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Truncated Gamow functions and the exponential decay law

by

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ABSTRACT. — For a quantum mechanical two body ℓ-wave resonance we prove that the evolution of sharp cut-off approximations of the Gamow function is outgoing and exponentially damped. An error estimate is given in terms of resonance energy and width, the time variable and finally an integral expressed by the potential V. Except for spherical symmetry, V is assumed to be a rather general short-range potential. We make use of the energy eigenfunction representation. The energy-transformed function is obtained explicitly (a Breit-Wigner form). The mathematical results are applied to α-decay to prove (within the usual simplified model) general validity of the exponential law for periods of several lifetimes.

RÉSUMÉ. — On montre, pour une résonance de moment angulaire ℓ du problème quantique à deux corps, que des approximations à cut off raide de la fonction de Gamow évoluent comme des ondes sortantes exponentiellement amorties. On donne une estimation d'erreur en termes de l'énergie et de la largeur de la résonance, du temps et d'une intégrale s'exprimant au moyen du potentiel V. En dehors de la symétrique sphérique, on suppose que V est un potentiel à courte portée assez général. On utilise la représentation en fonctions propres de l'énergie. On obtient explicitement la fonction transformée en énergie, qui a une forme de Breit Wigner. On applique les résultats mathématiques à la désintégration α pour démontrer (dans le modèle simplifié habituel) la validité générale de la loi exponentielle pour des durées de plusieurs vies moyennes.

1. INTRODUCTION

In a recent paper [*19*] we proved rigorous results concerning the evolution of truncated Gamow functions. We consider a spherically symmetric, compactly supported potential $V = V(r)$, the s-wave Hamiltonian $H^0 = -\frac{d^2}{dr^2} + V(r)$, a resonance $E - i\Gamma/2$ $(= k_0^2, k_0 = \alpha - i\beta)$ and finally the corresponding Gamow function $f_0(k_0, r)$. This function is regular at $r = 0$ and equal to $e^{ik_0 r}$ for r larger than R_s, a number satisfying $V(r) = 0$ for $r > R_s$. Introducing the truncated Gamow functions $f_R := f_0(k_0, \cdot)\chi_{(0,R)}$ the main result of [*19*] now is as follows (here stated in an imprecise way).

Let $R_1 > R_s$ be given and define $R_2(t) = 2\alpha t + R_1$, $t \geq 0$. Then for a period of many lifetimes we have (measured by the L^2-norm):

$$e^{-itH^0} f_{R_1} \simeq e^{-itk_0^2} f_{R_2(t)} . \tag{1.1}$$

It is remarked that in our units the (reduced) mass of systems described by H^0 is equal to $\frac{1}{2}$. Hence $R_2(t) = V_{cl}t + R_1$, where V_{cl} is the classical speed corresponding to momentum α. The resonance considered is assumed to be narrow in the sense that $1 \gg \frac{\beta}{\alpha}$, so the energy E is given approximately by α^2.

In [*19*] we apply (1.1) to α-decay. An α-particle is (as usual) considered in a spherically symmetric barrier potential $V = V(r)$. The success of Gamow [*10*], in the framework of this model to connect the observed relatively small energy differences of α-particles escaping from heavy nuclei (RaA, RaC′, Ur, etc.) with the observed extremely large decay-rate differences (the decay-rate formula), is well-known. It is claimed in [*10*] that the function $f_0(k_0, r)$ should describe the α-particle state. In this way also the exponential law follows (unrigorously). However, Gamow is somewhat imprecise. Explicitly he does not worry much about the fact that $f_0(k_0, r)$ is exponentially growing at infinity and hence not square integrable. Taking (1.1) into account we complete the theory of Gamow. The α-particle is at $t = 0$ described by some f_{R_1}. According to (1.1) the evolution of f_{R_1} is outgoing and exponentially damped. Beyond the (free) classical evolution radius the position probability is zero. These properties of a radioactive state should be expected. For instance the exponential decay law almost immediately follows. It is remarked that one can as well use smooth cut-off approximations of the Gamow function (close to f_{R_1}) without any significant alteration of the statement (1.1).

Some attempts to normalize the function $f_0(k_0, r)$ in some way and prove exponential decay have been done, see for instance [*5*] [*6*] and [*7*].

