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The finitary isomorphism problem for one-dimensional Gibbs systems of molecular type

by

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ABSTRACT. — For one-dimensional mixing Gibbs systems of molecular type it is proved that two such systems are finitarily isomorphic if and only if they have equal entropy. Using the notion of approximately finite groups of non-singular transformations of a measure space, a necessary condition is given for these finitary isomorphisms to have finite expected code-length.

RÉSUMÉ. — On démontre que deux systèmes de Gibbs moléculaires unidimensionnels mélangeants sont presque partout homéomorphes si, et seulement si, leurs entropies sont égales. En utilisant la notion de groupe de transformations non singulières approximativement fini d'un espace mesurable, nous donnons une condition nécessaire et suffisante pour que l'isomorphisme ait une longueur de code finie.

1. INTRODUCTION

Markov chains are the best understood examples of measure-theoretic dynamical systems. Besides their importance as mathematical objects, Markov chains can be used as models in classical statistical mechanics (lattice gas Ising model with finite range interactions).

The configuration space Ω of a Markov chain is the set of all two-sided infinite sequence of elements of a finite set Ω_0

$$\Omega = \prod_{i \in \mathbb{Z}} \Omega_0 .$$

In the lattice gas Ising model we have $\Omega_0 = \{ 0, 1 \}$, where 0 represents the absence of a particle and 1 represents the presence of a particle. The elements $\omega \in \Omega$ can be interpreted as arrangements of indistinguishable point particles on the lattice $\mathbb{Z}$.

From the point of view of classical statistical mechanics there is an interest to have models for systems of extended particles, e. g. hard rods or particles with hard cores.

For the description of such situations configuration spaces of molecular type, that are introduced in [*9*] and [*18*], can be used. The configurations in these configuration spaces are, roughly speaking, sequences of extended particles and empty places on the lattice $\mathbb{Z}$.

On configuration spaces of molecular type Markov measures μ can be defined by imposing a Markov property on the conditional probabilities of μ. These Markov measures are, by a result of the autor in [*17*], Gibbs measures for nearest neighbour potentials. These potentials describe the interactions between particles, that are close to each other.

All these notations are introduced in the sections 2 and 3.

Both the configuration space Ω and the Gibbs measure μ are supposed to be invariant with respect to the shift operation τ. Thus we have a measure-theoretic dynamical system

$$(\Omega, \mu, \tau)$$

and it arises the isomorphism problem. In section 4 an answer is given to this problem by the following theorem.

THEOREM. — Two topologically mixing Gibbs systems (Ω_1, μ_1, τ) and (Ω_2, μ_2, τ) with equal measure-theoretic entropy are finitarily isomorphic, i. e. there is an invertible measure-preserving map $I : \Omega_1 \to \Omega_2$ that commutes with the shift and there are null-sets $\tilde{\Omega}_i \subset \Omega_i$, $i = 1, 2$ such that the restriction of I to $\Omega_1 \backslash \tilde{\Omega}_1$ and of I^{-1} to $\Omega_2 \backslash \tilde{\Omega}_2$ are continuous.

By this result the entropy is a complete isomorphism invariant for such systems. A similar result was proved by Keane and Smorodinsky in [*5*] for Markov chains.

The fifth section of this paper is devoted to the examination of the finitary isomorphisms between Gibbs systems. There is given a necessary condition for the finitary isomorphism to have finite expected code length.

2. CONFIGURATION SPACES OF MOLECULAR TYPE

The systems under consideration are one-dimensional, i. e. they are defined on the lattice $\mathbb{Z}$. On $\mathbb{Z}$ the group $\Theta = \{ \Theta_s \mid s \in \mathbb{Z} \}$ of translations is defined by

$$\Theta_s t = t - s \qquad (t \in \mathbb{Z})$$
$$\Theta_s \mathrm{V} = \{ t \in \mathbb{Z} \mid t + s \in \mathrm{V} \} \qquad (\mathrm{V} \in \mathscr{P}_f(\mathbb{Z}))$$
$$\Theta_s \mathscr{V} = \{ \mathrm{W} \in \mathscr{P}_f(\mathbb{Z}) \mid \exists \mathrm{V} \in \mathscr{V} \text{ such that } \mathrm{W} = \Theta_s \mathrm{V} \} \qquad (\mathscr{V} \subset \mathscr{P}_f(\mathbb{Z})),$$

where $\mathscr{P}_f(\mathbb{Z})$ denotes the set of all nonempty finite subsets of $\mathbb{Z}$. We construct now the configuration space of the system. The following are supposed to be given:

$$\mathscr{G}_0 = \{ \mathrm{G}_1, \ldots, \mathrm{G}_n \mid \mathrm{G}_i = \{ 0, 1, \ldots, j_i - 1 \} \subset \mathbb{Z} \}$$

and

$$\Omega_\mathrm{G} \text{ a finite set for each } \mathrm{G} \in \mathscr{G}_0 .$$

The set $\mathscr{G}$ is generated by shifting $\mathscr{G}_0$ over the whole lattice, i. e.

$$\mathscr{G} = \bigcup_{s \in \mathbb{Z}} \Theta_s \mathscr{G}_0 .$$

For each $\mathrm{G} \in \mathscr{G}$ we set

$$\Omega_\mathrm{G} = \Omega_{\mathrm{G}_0}, \qquad \text{if} \qquad \mathrm{G} = \Theta_s \mathrm{G}_0 \quad \text{for} \quad \mathrm{G}_0 \in \mathscr{G}_0 \text{ and some } s \in \mathbb{Z}$$

and

$$\Omega^*_\mathrm{G} = \Omega_\mathrm{G} \cup \{ 0 \}, \qquad \text{where 0 is a fixed symbol}.$$

The element $0 \in \Omega^*_\mathrm{G}$ will represent the absence of a particle of the shape G.

For each $s \in \mathbb{Z}$ we introduce the notations

$$\mathscr{V}(s) = \{ \mathrm{G} \in \mathscr{G} \mid s \in \mathrm{G} \} \tag{1}$$

and

$$\overline{\Omega}_{\mathscr{V}(s)} = \left\{ (0, \ldots, 0, \omega_\mathrm{G}, 0, \ldots, 0) \in \prod_{\mathrm{G} \in \mathscr{V}(s)} \Omega^*_\mathrm{G} \mid \omega_\mathrm{G} \in \Omega_\mathrm{G} \right\}.$$

The configuration space of the system is defined by

$$\Omega = \left\{ \omega \in \prod_{\mathrm{G} \in \mathscr{G}} \Omega^*_\mathrm{G} \mid \text{for all } s \in \mathbb{Z} \ \omega \mid_{\mathscr{V}(s)} \in \overline{\Omega}_{\mathscr{V}(s)} \text{ holds} \right\}. \tag{2}$$

Let $\mathscr{P}_f(\mathscr{G})$ be the set of all nonempty finite subsets of $\mathscr{G}$. Then we use for $\mathscr{W} \in \mathscr{P}_f(\mathscr{G})$ the notations

$$\Omega_\mathscr{W} = \left\{ \omega \in \prod_{\mathrm{G} \in \mathscr{W}} \Omega^*_\mathrm{G} \mid \text{ if } \mathscr{V}(s) \subset \mathscr{W} \text{ then } \omega \mid_{\mathscr{V}(s)} \in \overline{\Omega}_{\mathscr{V}(s)} \right\}$$

and

$$X_{\mathscr{W}} : \Omega \rightarrow \Omega_{\mathscr{W}}, \text{ the natural projection operator}.$$

Note that in general $X_{\mathscr{W}}\Omega \neq \Omega_{\mathscr{W}}$!

