}$$

with some $\delta_0 > 0$ small enough. Throughout the proof, γ and β are used with the meanings ascribed above. We further introduce three smooth cut-off functions $\chi_j = \chi_j(x; h)$, $1 \leqq j \leqq 3$, with the following properties:

a) $0 \leqq \chi_j \leqq 1$ and $\Sigma_{j=1}^3 \chi_j = 1$ on R_x^n;
b) χ_1 is supported in $\{ x : |x| < 2h^{-\gamma} \}$ and $\chi_1 = 1$ for $|x| < h^{-\gamma}$;
c) χ_2 is supported in

$$B_{\gamma\beta} = \{ x : h^{-\gamma} < |x| < 3h^{-\beta} \} \tag{2.2}$$

and $\chi_2 = 1$ for x, $2h^{-\gamma} < |x| < 2h^{-\beta}$;
d) χ_3 is supported in $\{ x : |x| > 2h^{-\beta} \}$ and $\chi_3 = 1$ for $|x| > 3h^{-\beta}$;
e) $|\partial_x^\alpha \chi_j| \leqq C_{j\alpha} \langle x \rangle^{-|\alpha|}$, $1 \leqq j \leqq 3$, uniformly in h.

Finally, for brevity, we write ϕ_0 and $\phi_\pm$ for $\phi_0(x; \lambda, \omega, h)$ and $\phi_\pm(x; \lambda, \omega, h)$, respectively, with the fixed incoming direction ω and energy λ.

STEP (1). — The first step is to rewrite $\sigma(\omega; \lambda, h)$ in a more convenient form.

LEMMA 2.1. — *Under the above notations*

$$\sigma(\omega; \lambda, h) = 2(2\lambda)^{-1/2}h^{-1}\Sigma_{j=1}^{3}\sigma_j(\omega; \lambda, h),$$

where $\sigma_1 = \mathrm{Im}\,(\chi_1\phi_0, V\phi_+)$ *and*

$$\sigma_j = \mathrm{Im}\,(R(\lambda + i0; H(h))V\phi_0, \chi_j V\phi_0), \qquad 2 \leqq j \leqq 3.$$

Proof. — It is sufficient to prove the lemma when $V(x)$ is of compact support. The general case in which $V(x)$ satisfies $(V)_\rho$ with $\rho > (n+1)/2$ can be proved by approximating $V(x)$ by a sequence of C_0^∞-potentials. Thus we assume that $V(x) \in C_0^\infty(R_x^n)$. Then, by (1.1) ,we have

$$\sigma(\omega; \lambda, h) = -2(2\lambda)^{-1/2}h^{-1}\Sigma_{j=1}^{3}\,\mathrm{Im}\,(\phi_\pm, \chi_j V\phi_0).$$

Therefore, the lemma follows immediately from the relation

$$\phi_+ = \phi_0 - R(\lambda + i0; H(h))V\phi_0. \qquad \square$$

We assert that

$$\sigma_j(\omega; \lambda, h) = 0(h^{1-(n-1)\gamma}), \qquad 1 \leqq j \leqq 3, \tag{2.3}$$

from which the theorem follows at once.

STEP (2). — We write $R(\lambda + i0)$ for $R(\lambda + i0; H(h))$ and denote by $[\,,]$ the commutator notation. By a simple calculation

$$R(\lambda + i0)V\phi_0 = \chi_1\phi_0 + R(\lambda + i0)\{\,[\chi_1, H_0(h)] + \chi_2 V + \chi_3 V\,\}\,\phi_0. \tag{2.4}$$

This relation, together with Theorem 2, plays an important role in proving (2.3).

LEMMA 2.2. — $\sigma_3(\omega; \lambda, h) = o(h^{1-(n-1)\gamma})$.

Proof. — By (2.4), $\sigma_j(\omega; \lambda, h)$, $2 \leqq j \leqq 3$, can be decomposed into $\sigma_j = \Sigma_{k=0}^{3}\sigma_{jk}(\omega; \lambda, h)$, where

$$\begin{aligned}
\sigma_{j0} &= \mathrm{Im}\,(\chi_1\phi_0, \chi_j V\phi_0)(=0),\\
\sigma_{j1} &= \mathrm{Im}\,(R(\lambda + i0)[\chi_1, H_0(h)]\phi_0, \chi_j V\phi_0),\\
\sigma_{j2} &= \mathrm{Im}\,(R(\lambda + i0)\chi_2 V\phi_0, \chi_j V\phi_0),\\
\sigma_{j3} &= \mathrm{Im}\,(R(\lambda + i0)\chi_3 V\phi_0, \chi_j V\phi_0).
\end{aligned}$$

We prove $\sigma_{31} = o(h^{1-(n-1)\gamma})$ only, because the other terms can be dealt with similarly. We write

$$[\chi_1, H_0(h)]\phi_0 = f(x; h)\phi_0,$$

where f has support in $\{x : h^{-\gamma} < |x| < 2h^{-\gamma}\}$ and $|f| < C\langle x\rangle^{-\rho}$ uniformly in h. Hence, by Theorem 2,

$$\sigma_{31} = 0(h^{-1})\,|\langle x\rangle^{\alpha} f|_0\,|\langle x\rangle^{\alpha}\chi_3 V|_0 = 0(h^{1-(n-1)\gamma})0(h^{\mu})$$

for $\alpha > 1/2$ (but close enough to $1/2$), where

$$\mu = (n-1)\gamma - 2 + (\beta + \gamma)(\rho - \alpha - n/2).$$

We can take $\delta_0 > 0$ (see (2.1)) and $\alpha - 1/2 > 0$ so small that $\mu > 0$. This completes the proof. □

We now make use of the relations $V\phi_+ = -(H_0(h) - \lambda)\phi_+$ and $\phi_+ = \phi_0 - R(\lambda + i0)V\phi_0$ to rewrite $\sigma_1(\omega; \lambda, h)$ as

$$\sigma_1 = \mathrm{Im}\,([\chi_1, H_0(h)]\phi_0, \phi_0 - R(\lambda + i0)V\phi_0).$$

Hence, by (2.4) again, $\sigma_1(\omega; \lambda, h)$ can be decomposed as follows:

$$\begin{aligned} \sigma_{10} &= \mathrm{Im}\,([\chi_1, H_0(h)]\phi_0, (1-\chi_1)\phi_0), \\ \sigma_{11} &= \mathrm{Im}\,(R(\lambda + i0)[\chi_1, H_0(h)]\phi_0, [\chi_1, H_0(h)]\phi_0), \\ \sigma_{12} &= \mathrm{Im}\,(R(\lambda + i0)\chi_2 V\phi_0, [\chi_1, H_0(h)]\phi_0), \\ \sigma_{13} &= \mathrm{Im}\,(R(\lambda + i0)\chi_3 V\phi_0, [\chi_1, H_0(h)]\phi_0). \end{aligned}$$

By an explicit calculation, we can easily see that $\sigma_{10} = o(h^{1-(n-1)\gamma})$ and in the same way as in the proof of Lemma 2.2, we can prove that $\sigma_{j3} = o(h^{1-(n-1)\gamma})$, $1 \leqq j \leqq 2$. Thus we have

$$(2.5)\quad \sigma(\omega; \lambda, h) = 2(2\lambda)^{-1/2}h^{-1}\Sigma_{1\leqq j,k\leqq 2}\sigma_{jk}(\omega; \lambda, h) + o(h^{-(n-1)\gamma}).$$

STEP (3). — By step (2), the proof is reduced to evaluating terms such as $R(\lambda + i0)f\phi_0$ with f supported in $B_{\gamma\beta}$, $B_{\gamma\beta}$ being as in (2.2). We first consider waves passing over the effective region of scatterer and we prove that such waves do not make an essential contribution to the asymptotic behavior as $h \to 0$ of $\sigma(\omega; \lambda, h)$.

We fix another positive constant κ as

$$\kappa = (1 - 2\delta_0)\gamma < \gamma$$

with the same δ_0 as in (2.1).

LEMMA 2.3. — *Recall the notation Π_ω. Let $f = f(x; h)$ be a function such that:*

i) f is supported in $\{x = y + s\omega \in B_{\gamma\beta} : |y| < 2h^{-\kappa}, y \in \Pi_\omega\}$;
ii) $|f| < C\langle x\rangle^{-\rho}$ uniformly in h.