In spite of the simplicity of our approach ([*19*]), however, there does not seem (in the otherwise extensive physical literature concerning unstable systems) to be any rigorous results on the subject.

The purpose of this paper is to generalize results of [*19*] in two directions. As previously we proceed rigorously. In this paper we do not require that V(r) has compact support. Instead of this condition, V(r) is assumed to be a rather general, radial and short-range potential. Explicitly of the form of an « exterior analytic » plus an exponentially decaying potential. Secondly we prove results for all angular momentum numbers $l \geq 0$ (not only for $\ell = 0$).

Considering the ℓ-wave Hamiltonian $H^{\ell} = -\frac{d^2}{dr^2} + \ell(\ell+1)/r^2 + V(r)$, a resonance $E - i\Gamma/2 \; (= k_0^2)$ and the Gamow function $f_\ell(k_0, r)$ (these two concepts will be carefully defined) we find a result completely similar to (1.1) concerning the states $e^{-itH^{\ell}} f_{R_1}$. The functions $f_R(r)$ are defined (as before) to be sharp cut-off approximations of $f_\ell(k_0, r)$. The procedure in the present paper leading to this (main) result and some corollaries is close to the one of [*19*]. For instance we consider the $\ell(\ell+1)/r^2$-potential (in an exterior domain) on the same footing as any other analytic potential. In that way the problem of this paper essentially concerns an s-wave resonance defined for a rather general, radial and short-range potential.

To state the main result in three dimensions we put $H_0 = -\Delta$ and $H = H_0 + V$ on $L^2(\mathbb{R}^3)$ and $F_R(\bar{r}) = f_\ell(k_0, r)\chi_{(0,R)}(r) r^{-1} Y_\ell^m(\hat{r})$, where $Y_\ell^m(\hat{r})$ is a spherical harmonic and $f_\ell(k_0, r)$ a Gamow function. Now the analogue of (1.1) is as follows ($k_0 = \alpha - i\beta$).

Let R_1 be given « large » and define $R_2(t) = 2\alpha t + R_1$, $t \geq 0$. Then for a period of « many » lifetimes we have (measured by the L^2-norm).

$$e^{-itH} F_{R_1} \simeq e^{-itk_0^2} F_{R_2(t)} \,. \tag{1.2}$$

To prove (1.2) it is crucial to know the transformed functions $\langle \Psi^+(\bar{k}, \cdot), F_R \| F_R \|^{-1} \rangle$ of the normed (essentially continuum) states $F_R \| F_R \|^{-1}$ expressed by the energy eigenfunctions $\psi^+(\bar{k}, \bar{r})$ ([*14*], p. 299). As $R \to \infty$ and $\frac{\beta}{\alpha} \to 0$ these transformed functions are found to approach (we measure by the L^2-norm)

$$-\,2^{1/2}\left(\frac{\beta}{\pi}\right)^{1/2} e^{i(\alpha-k)R} \frac{S_\ell(-k)k^{1/2}}{k^2 - k_0^2} Y_\ell^m(\hat{k}) \,, \tag{1.3}$$

where $S_\ell(k)$ for $k \in \mathbb{R} \backslash \{0\}$ is the ℓ-wave S-matric element.

It is remarked that for α-decay $1 \gg \frac{\beta}{\alpha}$ is certainly satisfied. Also of course R_1, interpreted as the radius of detection if a decay measurement is started at $t = 0$, is large compared with atomic units.

We now explain in a heuristic way why (1.2) follows from (1.3). Firstly we note that it can be (easily) proved that $\| e^{-itk_0^2} F_{R_{2(t)}} \| \simeq \| F_{R_1} \|$. This fact together with (1.3) provides that the energy-transform of $e^{-itk_0^2} F_{R_{2(t)}} \| F_{R_1} \|^{-1}$ is given by

$$- e^{-itE} 2^{1/2} \left(\frac{\beta}{\pi} \right)^{1/2} e^{i(\alpha - k)R_{2(t)}} \frac{S_\ell(-k) k^{1/2}}{k^2 - k_0^2} Y_\ell^m(\hat{k}) .$$