The space Ω consists of configurations ω having the property: In any point $s \in \mathbb{Z}$ there is at most one $G \in \mathscr{V}(s)$ with $\omega|_G \neq 0$. So it will be convenient to interpret the configurations as arrangements of nonoverlapping extended particles on the lattice.

One has to pay attention to the following interesting case: Suppose $0 \in \Omega_G$ for all $G \in \mathscr{G}$, i. e. $\Omega_G^* = \Omega_G$. In this case it is possible to have configurations ω such that for some points $s \in \mathbb{Z}$ there is $\omega|_G = 0$ for all $G \in \mathscr{V}(s)$. Such points are called empty for ω.

But without loss of generality one can assume $0 \notin \Omega_G$ for all $G \in \mathscr{G}$. To realize the above situation one has to add to $\mathscr{G}_0$ the set $\{0\}$ and to assume $\Omega_{\{0\}} = \{\alpha\}$. If $\{0\}$ was already an element of $\mathscr{G}_0$ one only has to add to the original $\Omega_{\{0\}}$ the element α. Empty points are then represented by one-point α-particles.

The classical situation of a configuration space of the lattice gas Ising model is realized by the following choice of the parameters: $\mathscr{G}_0 = \{\{0\}\}$, $\Omega_{\{0\}} = \{\alpha, 1\}$.

The configuration space Ω will be equipped with the topology induced on $\Omega \subset \prod_{G \in \mathscr{G}} \Omega_G^*$ by the product topology on $\prod_{G \in \mathscr{G}} \Omega_G^*$. On the finite sets Ω_G^* the discrete topology is assumed. With this topology Ω is a compact space. On Ω the groupe $\tau = \{\tau_s \mid s \in \mathbb{Z}\}$ of homeomorphisms τ_s is defined by

$$\tau_s\omega|_G = \omega|_{\Theta_{-s}G} \qquad (\omega \in \Omega,\ s \in \mathbb{Z},\ G \in \mathscr{G}).$$

Michel and Schwenzfeger studied the pairs (Ω, τ) as topological dynamical systems. Michel proved in [*9*] that in the special case mentioned above, τ is a topologically mixing transformation group on Ω. In [*16*] Schwenzfeger proved: If $0 \notin \Omega_G$ for $G \in \mathscr{G}$, then τ is a topologically mixing transformation group on Ω, if and only if the greatest common divisor of the natural numbers j_i, that uniquely define the set $\mathscr{G}_0$, is equal to one. Furthermore he showed there that two systems (Ω', τ) and (Ω'', τ) are topologically conjugated, if and only if the sets $\mathscr{G}_0'$ and $\mathscr{G}_0''$ are identical and card Ω_G' = card Ω_G'' ıor all $G \in \mathscr{G}_0'$.

In the present paper we add to the configuration space Ω a τ-invariant Gibbs measure μ and study measure-theoretic isomorphisms between such measure-theoretic dynamical systems (Ω, μ, τ).

3. GIBBS MEASURES

The most instructive and most examinated class of examples in the isomorphism theory of abstract measure-theoretic dynamical systems is the class of Markov chains (see for example [3] [12]). From the papers of Averintzev [1] [2] we know that there is an equivalent description of the Markov property of measures with the help of nearest neighbour interactions, that allows an effortless characterization of measures on the configuration space, having the Markov property. In fact Averintzev showed this equivalence for lattices $\mathbb{Z}^d$ with $d \geqslant 1$. In [17] such an equivalent description by interactions was obtained for configuration spaces of the structure described above. In what follows, the interactions will have the same importance as the transition matrices have in the case of the classical theory of Markov chains.

An interaction U is a map $U: \mathscr{P}_f(\mathscr{G}) \times \Omega \to \mathbb{R}$ with the property: For fixed $\mathscr{V} \in \mathscr{P}_f(\mathscr{G})$ the value $U(\mathscr{V}, \omega)$ depends only on $\omega|_{\mathscr{V}}$. An interaction is translation invariant, if

$$U(\mathscr{V}, \omega) = U(\Theta_s\mathscr{V}, \tau_s\omega) \qquad (\mathscr{V} \in \mathscr{P}_f(\mathscr{G}), \omega \in \Omega, s \in \mathbb{Z}).$$

It is called to be of finite range, if there is a set $\mathscr{D} \in \mathscr{P}_f(\mathscr{G})$ with $\mathscr{D} \cap \mathscr{G}_0 \neq \phi$ such that $U(\mathscr{V}, .) \not\equiv 0$ implies the existence of an $s \in \mathbb{Z}$ with $\Theta_s\mathscr{V} \subset \mathscr{D}$.

For $\mathscr{V} \in \mathscr{P}_f(\mathscr{G})$ and $\omega \in \Omega_{\mathscr{V}}$ we denote

$$\Lambda_{\omega,\mathscr{V}} = \{ \eta \in \Omega \mid \exists \xi \in \Omega \text{ with } \xi|_{\mathscr{V}} = \omega \text{ and } \xi|_{\mathscr{G}\setminus\mathscr{V}} = \eta|_{\mathscr{G}\setminus\mathscr{V}} \}. \tag{3}$$

A measure is called a Gibbs measure for the interaction U, if for all $\mathscr{V} \in \mathscr{P}_f(\mathscr{G})$, $\omega \in \Omega_{\mathscr{V}}$, $\eta \in \Omega$ the conditional probability that on $\mathscr{V}$ the configuration is equal to ω under the condition that on $\mathscr{G}\setminus\mathscr{V}$ the configuration is equal to $\eta|_{\mathscr{G}\setminus\mathscr{V}}$ is

$$\mu_{\mathscr{V}}(\omega \mid \eta) = Z^{-1}_{\mathscr{V},\eta}(U) \exp\Big(- \sum_{\substack{\mathscr{Y} \in \mathscr{P}_f(\mathscr{G}) \\ \mathscr{Y} \cap \mathscr{V} \neq \phi}} U(\mathscr{Y}, \omega\eta|_{\mathscr{G}\setminus\mathscr{V}}) \Big) \qquad (\omega \in X_{\mathscr{V}}\Omega, \eta \in \Lambda_{\omega,\mathscr{V}}) \tag{4}$$

$\mu_{\mathscr{V}}(\omega \mid \eta) = 0$ otherwise.