Then, for any $\alpha > 1/2$ close enough to $1/2$,

$$|\langle x\rangle^{-\alpha}R(\lambda + i0)f\phi_0|_0 = o(h^{(\alpha - n/2)\gamma}).$$

Proof. — By Theorem 2, the term under consideration is of order $O(h^{(\alpha-n/2)\gamma})O(h^{\mu})$ with

$$\mu = \gamma\rho - 1 - (\alpha + 1/2)\beta - \kappa(n-1)/2 - (\alpha - n/2)\gamma .$$

By the choice of β and κ, we can take $\alpha - 1/2 > 0$ so small that $\mu > 0$, which completes the proof. □

Let f be as in Lemma 2.3. If $r = r(x; h)$ has support in $V_{\gamma\beta}$ and satisfies $|r| < C\langle x \rangle^{-\rho}$ uniformly in h, then it follows from Lemma 2.3 that

$$(2.6) \qquad (R(\lambda + i0)f\phi_0, r\phi_0) = o(h^{1-(n-1)\gamma}) .$$

STEP (4). — Next we consider waves passing over an outside of the effective region of scatterer and we see that only such waves make an essential contribution of the asymptotic behavior as $h \to 0$ of $\sigma(\omega; \lambda, h)$.

Let $f(x) = f(x; h) \in C_0^\infty(R_x^n)$ be a function such that:

(f_1) $\quad f$ has support in $\{ x = y + s\omega \in B_{\gamma\beta} : |y| > h^{-\kappa} \}$;

(f_2) $\quad |\partial_x^\alpha f| \leqq C_\alpha h^{|\alpha|\kappa} \langle x \rangle^{-\rho}$ uniformly in h.

We shall evaluate the term $R(\lambda + i0)f\phi_0$ with f as above by constructing an approximate representation. We fix $\tau = \tau(\lambda, h)$ as

$$(2.7) \qquad \tau = 9(2\lambda)^{-1/2}h^{-\beta}$$

and define $g_0 = g_0(x; h)$ by

$$g_0 = \int_0^\tau F(x, t; h)dt ,$$

where

$$F = f(x - \sqrt{2\lambda}\omega t) \exp\left(- ih^{-1} \int_0^t V(x - \sqrt{2\lambda}\omega(t-s))ds \right).$$

By a direct differentiation,

$$\langle \sqrt{2\lambda}\omega, \nabla \rangle g_0 + ih^{-1}Vg_0 = - \int_0^\tau (d/dt)F(x, t; h)dt .$$

Making use of this relation, we calculate

$$(H_0(h) + V - \lambda)(ih^{-1}g_0\phi_0) = f\phi_0 - r_1\phi_0 - r_2\phi_0 ,$$

where

$$r_1(x; h) = f(x - \sqrt{2\lambda}\omega\tau) \exp\left(- ih^{-1} \int_0^\tau V(x - \sqrt{2\lambda}\omega(\tau - s))ds \right),$$

$$r_2(x; h) = i(1/2)h(\Delta g_0)(x; h) .$$

Thus we can represent $R(\lambda + i0)f\phi_0$ as

$$R(\lambda + i0)f\phi_0 = ih^{-1}g_0\phi_0 + R(\lambda + i0)r_1\phi_0 + R(\lambda + i0)r_2\phi_0 .$$

We evaluate the remainder terms $R(\lambda + i0)r_j\phi_0$, $1 \leqq j \leqq 2$.

LEMMA 2.4. — *For any* $\alpha > 1/2$ *close enough to* $1/2$,

$$|\langle x \rangle^{-\alpha} R(\lambda + i0) r_2 \phi_0 |_0 = o(h^{(\alpha - n/2)\gamma}).$$

Proof. — By the choice of τ, r_2 has support in $\{ x : |x| < 12h^{-\beta}, |y| > h^{-\kappa} \}$. If $x \in \operatorname{supp} r_2$, then it follows from $(V)_\rho$ that

$$\int_{-\infty}^{\infty} |(\nabla V)(x - \sqrt{2\lambda}\omega(t - s))| \, ds = 0(h^{\kappa\rho})$$

uniformly in $t \in R^1$. Similarly

$$\int_{-\infty}^{\infty} |(\Delta V)(x - \sqrt{2\lambda}\omega(t - s))| \, ds = 0(h^{\kappa(\rho+1)}).$$

By choosing δ_0 small enough, we may assume that $\kappa > \kappa\rho - 1 > 0$. Hence, by property $(f - 2)$,

$$|r_2| \leqq Ch^{1+2(\kappa\rho-1)}(h^{-\gamma} + |y|)^{1-\rho} = 0(h^{2\kappa\rho}).$$

Hence, by Theorem 2, the term under consideration is of order $0(h^{(\alpha-n/2)\gamma})0(h^{\mu})$ with

$$\mu = 2\kappa\rho - 1 - (\alpha + n/2)\beta - (\alpha - n/2)\gamma.$$

We can take $\delta_0 > 0$ and $\alpha - 1/2 > 0$ so small that $\mu > 0$. This proves the lemma. □

LEMMA 2.5. — *Let* $\chi = \chi(x; h)$ *be the characteristic function of the set* $\{ x : |x| < 3h^{-\beta} \}$. *Then, for any* $\alpha > 1/2$ *close enough to* $1/2$,

$$|\chi \langle x \rangle^{-\alpha} R(\lambda + i0) r_1 \phi_0 |_0 = o(h^{(\alpha - n/2)\gamma}).$$

Remark. — By the choice of τ, r_1 has support in $\{ x : |x - 9h^{-\beta}\omega| < 3h^{-\beta} \}$ and hence the support of r_1 does not intersect with the support of χ. Roughly speaking, by the out-going property, the classical particles with momentum $\sqrt{2\lambda}\omega$ starting from the support of r_1 never pass over the support of χ. Thus, it is possible to prove that the term in the lemma is of order $0(h^N)$ for any $N \gg 1$. However, we do not intend to go into details here, because the weak bound as above is enough to our later application.

To prove Lemma 2.5, we prepare

LEMMA 2.6. — *Let* $R_0(\lambda + i0) = R(\lambda + i0; H_0(h))$. *Under the same notations as in Lemma 2.5,*

$$|\chi \langle x \rangle^{-\alpha} R_0(\lambda + i0) r_1 \phi_0 |_0 = 0(h^N)$$

for any $N \gg 1$.

Proof. — The lemma will be intuitively obvious because of the above remark and also the rigorous justification can be easily done. We give only a sketch for the proof.

We can write $\chi \langle x \rangle^{-\alpha} R_0(\lambda + i0) r_1 \phi_0$ as

$$ih^{-1} \int_0^\infty \chi \langle x \rangle^{-\alpha} \exp(ih^{-1} t\lambda) \exp(-ih^{-1} t H_0(h)) r_1 \phi_0 dt.$$

Making use of the Fourier transform, we further write the integral above in the explicit form. Then a partial integration proves the lemma. It is also possible to prove the lemma more directly by use of the explicit form of the Green function $G_0(x - y; \lambda, h)$ of $R_0(\lambda + i0)$. For example, when $n=3$,

$$R_0(\lambda+i0) r_1 \phi_0 = (2\pi h^2)^{-1} \int_{R^n} \exp(ih^{-1}\sqrt{2\lambda}\psi(x, y, \omega)) \, | x-y |^{-1} r_1(y) dy,$$

where $\psi = |x-y| + \langle y, \omega \rangle$. If $x \in \operatorname{supp} \chi$ and $y \in \operatorname{supp} r_1$, then $| \nabla_y \psi | \sim 2 | \omega |$ as is easily seen. Hence the lemma is proved by integration by parts. □

Proof of Lemma 2.5. — For brevity, we write R_0 and R for $R_0(\lambda+i0)$ and $R(\lambda+i0)$, respectively. By the resolvent identity,

$$\chi \langle x \rangle^{-\alpha} R r_1 \phi_0 = \chi \langle x \rangle^{-\alpha} \{ R_0 - R\chi V R_0 - R(1 - \chi) V R_0 \} r_1 \phi_0 .$$

By Theorem 2 and Lemma 2.6, the L^2 norm of the first and second terms on the right side is of order $0(h^N)$. To deal with the third term, we use the well-known resolvent estimate for $H_0(h)$ (see [1]); $\| \langle x \rangle^{-\alpha} R_0 \langle x \rangle^{-\alpha} \| = 0(h^{-1})$, $\alpha > 1/2$. Since

$$| \langle x \rangle^{\alpha} r_1 |_0 = 0(h^{-\beta\alpha}) \, | r_1 |_0 = 0(h^{-\beta\alpha + (\rho - n/2)\gamma}),$$

the L^2 norm of the third term is of order $0(h^{(\alpha - n/2)\gamma}) 0(h^{\mu})$ with

$$\mu = (\rho - 3\alpha)\beta - 2 + (\rho - n/2)\gamma - (\alpha - n/2)\gamma .$$

