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Feynman diagrams and large order estimates for the exponential anharmonic oscillator

by

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ABSTRACT. — Upper and lower bounds are derived which prove the conjectured divergence rates for the perturbation coefficients of the lowest eigenvalue (ground state energy) and of the trace of the semigroup for the exponential anharmonic oscillator. Our methods involve Feynman diagram representations for the exponential interaction and path integral estimates. We also prove a bracketing inequality for the ground state energy perturbation coefficients which has appeared in several other examples.

RÉSUMÉ. — Nous démontrons des bornes supérieures et inférieures qui prouvent les vitesses de divergence attendues pour les coefficients de perturbation de l'énergie la plus basse et de la trace du semi-groupe, pour l'oscillateur anharmonique exponentiel. Nos méthodes utilisent des représentations en diagrammes de Feynman pour l'interaction exponentielle et des estimations d'intégrales de chemins. Nous prouvons aussi une inégalité encadrant les coefficients de perturbation de l'énergie la plus basse, qui est apparue dans plusieurs autres exemples.

1. INTRODUCTION

In two recent articles, Grecchi and Maioli [*1*] [*2*] have introduced a generalized Borel summation method for divergent power series whose

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coefficients grow faster than any factorial power. Specifically, if a function has an asymptotic power series $\sum_k a_k \lambda^k$ with the coefficients a_k growing like $(nk)!\exp(ck^2)$ with $c > 0$, then [1] [2] give sufficient conditions for the series to be resummable to the function by an extension of the usual Borel method [5] [6].

The main proposed application of this method is to the eigenvalues of anharmonic oscillator Hamiltonians of the form

$$H(\lambda) = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 - 1 \right) + \lambda x^m e^{\alpha x} \tag{1.1}$$

with $\lambda \geqslant 0$, $\alpha \neq 0$, $m = 0, 1, 2, 3, \ldots$ However, there has not yet been any proof that the eigenvalues of (1.1) even have divergent perturbation series, or more importantly that the rate of divergence is faster than any factorial power. In [3], Grecchi *et al.* study the $m = 0$ case of (1.1) and give strong arguments, but not a rigorous proof, that all eigenvalues have perturbation coefficients a_k which grow like $\exp(c\alpha^2 k^2)$ for $c > 0$. In this paper, we prove that the perturbation series for the lowest eigenvalue (ground state energy) of (1.1) with $m = 0$ has the conjectured divergence rate. We also prove that a bracketing inequality for the k^{th} perturbation coefficient due to Spencer, which has been much used for large order estimates [7] [8] [9] [10], holds for this case. Finally, we consider the trace, $\mathrm{Tr}\,[e^{-\mathrm{TH}(\lambda)}]$, for $m=0, 1, 2, 3, \ldots$, and show that its perturbation series has coefficients b_k which grow like $k^m \exp(c\alpha^2 k^2)/k\,!$, $c > 0$. The trace was also considered in [2] as a suitable example for the new summability method, but no proof was given that the b_k had the desired divergence rate.

We will prove these results by using path integral methods and Feynman graph representations for the perturbation coefficients. Section 2 contains statements of our results and some definitions. The divergence rate for the ground state energy will be proved in section 3, along with the bracketing inequality. Also, the Feynman graph representations for the exponential interaction will be derived in section 3. Section 4 contains the proof of the divergence rate for the perturbation coefficients of the trace of the semigroup. Some comments on our results are contained in section 5.

2. PRELIMINARIES AND STATEMENT OF RESULTS

We let $E(\lambda)$ be the ground state energy of the Hamiltonian

$$H_1(\lambda) = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 - 1 \right) + \lambda e^{\alpha x},$$

$\lambda \geqslant 0, \alpha \neq 0$. In [4], Maioli studied $H_1(\lambda)$ and proved, among other things, that this operator had discrete spectrum with nondegenerate eigenvalues to each of which the Rayleigh-Schrodinger perturbation series was asymptotic. If

$$E(\lambda) \sim \sum_{k=0}^{\infty} a_k \lambda^k$$

is the asymptotic perturbation series as $\lambda \to 0^+$ for the ground state energy, then our first result is the following.

THEOREM 2.1. — There exist positive constants A_1, A_2, B_1, B_2 independent of k, such that for all $k \geqslant 1$,

$$A_1 B_1^k \frac{C_1^{k^2}}{k!} \leqslant (-1)^{k+1} a_k \leqslant A_2 B_2^k \frac{C_2^{k^2}}{k!}, \tag{2.1}$$

with $C_1 = \exp(\alpha^2/4e)$, and $C_2 = \exp(\alpha^2/4)$.

Remark. — The numerical values for C_1 and C_2 are reasonably accurate. That is, the dominant contribution to the large k asymptotics of a_k is expected to be $\exp(\alpha^2 k^2/4)$ [11].

Path integrals will enter in the proof of Theorem 2.1 by the well-known formula

$$E(\lambda) = -\lim_{T \to \infty} \frac{1}{T} \ln Z_X(\lambda),$$

where

$$Z_X(\lambda) = \int e^{-\lambda V(\phi)} d\mu_X(\phi), \tag{2.2}$$

with $V(\phi) = \int_{-T/2}^{T/2} \exp(\alpha\phi(s))ds$, and $d\mu_X(\phi)$ the mean zero Gaussian measure with covariance $G_X(s, t)$ for $X = p$ (periodic), D (Dirichlet), 0 (free). Explicitly,

$$G_0(s, t) = \frac{1}{2} e^{-|s-t|}, \tag{2.3}$$

$$G_p(s, t) = G_0(s, t) + (1 - e^{-T})^{-1} e^{-T} \cosh(s - t), \tag{2.4}$$

and

$$G_D(s, t) = G_0(s, t) - (1 - e^{-2T})^{-1} e^{-T} (\cosh(s-t) - e^{-T} \cosh(s+t)), \tag{2.5}$$

for $s, t \in [-T/2, T/2]$. Notice that $G_X(s, t)$ is the kernel of the integral operator $(-\Delta_X + 1)^{-1}$ in which Δ_X is the Laplacian obeying $X = p$, D, 0 boundary conditions on $[-T/2, T/2]$.

In proving Theorem 2.1, and in proving the bracketing inequality, we will need the finite T quantities

$$a_k^{\mathrm{X}}(\mathrm{T}) = -\frac{1}{\mathrm{T}k!}\frac{d^k}{d\lambda^k}\bigg|_{\lambda=0} \ln \mathrm{Z_X}(\lambda), \quad \mathrm{X} = p, \mathrm{D}, 0\,. \tag{2.6}$$

The bracketing inequality is our second result.