In the equation (4) the factor $Z_{\mathscr{V},\eta}(U)$ is a normalizing constant, making $\mu_{\mathscr{V}}(. \mid \eta)$ a probability measure on $\Omega_{\mathscr{V}}$. The family

$$\{ \mu_{\mathscr{V}}(\omega \mid \eta) \mid \mathscr{V} \in \mathscr{P}_f(\mathscr{G}), \omega \in \Omega_{\mathscr{V}}, \eta \in \Omega \}$$

is called the family of Gibbs distributions in the finite volumes for the interaction U. For finite range interactions the Gibbs distributions in the finite volumes are well defined. Moreover it is known (see [13] [14]) that for each finite range interaction there exists a Gibbs measure.

For the formulation of a Markov property it is necessary to have a neighbourhood relation. Thus a set $\mathfrak{N} \subset \{ \{ G, K \} \mid G, K \in \mathscr{G} \}$ is supposed to be given, that has the properties:

i) If $\{ G, K \} \in \mathfrak{N}$, then $\{ \Theta_s G, \Theta_s K \} \in \mathfrak{N}$ for each $s \in \mathbb{Z}$.

ii) If $\{ G, K \} \in \mathfrak{N}$, then $G \cap K = \phi$.

iii) For each pair $\{ G, K \} \in \mathfrak{N}$ $\min\{ |s-t| \mid s \in G, t \in K \} \leqslant d_0$ holds for some fixed $d_0 \in \mathbb{N}$.

If $\{ G, K \} \in \mathfrak{N}$, then G and K are called neighbours. For $\mathcal{V} \in \mathcal{P}_f(\mathcal{G})$ the set $\partial\mathcal{V} = \{ G \in \mathcal{G}\backslash\mathcal{V} \mid \exists K \in \mathcal{V} \text{ with } \{ G, K \} \in \mathfrak{N} \}$ is called the boundary of $\mathcal{V}$.

In the classical case of the Ising model with nearest neighbour interaction one has $\mathfrak{N} = \{ (s, t) \in \mathbb{Z} \times \mathbb{Z} \mid |s - t| = 1 \}$. A translation invariant interaction U is called a nearest neighbour interaction (n. n. interaction), if

$$j) \qquad U(\mathcal{V}, .) \equiv 0 \quad \text{for} \quad \mathcal{V} \notin \mathfrak{N} \cup \{ \{ G \} \mid G \in \mathcal{G} \} \tag{5}$$

$$jj) \qquad U(\mathcal{V}, \omega) = 0 \quad \text{if} \quad \mathcal{V} \neq \{ G \in \mathcal{V} \mid \omega|_G \neq 0 \}. \tag{6}$$

Each n. n. interaction is obviously of finite range. The condition *jj*) is introduced, because it ensures the uniqueness in the theorem 1 below. It is easily seen that for n. n. interactions the Gibbs distributions in the finite volumes have the following Markov property:

For all $\mathcal{V} \in \mathcal{P}_f(\mathcal{G})$, $\omega \in \Omega_{\mathcal{V}}$ and μ-a. a. $\eta \in \Lambda_{\omega,\mathcal{V}}$

$$\mu_{\mathcal{V}}(\omega \mid \eta) = \mu_{\mathcal{V}}(\omega \mid \xi) \tag{7}$$

holds for μ-a. a. $\xi \in \Lambda_{\omega,\mathcal{V}}$ with $\eta|_{\partial\mathcal{V}} = \xi|_{\partial\mathcal{V}}$, i. e. the conditional probability of finding in $\mathcal{V}$ the configuration ω under the condition that on $\mathcal{G}\backslash\mathcal{V}$ there is the configuration $\eta|_{\mathcal{G}\backslash\mathcal{V}}$ depends only on the configuration on the boundary of $\mathcal{V}$. The importance of Gibbs measures for n. n. interactions, from the mathematical point of view, lies in the following theorem that is proved in [*17*] for the class of configuration spaces under consideration.

THEOREM 1. — For each measure μ such that

$$\mu_{\mathcal{V}}(\omega) > 0 \qquad (\mathcal{V} \in \mathcal{P}_f(\mathcal{G}), \omega \in X_{\mathcal{V}}\Omega)$$

and such that (7) is fulfilled, there exists a unique n. n. interaction U such that μ is a Gibbs measure for U.

At the end of this section we want to give an example of a system of two kinds of hard rods. Each hard rod covers exactly two points of the lattice, and we allow empty points between hard rods. For this situation we choose:

$$\mathcal{G}_0 = \{ G_0 \}, \qquad \text{with} \qquad G_0 = \{ 0, 1 \} \subset \mathbb{Z}$$
$$\Omega_{G_0} = \{ 0, A, B \},$$

where 0 is the fixed symbol in the definition of the configuration space. The symbols A and B describe the two different kinds of hard rods. Let us assume that the particles have a chemical potential and that hard rods of the same kind attract each other, if they are close together, and that hard rods of different kind repel each other. We describe this situation by introducing the neighbourhood relation $\mathfrak{N}$ and the interaction U in the following way:

$$\{G, K\} \in \mathfrak{N} \text{ if and only if either } G = \{i, i+1\}, \quad K = \{i+2, i+3\}$$
$$\text{or } K = \{i, i+1\}, \quad G = \{i+2, i+3\} \text{ for } i \in \mathbb{Z}.$$

$$U(\{G\}, \omega) = \begin{cases} \mu_A, & \text{if } \omega|_{\{G\}} = A \\ \mu_B, & \text{if } \omega|_{\{G\}} = B \\ 0, & \text{if } \omega|_{\{G\}} = 0 \end{cases} \quad (G \in \mathcal{G}, \omega \in \Omega)$$

$$U(\{G, K\}, \omega) = \begin{cases} a \leqslant 0, & \text{if } \omega|_{\{G\}} = \omega|_{\{K\}} = A \\ b \leqslant 0, & \text{if } \omega|_{\{G\}} = \omega|_{\{K\}} = B \\ c > 0, & \text{if } 0 \neq \omega|_{\{G\}} \neq \omega|_{\{K\}} \neq 0 \\ 0, & \text{if either } \omega|_{\{G\}} = 0 \\ & \quad \text{or } \omega|_{\{K\}} = 0 \end{cases} \quad (\{G, K\} \in \mathfrak{N}, \omega \in \Omega)$$

$U(\mathcal{V}, .) \equiv 0$ in all other cases of $\mathcal{V} \in \mathcal{P}_f(\mathcal{G})$.