Because $e^{-itE} e^{i(\alpha-k)R_{2(t)}} = e^{-it(\alpha^2 - \beta^2 - (\alpha - k)2\alpha) + i(\alpha - k)R_1} = e^{-itk^2} e^{i(\alpha-k)R_1} e^{it|k-k_0|^2}$, we conclude that the energy-transform of $e^{-itk_0^2} F_{R_{2(t)}}$ and $e^{-itH} F_{R_1}$ are essentially connected as follows:

$$\langle \Psi^+(\bar{k}, \cdot), e^{-itk_0^2} F_{R_{2(t)}} \rangle \simeq \Psi^+ \langle (\bar{k}, \cdot), e^{-itH} F_{R_1} \rangle e^{it|k-k_0|^2} .$$

For a long time and for k near α, that is where the functions (1.3) are concentrated, we have that $e^{it|k-k_0|^2} \simeq 1$. Hence, $e^{-itH} F_{R_1} \simeq e^{-itk_0^2} F_{R_{2(t)}}$ for a long time.

In [*13*] and [*16*] the cross section for reactions as $0^{16}(d,p)\ {}^{17}0^*$ is calculated using the distorted-wave Born approximation. To do this it is necessary to know the neutron wave function representing the resonance state ${}^{17}0^*$. It seems that the authors to overcome this problem are forced to « guess », cleverly, however still to some extent arbitrarily. In spite of the fact that the resonance considered is not very sharp as for α-decay, such that the energy-transform of F_{R_1} is now given more approximately by (1.3), it is tempting to claim that (1.3) represents the « right » energy-transform of the state ${}^{17}0^*$. Here we take the point of view (as for α-decay) that F_{R_1} is the « best candidate » to represent unstable systems. Making this assumption the above claim is justified by a direct computation ((1.3) is the leading term of the exact transformed function) and we avoid to « guess » as it is done in [*13*] [*16*] and [*8*].

For s-waves and compactly supported potentials the limit transition $R \to \infty$, leading asymptotically $\left(\text{together with } \frac{\beta}{\alpha} \to 0\right)$ to (1.3), is superfluous. This is so because for R finite the deviation from (1.3) (precisely, of modification of (1.3) due to $\left.\frac{\beta}{\alpha} > 0\right)$ is estimated in terms of a parameter $\varepsilon = \varepsilon(R)$ typically given (see (3.18)) by

$$\exp\left(2^{1/2} \alpha^{-1} \int_R^\infty dr\, | V(r) + \ell(\ell + 1)/r^2 | \right) - 1 .$$

Hence the analysis in this paper is more lengthy than in [*19*] due to « ε-modifications ». Also we remark that the proof of Lemma 4.3 is quite different from the proof of the analogous Lemma in [*19*]. A better estimate is obtained, and this is one of the reasons why the main result (here confining ourselves to the potentials of [19]) is in fact slightly improved.

The mathematics in this paper is self-contained. However we refer to [19] for a more detailed discussion concerning application to α-decay and physical interpretation.

In Section 3 (2) we define the notion of resonance and Gamow function. We prove asymptotic estimates to be used in Section 4, where all other mathematical results are derived. The results corresponding to (1.2) and (1.3) are Theorem 4.7 and Lemmas 4.2 and 4.1, respectively. In Section 5 we apply our results to α-decay. We prove general validity of the exponential law for periods of several lifetimes.

2. DEFINITIONS AND ASSUMPTIONS ON V

We consider a multiplicative, radial and real potential $V = V(r)$ satisfying $\int_0^\infty \frac{r}{1+r} |V(r)| \, dr < \infty$. It is then known that V is infinitesimally form-bounded with respect to H_0, where H_0 denotes the free Hamiltonian on $L^2(\mathbb{R}^3)$. Hence the total Hamiltonian $H = H_0 + V$ can be constructed by the standard quadratic form technique, see e. g. [17].

We decompose H_0 and H in a standard way corresponding to the decomposition $L^2(\mathbb{R}^3) = \sum_\ell \sum_m \oplus \{ Y_\ell^m \otimes L^2(\mathbb{R}, r^2 dr) \}$, where Y_ℓ^m are the spherical harmonics. As usual we map $L^2(\mathbb{R}^+, r^2 dr)$ onto $L^2(\mathbb{R}^+)$ by the unitary operator $u(r) \to ru(r)$. In this way we finally obtain the operators to be studied, H_0^ℓ and H^ℓ on $L^2(\mathbb{R}^+)$ given for all angular momentum numbers by $H_0^\ell = -\frac{d^2}{dr^2} + \ell(\ell+1)/r^2$ and $H^\ell = H_0^\ell + V$ respectively.