THEOREM 2.2. — For all $k \geqslant 1$ and T,

$$(-1)^{k+1}a_k^{\mathrm{D}}(\mathrm{T}) \leqslant (-1)^{k+1}a_k \leqslant (-1)^{k+1}a_k^p(\mathrm{T})\,. \tag{2.7}$$

Remark. — This bracketing inequality holds in a number of other cases. It was first proven by Spencer [*7*] for the ground state energy of an x^{2m} anharmonic oscillator. It also holds for the pressure, with $(-1)^{k+1}$ changed to $(-1)^k$ of a ϕ^{2m} Euclidean lattice field theory in any dimension, and of a ϕ^{2m} Euclidean continuum field theory in 2 dimensions. See [*7*] [*8*] [*9*] [*10*] for applications to asymptotics of large order perturbation theory.

Our last result is motivated by considering

$$\mathrm{Tr}\,[\exp(-\mathrm{TH}_2(\lambda))],$$

where

$$\mathrm{H}_2(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda x^m e^{\alpha x},$$

$\lambda \geqslant 0$, $m = 0, 1, 2, \ldots$, and we assume $\alpha > 0$. The Feynman-Kac formula yields

$$\mathrm{Tr}\,[\exp(-\mathrm{TH}_2(\lambda))] = \int e^{-\lambda \mathrm{W}(\phi)} d\mu_p(\phi) \tag{2.8}$$

with

$$\mathrm{W}(\phi) = \int_{-\mathrm{T}/2}^{\mathrm{T}/2} \phi^m(s) e^{\alpha\phi(s)} ds,$$

and so the coefficients $b_k^p(\mathrm{T})$ in the asymptotic series

$$\mathrm{Tr}\,[\exp(-\mathrm{TH}_2(\lambda))] \sim \sum_{k=0}^{\infty} b_k^p(\mathrm{T})\lambda^k$$

have the representation

$$b_k^p(\mathrm{T}) = \frac{(-1)^k}{k!}\int \mathrm{W}^k(\phi) d\mu_p(\phi)\,.$$

We wish to find upper and lower bounds to $b_k^p(\mathrm{T})$, which will prove the desired divergence rate. However, we will consider the case of more general boundary conditions by looking at

$$\tilde{\mathrm{Z}}_{\mathrm{X}}(\lambda) = \int e^{-\lambda \mathrm{W}(\phi)} d\mu_{\mathrm{X}}(\phi), \quad \mathrm{X} = p, \mathrm{D}, 0\,, \tag{2.9}$$

and its asymptotic perturbation series

$$\tilde{Z}_X(\lambda) \sim \sum_{k=0}^{\infty} b_k^X(T)\lambda^k,$$

in which

$$b_k^X(T) = \frac{(-1)^k}{k!} \int W^k(\phi) d\mu_X(\phi). \tag{2.10}$$

The next theorem is our final result.

THEOREM 2.3. — For all $k \geqslant 1$ and T, there exist positive constants A, B, C, a, b ($a < b$) independent of k, such that

$$AB^k k^{mk} \frac{e^{a\alpha^2 k^2}}{k!} \leqslant (-1)^k b_k^X(T) \leqslant C^k k^{mk/2} \frac{e^{b\alpha^2 k^2}}{k!}. \tag{2.11}$$

Remarks. — 1. The constants may depend on T and X.

2. Theorems 2.2 and 2.3 may be combined to give a second proof of Theorem 2.1 independent of the proof given in section 3, but with different values of the constants $A_1, A_2, B_1, B_2, C_1, C_2$. We will discuss this at the end of section 4.

3. The assumption $\alpha > 0$ seems necessary for $m > 0$, since if both m and k are odd, $(-1)^k b_k^X(T)$ is negative for $\alpha < 0$, so (2.11) cannot be true.

3. FEYNMAN DIAGRAMS AND THE BRACKETING INEQUALITY

Our first objective is to establish Feynman graph representations for $b_k^X(T)$, $a_k^X(T)$, and a_k. The proof of Theorem 2.1 will then follow by adapting work of Simon on the quartic oscillator [*12*, sec. 20]. To begin, define

$\Gamma_k = \{$ graphs | k vertices, no more than 1 line between each distinct pair of vertices, no self loops $\}$,

$\Gamma_k^P = \{$ graphs | k vertices, no more than 1 line between each distinct pair of vertices, no more than 1 self loop to a vertex $\}$,

and Γ_k^c, $\Gamma_k^{P,c}$ are defined in the same way, but with the additional restriction that all graphs are connected. For fixed k, we note that the number of lines in different graphs may be different. In particular, if L is the number of lines in a graph, than for Γ_k, $0 \leqslant L \leqslant k(k-1)/2$, while for Γ_k^c, $k-1 \leqslant L \leqslant k(k-1)/2$. Similarly, for $\Gamma_k^P, 0 \leqslant L \leqslant k(k+1)/2$, and for $\Gamma_k^{P,c}$, $k-1 \leqslant L \leqslant k(k+1)/2$. The reason for the separate definitions for

$\Gamma_k^{\rm D}$, $\Gamma_k^{{\rm D},c}$ is that for $X = p, 0$ we will be able to factor out contributions to the graphs from self loops.

Each line in a graph may be thought of as having a direction, with an initial and final vertex. For the line l, we label the variable corresponding to its initial vertex as s_{l_i} and the variable corresponding to its final vertex as s_{l_f}. Our last definitions are

$$h_{\rm X}(s, t) = \exp(\alpha^2 G_{\rm X}(s, t)) - 1, \quad X = p, D, 0,$$

and

$$f_k^{\rm X}(\alpha) = \exp(\alpha^2 k G_{\rm X}(0)/2), \quad X = p, 0.$$

PROPOSITION 3.1. — For all $k \geqslant 1$ and T,

$$b_k^{\rm X}(T) = \frac{(-1)^k}{k!} f_k^{\rm X}(\alpha) \sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} h_{\rm X}(s_{l_i} - s_{l_f}), \quad X = p, 0, \tag{3.1}$$

$$b_k^{\rm D}(T) = \frac{(-1)^k}{k!} \sum_{\gamma \in \Gamma_k^{\rm D}} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} h_{\rm D}(s_{l_i}, s_{l_f}), \tag{3.2}$$

$$a_k^{\rm X}(T) = \frac{(-1)^{k+1}}{Tk!} f_k^{\rm X}(\alpha) \sum_{\gamma \in \Gamma_k^c} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} h_{\rm X}(s_{l_i} - s_{l_f}), \quad X = p, 0, \tag{3.3}$$

$$a_k^{\rm D}(T) = \frac{(-1)^{k+1}}{Tk!} \sum_{\gamma \in \Gamma_k^{{\rm D},c}} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} h_{\rm D}(s_{l_i}, s_{l_f}), \tag{3.4}$$

and

$$a_k = \frac{(-1)^{k+1}}{k!} f_k^0(\alpha) \sum_{\gamma \in \Gamma_k^c} \int_{\mathbb{R}} d^{k-1} s \prod_{l \in \gamma} h_0(s_{l_i} - s_{l_f}). \tag{3.5}$$