The interaction U is obviously a nearest neighbour interaction. It is not hard to see that for fixed $\mathcal{V} \in \mathcal{P}_f(\mathcal{G})$, $\omega \in \Omega_{\mathcal{V}}$ the Gibbs distribution

$$\mu_{\mathcal{V}}(\omega \mid \eta) = Z_{\mathcal{V},\eta}^{-1}(U) \exp\Bigg(- \sum_{\{G\} \in \mathcal{V}} U(\{G\}, \omega\eta|_{\mathcal{G}\setminus\mathcal{V}}) - \frac{1}{2} \sum_{\substack{\{G,K\} \in \mathfrak{N} \\ \{G,K\} \subset \mathcal{V}}} U(\{G, K\}, \omega\eta|_{\mathcal{G}\setminus\mathcal{V}}) - \sum_{\substack{\{G,K\} \in \mathfrak{N} \\ \{G\} \in \mathcal{V}, \{K\} \in \mathcal{G}\setminus\mathcal{V}}} U(\{G, K\}, \omega\eta|_{\mathcal{G}\setminus\mathcal{V}})\Bigg)$$

depends for $\eta \in \Lambda_{\omega,\mathcal{V}}$ only on $\eta|_{\partial\mathcal{V}}$.

4. THE FINITARY ISOMORPHISM PROBLEM

We are going now to look at (Ω, μ, τ) as a measure-theoretic dynamical system. We call it a Gibbs system, if

Ω is the configuration space determined by $\mathcal{G}_0$ and $(\Omega_G)_{G \in \mathcal{G}_0}$,

μ is a τ-invariant Gibbs measure for the n. n. interaction U on Ω with respect to the neighbourhood structure $\mathfrak{N}$ and

τ is the shift transformation group on Ω.

A Gibbs system (Ω, μ, τ) is said to be topologically mixing, if τ is a topologically mixing transformation group on Ω.

It is well known that a necessary condition for two abstract measure-theoretic dynamical systems to be measure-theoretically isomorphic,

i. e. for the existence of an invertible measure-preserving map that commutes with the transformation group almost everywhere, is the equality of the entropies of the measures. In [4] Friedman and Ornstein proved that in the case of finite state mixing Markov chains this condition is sufficient, too. In [5] is proved that two finite state mixing Markov chains (Ω_1, μ_1, τ) and (Ω_2, μ_2, τ) with finite memory and equal entropy are finitarily isomorphic, i. e. there exists an isomorphism $I: \Omega_1 \to \Omega_2$ and there exist null-sets $\tilde{\Omega}_i \subset \Omega_i$, $i = 1, 2$ such that the restriction of I to $\Omega_1 \backslash \tilde{\Omega}_1$ and the restriction of I^{-1} to $\Omega_2 \backslash \tilde{\Omega}_2$ are continuous. The notion of finitary isomorphism can be applied to Gibbs systems in the same sense. Then, if I is a finitary isomorphism, it maps finite dimensional cylinder sets onto finite dimensional cylinder sets with the exception of nullsets. We are now in the situation to formulate the main theorem of this section.

THEOREM 2. — Let Ω_1, Ω_2 be two configuration spaces such that the shift transformation group τ acts on both of them as a topologically mixing transformation group. Let furthermore for each $i = 1, 2$ μ_i be a τ-invariant Gibbs measure on Ω_i with respect to a nearest neighbour interaction U_i. Then, if the measures μ_1 and μ_2 have equal entropy, the measure-theoretic dynamical systems (Ω_1, μ_1, τ) and (Ω_2, μ_2, τ) are finitarily isomorphic.

The proof proceeds in the following way: We associate at first to each pair (Ω_i, τ), $i = 1, 2$ a suitable subshift of finite type (Ω_i', τ'). The map $I_i': \Omega_i \to \Omega_i'$ generates a measure $I_i'\mu_i$ on Ω_i'. Moreover I_i' is chosen to be a finitary isomorphism between (Ω_i, μ_i, τ) and $(\Omega_i', I_i'\mu_i, \tau')$. Then we show that the system $(\Omega_i', I_i'\mu_i, \tau')$ is a finite state mixing Markov chain with finite memory. This, together with the theorem of Keane and Smorodinsky in [5], will complete the proof.

Proof. — Let (Ω, μ, τ) be a topologically mixing Gibbs system. Without loss of generality we assume $0 \notin \Omega_G$ for all $G \in \mathcal{G}$. To this Gibbs system we associate the subshift of finite type (Ω', τ'). The alphabet of Ω' is the set

$$\Omega_0' = \{ (y, G, \omega) \mid y \in G \in \mathcal{G}_0, \omega \in \Omega_G \}.$$

Let for $s \in \mathbb{Z}$

$$V(s) = \bigcup_{s \in G \in \mathcal{G}} G.$$

The system of blocks occuring in Ω is

$$\overline{\Omega}'_{V(s)} = \left\{ \xi \in \prod_{z \in V(s)} \Omega_0' \,\middle|\, \begin{array}{l} \xi|_z = (y, G, \omega) \quad \text{and} \quad z' \in \Theta_{z-y} G \\ \text{imply} \quad \xi|_{z'} = (\Theta_{z'-z} y, G, \omega) \end{array} \right\} \qquad (s \in \mathbb{Z}).$$

The set Ω' is then defined by

$$\Omega' = \left\{ \omega' \in \prod_{s \in \mathbb{Z}} \Omega'_0 \mid \text{for all } s \in \mathbb{Z}\ \omega' |_{V(s)} \in \overline{\Omega}'_{V(s)} \text{ holds} \right\}. \tag{8}$$

The elements $\omega' \in \Omega'$ are compositions of blocks of the form

$$((0, G_i, \omega), (1, G_i, \omega), \ldots, (j_i - 1, G_i, \omega)).$$

The notations Ω'_V and $X_V\Omega'$ for $V \subset \mathbb{Z}$ will be used in the same sense as described above for Ω.

Let now $(I_G)_{G \in \mathscr{G}}$ be a family of maps

$$I_G : \Omega'_G \to \Omega^*_G$$

with

$$I_G(\xi') = \begin{cases} \omega \in \Omega^*_G, & \text{if } \xi' = ((0, G_i, \omega), (1, G_i, \omega), \ldots, (j_i - 1, G_i, \omega)) \\ & \qquad \text{and} \quad G = \Theta_s G_i \quad \text{for some } s \in \mathbb{Z}, \\ 0 \in \Omega^*_G & \text{otherwise}. \end{cases}$$

It has the property: For each $s \in \mathbb{Z}$ and each $\xi' \in \Omega'_{V(s)}$

$$(I_G(\xi' |_G))_{G \in \mathscr{V}(s)} \in \Omega_{\mathscr{V}(s)}$$

holds.