We can take $\alpha - 1/2 > 0$ so small that $\mu > 0$, which completes the proof. □

Let f satisfy (f_1) and (f_2). If $r = r(x; h)$ has support in $B_{\gamma\beta}$ and satisfies $| r | < C \langle x \rangle^{-\rho}$ uniformly in h, then it follows from Lemmas 2.5 and 2.5 that

$$(R(\lambda + i0) f \phi_0, r\phi_0) = (ih^{-1} g_0, r) + o(h^{1-(n-1)\gamma}).$$

As is easily seen, $(ih^{-1} g_0, r) = 0(h^{1-(n-1)\gamma})$ and hence

$$(R(\lambda + i0) f \phi_0, r\phi_0) = 0(h^{1-(n-1)\gamma}). \tag{2.9}$$

STEP (5). — We shall complete the proof. Let $\sigma_{jk}(\omega; \lambda, h)$, $1 \leqq j, k \leqq 2$, be as in step (2). By (2.6) and (2.9), $\sigma_{jk} = 0(h^{1-(n-1)\gamma})$. This proves (2.3) and the proof is now complete. □

For given $f(x) = f(x; h)$ with bound $| f | < C \langle x \rangle^{-\rho}$ uniformly in h, we define $F(x, t; h)$ by (2.8) and $g = g(x; h)$ by

$$g = \int_0^\infty F(x, t; h) dt. \tag{2.10}$$

Let $r = r(x; h)$ be as above. If f has support in $B_{\gamma\beta}$, then the support of $\int_\tau^\infty F(x, t; h)dt$, τ being as in (2.7), does not intersect with $B_{\gamma\beta}$ and if f has support in $\{ x \in B_{\gamma\beta} : | y | < h^{-\kappa}, y \in \Pi_\omega \}$, then $(ih^{-1}g, r) = o(h^{1-(n-1)\gamma})$. Thus, by the same arguments as in steps (3) and (4), we can prove

$$(2.11) \qquad (R(\lambda + i0) f \phi_0, r\phi_0) = (ih^{-1}g, r) + o(h^{1-(n-1)\gamma}),$$

if $f = f(x; h)$ is supported in $B_{\gamma\beta}$ and satisfies $| \partial_x^\alpha f | < C_\alpha \langle x \rangle^{-\rho - |\alpha|}$ uniformly in h. Relation (2.11) plays an important role in proving the main theorem.

§ 3. ASYMPTOTICS FOR TOTAL CROSS-SECTIONS

In this section we shall prove the main theorem. By the arguments in the previous section, the main contribution to the asymptotic behavior as $h \to 0$ of $\sigma(\omega; \lambda, h)$ comes from the terms $\sigma_{jk}(\omega; \lambda, h)$, $1 \leqq j, k \leqq 2$, and also the leading terms of these terms are explicitly representable by construction in step (4). The proof of the main theorem is done by looking at the cancellations of the cut-off functions χ_j, $1 \leqq j \leqq 3$, carefully.

Proof of Theorem 1. — The proof is divided into several steps.

STEP (1). — We keep the same notations as in section 2. We set

$$f_0 = f_0(x; h) = i\sqrt{2\lambda}h \langle \omega, \nabla \rangle \chi_1$$
$$f_j = f_j(x; h) = \chi_j V, \qquad 1 \leqq j \leqq 3.$$

We define $F_j = F_j(x, t; h)$ and $g_j = g_j(x; h)$ by (2.8) and (2.10) with $f = f_j$, $0 \leqq j \leqq 3$, respectively. Then, by (2.5) and (2.11),

$$(3.1) \qquad \sigma(\omega; \lambda, h) = 2(2\lambda)^{-1/2}h^{-1}s(\omega; \lambda, h) + o(h^{-(n-1)\gamma}),$$

where

$$s(\omega; \lambda, h) = \operatorname{Im} (ih^{-1}(g_0 + g_2), f_0 + f_2).$$

STEP (2). — *Lemma 3.1. — For $j \neq 1, 0 \leqq j \leqq 3$,*

a) $$(ih^{-1}g_j, f_3) = o(h^{1-(n-1)\gamma}),$$

b) $$(ih^{-1}g_3, f_j) = o(h^{1-(n-1)\gamma}).$$

Proof. — The proof is easy. □

LEMMA 3.2. — $ih^{-1}g_0 = -\chi_1 + ih^{-1}g_1$.

Proof. — If we take account of the simple relation

$$(d/dt)f(x - \sqrt{2\lambda}\omega t) = -\sqrt{2\lambda} \langle \omega, \nabla_x \rangle f(x - \sqrt{2\lambda}\omega t),$$

the lemma is proved by partial integration. The calculation is slightly tedious but direct. □

We now define $G = G(x; h)$ by

$$G = \Sigma_{j=1}^{3} g_j = \int_0^\infty K(x, t; h)dt,$$

where

$$K = V(x - \sqrt{2\lambda}\omega t)\exp\left(-ih^{-1}\int_0^t V(x - \sqrt{2\lambda}\omega(t-s))ds\right).$$

Then, by Lemmas 3.1 and 3.2,

$$s(\omega; \lambda, h) = \mathrm{Im}\,(ih^{-1}G, f_0 + f_2 + f_3) + o(h^{1-(n-1)\gamma}).$$

LEMMA 3.3. — $\mathrm{Im}\,(ih^{-1}G, f_0) = \mathrm{Im}\,(ih^{-1}G, f_1)$.

Proof. — By partial integration,

$$\mathrm{Im}\,(ih^{-1}G, f_0) = \mathrm{Im}\,(-2\sqrt{2\lambda}\langle \omega, \nabla \rangle G, \chi_1).$$

Hence, the lemma follows from the relation

$$\sqrt{2\lambda}\langle \omega, \nabla \rangle G + ih^{-1}VG = -\int_0^\infty (d/dt)K(x, t; h)dt$$

which is verified by a direct calculation. □

By Lemma 3.3,

$$s(\omega; \lambda, h) = \mathrm{Im}\,(ih^{-1}G, V) + o(h^{1-(n-1)\gamma}).$$

STEP (3). — *Lemma 3.4. We write $x = y + s\omega$, $y \in \Pi_\omega$. Then*

$$ih^{-1}G = 1 - \exp\left(-i(2\lambda)^{-1/2}h^{-1}\int_{-\infty}^{s} V(y + \tau\omega)d\tau\right).$$

Proof. — Define $K_1 = K_1(x, t; h)$ by

$$K_1 = \exp\left(-ih^{-1}\int_0^t V(x - \sqrt{2\lambda}\omega(t-\tau))d\tau\right).$$

Then, we have

$$ih^{-1}G = -\int_0^\infty (d/dt)K_1(x, t; h)dt$$

from which the lemma follows after a simple calculation. □

STEP (4). — We are now able to complete the proof. By Lemma 3.4,

$$s(\omega; \lambda, h) = \int_{\Pi_\omega}\int_{-\infty}^{\infty} W(y, s; h)ds\,dy + o(h^{1-(n-1)\gamma}),$$

where

$$W = V(y + s\omega)\sin\left[(2\lambda)^{-1/2}h^{-1}\int_{-\infty}^{s} V(y + \tau\omega)d\tau\right].$$

Hence, the desired asymptotic formula is obtained from (3.1) and the proof is now complete. □

The same arguments as used for $\sigma(\omega; \lambda, h)$ apply to the semi-classical asymptotics for the forward scattering amplitude $f(\omega \to \omega; \lambda, h)$. We here mention only the result without proof.

THEOREM 3.1. — *Assume* $(V)_\rho$ *with* $\rho > n$ *and that* $\lambda > 0$ *is fixed in a non-trapping energy range. Then*

$$f(\omega \to \omega; \lambda, h) = c(\lambda) h^{-(n-1)/2} \int_{\Pi_\omega} U(y; \lambda, \omega, h) dy + o(h^{-\mu})$$

with $\mu = (1/2)(n-1)(\rho+1)/(\rho-1)$, *where*

$$c(\lambda) = i(2\pi)^{-(n-1)/2}(2\lambda)^{(n-1)/4} \exp(-i(n-3)(\pi/4))$$

and

$$U = 1 - \exp\left(- i(2\lambda)^{-1/2} h^{-1} \int_{-\infty}^{\infty} V(y + \tau\omega) d\tau \right).$$

By the above theorem, the forward amplitude $f(\omega \to \omega; \lambda, h)$ is of order $0(h^{-\mu})$. Thus the amplitude $f(\omega \to \theta; \lambda, h)$ has a strong peak in a neighborhood of ω. See [*15*] [*17*] and [*18*] for the asymptotic behavior as $h \to 0$ of $f(\omega \to \theta; \lambda, h)$ with $\theta \neq \omega$.