Remarks. — 1. Recall that for Theorem 2.1, we are considering the $m = 0$ case of (1.1), so the $b_k^{\rm X}(T)$ of this proposition will be

$$b_k^{\rm X}(T) = \frac{(-1)^k}{k!} \int V^k(\phi) d\mu_{\rm X}(\phi)$$

and not the $m \neq 0$ case of (2.10).

2. For the case $X = p$, we may use the periodicity and translation invariance of $h_p(s - t)$ to obtain

$$b_k^p(T) = \frac{(-1)^k T}{k!} f_k^p(\alpha) \sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^{k-1} s \prod_{l \in \gamma} h_p(s_{l_i} - s_{l_f}) \tag{3.6}$$

and

$$a_k^p(T) = \frac{(-1)^{k+1}}{k!} f_k^p(\alpha) \sum_{\gamma \in \Gamma_k^c} \int_{-T/2}^{T/2} d^{k-1} s \prod_{l \in \gamma} h_p(s_{l_i} - s_{l_f}). \tag{3.7}$$

3. In (3.5), (3.6), and (3.7) one vertex has been set equal to zero in the integrands.

4. From the calculation (3.9), it might seem that the functions $\exp(\alpha^2 G_X(s, t))$ should appear in our Feynman graph representations instead of $h_X(s, t)$. However, the use of $h_X(s, t)$ makes explicit the cancellations between terms which appear in $a_k^X(T)$ that diverge for large T. For example, if $X = p$, then

$$\begin{aligned} a_2^p(T) &= -\frac{1}{T}\left[b_2^p(T) - \frac{1}{2}(b_1^p(T))^2\right] \\ &= -\frac{1}{T}\left[\frac{1}{2}\exp(\alpha^2 G_p(0)) \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} ds_1 ds_2 \exp(\alpha^2 G_p(s_1, s_2)) \right. \\ &\quad \left. -\frac{1}{2} T^2 \exp(\alpha^2 G_p(0))\right] \\ &= \frac{-\exp(\alpha^2 G_p(0))}{2}\left[\int_{-T/2}^{T/2} \exp(\alpha^2 G_p(s)) ds - T\right] \\ &= \frac{-\exp(\alpha^2 G_p(0))}{2} \int_{-T/2}^{T/2} h_p(s) ds, \end{aligned} \tag{3.8}$$

and the bound $h_p(s) \leqslant \text{const.}\, G_p(s)$ shows that $a_2^p(T)$ has a finite limit as $T \to \infty$.

Proof. — Evaluating the integral in (3.8), we have

$$\begin{aligned} \int V^k(\phi) d\mu_X(\phi) &= \int_{-T/2}^{T/2} d^k s \int \exp\left(\alpha \sum_{i=1}^{k} \phi(s_i)\right) d\mu_X(\phi) \\ &= \int_{-T/2}^{T/2} d^k s \exp\left(\frac{\alpha^2}{2} \sum_{i,j=1}^{k} G_X(s_i, s_j)\right) \\ &= \int_{-T/2}^{T/2} d^k s \exp\left(\frac{\alpha^2}{2} \sum_{i=1}^{k} G_X(s_i, s_i) + \alpha^2 \sum_{i<j} G_X(s_i, s_j)\right) \\ &= \int_{-T/2}^{T/2} d^k s \prod_{i=1}^{k} e^{\alpha^2 G_X(s_i, s_i)/2} \prod_{i<j} e^{\alpha^2 G_X(s_i, s_j)}. \end{aligned} \tag{3.9}$$

If $X = p, 0$, then $G_X(s_i, s_i) = G_X(0)$ and (3.9) becomes

$$\int V^k(\phi) d\mu_X(\phi) = f_k^X(\alpha) \int_{-T/2}^{T/2} d^k s \prod_{i<j} \exp(\alpha^2 G_X(s_i - s_j)). \tag{3.10}$$

For simplicity, we will work the rest of the proof for the cases $X = p, 0$. The $X = D$ case is a trivial modification.

Next, write $\exp(\alpha^2 G_X(s_i - s_j)) = h_X(s_i - s_j) + 1$ and substitute into (3.10) to obtain

$$\int_{-T/2}^{T/2} d^k s \prod_{i<j} \exp(\alpha^2 G_X(s_i - s_j)) = \int_{-T/2}^{T/2} d^k s \prod_{i<j} (h_X(s_i - s_j) + 1)$$

$$= \int_{-T/2}^{T/2} d^k s \prod_{l=0}^{k(k-1)/2} (h_X(s_{l_i} - s_{l_f}) + 1)$$

$$= \sum_{l=0}^{k(k-1)/2} \sum_{m_1 < \ldots < m_l} \int_{-T/2}^{T/2} d^k s \prod_{n \neq m_1, \ldots, m_l} h_X(s_{n_i} - s_{n_f}), \quad (3.11)$$

where it is understood that when $l = 0$ there is no sum on $m_1, \ldots, m_l$ and when $l = k(k-1)/2$ the empty product is 1. To obtain (3.1) from (3.11), use that

$$\sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} h_X(s_{l_i} - s_{l_f}) = \sum_{l=0}^{k(k-1)/2} \sum_{\gamma \in \Gamma_{k,l}} \int_{-T/2}^{T/2} d^k s \prod_{n \in \gamma} h_X(s_{n_i} - s_{n_f})$$

where $\Gamma_{k,l}$ is the set of those graphs in Γ_k with exactly l lines. It is then easy to verify that

$$\sum_{\gamma \in \Gamma_{k,k(k-1)/2-l}} \int_{-T/2}^{T/2} d^k s \prod_{n \in \gamma} h_X(s_{n_i} - s_{n_f})$$

$$= \sum_{m_1 < \ldots < m_l} \int_{-T/2}^{T/2} d^k s \prod_{n \neq m_1, \ldots, m_l} h_X(s_{n_i} - s_{n_f}). \quad (3.12)$$

Indeed, the right side of (3.12) is an enumeration of all possible ways of choosing exactly $k(k-1)/2 - l$ lines from amongst the maximum $k(k-1)/2$ lines possible for a graph in Γ_k. Therefore, (3.10), (3.11), and (3.12) prove (3.1).