Thus this family defines a continuous map

$$I : \Omega' \to \Omega$$

by

$$(I\xi') |_G = I_G(\xi' |_G) \qquad (G \in \mathscr{G}).$$

The map I is a bijection. To see this we construct a map $I' : \Omega \to \Omega'$ with

$$I \circ I' = id \qquad \text{and} \qquad I' \circ I = id. \tag{9}$$

We define a family $(I'_s)_{s \in \mathbb{Z}}$ of maps

$$I'_s : \Omega_{\mathscr{V}(s)} \to \Omega'_{\{s\}} = \Omega'_0.$$

Let for $\omega = (0, \ldots, 0, \omega_G, 0, \ldots, 0) \in \Omega_{\mathscr{V}(s)}$

$$I'_s(\omega) = (y, G_0, \omega_G)$$

with $G_0 = \Theta_z G$ for some $z \in \mathbb{Z}$ and $y = s + z$.

This family has the property: For each $s \in \mathbb{Z}$ and each $\omega \in \Omega_{\cup\{\mathscr{V}(t) | t \in V(s)\}}$

$$(I'_t(\omega |_{\mathscr{V}(t)}))_{t \in V(s)} \in \Omega'_{V(s)}$$

holds. Thus it defines a continuous map

$$I' : \Omega \to \Omega'$$

by

$$(I'\omega) |_s = I'_s(\omega |_{\mathscr{V}(s)}) \qquad (s \in \mathbb{Z}).$$

It is not hard to see that (9) is fulfilled.

For $\omega' \in \Omega'$ let $\Lambda'_{\omega'} \subset \Omega'$ consist of all those η' such that $\omega'|_s = \eta'|_s$ except for finitely many $s \in \mathbb{Z}$, and define similarly $\Lambda_\omega \subset \Omega$ for $\omega \in \Omega$. Then, I maps $\Lambda'_{\omega'}$ into $\Lambda_{I\omega'}$ by construction. Since I′ has a similar property and since the equations (9) hold, the restriction of I to $\Lambda'_{\omega'}$ is a bijection to $\Lambda_{I\omega'}$ for all $\omega' \in \Omega'$.

Hence, I is endowed with all properties to be an isomorphism between two configuration spaces in the sense defined by Ruelle in [*14*, p. 25], and so is I′.

On Ω' one can introduce the notions of interactions, Gibbs distributions in the finite volumes and Gibbs measures in the same way as in the case of Ω. In the formalism $\mathcal{G}$ has to be replaced by $\mathbb{Z}$. The shift on Ω' is denoted by τ'. Obviously $I \circ \tau' = \tau \circ I$ holds everywhere on Ω'.

Following Ruelle [*14*, p. 26], the isomorphism $I: \Omega' \to \Omega$ defines for each interaction U on Ω an interaction I*U on Ω' by

$$I^*U(V, \omega') = \sum_{\mathcal{V} : \cup_{G \in \mathcal{V}} G = V} U(\mathcal{V}, I\omega') \qquad (\omega' \in \Omega', V \in \mathcal{P}_f(\mathbb{Z})).$$

Then, if U is a n. n. interaction, I*U is of finite range. We quote now from [*14*, p. 27, 29] the following proposition.

PROPOSITION. — Let ν' be a Gibbs measure on Ω' for the interaction I*U. Then $I\nu'$ is a Gibbs measure on Ω for U. Furthermore, I is a bijection of the set of Gibbs measures for I*U on Ω' to the set of Gibbs measures for U on Ω.

Using this proposition, we conclude that I is a measure-theoretic finitary isomorphism

$$I: (\Omega', I^{-1}\mu, \tau') \to (\Omega, \mu, \tau),$$

where $I^{-1}\mu$ is a τ'-invariant Gibbs measure for I*U. Since U is assumed to be a n. n. interaction for Ω equipped with the neighbourhood structure $\mathfrak{N}$, I*U is of the form

$$\begin{aligned} I^*U(G, \omega') &= U(\{G\}, I\omega') && (G \in \mathcal{G}, \omega' \in \Omega'), \\ I^*U((G \cup K), \omega') &= U(\{G, K\}, I\omega') && (\{G, K\} \in \mathfrak{N}, \omega' \in \Omega'), \\ I^*U(V, .) &\equiv 0 && \text{in all other cases of } V \in \mathcal{P}_f(\mathbb{Z}). \end{aligned}$$

It follows immediately that $(\Omega', I^{-1}\mu, \tau')$ is a finite state Markov chain with a memory smaller or equal to the number

$$\max \{ |s - t| \mid s \in G, t \in K, \{G, K\} \in \mathfrak{N} \},$$

that is finite by the definition of $\mathfrak{N}$. We are going now to show that this Markov chain is mixing.

By the assumption of the theorem (Ω', τ') is a topologically mixing topological dynamical system. The measure-theoretic mixing property follows from a proposition, that is proved not only for interactions of finite range.

PROPOSITION [*14*]. — Let U′ be an interaction of finite range for the topologically mixing subshift of finite type Ω' and ν' the unique Gibbs measure. Then, if Ω' consists of more than one point, the measure-theoretic dynamical system (Ω', ν', τ') is isomorphic to a Bernoulli shift and thus measure-theoretically mixing.

As a consequence we get the following diagram, that completes the proof of the theorem.

$$(\Omega_1, \mu_1, \tau) \xrightarrow{I_1^{-1}=I_1} (\Omega_1', I_1^{-1}\mu_1, \tau') \overset{I_{KS}}{\to} (\Omega_2', I_2^{-1}\mu_2, \tau') \overset{I_2}{\to} (\Omega_2, \mu_2, \tau).$$

Here I_{KS} is the finitary isomorphism, existing by the theorem of Keane and Smorodinsky [*5*].

5. FINITE EXPECTED CODE LENGTH

The remarkable result that mixing Markov chains of the same entropy are finitarily isomorphic induced several authors [*8*] [*10*] [*11*] [*15*] to look for isomorphisms between such Markov chains that reflect the sequential structure of the underlying configuration space, especially to look for isomorphisms with finite expected code length. Since we know from Theorem 2 that two topologically mixing Gibbs systems with equal entropy are finitarily isomorphic, we will do this for isomorphisms of topologically mixing Gibbs systems. For this we introduce the notation of code length.