§ 4. FINITE RANGE CASES

The aim here is to prove that

$$\sigma(\omega; \lambda, h) = 2\sigma_{cl}(\omega) + o(1), \qquad h \to 0,$$

for a class of finite range potentials.

Let $V(x) \in C_0^\infty(R_x^n)$;

$$\operatorname{supp} V \subset \{ x : |x| < R_0 \}$$

for some $R_0 > 0$. We fix the incoming direction ω and assume that $\lambda > 0$ is fixed in a non-trapping energy range. We denote by Σ_ω the projection of supp V to the hyperplane Π_ω.

ASSUMPTION (A.1). — The boundary $\partial\Sigma_\omega$ is C^∞-smooth.

Let dist (x, A) denote the distance from x to the set A. For $\varepsilon > 0$ small enough, we define

$$\Sigma_\omega^i(\varepsilon) = \{ y \in \Pi_\omega : y \in \Sigma_\omega, \operatorname{dist}(y, \partial\Sigma_\omega) > \varepsilon \},$$
$$\Sigma_\omega^b(\varepsilon) = \{ y \in \Pi_\omega : \operatorname{dist}(y, \partial\Sigma_\omega) < \varepsilon \},$$
$$\Sigma_\omega^e(\varepsilon) = \{ y \in \Pi_\omega : y \notin \Sigma_\omega, \operatorname{dist}(y, \partial\Sigma_\omega) > \varepsilon \}.$$

Let $\{ x(t; z, \eta), \xi(t; z, \eta) \}$ be the classical phase trajectory with initial state (y, η) for the hamiltonian function $p(x, \xi) = |\xi|^2/2 + V(x)$.

ASSUMPTION (A.2). — There exists $T_0 > 0$ such that

$$\xi(t; z, \sqrt{2\lambda}\omega) \neq \sqrt{2\lambda}\omega, \qquad t > T_0,$$

when $z = y - R_0\omega$ with $y \in \Sigma_\omega^i(\varepsilon)$, $\varepsilon > 0$.

The above assumption means that classical particles have momentum different from $\sqrt{2\lambda}\omega$ (initial momentum) after they are scattered by the potential $V(x)$.

THEOREM 4.1. — *Let $V(x)$ be a real C^∞-smooth potential with compact support and let $\lambda > 0$ be fixed in a non-trapping energy range. Assume* (A.1) *and* (A.2) *for incoming direction ω. Then*

$$\sigma(\omega; \lambda, h) = 2\sigma_{cl}(\omega) + o(1), \qquad h \to 0.$$

Proof. — The proof is done through several steps.

STEP (0). — We begin by fixing notations. Let R_0 be as above. We introduce three smooth cut-off functions $a(y) \in C_0^\infty(\Pi_\omega)$, $0 \leqq a \leqq 1$, and $b_\pm(s) \in C^\infty(R^1)$, $0 \leqq b_\pm \leqq 1$, such that:

a) a has support in $\{ y : |y| < 2R_0 \}$ and $a = 1$ for $|y| < R_0$;
b_+) b_+ has support in $\{ s : s > 4R_0 \}$ and $b_+ = 1$ for $s > 5R_0$;
b_-) b_- has support in $\{ s : s > -5R_0 \}$ and $b_- = 1$ for $s > -4R_0$.

We set $b(s) = b_-(s) - b_+(s)$, so that b has support in $\{ s : |s| < 5R_0 \}$ and $b = 1$ for $|s| < 4R_0$. We further define $\chi_0 \in C_0^\infty(R_x^n)$ by

$$\chi_0(x) = \chi_0(y, s) = a(y)b(s).$$

We again write $\phi_0 = \exp(ih^{-1}\sqrt{2\lambda}\langle x, \omega \rangle)$ and $R(\lambda \pm i0) = R(\lambda \pm i0; H(h))$.

STEP (1). — By the same argument as in the proof of Theorem 2.1, we have

$$\sigma(\omega; \lambda, h) = 2(2\lambda)^{-1/2}h^{-1}\,\mathrm{Im}\,(R(\lambda + i0)[\chi_0, H_0(h)]\phi_0, [\chi_0, H_0(h)]\phi_0) + o(1)$$

as $h \to 0$. We now define

$$f_0 = f_0(x; h) = i\sqrt{2\lambda}h\langle \omega, \nabla_x \rangle \chi_0.$$

Then, by Theorem 2, we obtain

$$\sigma(\omega; \lambda, h) = 2(2\lambda)^{-1/2}h^{-1}\,\mathrm{Im}\,(R(\lambda + i0)f_0\phi_0, f_0\phi_0) + o(1).$$

Recall the notations $\Sigma_\omega^i(\varepsilon)$, $\Sigma_\omega^b(\varepsilon)$ and $\Sigma_\omega^e(\varepsilon)$. We fix ε, $0 < \varepsilon \ll 1$, arbitrarily and introduce three smooth cut-off functions $\psi_j(y;\varepsilon) \in C^\infty(\Pi_\omega)$, $1 \leqq j \leqq 3$, with the following properties:

a) $0 \leqq \psi_j \leqq 1$ and $\Sigma_{j=1}^3 \psi_j = 1$ on Π_ω;
b) ψ_1 has support in $\Sigma_\omega^i(\varepsilon)$ and $\psi_1 = 1$ on $\Sigma_\omega^i(2\varepsilon)$;
c) ψ_2 has support in $\Sigma_\omega^b(2\varepsilon)$ and $\psi_2 = 1$ on $\Sigma_\omega^b(\varepsilon)$;
d) ψ_3 has support in $\Sigma_\omega^e(\varepsilon)$ and $\psi_3 = 1$ on $\Sigma_\omega^e(2\varepsilon)$.

We decompose f_0 into $f_0 = \Sigma_{j=1}^3 f_j$, where

$$f_j = f_j(x;\varepsilon,h) = \psi_j f_0 = i\sqrt{2\lambda}h\psi_j \langle \omega, \nabla \rangle \chi_0 .$$

STEP (2). — *Lemma 4.1.* $(R(\lambda \pm i0)f_2\phi_0, f_j\phi_0) = O(\varepsilon)h$, $1 \leqq j \leqq 3$, *where the order relation* $O(\varepsilon)$ *is uniform in* h.

Proof. — The lemma follows from Theorem 2. □

LEMMA 4.2. — *For* $j \neq 2$, $1 \leqq j \leqq 3$,

$$(R(\lambda \pm i0)f_3\phi_0, f_j\phi_0) = o(h), \tag{4.1}$$

where the order relation $o(h)$ *may depend on* ε.

Proof. — We give only a sketch for the proof. The proof is based on the same arguments as in step (4) of section 2 and is done by constructing approximate representations for $R(\lambda \pm i0)f_3\phi_0$. In this case, such a construction is simpler, because classical particles starting from the support of f_3 with momentum $\sqrt{2\lambda}\omega$ never pass over the scatterer (support of V).

We consider the + case only. Let χ be the characteristic function of the set $\{x : |x| < 10R_0\}$. Then, $\chi R(\lambda + i0)f_3\phi_0$ is approximately represented in the form

$$\chi R(\lambda + i0)f_3\phi_0 = ih^{-1}\chi g_3\phi_0 + o(1),$$

where

$$g_3 = \int_0^\infty f_3(x - \sqrt{2\lambda}\omega t)dt$$

and the order relation $o(1)$ means that the L^2 norm of the remainder term is of order $o(1)$ as $h \to 0$. The bound (4.1) with $j = 1$ is obvious, because the supports of g_3 and f_1 do not intersect with each other. The bound with $j = 3$ follows from the relation

$$(ih^{-1}g_3\phi_0, \quad \phi_0) = i\sqrt{2\lambda}h(\psi_3^2\chi_0, \langle \omega, \nabla \rangle \chi_0) = 0. \qquad \square$$

STEP (3). — We can write f_1 as $f_1 = f_- - f_+$, where

$$f_\pm = f_\pm(x;\varepsilon,h) = i\sqrt{2\lambda}h\psi_1 \langle \omega, \nabla \rangle b_\pm .$$

By the same argument as in the proof of Lemma 4.2, we can prove the following two lemmas.