The representation (3.3) is an immediate consequence of (3.1) since it is a well-known property of Feynman graph representations for perturbation coefficients that in taking logarithmic derivatives, as in (2.6), we pass from the sum of all graphs for $b_k^X(T)$ to the sum of all connected graphs for $a_k^X(T)$.

In order to prove (3.5), we first note that using (3.8) for $a_k^0(T)$, it is easy to see that

$$\lim_{T \to \infty} a_k^0(T) = \text{right side of (3.5)}.$$

Therefore, we must show that $a_k^0(\mathrm{T}) \to a_k$ as $\mathrm{T} \to \infty$, where we use the definition (2.6) for $a_k^0(\mathrm{T})$. This may be easily done by adapting Simon's work on the x^4 anharmonic oscillator [*12*, p. 213-215]. In particular, if $\mathrm{U}_{k,\lambda}(x_1, \ldots, x_k)$ is the k^{th} Ursell function (connected Schwinger function) [*12*, p. 129] with respect to the measure $\exp(-\lambda \mathrm{V}(\phi))d\mu_0(\phi)/\mathrm{Z}_0(\lambda)$ then

$$a_k^0(\mathrm{T}) = \frac{(-1)^{k+1}}{\mathrm{T}} \int_{-\mathrm{T}/2 < s_1 < \ldots < s_k < \mathrm{T}/2} d^k s\, \mathrm{U}_{k,0}(e^{\alpha\phi(s_1)}, \ldots, e^{\alpha\phi(s_k)}),$$

and from [*12*] it will suffice to show

$$\mathrm{U}_{k,\lambda}(e^{\alpha\phi(s_1)}, \ldots, e^{\alpha\phi(s_k)}) \leqslant \mathrm{C}_k \exp(-\mathrm{D}_k \mid s_k - s_1 \mid), \qquad (3.14)$$

for $s_1 < \ldots < s_k$ and $0 \leqslant \lambda \leqslant 1$. If we define

$$\tilde{\mathrm{X}} = \sum_{j=0}^{k-1} \omega^j e^{\alpha x_j},$$

then (3.14) will follow from Simon's proof, since the necessary estimates, $\exp(\alpha\phi(s)) \geqslant 0$,

$$\int \mid \tilde{\mathrm{X}}(\phi(0)) \mid^m d\mu_0(\phi) < \infty,$$

and $\mid \tilde{\mathrm{X}} \mid^m \exp(-\tilde{\mathrm{H}}_1(0))$ a bounded operator, still hold ($\tilde{\mathrm{H}}_1(0)$ is the sum of k copies of $\mathrm{H}_1(0)$ acting on $\mathrm{L}^2(\mathbb{R}^k)$). This finishes the proof of (3.5).

Proof of Theorem 2.1. — Our proof is similar to Simon's proof [*12*] of upper and lower bounds for the ground state energy perturbation coefficients of the x^4 anharmonic oscillator. We will obtain our upper and lower bounds by rewriting the sum in (3.5) as

$$\sum_{\gamma\in\Gamma_k^c} \int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i} - s_{l_f}) = \sum_{l=0}^{(k-1)(k-2)/2} \sum_{\gamma\in\Gamma_{k,k-1+l}^c} \int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i} - s_{l_f}), \qquad (3.15)$$

where $\Gamma_{k,k-1+l}^c$ is the set of all graphs in Γ_k^c with exactly $k-1+l$ lines. The lower bound in (2.1) will be obtained by restricting each integration in (3.15) to the interval $[0, 1]$. This will give us

$$\sum_{\gamma\in\Gamma_{k,k-1+l}^c} \int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i} - s_{l_f}) \geqslant \#(\Gamma_{k,k-1+l}^c)[h_0(1)]^l, \qquad (3.16)$$

in which $\#(\Gamma_{k,k-1+l}^c)$ is the number of graphs in $\Gamma_{k,k-1+l}^c$. A graph in $\Gamma_{k,k-1+l}^c$ can be constructed by first choosing $k-1$ lines which connect the graph. Next pick only l lines from the remaining $(k-1)(k-2)/2$ possible choices for lines. Therefore,

$$\#\left(\Gamma^c_{k,k-1+l}\right) \geqslant \binom{(k-1)(k-2)/2}{l},$$

and so

$$\sum_{l=0}^{(k-1)(k-2)/2} \sum_{\gamma\in\Gamma^c_{k,k-1+l}} \int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i}-s_{l_f})$$
$$\geqslant \sum_{l=0}^{(k-1)(k-2)/2} \binom{(k-1)(k-2)/2}{l} (h_0(1))^l$$
$$= [1+h_0(1)]^{(k-1)(k-2)/2} = [\exp(\alpha^2 G_0(1))]^{(k-1)(k-2)/2}.$$

This gives us the lower bound of (2.1). The upper bound will follow by picking, for each graph $\gamma\in\Gamma_{k,k-1+l}$, $k-1$ lines which connect γ. If these lines are labeled $l=1,\ldots,k-1$, then the change of variables $x_l = s_{l_i}-s_{l_f}$ will have Jacobian equal to 1, as in [*12*, p. 221]. We obtain an upper estimate of

$$\int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i}-s_{l_f}) \leqslant (e^{\alpha^2/2}-1)^l \left[\int_{\mathbb{R}} h_0(x)dx\right]^{k-1},$$

since $h_0(s-t) \leqslant \exp(\alpha^2/2)-1$. Also

$$\int_{\mathbb{R}} h_0(x)dx \leqslant 2(\exp(\alpha^2/2)-1),$$

so combining with (3.15) yields

$$\sum_{l=0}^{(k-1)(k-2)/2} \sum_{\gamma\in\Gamma^c_{k,k-1+l}} \int_{\mathbb{R}} d^{k-1}s \prod_{l\in\gamma} h_0(s_{l_i}-s_{l_f})$$
$$\leqslant [2(e^{\alpha^2/2})-1]^{k-1} \sum_{l=0}^{(k-1)(k-2)/2} \#\left(\Gamma^c_{k,k-1+l}\right)(e^{\alpha^2/2}-1)^l$$
$$\leqslant [2(e^{\alpha^2/2})-1]^{k-1} \sum_{l=0}^{k(k-1)/2} \#\left(\Gamma_{k,l}\right)(e^{\alpha^2/2}-1)^{l-(k-1)}$$
$$= 2^{k-1} \sum_{l=0}^{k(k-1)/2} \binom{k(k-1)/2}{l} (e^{\alpha^2/2}-1)^l$$
$$= 2^{k-1}[\exp(\alpha^2/2)]^{k(k-1)/2}.$$

This finishes the proof of the upper bound for Theorem 2.1.