In this section we assume $0 \notin \Omega_G$ for all $G \in \mathcal{G}$, too. Then, if $\omega \in \Omega$ is a configuration, it determines uniquely a set $\mathcal{W}(\omega) \subset \mathcal{G}$ by

$$G \in \mathcal{W}(\omega) \Leftrightarrow \omega|_G \neq 0.$$

The elements of $\mathcal{W}(\omega)$ are by the definition of Ω pairwise disjoint. It is intuitively clear, what it means to say: « G lies on the left or on the right of K » for $G, K \in \mathcal{W}(\omega)$. For $\omega \in \Omega$ and $n \in \mathbb{N}$ we define $t_r(\omega, n)$ to be the smallest integer such that there are on the right of the origin and on the left of $t_r(\omega, n)$ n sets $G \in \mathcal{W}(\omega)$. The integer $t_1(\omega, n)$ is the greatest one such that there are on the left of the origin and on the right of $t_1(\omega, n)$ n sets $G \in \mathcal{W}(\omega)$. Furthermore we denote

$$\begin{aligned} W_r(\omega, n) &= \{ s \in \mathbb{Z} \mid 0 \leqslant s \leqslant t_r(\omega, n) \} \\ W_1(\omega, n) &= \{ s \in \mathbb{Z} \mid t_1(\omega, n) \leqslant s \leqslant 0 \}. \end{aligned} \tag{10}$$

Let I be the finitary isomorphism between the two topologically mixing Gibbs systems (Ω_i, μ_i, τ), $i = 1, 2$, of equal entropy and $\hat{\Omega}_i \subset \Omega_i$ the two sets of measure one, where I respectively I^{-1} are continuous. Then there exist obviously two maps

$$m: \hat{\Omega}_1 \to \mathbb{N}$$
$$a: \hat{\Omega}_1 \to \mathbb{N}$$

such that $\omega, \omega' \in \hat{\Omega}_1$ with $\omega|_{\mathscr{V}(s)} = \omega'|_{\mathscr{V}(s)}$ for all $s \in W_r(\omega, a(\omega)) \cup W_1(\omega, m(\omega))$ implies

$$I\omega|_{\mathscr{V}(0)} = I\omega'|_{\mathscr{V}(0)}. \tag{11}$$

The map m is called the memory of I; a the anticipation of I. The isomorphism I is called to have finite expected code length, if

$$\int (a + m) d\mu_1 < \infty. \tag{12}$$

It is the aim of this section to give a necessary condition for the isomorphism to have finite expected code length. Our approach to this problem and the result are near to that of Krieger in [8] in the case of mixing Markov chains. It is based on some notations concerning countable groups of non-singular transformations of a measure space, that are developed in [6] [7]. We quote them here now.

Let (X, ν) be a Lebesque measure space and $\mathscr{F}$ a countable group of non-singular transformations of (X, ν).

It is called approximately finite, if there exists an increasing sequence $(\mathscr{F}_n)$ of finite groups $\mathscr{F}_n$ of non-singular transformations of (X, ν) such that

$$\mathscr{F}x = \bigcup_{n \in \mathbb{N}} \mathscr{F}_n x \qquad \text{for} \quad \nu\text{-a. a.} \quad x \in X.$$

The group $[\mathscr{F}]$ of all non-singular transformations T of (X, ν) such that

$$Tx \in \mathscr{F}x \qquad \text{for} \quad \nu\text{-a. a.} \quad x \in X$$

is called the full group.

We say that the group $\mathscr{F}$ contains the measure ν, if the group

$$\{ T \in [\mathscr{F}] \mid T\nu = \nu \}$$

is ergodic and if for all $T \in \mathscr{F}$ there is a countable set $\mathscr{R}_T \subset (0, \infty)$ such that

$$\begin{aligned} &\nu\left\{ x \in X \,\middle|\, \frac{dT^{-1}\nu}{d\nu}(x) = w \right\} > 0 \qquad (w \in \mathscr{R}_T) \\ &\nu\left\{ x \in X \,\middle|\, \frac{dT^{-1}\nu}{d\nu}(x) \notin \mathscr{R}_T \right\} = 0. \end{aligned} \tag{13}$$

The set $\Delta(\mathscr{F}, \nu) = \bigcup_{T \in \mathscr{F}} \mathscr{R}_T$ is then a subgroup of $(0, \infty)$.

We construct now for each Gibbs system (Ω, μ, τ) an approximately finite group $\mathscr{F}_\mu$ that contains the measure μ. For all pairs a, $b \in \mathbb{Z}$ we use the notations

$$\mathscr{V}(a, b) = \{ G \in \mathscr{G} \mid G \cap \{ s \in \mathbb{Z} \mid a \leqslant s \leqslant b \} \neq \phi \}$$

and

$\mathbb{T}(a, b)$ the group of all permutations π of elements

$$\omega \in X_{\mathscr{V}(a,b)}\Omega \quad \text{with} \tag{14}$$

$$\pi\omega \mid_{\mathscr{V}(a)} = \omega \mid_{\mathscr{V}(a)} \qquad \pi\omega \mid_{\mathscr{V}(b)} = \omega \mid_{\mathscr{V}(b)} .$$

Each $\pi \in \mathbb{T}(a, b)$ defines a non-singular transformation T_π of Ω by

$$T_\pi\omega = \eta \qquad \text{with} \qquad \begin{cases} \eta \mid_{\mathscr{V}(a,b)} = \pi(\omega \mid_{\mathscr{V}(a,b)}) \\ \eta \mid_{\mathscr{G}\setminus\mathscr{V}(a,b)} = \omega \mid_{\mathscr{G}\setminus\mathscr{V}(a,b)} . \end{cases} \tag{15}$$

The condition (14) ensures that T_π is well defined by (15).

THEOREM 3. — Let (Ω, μ, τ) be a topologically mixing Gibbs system. Then the approximately finite group

$$\mathscr{F}_\mu = \bigcup_{a \in \mathbb{N}} \{ T_\pi \mid \pi \in \mathbb{T}(-a, a) \}$$

contains the Gibbs measure μ.

Proof. — Following the ideas of Krieger in [*8*], we prove at first that the group

$$\{ T \in \mathscr{F}_\mu \mid T\mu = \mu \}$$

acts ergodically on (Ω, μ, τ).

Let $\Phi, \Phi' \subset \Omega$ be two sets of positive μ-measure. Let $a \in \mathbb{N}$ and $\{ \delta \}$, $\{ \delta' \} \subset X_{\mathscr{V}(-a,a)}\Omega$ be such that with the notations

$$\Delta = X^{-1}_{\mathscr{V}(-a,a)} \{ \delta \} \qquad \Delta' = X^{-1}_{\mathscr{V}(-a,a)} \{ \delta' \}$$

$$\begin{gathered} \mu(\Delta \triangle \Phi) < \frac{1}{2}(1 - 2^{-1/5})\mu(\Phi) \\ \mu(\Delta' \triangle \Phi') < \frac{1}{2}(1 - 2^{-1/5})\mu(\Phi') \end{gathered} \tag{16}$$

holds.