LEMMA 4.3. — $(R(\lambda + i0)f_+\phi_0, f_-\phi_0) = o(h)$.

LEMMA 4.4.

$$\operatorname{Im}(R(\lambda + i0)f_\pm\phi_0, f_\pm\phi_0) = (1/2)(2\lambda)^{1/2}h\int_{\Pi_\omega}\psi_1(y;\varepsilon)^2dy + o(h).$$

The next lemma, together with Lemmas 4.1-4.4, completes the proof of the theorem. Assumption (A.2) is used only for the proof of this lemma.

LEMMA 4.5. — $(R(\lambda + i0)f_-\phi_0, f_+\phi_0) = 0(h)$.

4) To prove Lemma 4.5, we prepare two lemmas. We again denote by $\{x(t;z,\eta), \xi(t;z,\eta)\}$ the classical phase trajectory with initial state (z,η) and define the canonical mapping Φ_t by

$$\Phi_t : (z,\eta) \to (x(t;z,\eta), \xi(t;z,\eta)).$$

LEMMA 4.6 (semi-classical Egorov theorem). — *Let $a(x,\xi) \in C_0^\infty(R_x^n \times R_\xi^n)$. Fix* T *arbitrarily and define*

$$B(T,h) = \exp(-ih^{-1}TH(h))a(x,hD_x)\exp(ih^{-1}TH(h)).$$

Then, $B(T,h)$ *is asymptotically expanded as*

$$B(T,h) = \Sigma_{j=1}^{N-1}b_{jT}(x,hD_x)h^j + h^N R_{NT}(h)$$

for any $N \gg 1$, *where the symbol* $b_{jT}(x,\xi)$ *has support in*

$$\Omega_T = \{(x,\xi) : (x,\xi) = \Phi_T(z,\eta), (z,\eta) \in \operatorname{supp} a\}$$

and the remainder operator $R_{NT}(h)$ *is bounded uniformly in* h *as an operator from* L^2 *into the weighted* L^2 *space* L^2_α *for any* $\alpha \ll 1$.

For a proof, see, for example, the literature [*11*]. Let $a(x,\xi)$ be as above. If $b(x,\xi) \in C_0^\infty(R_x^n \times R_\xi^n)$ vanishes on Ω_T, then it follows from Lemma 4.6 that

$$\| b(x,hD_x)\exp(-ih^{-1}TH(h))a(x,hD_x)\| = 0(h^N)$$

for any $N \gg 1$.

LEMMA 4.7. — *Let* $U(x) \in C_0^\infty(R_x^n)$ *and let* $a(x,\xi) \in C_0^\infty(R_x^n \times (R_\xi^n\backslash\{0\}))$. *Assume that free particles starting from z with momentum* η, $(z,\eta) \in \operatorname{supp} a$, *never pass over the support of* U *for time* $t > 0$. *Then*

$$\| UR(\lambda + i0; H_0(h))a(x,hD_x)\| = 0(h^N)$$

for any $N \gg 1$.

Proof. — By the resolvent estimate

$$\| \langle x\rangle^{-\alpha}R(\lambda + i0; H_0(h))\langle x\rangle^{-\alpha}\| = 0(h^{-1}), \qquad \alpha > 1/2,$$

and by the calculus of pseudodifferential operators, it suffices to prove the lemma for the conjugate pseudodifferential operator $a(hD_x, x)$ defined by

$$(a(hD_x, x)f)(x) = (2\pi h)^{-n} \iint \exp(ih^{-1} \langle x-z, \xi \rangle) a(z, \xi) f(z) dz d\xi .$$

The lemma can be proved in the standard way using a partial integration. □

STEP (5). — *Proof of Lemma 4.5.* — Let $\chi_\pm \in C_0^\infty(R_x^n)$ be such that $\chi_\pm = 1$ on the support of $f_\pm$. We may assume that the supports of $\chi_\pm$ do not intersect with each other. Let $\zeta_0 \in C_0^\infty(R_\xi^n)$, $0 \leqq \zeta_0 \leqq 1$, be a function such that ζ_0 has support in $\{ \xi : | \xi - \sqrt{2\lambda}\omega | < 2\delta \}$ and $\zeta_0 = 1$ on $\{ \xi : | \xi - \sqrt{2\lambda}\omega | < \delta \}$. The choice of δ depends on ε. We can easily obtain

$$\chi_\pm \phi_0 = \zeta_0(hD_x)\chi_\pm \phi_0 + o(1),$$

where the abbreviation $o(1)$ is used with the same meaning as before. Define the symbols $d_\pm(x, \xi ; \varepsilon, h)$ by

$$d_\pm(x, \xi ; \varepsilon, h) = f_\pm(x ; \varepsilon, h)\zeta_0(\xi) .$$

By assumption (A.2), we can choose δ so small that the classical phase trajectory $\{ x(t; z, \eta), \xi(t; z, \eta) \}$ starting from $(z, \eta) \in \operatorname{supp} d_-$ never pass over the support of d_+. We have only to show that

$$(R(\lambda + i0)d_-(x, hD_x)\chi_- \phi_0, d_+(x, hD_x)\chi_+ \phi_0) = o(h) .$$

To prove this, we write

$$R(\lambda + i0) = ih^{-1} \int_0^\infty \exp(ih^{-1}t\lambda) \exp(- ih^{-1}tH(h))dt .$$

We take T sufficiently large and decompose the above integral into two parts; $R(\lambda + i0) = ih^{-1}\left\{ \int_0^T + \int_T^\infty \right\} dt$. We denote by Q_1 and Q_2 the first and second integrals, respectively. By the choice of δ and by Lemma 4.6,

$$\| d_+(x, hD_x)^* Q_1 d_-(x, hD_x) \| = 0(h^N)$$

for any $N \gg 1$. We can rewrite $d_+^* Q_2 d_-$ as

$$d_+^* Q_2 d_- = \exp(ih^{-1}T\lambda) d_+^* R(\lambda + i0) \exp(- ih^{-1}TH(h)) d_- .$$

For brevity, we write $R = R(\lambda + i0)$ and $R_0 = R(\lambda + i0; H_0(h))$. By the resolvent identity,

$$d_+^* Q_2 d_- = \exp(ih^{-1}T\lambda)(P_1 - P_2),$$

where

$$P_1 = d_+^* R_0 \exp(-ih^{-1}TH(h))d_- ,$$
$$P_2 = d_+^* RVR_0 \exp(-ih^{-1}TH(h))d_- .$$

By Theorem 2 and by Lemmas 4.6 and 4.7, the norm of both the operators P_1 and P_2 is bounded by $0(h^N)$. This proves the lemma. □

§ 5. OUT-GOING PARAMETRICES

Throughout this section, the potential $V(x)$ is assumed to satisfy $(V)_\rho$ with $\rho > 0$. Without loss of generality, we may assume that $0 < \rho < 1$. One of basic tools which we use for the proof of Theorem 2 is the out-going parametrices constructed globally in time $t \geqq 0$ for the non-stationary Schrödinger equations. Such parametrices have been constructed by [5] and [6], etc., and have played an important role in proving the completeness of wave operators in long-range scattering problems by the time-dependent method (Enss method). We here follow the idea due to Isozaki-Kitada [5] to construct such parametrices.

We start by introducing a certain class of symbols.

DEFINITION (A_μ^m). — For given $\Omega \subset R_x^n \times R_\xi^n$, we denote by $A_\mu^m(\Omega)$ the set of all $a(x, \xi)$, $(x, \xi) \in \Omega$, such that

$$| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) | \leqq C_{\alpha\beta} \langle x \rangle^{\mu - |\alpha|} \langle \xi \rangle^{m - |\beta|} .$$

If, in particular, $\Omega = R_x^n \times R_\xi^n$, we write A_μ^m for $A_\mu^m(\Omega)$.

We say that a family of $a(x, \xi : \varepsilon)$ with parameter ε belongs to $A_\mu^m(\Omega)$ uniformly in ε, if the constants $C_{\alpha\beta}$ above can be taken uniformly in ε. We further define $A_\mu(\Omega)$ by

$$A_\mu(\Omega) = \cap_{-\infty < m < \infty} A_\mu^m(\Omega) .$$

Most of symbols we consider in later application have compact support in ξ and hence are of class $A_\mu(\Omega)$.