Proof of Theorem 2.2. — The proof follows Spencer's argument for the x^4 case [7]. We will first do the lower bound in (2.7). For purpose of comparison with $a_k^{\mathrm{D}}(\mathrm{T})$, we write a_k as

$$(-1)^{k+1}a_k = \frac{1}{k!}\lim_{\mathrm{V}\to\infty}\frac{1}{\mathrm{V}}\sum_{\gamma\in\Gamma_k^{\mathrm{D},c}}\int_{-\mathrm{V}/2}^{\mathrm{V}/2} d^k s \prod_{l\in\gamma} h_0(s_i - s_j), \qquad (3.17)$$

where $\mathrm{V} = \mathrm{MT}$ for fixed T, $\mathrm{M} \to \infty$ through positive integers, and we are indexing the integration variables by the vertices instead of the lines. By using $\Gamma_k^{\mathrm{D},c}$, we have absorbed the factor $f_k^0(\alpha)$ of (3.5) into the integrand in (3.17). We can rewrite the integral in (3.17) as

$$\begin{aligned}
\frac{1}{\mathrm{V}}\int_{-\mathrm{V}/2}^{\mathrm{V}/2} d^k s \prod_{l\in\gamma} h_0(s_i - s_j) &= \frac{1}{\mathrm{MT}}\sum_{\substack{n_i=-(\mathrm{M}-1)/2\\ i=1,\dots,k}}^{(\mathrm{M}-1)/2}\int_{(n_i-1/2)\mathrm{T}}^{(n_i+1/2)\mathrm{T}} d^k s \prod_{l\in\gamma} h_0(s_i - s_j)\\
&= \frac{1}{\mathrm{MT}}\sum_{\substack{n_i=-(\mathrm{M}-1)/2\\ i=1,\dots,k}}^{(\mathrm{M}-1)/2}\int_{-\mathrm{T}/2}^{\mathrm{T}/2} d^k s \prod_{l\in\gamma} h_0(s_i - s_j + (n_i - n_j)\mathrm{T})\\
&\geqslant \frac{1}{\mathrm{T}}\int_{-\mathrm{T}/2}^{\mathrm{T}/2} d^k s \prod_{l\in\gamma} h_0(s_i - s_j)\\
&\geqslant \frac{1}{\mathrm{T}}\int_{-\mathrm{T}/2}^{\mathrm{T}/2} d^k s \prod_{l\in\gamma} h_{\mathrm{D}}(s_i, s_j), \qquad (3.18)
\end{aligned}$$

and this shows that $(-1)^{k+1}a_k \geqslant (-1)^{k+1}a_k^{\mathrm{D}}(\mathrm{T})$. In the third line of (3.18), we have dropped those terms for which $n_i \neq n_j$, and in the last line we use $\mathrm{G}_0(s_i - s_j) \geqslant \mathrm{G}_{\mathrm{D}}(s_i, s_j)$.

For the upper bound of (2.7), we need only compare the integrals in (3.5) and (3.7) since $f_k^0(\alpha) \leqslant f_k^p(\alpha)$. Expanding the exponentials in the integrand of (3.7) gives us

$$\int_{-\mathrm{T}/2}^{\mathrm{T}/2} d^{k-1}s \prod_{l\in\gamma} h_p(s_{l_i} - s_{l_f}) = \sum_{\substack{n_l=1\\ l\in\gamma}}^{\infty} \frac{\alpha^{2|n|}}{\prod_{l\in\gamma}(n_l)!}\int_{-\mathrm{T}/2}^{\mathrm{T}/2} d^{k-1}s \prod_{l\in\gamma} \mathrm{G}_p^{n_l}(s_{l_i} - s_{l_f}),$$

in which $|n| = \Sigma n_l$. The method of images formula

$$\mathrm{G}_p(s_{l_i} - s_{l_f}) = \sum_{m\in\mathbb{Z}} \mathrm{G}_0(s_{l_i} - s_{l_f} + m\mathrm{T})$$

then yields

$$G_p^{n_l}(s_{l_i} - s_{l_f}) \geqslant \sum_{m_l \in \mathbb{Z}} G_0^{n_l}(s_{l_i} - s_{l_f} + m_l T),$$

and so

$$\int_{-T/2}^{T/2} d^{k-1}s \prod_{l \in \gamma} G_p^{n_l}(s_{l_i} - s_{l_f}) \geqslant \sum_{\substack{m_l \in \mathbb{Z} \\ l \in \gamma}} \int_{-T/2}^{T/2} d^{k-1}s \prod_{l \in \gamma} G_0^{n_l}(s_{l_i} - s_{l_f} + m_l T). \quad (3.19)$$

Similarly,

$$\int_{\mathbb{R}} d^{k-1}s \prod_{l \in \gamma} h_0(s_i - s_j) = \sum_{\substack{n_l = 1 \\ l \in \gamma}}^{\infty} \frac{\alpha^{2|n|}}{\prod_{l \in \gamma} (n_l)!} \int_{\mathbb{R}} d^{k-1}s \prod_{l \in \gamma} G_0^{n_l}(s_i - s_j),$$

where we are again indexing the integration variables by the vertices. Splitting up the integration and then translating yields

$$\int_{\mathbb{R}} d^{k-1}s \prod_{l \in \gamma} G_0^{n_l}(s_i - s_j) = \sum_{\substack{n_i \in \mathbb{Z} \\ i = 1, \ldots, k-1}} \int_{(n_i - 1/2)T}^{(n_i + 1/2)T} d^{k-1}s \prod_{l \in \gamma} G_0^{n_l}(s_i - s_j)$$

$$= \sum_{\substack{n_i \in \mathbb{Z} \\ i = 1, \ldots, k-1}} \int_{-T/2}^{T/2} d^{k-1}s \prod_{l \in \gamma} G_0^{n_l}(s_i - s_j + (n_i - n_j)T). \quad (3.20)$$

The right side of (3.19) contains all terms of the right side of (3.20) plus additional positive terms, and so $(-1)^{k+1} a_k^p(T) \geqslant (-1)^{k-1} a_k$.