For some $\frac{\pi}{2} > \sigma > 0$, $R_\sigma > 0$ and $a > 0$ we furthermore impose the condition that for $r \geq R_\sigma$

$$\ell(\ell+1)/r^2 + V(r) = V_1(r) + V_2(r),$$

where

$$\int_{R_\sigma}^\infty |V_2(r)| e^{2ar} dr < \infty \qquad \text{and}$$

$V_1(r)$ has a continuous extension to $M_\sigma := \{ z \,|\, |z| \geq R_\sigma \text{ and } |\operatorname{Arg} z| \leq \sigma \}$, analytic in the interior of M_σ. Also $zV_1(z) \to 0$ for $z \to \infty$ in M_σ and $\sup_{-\sigma \leq \theta \leq \sigma} \int_{e^{i\theta}R_\sigma}^{e^{i\theta}\infty} |dz| \, |V_1(z)| < \infty$ are assumed.

The following functions can all be found in Newton [14] Section 12.2. We consider solutions $\varphi_\ell(k,r)$, $f_\ell(k,r)$ and $\Psi_\ell^+(k,r)$ of the equation $\left(-\frac{d^2}{dr^2}+\ell(\ell+1)/r^2+V(r)-k^2\right)\Psi(r)=0$. $\varphi_\ell(k,r)$ is for all k the regular solution satisfying $\lim_{r\to 0} r^{-(\ell+1)}\varphi_\ell(k,r)=1$. $\varphi_\ell(k,r)$ can be constructed by iteration of an integral equation cf. [14] (12.133). $f_\ell(k,r)$ is for $k\in\mathbb{C}^+=\{\zeta\neq 0\mid \operatorname{Im}\zeta\geq 0\}$ the outgoing solution defined uniquely by

$$f_\ell(k,r)=e^{ikr}-\frac{1}{k}\int_r^\infty dr'\sin k(r-r')\{\ell(\ell+1)/r'^2+V(r')\}f_\ell(k,r'). \quad (2.1)$$

The equation can be solved by iteration, cf. [14] Section 12.1. For $k\in\mathbb{C}^+$ the Jost function $F_\ell(k)$ is given by

$$F_\ell(k)=W(f_\ell(k,r),\varphi_\ell(k,r))=\frac{d}{dr}\varphi_\ell(k,r)f_\ell(k,r)-\varphi_\ell(k,r)\frac{d}{dr}f_\ell(k,r),$$

i. e. the Wronskian between $\varphi_\ell(k,r)$ and $f_\ell(k,r)$.

For $k\in\mathbb{R}\setminus\{0\}$ we have

$$\varphi_\ell(k,r)=\frac{1}{2ik}(F_\ell(-k)f_\ell(k,r)-F_\ell(k)f_\ell(-k,r)). \quad (2.2)$$

Using that $\overline{F_\ell(k)}=F_\ell(-k)\,(k\in\mathbb{R}\setminus\{0\})$ we find from (2.2) that $F_\ell(k)\neq 0$. For $k\in\mathbb{R}^+$ the physical wave function $\Psi_\ell^+(k,r)$ is defined to be equal to $\frac{k}{F_\ell(k)}\varphi_\ell(k,r)e^{i\ell\pi/2}$.

The « unitary property » of $S_\ell(k):=(-1)^\ell\frac{F_\ell(-k)}{F_\ell(k)}$ for $k\in\mathbb{R}\setminus\{0\}$

$$|S_\ell(k)|=1, \quad (2.3)$$

and the equation

$$\overline{\Psi_\ell^+(k,r)}=\frac{k\varphi_\ell(k,r)}{F_\ell(-k)}e^{-i\ell\pi/2}, \quad (2.4)$$

are also simple consequences of $\overline{F_\ell(k)}=F_\ell(-k)$.