It follows from (16) that

$$\begin{gathered} \mu(\Delta \cap \Phi) > 2^{-1/5}\mu(\Delta) \\ \mu(\Delta' \cap \Phi') > 2^{-1/5}(\Delta') \end{gathered} \tag{17}$$

holds.

Fix now a $d \in \mathbb{N}$ such that

$$\mathscr{V}(-d, d) \supset \bigcup_{G \in \mathscr{G}_0} \{ K \in \mathscr{G} \mid \{ G, K \} \in \mathfrak{N} \} . \tag{18}$$

This choice of d has the consequence that two elements $G, K \in \mathscr{G} \backslash \mathscr{V}(-d, d)$, that are situated on different sides of the origin, cannot be neighbours, i. e. $U(\{G, K\}, .) \equiv 0$. Choose a $\sigma \in X_{\mathscr{V}(-d,d)}\Omega$ and set

$$\Sigma = X^{-1}_{\mathscr{V}(-d,d)}(\sigma) \qquad \text{and} \qquad p = \mu(\Sigma) > 0\,.$$

Then the measure-theoretic mixing property of the Gibbs system permits us to choose a $b > a + d$ such that with the notations

$$\Gamma = \tau_b\Sigma \cap \Delta \cap \tau_{-b}\Sigma \qquad \text{and} \qquad \Gamma' = \tau_b\Sigma \cap \Delta' \cap \tau_{-b}\Sigma$$

the inequalities

$$\mu(\Gamma) < 2^{1/5}p^2\mu(\Delta) \qquad \text{and} \qquad \mu(\Gamma') < 2^{1/5}p^2\mu(\Delta') \tag{19}$$

and

$$\begin{aligned} \mu(\Gamma \cap \Phi) &> 2^{-1/5}p^2\mu(\Delta \cap \Phi) \\ \mu(\Gamma' \cap \Phi') &> 2^{-1/5}p^2\mu(\Delta' \cap \Phi') \end{aligned} \tag{20}$$

hold. Utilizing once more the mixing property, one can choose a natural number $c > 2b + 2d$ such that with the notations

$$\Lambda = \Gamma \cap \tau_{-c}\Gamma' \qquad \text{and} \qquad \Lambda' = \Gamma' \cap \tau_{-c}\Gamma$$

the inequalities

$$\begin{aligned} \mu(\Lambda) &< 2^{1/5}\mu(\Gamma)\mu(\Gamma') \\ \mu(\Lambda') &< 2^{1/5}\mu(\Gamma')\mu(\Gamma) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \mu(\Lambda \cap \Phi) &> 2^{-1/5}\mu(\Gamma \cap \Phi)\mu(\Gamma') \\ \mu(\Lambda' \cap \Phi') &> 2^{-1/5}\mu(\Gamma' \cap \Phi')\mu(\Gamma) \end{aligned} \tag{22}$$

hold. Starting with the last inequalities and then utilizing subsequently (20), (17), (19) and (21) one gets

$$\mu(\Lambda \cap \Phi) > \frac{1}{2}\mu(\Lambda) \qquad \text{and} \qquad \mu(\Lambda' \cap \Phi') > \frac{1}{2}\mu(\Lambda')\,. \tag{23}$$

We select now a non-singular transformation $T_\pi \in \mathscr{F}_\mu$ that maps Λ onto Λ' and *vice versa*. Let us denote $\mathscr{V} = \mathscr{V}(-c-b-d,\ c+b+d)$. The corresponding to T_π permutation π that is in $\mathbb{T}(-c-b-d,\ c+b+d)$ is given for $\omega \in X_{\mathscr{V}}(\Lambda \cup \Lambda')$ by

$$\pi\omega\,|_G = \begin{cases} \omega\,|_{\Theta_c G} & \text{for } G \in \mathscr{V}(s) \text{ with } s \in \{-b-d, \dots, b+d\} \\ \omega\,|_{\Theta_{-c} G} & \text{for } G \in \mathscr{V}(s) \text{ with } s \in \{c-b-d, \dots, c+b+d\} \\ \omega\,|_G & \text{for } G \in \mathscr{V}(s) \text{ with } s \in \{-c-b-d, \dots, -b-d+1\} \\ & \qquad \cup \{b+d+1, \dots, c-b-d+1\}. \end{cases}$$

For $\omega \notin X_{\mathscr{V}}(\Lambda \cup \Lambda')$ it is given by $\pi\omega = \omega$.

Since for each $\omega \in \Lambda \cup \Lambda'$

$$\omega\,|_{\mathscr{V}(b+d)} = \omega\,|_{\mathscr{V}(c+b+d)},$$

π is indeed an element of $\mathbb{T}(-c-b-d, c+b+d)$. Furthermore it follows that

$$T_\pi^2 = id \qquad T_\pi\Lambda = \Lambda' \qquad T_\pi\Lambda' = \Lambda. \tag{24}$$

We show now that T_π preserves μ. By definition of μ as a Gibbs measure (see (4)) we have for $\omega \in X_{\mathscr{V}}(\Lambda \cup \Lambda')$

$$\begin{aligned} X_{\mathscr{V}}\mu(\omega) &= \int_\Omega \mu_{\mathscr{V}}(\omega \mid \eta)d\mu(\eta) = \int_{\Lambda_{\omega,\mathscr{V}}} \mu_{\mathscr{V}}(\omega \mid \eta)d\mu(\eta) \\ &= \sum_{\eta^* \in X_{\partial\mathscr{V}}\Lambda_{\omega,\mathscr{V}}} \Bigg[Z_{\mathscr{V},\eta}^{-1}(U)\exp\Bigg(-\sum_{G \in \mathscr{V}} U(\{G\}, \omega\eta\,|_{\mathscr{G}\setminus\mathscr{V}}) \\ &\quad - \sum_{\substack{\{G,K\}\in\mathfrak{N} \\ \{G,K\}\cap\mathscr{V}\neq\emptyset}} U(\{G,K\}, \omega\eta\,|_{\mathscr{G}\setminus\mathscr{V}})\Bigg)\Bigg]\mu(\{X_{\partial\mathscr{V}} = \eta^*\}), \end{aligned}$$

where η is an arbitrary element of $X_{\partial\mathscr{V}}^{-1}(\eta^*)$.