4. PARTITION FUNCTION PERTURBATION COEFFICIENTS

Our objective in this section is to prove upper and lower bounds on the perturbation coefficients $b_k^X(T)$ of $Z_X(\lambda)$, which are defined, respectively, in (2.10) and (2.9).

Proof of Theorem 2.3. — The theorem requires upper and lower bounds on the non-negative integral

$$\int \left[\int_{-T/2}^{T/2} \phi^m(s) e^{\alpha\phi(s)} ds \right]^k d\mu_X(\phi).$$

We will do the upper bound first. Applying Schwartz inequality in the s integral and then in the ϕ integral gives us

$$\int\left[\int_{-T/2}^{T/2}\phi^m(s)e^{\alpha\phi(s)}ds\right]^k d\mu_X(\phi)$$
$$\leqslant \int\left[\int_{-T/2}^{T/2}\phi^{2m}(s)ds\right]^{k/2}\left[\int_{-T/2}^{T/2}e^{2\alpha\phi(s)}ds\right]^{k/2} d\mu_X(\phi)$$
$$\leqslant\left(\int\left[\int_{-T/2}^{T/2}\phi^{2m}(s)ds\right]^k d\mu_X(\phi)\right)^{1/2}\left(\int\left[\int_{-T/2}^{T/2}e^{2\alpha\phi(s)}ds\right]^k d\mu_X(\phi)\right)^{1/2}. \quad (4.1)$$

If we let $V_1(\phi) = \int_{-T/2}^{T/2}\phi^{2m}(s)ds$, then

$$\left(\int\left[\int_{-T/2}^{T/2}\phi^{2m}(s)ds\right]^k d\mu_X(\phi)\right)^{1/2} = \| V_1 \|_k^{k/2} \leqslant (k-1)^{mk/2} \| V_1 \|_2^{k/2}, \quad (4.2)$$

in which $\| \,.\, \|_k$ is the $L^k(d\mu_X(\phi))$ norm and we have used hypercontractivity [*13*, Th. I.22] in the last inequality. For the second integral in (4.1),

$$\int\left[\int_{-T/2}^{T/2}e^{2\alpha\phi(s)}ds\right]^k d\mu_X(\phi) = \int_{-T/2}^{T/2}d^k s\,\exp\left(2\alpha^2\sum_{i,j=1}^{k}G_X(s_i, s_j)\right)$$
$$\leqslant T^k\exp(2\alpha^2k^2\max G_X(s, s)). \quad (4.3)$$

If $X = p, 0$, then $\max G_X(s, s) = G_X(0)$. Combining (4.2) and (4.3) gives us

$$\int\left[\int_{-T/2}^{T/2}\phi^m(s)e^{\alpha\phi(s)}ds\right]^k d\mu_X(\phi) \leqslant T^k \| V_1 \|_2^{k/2}k^{mk/2}\exp(2\alpha^2k^2\max G_X(s, s))$$

and this finishes the proof of the upper bound.

In doing the lower bound, we first consider the case of m even, $m = 2p$. The lower bound follows, as in [*8*], by translating $\phi \to \phi + k\phi_0$ and then using Jensen's inequality. The function ϕ_0 will be given later, and depends on the boundary conditions. We obtain

$$\int W^k(\phi)d\mu_X(\phi) = e^{-(k^2/2)\langle\phi_0, A_X\phi_0\rangle}\int W^k(\phi + k\phi_0)e^{-k\langle\phi, A_X\phi_0\rangle}d\mu_X(\phi)$$
$$= e^{-(k^2/2)\langle\phi_0, A_X\phi_0\rangle}\int [W(\phi + k\phi_0)e^{-\langle\phi, A_X\phi_0\rangle}]^k d\mu_X(\phi)$$
$$\geqslant e^{-(k^2/2)\langle\phi_0, A_X\phi_0\rangle}\left[\int W(\phi + k\phi_0)e^{-\langle\phi, A_X\phi_0\rangle}d\mu_X(\phi)\right]^k \quad (4.4)$$

where $A_X = -\Delta_X + 1$. The integral in the last inequality of (4.4) is explicitly

$$\int W(\phi + k\phi_0)e^{-\langle\phi, A_X\phi_0\rangle}d\mu_X(\phi)$$

$$= \iint_{-T/2}^{T/2} [\phi(s) + k\phi_0(s)]^{2p} e^{\alpha\phi(s)} e^{\alpha k\phi_0(s)} ds\, e^{-\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi)$$

$$= k^{2p} \sum_{j=0}^{2p} \binom{2p}{j} \iint_{-T/2}^{T/2} (\phi(s)/k)^j (\phi_0(s))^{2p-j} e^{\alpha\phi(s)} e^{\alpha k\phi_0(s)} ds\, e^{-\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi)$$

$$= k^{2p}\Bigg[\int_{-T/2}^{T/2} \phi_0^{2p}(s) e^{\alpha k\phi_0(s)} \int e^{\alpha\phi(s)} e^{-\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi) ds$$

$$+ \sum_{j=1}^{2p} \binom{2p}{j} \iint_{-T/2}^{T/2} (\phi(s)/k)^j (\phi_0(s)^{2p-j}) e^{\alpha\phi(s)} e^{\alpha k\phi_0(s)} ds\, e^{-\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi)\Bigg]$$

$$= k^{2p}\Bigg[\int_{-T/2}^{T/2} \phi_0^{2p}(s) e^{\alpha k\phi_0(s)} e^{(\alpha^2/2)G_X(s,s)} e^{(1/2)\langle\phi_0, A_X\phi_0\rangle} e^{-(\alpha/2)\phi_0(s)} ds$$

$$+ \sum_{j=1}^{2p} \binom{2p}{j} \iint_{-T/2}^{T/2} ds (\phi(s)/k)^j \phi_0^{2p-j}(s) e^{\alpha\phi(s)} e^{\alpha k\phi_0(s)} e^{-\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi)\Bigg].$$