In this paper we make use of the following expression for the kernel of the spectral density $\frac{dE_\lambda^\ell}{d\lambda}$ of H^ℓ, where we put $\lambda=k^2$ and $k>0$.

$$\frac{d}{d\lambda}E_\lambda^\ell(r,r')=\frac{1}{k\pi}\Psi_\ell^+(k,r)\overline{\Psi_\ell^+(k,r')}.$$

The expression is found utilizing the identity

$$2\pi i\frac{d}{d\lambda}E_\lambda^\ell=\lim_{\varepsilon\downarrow 0}\{H^\ell-\lambda-i\varepsilon)^{-1}-(H^\ell-\lambda+i\varepsilon)^{-1}\},$$

the fact that the kernel of $(H^\ell - \lambda^2 \mp io)^{-1}$ is given (cf. [14] (12.146)) by

$$\frac{\varphi_\ell(k,r)f_\ell(\pm k,r')}{F_\ell(\pm k)} \quad \text{for} \quad r \leq r'$$

and
$$\frac{\varphi_\ell(k,r')f_\ell(\pm k,r)}{F_\ell(\pm k)} \quad \text{for} \quad r > r',$$

and finally the formulas (2.2), (2.4).

The well-known eigenfunction expansion theorem (see for instance Agmon [1]) is closely related to the above formula.

For $\delta > 0$ we denote by P_δ the spectral projection

$$I - E^\ell_{\delta^2} = \chi_{(\delta^2,\infty)}(H^\ell). \tag{2.5}$$

It is easy to prove for fixed $r > 0$ that $\varphi_\ell(k,r)$ and $\frac{d}{dr}\varphi_\ell(k,r)$ are entire functions in k and that $f_\ell(k,r)$, $\frac{d}{dr}f_\ell(k,r)$ and $F_\ell(k)$ are continuous on $\mathbb{C}^+$, analytic in the interior. Letting $S_\sigma = \{\zeta \neq 0 \mid 0 > \operatorname{Arg}\zeta > -\sigma\}$ and $T_a = \{\zeta \mid 0 > \operatorname{Im} > -a\}$ we continue $f_\ell(k,r)$ $(r \geq R_\sigma)$ and $F_\ell(k)$ analytically in k to $\mathbb{C}^+ \cup (S_\sigma \cap T_a)$. This is done in Section 3. Of course (2.2) then also holds true for $k \in S_\sigma \cap T_a$. A resonance is defined to be a point $k_0 = \alpha - i\beta$ $(\alpha, \beta > 0)$ where $F_\ell(k_0) = 0$. For convenience also k_0^2 is called a resonance. We define the resonance energy E and the width Γ by $k_0^2 = \alpha^2 - \beta^2 - i2\alpha\beta = E - i\Gamma/2$. We introduce truncated Gamow functions $f_R = f_\ell(k_0,\cdot)\chi_{(0,R)}$ for $R > R_\sigma$.

The following Wronski formulas are useful.

$$\frac{d}{dr}W(\overline{f_\ell(k_0,r)}, f_\ell(k_0,r)) = (\overline{k_0^2} - k_0^2)\,|\,f_\ell(k_0,r)\,|^2, \tag{2.6}$$

and

$$\frac{d}{dr}W(\varphi_\ell(k,r), f_\ell(k_0,r)) = (k^2 - k_0^2)\varphi_\ell(k,r)f_\ell(k_0,r). \tag{2.7}$$

3. THE GAMOW FUNCTION, CONSTRUCTION AND ESTIMATES

For $r \geq R_\sigma$ fixed we shall continue the outgoing solution $f_\ell(k,r)$ analytically from $k \in \mathbb{C}^+$ to $\mathbb{C}^+ \cup (S_\sigma \cap T_a)$. It is not clear how to do this from the defining equation (2.1). However for k positive $f_\ell(k,r)$ is also given uniquely by the equation

$$f_\ell(k,r) = f_\ell^1(k,r) - \int_r^\infty dr' G(r,r')V_2(r')f_\ell(k,r'), \tag{3.1}$$

where $f_\ell^1(k,r)$ for $k \in \mathbb{C}^+$ is the solution of

$$f_\ell^1(k,r) = e^{ikr} - \frac{1}{k}\int_r^\infty dr' \sin k(r-r')\mathrm{V}_1(r')f_\ell^1(k,r'), \tag{3.2}$$

and

$$\mathrm{G}(r,r') = \frac{1}{2ik}\{ f_\ell^1(-k,r')f_\ell^1(k,r) - f_\ell^1(-k,r)f_\ell^1(k,r') \}. \tag{3.3}$$

It is remarked that $\left(-\frac{d^2}{dr^2} + \mathrm{V}_1(r) - k^2\right)\chi_{(0,r')}(r)\mathrm{G}(r,r') = \delta(r-r')$ and that (3.2) and (3.1) can be solved by iteration, cf. [*14*].