Since $\omega\,|_{\mathscr{V}(s,t)} = \pi\omega\,|_{\mathscr{V}(s,t)} = \sigma$ for all pairs

$$(s,t) \in \{(-b-d, -b+d), (b-d, b+d), (c-b-d, c-b+d), (c+b-d, c+b+d)\}$$

and by the special choice of d in (18) one has for all $\omega \in \Omega_{\mathscr{V}}$, $\eta \in \Lambda_{\omega,\mathscr{V}}$

$$\Lambda_{\omega,\mathscr{V}} = \Lambda_{\pi\omega,\mathscr{V}}$$

and

$$\begin{aligned} &\sum_{G\in\mathscr{V}} U(\{G\}, \omega\eta\,|_{\mathscr{G}\setminus\mathscr{V}}) + \sum_{\substack{\{G,K\}\in\mathfrak{N} \\ \{G,K\}\cap\mathscr{V}\neq\emptyset}} U(\{G,K\}, \omega\eta\,|_{\mathscr{G}\setminus\mathscr{V}}) \\ &\quad = \sum_{G\in\mathscr{V}} U(\{G\}, (\pi\omega)\eta\,|_{\mathscr{G}\setminus\mathscr{V}}) + \sum_{\substack{\{G,K\}\in\mathfrak{N} \\ \{G,K\}\cap\mathscr{V}\neq\emptyset}} U(\{G,K\}, (\pi\omega)\eta\,|_{\mathscr{G}\setminus\mathscr{V}}). \end{aligned}$$

As a consequence we get the invariance of the Gibbs measure μ with respect to T_π. Hence from (23) and (24) it follows

$$\begin{aligned} &\mu(T_\pi(\Lambda \cap \Phi) \cap (\Lambda' \cap \Phi')) \\ &\quad = \mu(T_\pi(\Lambda \cap \Phi)) + \mu(\Lambda' \cap \Phi') - \mu(T_\pi(\Lambda \cap \Phi) \cup (\Lambda' \cap \Phi')) \\ &\quad \geqslant \mu(T_\pi(\Lambda \cap \Phi)) + \mu(\Lambda' \cap \Phi') - \mu(\Lambda') \\ &\quad > \frac{1}{2}\mu(\Lambda') + \frac{1}{2}\mu(\Lambda') - \mu(\Lambda') = 0. \end{aligned}$$

Thus we get $\mu(T_\pi\Phi \cap \Phi') > 0$. The ergodicity is proved.

The existence of the countable sets $\mathscr{R}_{T_\pi}$ for $T_\pi \in \mathscr{F}_\mu$ (see (13)) follows immediately from the fact that the maps T_π are permutations of finite dimensional cylinder sets. The theorem is proved.

We formulate now the main result of this section.

THEOREM 4. — Let (Ω_1, μ_1, τ) and (Ω_2, μ_2, τ) be two topologically mixing Gibbs systems of equal entropy. Then, if the isomorphism I between them has finite expected code length,

$$I[\mathscr{F}_{\mu_1}]I^{-1} \subset [\mathscr{F}_{\mu_2}]. \tag{25}$$

Proof. — Let $h = \max_{G_0 \in \mathscr{G}_0} G_0$. Then it follows from the finiteness of

$$\int (a + m) d\mu_1$$

that

$$\int ha \, d\mu_1 < \infty \qquad \text{and} \qquad \int hm \, d\mu_1 < \infty .$$

It is then not hard to deduce from the individual ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n} ha(\tau_n\omega) = 0 \qquad \text{for} \qquad \mu_1\text{-a. a. } \omega \in \Omega_1 ,$$

$$\lim_{n\to\infty} \frac{1}{n} hm(\tau_n\omega) = 0 \qquad \text{for} \qquad \mu_1\text{-a. a. } \omega \in \Omega_1 . \tag{26}$$

Hence there exists a set $\Gamma_1 \subset \Omega_1$ of μ_1-measure one such that for all $\omega \in \Gamma_1$ and all $N \in \mathbb{N}$ there is a $S(\omega, N) \in \mathbb{N}$ with

$$\begin{aligned} s + ha(\tau_s\omega) &< -N \qquad (s < -S(\omega, N)) \\ s - hm(\tau_s\omega) &> N \qquad (s > S(\omega, N)). \end{aligned} \tag{27}$$

Let now $\omega, \omega' \in \Gamma_1$ and $N \in \mathbb{N}$ be such that

$$\omega|_{\mathscr{V}(s)} = \omega'|_{\mathscr{V}(s)} \qquad (|s| > N). \tag{28}$$

Then, if $S = S(\omega, N)$ and $s > S > N$,

$$(\tau_s\omega)|_{\mathscr{V}(t)} = (\tau_s\omega')|_{\mathscr{V}(t)} \qquad \text{for} \qquad t = N - s, \ldots, 0, 1, \ldots .$$

Since $hm(\tau_s\omega) < -N + s$ for all $s > S$ it follows

$$(\tau_s\omega)|_{\mathscr{V}(t)} = (\tau_s\omega')|_{\mathscr{V}(t)} \qquad \text{for all} \qquad t \in W_r(\omega, a(\omega)) \cup W_1(\omega, m(\omega)),$$

i. e.

$$I(\tau_s\omega)|_{\mathscr{V}(0)} = I(\tau_s\omega')|_{\mathscr{V}(0)} \qquad \text{for all} \quad s > S .$$

If $-s < -S < -N$ then

$$(\tau_s\omega)|_{\mathscr{V}(t)} = (\tau_s\omega')|_{\mathscr{V}(t)} \qquad \text{for} \qquad t = -N - s, \ldots, 0, 1, \ldots .$$

Since $ha(\tau_s\omega) < -N - s$ for all $s < -S$ it follows

$$(\tau_s\omega)|_{\mathscr{V}(t)} = (\tau_s\omega')|_{\mathscr{V}(t)} \quad \text{for} \quad t \in W_r(\omega, a(\omega)) \cup W_1(\omega, m(\omega)),$$

i. e.

$$I(\tau_s\omega)|_{\mathscr{V}(0)} = I(\tau_s\omega')|_{\mathscr{V}(0)} \quad \text{for all} \quad s < -S.$$

Thus we have shown that it follows from (28) the existence of a natural number S such that

$$(I\omega)|_{\mathscr{V}(s)} = (I\omega')|_{\mathscr{V}(s)} \qquad (|s| > S).$$

The theorem is proved.

COROLLARY. — If

$$\int (a + m)d\mu_1 < \infty$$

then

$$\Delta(\mathscr{F}_{\mu_1}, \mu_1) \subset \Delta(\mathscr{F}_{\mu_2}, \mu_2). \tag{29}$$

It arises the question whether the inclusions in (25) and (29) are strong, i. e. the question whether the isomorphism I and its inverse I^{-1} can have both finite expected code length. Parry was able to show in [*10*] that there are many cases of Markov chain isomorphisms in which the finitary isomorphism and its inverse cannot have both finite expected code length. According to the fact that Markov chains are examples of Gibbs systems this is true in our situation, too.

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