For given (σ, d), $-1 < \sigma < 1$, $d > 1$, we use the notation

$$\Gamma_+(\sigma, R, d) = \{ (x, \xi) : x \in \Sigma_+(\sigma, R; \xi), d^{-1} < |\xi| < d \} ,$$

where

$$\Sigma_+(\sigma, R; \xi) = \{ x : |x| > R, \langle x, \xi \rangle > \sigma |x| . |\xi| \} .$$

Now, assume that ξ ranges over $\{ \xi : d_0^{-1} < |\xi| < d_0 \}$ for some $d_0 > 1$. Then, according to the result (Proposition 2.4) of [5] (see also [4]), we can construct a real C^∞-smooth function $\phi(x, \xi)$ with the following properties: given σ_0, there exists R_0 large enough such that: *i)* ϕ solves the equation

$$(1/2) | \nabla_x \phi(x, \xi) |^2 + V(x) = (1/2) | \xi |^2$$

in $\Gamma_+(\sigma_0, R_0, d_0)$; *ii)* $\phi(x, \xi) - \langle x, \xi \rangle$ is of class $A_{1-\rho}$; *iii)* $\phi(x, \xi)$ satisfies $|(\partial^2\phi/\partial x_j\partial\xi_k) - \delta_{jk}\phi| < c(R_0)$, δ_{jk} being the Kronecker notation, where we can make $c(R_0)$ as small as we desire by taking R_0 sufficiently large.

Let $\phi(x, \xi)$ be as above and let $a(x, \xi)$ be of class A^m_μ. Then we define the Fourier integral operator $I_\phi(a; h): S(R^n_x) \to S(R^n_x)$ by

$$(I_\phi(a; h)f)(x) = (2\pi h)^{-n} \iint \exp(ih^{-1}(\phi(x, \xi) - \langle y, \xi \rangle))a(x, \xi)f(y)dyd\xi ,$$

where the integration with no domain attached is taken over the whole space. We use this abbreviation through the entire discussion.

We now take σ_j, R_j and d_j, $1 \leqq j \leqq 3$, as follows: $\sigma_3 > \sigma_2 > \sigma_1 > \sigma_3$, $R_3 > R_2 > R_1 > R_0$ and $d_3 < d_2 < d_1 < d_0$. Let $\omega(x, \xi) \in A_0$ be supported in $\Gamma_+(\sigma_3, R_3, d_3)$. We shall construct a parametrix (approximate representation) for the operator

$$U(t; h, \omega) = \exp(-ih^{-1}tH(h))\omega(x, hD_x), \qquad t \geqq 0,$$

in the above form of oscillatory integral operators.

We first determine the symbol $a(x, \xi; h)$ to satisfy

$$\exp(-ih^{-1}\phi)(-(1/2)h^2\Delta + V - (1/2)|\xi|^2)\exp(ih^{-1}\phi)a \sim 0 .$$

We formally set

$$a(x, \xi; h) \sim \Sigma_{j=0}^{\infty} a_j(x, \xi)h^j .$$

The symbols a_j are inductively determined by solving the transport equations;

$$(5.1) \qquad \begin{aligned} &\nabla_x\phi\cdot\nabla_x a_0 + (1/2)(\Delta_x\phi)a_0 = 0 \\ &\nabla_x\phi\cdot\nabla_x a_j + (1/2)(\Delta_x\phi)a_j - (i/2)\Delta_x a_{j-1} = 0, \qquad j \geqq 1 \end{aligned}$$

under the conditions; $a_0 \to 1$ and $a_j \to 0$, $j \geqq 1$, as $|x| \to \infty$. Since $\nabla_x\phi = \xi + 0(|x|^{-\rho})$ as $|x| \to \infty$ by construction, the standard characteristic curve method enables us to solve (5.1) under the above conditions. Indeed, we may assume that $|\nabla_x\phi - \xi| \leqq \delta$ for $\delta > 0$ small enough. Consider the characteristic curve $q(t; x, \xi)$, $t \geqq 0$, $\dot{q} = \nabla_x\phi(q, \xi)$, $q(0) = x$. If (x, ξ) lies in $\Gamma_+(\sigma_1, R_1, d_1)$, then we can prove that

$$|q(t; x, \xi)| \geqq C(1 + |t| + |x|)$$

and

$$|\partial_x^\alpha q(t; x, \xi)| \leqq C_\alpha \langle x \rangle^{1-|\alpha|}, \qquad |\alpha| \geqq 1 .$$

This proves that the solution $a_j(x, \xi)$ to (5.1) belongs to $A_{-j}(\Gamma_+(\sigma_1, R_1, d_1))$. We extend a_j to the whole space $R^n_x \times R^n_\xi$ in the following way: *i)* $a_j \in A_{-j}$; *ii)* a_j has support in $\Gamma_+(\sigma_0, R_0, d_0)$.

We fix N large enough and define

$$a_N(x, \xi; h) = \Sigma_{j=0}^{N} a_j(x, \xi)h^j$$

and

$$r_N(x, \xi; h) = h^{-(N+2)} \exp(-ih^{-1}\phi)[(1/2)h^2\Delta - V + (1/2)|\xi|^2] \exp(ih^{-1}\phi)a_N.$$

By construction of ϕ and a_j, it follows that

$$h^{N+1}r_N \in A_{-1} \quad \text{and} \quad r_N \in A_{-(N+2)}(\Gamma_+(\sigma_1, R_1, d_1))$$

uniformly in h.

Let the symbol $\omega(x, \xi)$ be as before. Since $a_0(x, \xi) = 1 + 0(|x|^{-\rho})$ as $|x| \to \infty$ in $\Gamma_+(\sigma_1, R_1, d_1)$, we can find a symbol $b_N(x, \xi; h) \in A_0$ with support in $\Gamma_+(\sigma_2, R_2, d_2)$ such that

$$\omega(x, hD_x) = I_\phi(a_N; h)(I_\phi(b_N; h))^* + h^N\omega_N(x, hD_x; h)$$

with $\omega_N \in A_{-N}$. This follows from the composite formula of Fourier integral operators.

We define $U_N(t; h)$ and $R_N(t; h)$, $t \geqq 0$, as follows:

$$U_N = I_\phi(a_N; h) \exp(-ih^{-1}tH_0(h))(I_\phi(b_N; h))^*,$$
$$R_N = I_\phi(r_N; h) \exp(-ih^{-1}tH_0(h))(I_\phi(b_N; h))^*.$$

Then the Duhamel principle yields

$$(5.2)\quad U(t; h, \omega) = U_N + h^N \exp(-ih^{-1}tH(h))\omega_N + h^{N+1}G_N(t; h),$$

where

$$G_N = i\int_0^t \exp(-ih^{-1}(t-s))H(h))R_N(s; h)ds.$$

LEMMA 5.1. — *If* N *is chosen large enough, then*

$$\|\langle x\rangle^{n+1}\langle D_x\rangle^{n+1}R_N(t; h)\| \leqq C_N h^{N/2}(1 + |t|)^{-2}$$

for $t \geqq 0$.

Proof. — We write $[R_N(t; h)f](x)$ as

$$(2\pi h)^{-n}\iint \exp(ih^{-1}S(t, x, \xi, y))r_N(x, \xi; h)\overline{b_N}(y, \xi; h)f(y)dyd\xi,$$

where

$$S = \phi(x, \xi) - \phi(y, \xi) - (1/2)t|\xi|^2.$$

Suppose that $(y, \xi) \in \operatorname{supp} b_N$ and $(x, \xi) \in \operatorname{supp} r_N$. If $(x, \xi) \in \Gamma_+(\sigma_1, R_1, d_1)$, then we integrate by parts in ξ, using

$$|\nabla_\xi(\phi(y, \xi) + (1/2)t|\xi|^2)| \geqq C(1 + |t| + |y|).$$

On the other hand, if $(x, \xi) \notin \Gamma_+(\sigma_1, R_1, d_1)$, then we use

$$|\nabla_\xi S| \geqq C(1 + |t| + |y| + |x|),$$

which follows by the choice of σ_2 and σ_1; $\sigma_2 > \sigma_1$. Thus, the famous

Calderón-Vaillancourt theorem on the L^2 boundedness of oscillatory integral operators proves the lemma. □

We introduce the notation

$$\Gamma_-(\sigma, R, d) = \{ (x, \xi) : x \in \Sigma_-(\sigma, R; \xi), d^{-1} < |\xi| < d \},$$

where

$$\Sigma_-(\sigma, R; \xi) = \{ x : |x| > R, \langle x, \xi \rangle < \sigma |x| \cdot |\xi| \}.$$

Let $\omega(x, \xi) \in A_0$ be supported in $\Gamma_-(\sigma, R, d)$. Then, by making use of the same arguments as above, we can construct a parametrix for the operator

$$\exp(-ih^{-1}tH(h))\omega(x, hD_x), \qquad t \leqq 0,$$

in a form similar to (5.2).