The procedure is now as follows:

A. The function $f_\ell^1(k,r)$ is for fixed $r \geq \mathrm{R}_\sigma$ continued analytically in k from $\mathbb{C}^+$ to $\mathbb{C}^+ \cup \mathrm{S}_\sigma$. We estimate $|f_\ell^1(k,r)|$ and $|\mathrm{G}(r,r')|$ (for $k \in \mathrm{S}_\sigma$, $\mathrm{G}(r,r')$ is also defined by (3.3)).

B. The equation (3.1) is solved for $k \in \mathrm{S}_\sigma \cap \mathrm{T}_a$ and $r \geq \mathrm{R}_\sigma$. It is proved that $f_\ell(k,r)$ is analytic in $k \in \mathbb{C}^+ \cup (\mathrm{S}_\sigma \cap \mathrm{T}_a)$ (1).

A

Using the conditions imposed on $\mathrm{V}_1(r)$ we can extend $f_\ell^1(k,r)$ for k positive and $r \geq \mathrm{R}_\sigma$ continuously in r to $\mathrm{N}_\sigma := \{ z \,|\, |z| \geq \mathrm{R}_\sigma \text{ and } 0 \leq \operatorname{Arg} z \leq \sigma \}$, analytically in the interior of N_σ.

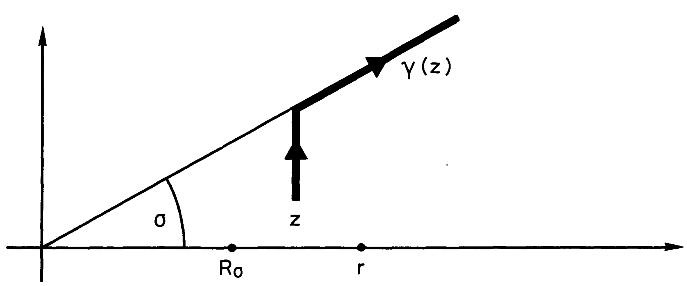

DIAGRAM 1.

(1) In the above procedure for construction of outgoing solutions the potential V_2 is considered as a perturbation of $-\frac{d^2}{dr^2} + \mathrm{V}_1$. This is also a basic strategy in Balslev's article in [*2*].

Explicitly, for $k > 0$ and $z \in \mathrm{N}_\sigma$ $f_\ell^1(k, z) = \sum_{n=0}^{\infty} f_\ell^{1,n}(k, z)$, where

$$e^{-ikz} f_\ell^{1,0}(k, z) = 1$$
$$\vdots \qquad \vdots$$
$$e^{-ikz} f_\ell^{1,n}(k, z) = \frac{-1}{2ik} \int_{\gamma(z)} dz_1 (1 - e^{2ik(z_1 - z)}) \mathrm{V}_1(z_1) e^{-ikz_1} f_\ell^{1,n-1}(k, z_1) \qquad (n \geq 1). \quad (3.4)$$

We remark that $f_\ell^{1,n}(k, z = r)$ $(r \geq \mathrm{R}_\sigma)$ coincides with the n'th iteration term corresponding to (3.2). This and the stated continuity and analyticity properties of $f_\ell^1(k, z)$ (the same properties hold true for $f_\ell^{1,n}(k, z)$, too) can be proved by induction using Cauchy's integral theorem. We refer to [*18*] Section 4B for more details. Consult also [*14*] p. 339 and [*15*] p. 145. The above properties of an analytic potential was first noted and utilized in [*4*].