For the cases $X = p, 0$, we choose $\phi_0(s) = \alpha/T$, and our lower bound becomes

$$\int W^k(\phi) d\mu_X(\phi) \geq e^{(\alpha^2 k^2/2T)} (k\alpha/2T)^{2pk} T^k e^{(\alpha^2 k/2)G_X(0)} \Bigg[1 + (T^{-1} e^{-(\alpha^2/2)G_X(0)})$$

$$\sum_{j=1}^{2p} \binom{2p}{j} \iint_{-T/2}^{T/2} ds (\phi(s)/k)^j (\alpha/T)^{2p-j} e^{\alpha\phi(s)} e^{-(\alpha/T)\int\phi(s)ds} d\mu_X(\phi)\Bigg]^k$$

$$\geq \exp\,[(\alpha^2 k^2/2T) + (\alpha^2 k G_X(0))/2](k\alpha/2T)^{2pk} T^k \tag{4.5}$$

and we are done. For $X = D$, let $\phi_0(s) = \alpha\psi(s)$, where $\psi(s)$ is a smooth, non-negative function obeying Dirichlet boundary conditions, and a similar analysis yields the required lower bound.

If m is odd, $m = 2p + 1$, but k is even, $k = 2j$, then we translate, as in (4.4) to get

$$\int W^k(\phi) d\mu_X(\phi) = e^{-(k^2/2)\langle\phi_0, A_X\phi_0\rangle} \int W^k(\phi + k\phi_0) e^{-k\langle\phi, A_X\phi_0\rangle} d\mu_X(\phi)\,. \tag{4.6}$$

The integral in (4.6) is

$$\int W^k(\phi+k\phi_0)e^{-k\langle\phi,A_X\phi_0\rangle}d\mu_X(\phi)=\int[W(\phi+k\phi_0)e^{-\langle\phi,A_X\phi_0\rangle}]^{2j}d\mu_X(\phi)$$
$$\geqslant\left[\int[W(\phi+k\phi_0)e^{-\langle\phi,A_X\phi_0\rangle}]^2d\mu_X(\phi)\right]^j, \quad (4.7)$$

again by Jensen's inequality. The argument leading up to (4.5) may now be repeated on the last integral of (4.7) to obtain the lower bound of (2.11).

Finally, if m and k are both odd, $m = 2p + 1$, $k = 2j + 1$, then

$$\int W^k(\phi)d\mu_X(\phi) = \int\left[\int_{-T/2}^{T/2}\phi^{2p+1}(s)e^{\alpha\phi(s)}ds\right]^{2j+1}d\mu_X(\phi)$$
$$=\int_{-T/2}^{T/2}d^{2j+1}s\int\phi^{2p+1}(s_1)e^{\alpha\phi(s_1)}\prod_{i=2}^{2j+1}\phi^{2p+1}(s_i)e^{\alpha\phi(s_i)}d\mu_X(\phi)$$
$$\geqslant\int_{-T/2}^{T/2}d^{2j+1}s\int\phi^{2p+1}(s_1)e^{\alpha\phi(s_1)}d\mu_X(\phi)\int\prod_{i=2}^{2j+1}\phi^{2p+1}(s_i)e^{\alpha\phi(s_i)}d\mu_X(\phi)$$
$$=\int W(\phi)d\mu_X(\phi)\int W^{2j}(\phi)d\mu_X(\phi) \quad (4.8)$$

where we have used the second GKS inequality [*13*, p. 274; *12*, p. 120] in the third line of (7.8). Using GKS requires expanding the exponentials in (4.8) and also noting that the lattice approximation to $d\mu_X(\phi)$ is ferromagnetic for $X = p, D, 0$ (see [*14*, sec. IX.1]. The integral

$$\int W^{2j}(\phi)d\mu_X(\phi)$$

may now be treated as in (4.7) and this finishes the proof of the theorem.

Theorems 2.3 and 2.2 may be combined to yield a second proof of Theorem 2.1. First, comparison of (3.6) and (3.7) shows that

$$(-1)^{k+1}a_k^p(T) \leqslant \frac{(-1)^k}{T}b_k^p(T) \quad (4.9)$$

since the sum over $\gamma\in\Gamma_k$ contains the sum over $\gamma\in\Gamma_k^c$. Combining (4.9) with the $m = 0$ upper bound of (2.11) and the bracketing inequality (2.7) yields

$$(-1)^{k+1}a_k \leqslant \frac{1}{T}D^k\frac{e^{\alpha^2bk^2}}{k!}\,.$$

For the lower bound, it follows from the definition (2.6) that

$$a_k^{\mathrm{D}}(\mathrm{T}) = -\frac{1}{\mathrm{T}}\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n}\,\mathrm{B}(k,n)$$
$$= -\frac{1}{\mathrm{T}}b_k^{\mathrm{D}}(\mathrm{T})\left[1 + (1/b_k^{\mathrm{D}}(\mathrm{T}))\sum_{n=2}^{k}\frac{(-1)^{n-1}}{n}\,\mathrm{B}(k,n)\right], \tag{4.10}$$

where

$$\mathrm{B}(k,n) = \sum_{\substack{k_1+\ldots+k_n=k\\ k_i\geqslant 1}}\prod_{i=1}^{n} b_{k_i}^{\mathrm{D}}(\mathrm{T}),$$

and recall that

$$b_k^{\mathrm{D}}(\mathrm{T}) = \frac{(-1)^k}{k!}\int \mathrm{V}^k(\phi)d\mu_{\mathrm{D}}(\phi).$$

Using (4.10) and the lower bound of Theorem 2.3 it is easy to see that

$$a_k^{\mathrm{D}}(\mathrm{T}) = -\frac{1}{\mathrm{T}}\,b_k^{\mathrm{D}}(\mathrm{T})[1 - 0(e^{-ck})] \tag{4.11}$$

for k large (see for ex., [9, Lemma 2.2]). Briefly, this follows by using

$$\mathrm{B}(k,n) = \sum_{m=1}^{k-n+1} b_m^{\mathrm{D}}(\mathrm{T})\mathrm{B}(k-m, n-1) \tag{4.12}$$

to prove inductively that

$$|\,\mathrm{B}(k,n)\,| \leqslant \mathrm{C}^{n-1}\,|\,b_{k-n+1}^{\mathrm{D}}(\mathrm{T})\,|\,. \tag{4.13}$$