§ 6. SEMI-CLASSICAL BOUNDS ON RESOLVENTS

In this section, we prove Theorem 2, following almost the same arguments as in [*12*], where similar estimates have been established for a class of C_0^∞-potentials. The method of proof is essentially based on the same idea as in the works [*14*] and [*8*], where the high energy estimates for resolvents have been obtained for general elliptic operators by the time-dependent method (hyperbolic equation method) and have been applied to investigate the asymptotic behavior as $|t| \to \infty$ of solutions to the corresponding non-stationary problems. In [*14*], such estimates have been obtained when perturbed coefficients have compact support, and in [*8*], these results have been extended to the case of non-compact support. We here use the Schrödinger equation method and the commutator method due to Mourre [*7*]. The Mourre method reduces the problem under consideration essentially to the one in the case of compact support perturbations and the proof seems to be slightly simplified.

Proof of Theorem 2. — The proof is done for the « + » case only by dividing it into several steps.

STEP (1). — We first fix a compact interval $I_0 = [a, b] (\subset (0, \infty))$ in non-trapping energy regions. We can take R_0 so large that $V(x)$ admits the following decomposition:

$$V(x) = V_1(x) + V_2(x), \tag{6.1}$$

where V_2 has support in $\{ x : |x| < R_0 \}$ and $V_1(x)$ satisfies the estimate

$$|V_1(x)| + (1/2)|(x \cdot \nabla)V_1(x)| < a/2. \tag{6.2}$$

LEMMA 6.1. — *Let* V_1 *be as above and define* $H_1(h)=(1/2)h^2\Delta + V_1$. *Then, for any* $\alpha > 1/2$,

$$\| \langle x \rangle^{-\alpha} R(\lambda \pm i0; H_1(h)) \langle x \rangle^{-\alpha} \| = 0(h^{-1})$$

uniformly in $\lambda \in I_0$. *Furthermore, if* z *ranges over*

$$\Sigma_\pm = \{ z = \lambda \pm i\kappa : \lambda \in I_0, 0 < \kappa \leqq 1 \},$$

then the same estimates as above hold for $R(z; H_1(h))$ *uniformly in* $z \in \Sigma_\pm$.

Proof. — The proof uses the commutator method due to Mourre [7] (see also [9]). Let $A(h)$ be the generator of the dilation unitary group;

$$A(h) = (1/4)(x \cdot hD_x + hD_x \cdot x), \qquad D_x = - i\nabla_x,$$

and let $f_0 \in C_0^\infty(R^1)$, $0 \leqq f_0 \leqq 1$, be such that $f_0 = 1$ on I_0 and f_0 is supported in $(a - \varepsilon, b + \varepsilon)$ for $\varepsilon > 0$ small enough. Then a direct calculation gives

$$i[H_1(h), A(h)] = hH_1(h) - h(V_1 + (1/2)(x \cdot \nabla)V_1).$$

Hence it follows from (6.2) that

$$if_0(H_1(h))[H_1(h), A(h)]f_0(H_1(h)) \geqq (a/3)hf_0(H_1(h))^2$$

in the form sense. This enables us to follow exactly the same arguments as in [7] and [9] and the lemma is proved, although we have to look at the h-dependence carefully. □

STEP (2). — Let $B_R = \{ x : |x| < R \}$ and let $\chi_R(x)$ be the characteristic function of B_R.

LEMMA 6.2.

$$\| \chi_R R(\lambda \pm i0; H(h))\chi_R \| = 0(h^{-1})$$

uniformly in $\lambda \in I_0$.

Lemmas 6.1 and 6.2 enable us to regard $H(h)$ as the perturbed operator to $H_1(h)$ with the compactly supported perturbed coefficient V_2. We here complete the proof of Theorem 2, accepting Lemma 6.2 as proved.

Completion of proof of Theorem 2. — Let

$$L^2(B_R) = \{ f \in L^2 : \text{supp } f \subset B_R \}.$$

Let $\psi \in C_0^\infty(R_x^n)$, $0 \leqq \psi \leqq 1$, be a cutt-off function such that $\psi = 1$ for $|x| < 1$ and $\psi = 0$ for $|x| > 2$. We take R to be $R > R_0$ for R_0 in step (1) and set $\zeta_R(x) = 1 - \psi(x/R)$. Assume that $f \in L^2(B_R)$ and define v_R by

$$v_R(x) = \zeta_R R(\lambda - i0; H(h))f.$$

Since $V(x) = V_1(x)$ for $|x| > R > R_0$ by (6.1) and $\zeta_R f = 0$, v_R satisfies the equation

$$(-(1/2)h^2\Delta + V_1 - \lambda)v_R = g_R$$

where

$$g_R(x) = [H_0(h), \zeta_R]R(\lambda - i0; H(h))f.$$

Hence, v_R is represented as

$$v_R = R(\lambda - i0; H_1(h))g_R .$$

By Lemma 6.2 and by elliptic estimate, $|\langle x \rangle^\alpha g_R|_0 \leqq C_R |f|_0$ and hence by Lemma 6.1, we have

$$\| \langle x \rangle^{-\alpha} R(\lambda - i0; H(h))\chi_R \| = 0(h^{-1})$$

for $\alpha > 1/2$, which yields

$$\| \chi_R R(\lambda + i0; H(h)) \langle x \rangle^{-\alpha} \| = 0(h^{-1}).$$

We repeat the same arguments as above for $f \in L^2_\alpha(R^n_x)$. This yields the desired estimate. □

The remaining steps are devoted to the proof of Lemma 6.2.

STEP (3). — Let $g_2 \in C_0^\infty(R^1)$, $0 \leqq g_2 \leqq 1$, be such that $g_2 = 1$ on $[a-\varepsilon, b+\varepsilon]$ and g_2 is supported in $(a-2\varepsilon, b+2\varepsilon)$. We set $g_1 = 1 - g_2$, so that $g_1 = 0$ on $[a-\varepsilon, b+\varepsilon]$. Let ψ be as above. We write $p(x, \xi) = (1/2)|\xi|^2 + V(x)$ and define the symbol $\omega_1(x, \xi) \in A_0^0$ by

$$\omega_1(x, \xi) = \psi(x/R)g_1(p(x, \xi))$$

for $R > R_0$.

We consider the equation

$$(H(h) - z)u = \omega_1(x, hD_x)f \tag{6.3}$$

for $f \in L^2$, where z is assumed to be in Σ_+. (See Lemma 6.1 for the notation Σ_+.) By definition, ω_1 vanishes in a neighborhood of $\{(x, \xi) : p(x, \xi) \in I_0\}$. Hence we can construct an approximate solution to (6.3) in the same way as parametrices for elliptic operators are constructed. The obtained results can be summarized as follows.

LEMMA 6.3. — *For given* N *large enough, there exist* $A_N(z; h)$ *and* $X_N(z; h)$ *such that:*

a) $(H(h) - z)A_N(z; h) = \omega_1(x, hD_x) + X_N(z; h)$;
b) $A_N(z; h) : L^2 \to L^2(B_{2R})$ *is bounded uniformly in h and* $z \in \Sigma_+$;
c) $X_N(z; h) : L^2 \to L^2(B_{2R})$ *is also bounded with bound* $0(h^N)$.

We define $B_N(z; h)$ by

$$B_N(z; h) = A_N(z; h) - R(z; H_1(h))X_N(z; h).$$

Since X_N maps L^2 into $L^2(B_{2R})$, it follows from Lemma 6.1 that

$$(6.4) \qquad \| \chi_R B_N(z; h) \| = 0(1), \qquad h \to 0.$$

and we have

$$(H(h) - z)B_N(z; h) = \omega_1(x, hD_x) + Y_N(z; h),$$

where $Y_N = - V_2 R(z; H_1(h)) X_N(z; h): L^2 \to L^2(B_R)$ is bounded and

$$(6.5) \qquad \| Y_N(z; h) \| = 0(h^{N-1}).$$

STEP (4). — We keep the same notations as in step (3). Define the symbol $\omega_2(x, \xi) \in A_0$ by

$$\omega_2(x, \xi) = \psi(x/R)\omega_2(p(x, \xi))$$

and consider the equation

$$(6.6) \qquad (H(h) - z)u = \omega_2(x, hD_x)f$$

for $f \in L^2(R^n_x)$, where z is again assumed to be in Σ_+. Let $\{ x(t; y, \eta), \xi(t; y, \eta) \}$ be the phase trajectory with initial state (y, η) for the hamiltonian function $p(x, \xi)$. We may assume that $\lambda \in [a - 2\varepsilon, b + 2\varepsilon]$ is non-trapping. Fix $R_1, R_1 > R$, large enough. By the non-trapping condition, there exists $T = T(R_1)$ such that $| x(t; y, \eta) | > R_1$ for $t > T$, if $(y, \eta) \in \operatorname{supp} \omega_2$. We introduce a function $\theta(t) \in C_0((-1, T+1))$, $0 \leqq \theta \leqq 1$, such that $\theta = 1$ on $[O,T]$ and decompose V as $V = V_3(t, x) + V_4(t, x)$, where $V_3 = V_1(x) + \theta(t)V_2(x)$ and $V_4 = (1 - \theta(t))V_2(x)$.