We let for $k \in \mathrm{S}_\sigma$ and $z \in \mathrm{N}_\sigma$ $f_\ell^1(k, z) = \sum_{n=0}^{\infty} f_\ell^{1,n}(k, z)$ be defined by (3.4). Remark that

$$| f_\ell^{1,n}(k, z) | \leq | e^{ikz} | \frac{1}{n!} \left(\frac{1}{|k|} \int_{\gamma(z)} | dz_1 | | \mathrm{V}_1(z_1) | \right)^n, \qquad (3.5)$$

due to the fact that $\mathrm{Im}\, k(z_1 - z) \geq 0$. (3.5) holds true for $(k, z) \in (\mathrm{S}_\sigma \cup \mathbb{R}^+) \times \mathrm{N}_\sigma$. By induction we can prove that $f_\ell^1(k, z)$ is continuous on this set and separately analytic in $k \in \mathrm{S}_\sigma$ and z belonging to the interior of N_σ.

Taking into account that $f_\ell^1(k, r)$ is continuous in $k \in \mathbb{C}^+$ and analytic in the interior, it is now clear that for $r \geq \mathrm{R}_\sigma$ fixed $f_\ell^1(k, r)$ is analytic in $k \in \mathbb{C}^+ \cup \mathrm{S}_\sigma$. Furthermore the analogue of (3.2) for $k \in \mathrm{S}_\sigma$ (valid for $k \in \mathbb{R}^+$, too) is as follows:

$$f_\ell^1(k, r) = e^{ikr} - \frac{1}{k} \int_{\gamma(r)} dz \sin k(r - z) \mathrm{V}_1(z) f_\ell^1(k, z). \qquad (3.6)$$

Estimates of $| f_\ell^1(k, r) |$ *and* $| \mathrm{G}(r, r') |$.

For $k \in \mathrm{S}_\sigma$ we find using (3.5):

$$| f_\ell^1(k, r) | \leq | e^{ikr} | \exp \left\{ \frac{1}{|k|} \int_{\gamma(r)} | dz | | \mathrm{V}_1(z) | \right\}.$$

For $k \in \mathbb{C}^+$:

$$| f_\ell^1(k, r) | \leq | e^{ikr} | \exp \left\{ \frac{1}{|k|} \int_r^{\infty} dr' | \mathrm{V}_1(r) | \right\}.$$

Introducing

$$\mathrm{B}_{\mathrm{V}_1}(r,k)=\exp\left\{\frac{1}{|k|}\max\left[\sup_{r'>r}\int_{\gamma(r')}|dz|\,|\mathrm{V}_1(z)|,\ \int_r^\infty dr'\,|\mathrm{V}_1(r')|\right]\right\}$$

we conclude that for all $k\in \mathrm{S}_\sigma \cup \mathbb{C}^+$ and $r\geq \mathrm{R}_\sigma$

$$|f_\ell^1(k,r)|\leq |e^{ikr}|\,\mathrm{B}_{\mathrm{V}_1}(r,k). \tag{3.7}$$

Concerning $\mathrm{G}(r,r')$ for $r'\geq r\geq \mathrm{R}_\sigma$ and $k\in \mathrm{S}_\sigma\cup\mathbb{R}^+$ we have

$$|\mathrm{G}(r,r')|\leq |e^{ik(r'-r)}|\frac{1}{|k|}\mathrm{B}_{\mathrm{V}_1}(r,k)^2. \tag{3.8}$$

B

For $k\in\mathbb{R}^+\cup(\mathrm{S}_\sigma\cap \mathrm{T}_a)$ and $r\geq \mathrm{R}_\sigma$ we will solve (3.1) by iteration. Each iteration term is denoted by $f_\ell^{(n)}(k,r)$. We introduce

$$h_\ell^{(n)}(k,r):=f_\ell^{(n)}(k,r)\,|e^{ikr}|^{-1}\mathrm{B}_{\mathrm{V}_1}(r,k)^{-1},\qquad n\geq 0.$$

We find using (3.7) and (3.8) that $|h_\ell^{(0)}(k,r)|\leq 1$ and for $n\geq 1$ $|h_\ell^{(n)}(k,r)|\leq \frac{1}{|k|}\mathrm{B}_{\mathrm{V}_1}(r,k)^2\int_r^\infty dr'\,|e^{2ik(r'-r)}|\,|\mathrm{V}_2(r')|\,|h_\ell^{(n-1)}(k,r')|$. Thus $\left|\sum_{n=0}^\infty h_\ell^{(n)}(k,r)\right|\leq \exp\left\{\frac{1}{|k|}\mathrm{B}_{\mathrm{V}_1}(r,k)^2\,|e^{-2ikr}|\int_r^\infty dr'\,|e^{2ikr'}|\,|\mathrm{V}_2(r')|\right\}$. Let $\mathrm{C}_{\mathrm{V}_2}(r)=\int_r^\infty dr' e^{2ar'}|\mathrm{V}_2(r')|$. Then for all $k\in\mathbb{R}^+\cup(\mathrm{S}_\sigma\cap\mathrm{T}_a)$ and $r\geq\mathrm{R}_\sigma$, $f_\ell(k,r):=|e^{ikr}|\,\mathrm{B}_{\mathrm{V}_1}(r,k)\sum_{n=0}^\infty h_\ell^{(n)}(k,r)$ satisfies