Eq. (4.11) follows from (4.10) and (4.13) since

$$\frac{\mathrm{B}(k,n)}{b_k^{\mathrm{D}}(\mathrm{T})} \leqslant \mathrm{C}^{n-1}\frac{|\,b_{k-n+1}^{\mathrm{D}}(\mathrm{T})\,|}{|\,b_k^{\mathrm{D}}(\mathrm{T})\,|} = \mathrm{C}^{n-1}\frac{k!}{(k-n+1)!}\frac{\|\,\mathrm{V}(\phi)\,\|_{k-n+1}^{k-n+1}}{\|\,\mathrm{V}(\phi)\,\|_k^k}$$
$$\leqslant \mathrm{C}^{n-1}\frac{k!}{(k-n+1)!}\frac{1}{\|\,\mathrm{V}(\phi)\,\|_k^{n-1}} \leqslant \mathrm{C}^{n-1}k^{-(n-1)}e^{-(n-1)a\alpha^2 k}. \tag{4.14}$$

The second line above uses that $\mathrm{V}(\phi) \geqslant 0$, so that

$$\int \mathrm{V}^k(\phi)d\mu_{\mathrm{D}}(\phi) = \|\,\mathrm{V}(\phi)\,\|_k^k,$$

while the last lines uses the $m = 0$ lower bound of (2.11). To prove (4.13), assume it is true for $n-1$, $k \geqslant 2$, and use (4.12) to find

$$|\,\mathrm{B}(k,n)\,| \leqslant \sum_{m=1}^{k-n+1}|\,b_m^{\mathrm{D}}(\mathrm{T})\,|\,\mathrm{C}^{n-2}\,|\,b_{k-m-n+2}^{\mathrm{D}}(\mathrm{T})\,|$$
$$= \mathrm{C}^{n-2}\,|\,\mathrm{B}(k-n+2,2)\,| \leqslant \mathrm{C}^{n-1}\,|\,b_{k-n+1}^{\mathrm{D}}(\mathrm{T})\,|\,,$$

which proves the induction step. The argument used in (4.14) can then be adapted to prove (4.13) when $n = 2$. For example, if k_0 is fixed but large, then

$$\sum_{j=k_0}^{k-k_0} | b_j^{\mathrm{D}}(\mathrm{T}) b_{k-j}^{\mathrm{D}}(\mathrm{T}) / b_{k-1}^{\mathrm{D}}(\mathrm{T}) | \leqslant \mathrm{C}2^k \frac{\int \mathrm{V}^{k_0}(\phi) d\mu_{\mathrm{D}}(\phi) \int \mathrm{V}^{k-k_0}(\phi) d\mu_{\mathrm{D}}(\phi)}{\int \mathrm{V}^{k-1}(\phi) d\mu_{\mathrm{D}}(\phi)}$$

$$\leqslant \mathrm{C}2^k \frac{1}{\| \mathrm{V}(\phi) \|_{k-1}^{k_0-1}} \leqslant \mathrm{C}2^k e^{-(k_0-1)a\alpha^2 k},$$

where the first line uses the log convexity in k of $\int \mathrm{V}^k(\phi) d\mu_{\mathrm{D}}(\phi)$. This completes the proof of (4.11). If the $m = 0$ lower bound of (2.11) is applied to (4.11) along with the bracketing inequality (2.7), then we obtain

$$(-1)^{k+1} a_k \geqslant \frac{(-1)^k}{\mathrm{T}} b_k^{\mathrm{D}}(\mathrm{T}) [1 - 0(e^{-ck})] \geqslant \frac{\mathrm{AB}^k}{\mathrm{T}} \frac{e^{a\alpha^2 k^2}}{k!}$$

which is the lower bound of Theorem 2.1 for k large.

The advantage of the Feynman graph proof of Theorem 2.1 is, first, that the lower bound obtained is valid for all k, and not just k large. Secondly, the upper and lower bounds of (2.1) do not depend on T, while the ones just derived do, and get worse as T increases.

5. DISCUSSION

We have shown that the groundstate energy $\mathrm{E}(\lambda)$ of $\mathrm{H}_1(\lambda)$ and the trace $\mathrm{Tr}\,[\exp(-\mathrm{TH}_2(\lambda))]$ both have divergent perturbation series with rates of divergence which require the generalized Borel method of Grecchi and Maioli for summation. Unfortunately, it has not yet been proven that either of these objects satisfies the hypotheses necessary for summation, although they are expected to. The problem is a technical one. In order to apply the generalized Borel summation to a function, a certain bound on the remainder term in its Taylor expansion must be proven uniformly on a sector of the Riemann surface of $\ln(z)$ [*1*] [*2*]. This seems to be hard to do (see [*1*] for a partial result for $\mathrm{E}(\lambda)$).

Secondly, it would be desirable to extend the divergence estimates for $\mathrm{E}(\lambda)$ to all eigenvalues of $\mathrm{H}_1(\lambda)$. In general, it would be interesting to know under what hypotheses on $\mathrm{V}(x)$, the divergence of the perturbation series for one eigenvalue of

$$\mathrm{H}(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda \mathrm{V}(x)$$

implies divergence of the perturbation series for all eigenvalues. Also, for the examples $V(x) = x^{2m}$ or $V(x) = \exp(\alpha x)$, if the $i^{\underline{\text{th}}}$ eigenvalue $E_i(\lambda)$ has asymptotic series $E_i(\lambda) \sim \Sigma a_k^i \lambda^k$, then the dominant contribution to the large k behavior of a_k^i is independent of i($[m-1)k]$! for x^{2m}, and $\exp(\alpha^2 k^2/4)$ for $\exp(\alpha x)$). It would be interesting to know if this connection can be proven without knowing the asymptotics of a_k^i for all i. In this way, the $\exp(ck^2)$ divergence rate for the ground state energy perturbation coefficients in Theorem 2.1 would extend to all higher eigenvalues of $H_1(\lambda)$.

In [3], Grecchi *et al.* give an argument which strongly suggests that Pade approximants will not converge to the eigenvalues of $H_1(\lambda)$. Our Theorem 2.1 provides further evidence for this claim, since it shows that the coefficients a_k violate the condition

$$| a_k | \leqslant C^{k+1}(2k)!$$

which is a sufficient condition for uniqueness of the Stieltjes moment problem associated with Pade approximants [15, Th. 1.3].

Finally, the two dimensional exponential interaction field theories were mentioned in [1] as possible condidates for generalized Borel summation. We wish to point out that this will not be true since these theories only have asymptotic perturbation series to a finite order. That is, derivatives in the coupling constant λ at $\lambda = 0^+$ fail to exist beyond a certain finite order (see [13, p. 313] for discussion).

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