We now construct an approximate solution operator to (6.6) by the time-dependent method (Schrödinger equation method). Let $Q(t; h)$, $t \geqq 0$, be the solution operator to

$$(6.7) \qquad (ih\partial/\partial t - H_0(h) - V_3(t))Q(t; h) = 0, \qquad t > 0.$$

with $Q |_{t=0} = \omega_2(x, hD_x)$, where $V_3(t)$ denotes the multiplication by $V_3(t, x)$. Define $\hat{Q}(z; h)$, $z \in \Sigma_+$, by

$$\hat{Q}(z; h) = ih^{-1} \int_0^\infty \exp(ih^{-1}tz)Q(t; h)dt$$

and $\hat{Z}(z; h)$ by

$$\hat{Z}(z; h) = ih^{-1} \int_0^\infty \exp(ih^{-1}tz)V_4(t)Q(t; h)dt.$$

Then we have

$$(H(h) - z)\hat{Q}(z; h) = \omega_2(x, hD_x) + \hat{Z}(z; h).$$

LEMMA 6.4. — *a)* $\| \chi_R \hat{Q}(z; h) \| = 0(h^{-1})$.
b) $\hat{Z}(z; h): L^2 \to L^2(B_R)$ *is bounded and* $\| \hat{Z}(z; h) \| = 0(h^N)$ *for any* $N \gg 1$.

STEP (5). — We first complete the proof of Lemma 6.2, accepting Lemma 6.4 as proved.

Proof of Lemma 6.2. — Let $B_N(z;h)$ and $Y_N(z;h)$ be as in step (3). Set $G_N(z;h)=B_N(z;h)+\hat{Q}(z;h)$ and $W_N(z;h)=Y_N(z;h)+\hat{Z}(z;h)$. If $f\in L^2(B_R)$, then, by definition

$$\omega_1(x: hD_x)f + \omega_2(x, hD_x)f = f$$

and also we have

$$(H(h) - z)G_N(z;h)f = f + W_N(z;h)f.$$

We regard W_N as an operator from $L^2(B_R)$ into itself. By (6.5) and Lemma 6.4-(*b*), there exists a bounded inverse $(Id+W_N)^{-1}: L^2(B_R) \to L^2(B_R)$ with bound $0(1)$ (uniformly in $z\in\Sigma_+$). Thus we have

$$R(z;H(h))\chi_R = G_N(z;h)(Id + W_N(z;h))^{-1}\chi_R.$$

By the principle of limiting absorption, Lemma 6.2 follows from (6.4) and Lemma 6.4-(*a*). □

STEP (6). — We shall prove Lemma 6.4 by making use of out-going parametrices constructed in section 5. By the definition of $V_3(t,x)$, we may write $Q(t;h)$ as

$$Q(t;h) = \exp(-ih^{-1}(t - T - 1)H_1(h))Q(T + 1;h)$$

for $t > T + 1$. Let $p_3(t, x, \xi) = (1/2)|\xi|^2 + V_3(t,x)$. We denote by $\{\tilde{x}(t;y,\eta), \tilde{\xi}(t;y,\eta)\}$ the phase trajectory with initial state (y,η) for the hamiltonian function $p_3(t,x,\xi)$. If $(y,\eta)\in\operatorname{supp}\omega_2$, then by the non-trapping condition, $\{\tilde{x}(t;y,\eta), \tilde{\xi}(t;y,\eta)\}$, $t > T$, lies in the out-going region $\Gamma_+(\sigma_0, R_1, d_0)$ with some $\sigma_0 > 0$ and $d_0 > 1$. (See section 5 for the notation $\Gamma_+(\sigma, R, d)$).

LEMMA 6.5. — *Let $\Gamma_+(\sigma_0, R_1, d_0)$ be as above. Assume that $\omega(x,\xi)\in A_0^0$ vanishes in $\Gamma_+(\sigma_0, R_1, d_0)$. Then*

$$\| \langle x \rangle^\alpha \omega(x, hD_x)Q(t;h) \| = 0(h^N), \qquad T < t < T + 1,$$

for any $N \gg 1$ and $\alpha > 0$.

Proof. — We omit the detailed proof, because the lemma is essentially the famous Egorov theorem. We have only to note that by assumption the phase trajectories starting from $(y,\eta)\in\operatorname{supp}\omega_2$ never pass over the support of ω. □

LEMMA 6.6. — *Let $\omega(x,\xi)$ be as in* Lemma 6.5. *Define*

$$W_3(z;h) = \int_{T+1}^{\infty} \exp(ih^{-1}tz)\exp(-ih^{-1}(t - T - 1)H_1(h))dt$$

for $z\in\Sigma_+$. Then, for any $N \gg 1$,

$$\| \chi_R W_3(z;h)\omega(x, hD_x)Q(T + 1;h) \| = 0(h^N).$$

Proof. — A simple computation gives

$$W_3(z;h) = -ih\exp(ih^{-1}(T+1)z)R(z;H_1(h)).$$

Hence the lemma follows immediately from Lemmas 6.1 and 6.5. □

Let $\omega_0(x,\xi)\in A_0$ be supported in $\Gamma_+(\sigma_0, R_1, d_0)$. Define

$$U(t;h) = \exp(-ih^{-1}(t-T-1)H_1(h))\omega_0(x, hD_x)$$

for $t > T+1$. According to (5.2), $U(t;h)$ can be represented as

$$U(t;h) = \Sigma_{j=0}^{2} U_{jN}(t;h)$$

for given N large enough, where U_{jN}, $0 \leqq j \leqq 2$, take the following forms:

$$U_{0N} = I_\phi(a_N;h)\exp(-ih^{-1}(t-T-1)H_0(h))(I_\phi(b_N;h))^*,$$
$$U_{1N} = h^N\exp(-ih)^{-1}(t-T-1)H_1(h))\omega_N(x, hD_x;h),$$
$$U_{2N} = ih^{N+1}\int_0^{t-T-1}\exp(-ih^{-1}(t-T-1-s)H_1(h))R_N(s;h)ds.$$

Here $\omega_N(x,\xi;h)$ belongs to A_{-N} uniformly in h and $R_N(t;h)$ has property similar to that in Lemma 5.1.

Define $W_{jN}(z;h)$, $0 \leqq j \leqq 2$, by

$$W_{jN}(z;h) = \int_{T+1}^{\infty}\exp(ih^{-1}tz)U_{jN}(t;h)dt$$

for $z\in\Sigma_+$. The next lemma, together with Lemmas 6.5 and 6.6, completes the proof of Lemma 6.4.

LEMMA 6.7. — *a)* $\chi_R W_{0N}(z;h)Q(T+1;h) = 0$

b) $\|\chi_R W_{1N}(z;h)Q(T+1;h)\| = 0(h^{N-1})$.

c) $\|\chi_R W_{2N}(z;h)Q(T+1;h)\| = 0(h^{N/2})$.

Proof. — *a)* Is obvious, because by construction the symbol $a_N(x,\xi;h)$ vanishes in $|x| < R$. Since $\omega_N(x, hD_x;h)$: $L^2 \to L^2_N$ is bounded uniformly in h, *b)* follows from Lemma 6.1 after a computation similar to that in the proof of Lemma 6.6. For the proof of *c)*, exchanging the order of integrations, we calculate;

$$W_{2N} = h^{N+2}\exp(ih^{-1}(T+1)z)R(z;H_1(h))\int_0^{\infty}\exp(ih^{-1}sz)R_N(s;h)ds.$$

Hence, *c)* follows from Lemmas 5.1 and 6.1. □

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