$$|f_\ell(k,r)|\leq |e^{ikr}|\,\mathrm{B}_{\mathrm{V}_1}(r,k)\exp\left\{\frac{1}{|k|}\mathrm{B}_{\mathrm{V}_1}(r,k)^2\mathrm{C}_{\mathrm{V}_2}(r)\right\}. \tag{3.9}$$

Clearly, $f_\ell(k,r)$ for $k\in\mathbb{R}^+\cup(\mathrm{S}_\sigma\cap\mathrm{T}_a)$ and $r\geq\mathrm{R}_\sigma$ is a solution of (3.1). Also for $r\geq \mathrm{R}_\sigma$ fixed $f_\ell^{(n)}(k,r)$ is continuous in $k\in\mathbb{R}^+\cup(\mathrm{S}_\sigma\cap\mathrm{T}_a)$ and analytic in the interior. Because of local uniform convergence cf. (3.9) the same conclusion is valid for $f_\ell(k,r)$. Taking into account also that $f_\ell(k,r)$, defined by (2.1) for $k\in\mathbb{C}^+$, is continuous and analytic in the interior of $\mathbb{C}^+$ ($r>0$ fixed), we conclude that $f_\ell(k,r)$ for $r\geq\mathrm{R}_\sigma$ fixed is analytic in $k\in\mathbb{C}^+\cup(\mathrm{S}_\sigma\cap\mathrm{T}_a)$.

The Jost function.

We shall continue $\mathrm{F}_\ell(k)$ analytically from $k\in\mathbb{C}^+$ to $\mathbb{C}^+\cup(\mathrm{S}_\sigma\cap\mathrm{T}_a)$.

We consider $\frac{d}{dr} f_\ell(k,r)$ for $r > R_\sigma$:

For $k \in \mathbb{R}^+ \cup (S_\sigma \cap T_a)$ and $r > R_\sigma$ we find differentiating (3.1) and (3.6) that

$$\frac{d}{dr} f_\ell(k,r) = ike^{ikr} - \int_{\gamma(r)} dz \cos k(r-z) V_1(z) f_\ell^1(k,z)$$
$$- \int_r^\infty dr' \frac{1}{2ik} \left\{ f_\ell^1(-k,r') \frac{d}{dr} f_\ell^1(k,r) - \frac{d}{dr} f_\ell^1(-k,r) f_\ell^1(k,r') \right\} V_2(r') f_\ell(k,r') . \tag{3.10}$$

The calculation is justified by (3.7) and (3.9).

From (3.10) we get that $\frac{d}{dr} f_\ell(k,r)$ $(r > R_\sigma)$ is continuous in $k \in \mathbb{R}^+ \cup (S_\sigma \cap T_a)$ and analytic in the interior. Because $\frac{d}{dr} f_\ell(k,r)$ $(r > 0)$ is continuous in $k \in \mathbb{C}^+$ and analytic in the interior, we conclude that $\frac{d}{dr} f_\ell(k,r)$ for $r > R_\sigma$ fixed is analytic in $\mathbb{C}^+ \cup (S_\sigma \cap T_a)$.

Taking into account that $\varphi_\ell(k,r)$ and $\frac{d}{dr}\varphi_\ell(k,r)$ for $r > 0$ fixed are entire analytic we now have proved that the Jost function $F_\ell(k) = W(f_\ell(k,r), \varphi_\ell(k,r))$ for $r > R_\sigma$ is analytic in $\mathbb{C}^+ \cup (S_\sigma \cap T_a)$.

Asymptotic estimates of the Gamow function.

We consider a resonance $k_0 = \alpha - i\beta$ (that is a point $k_0 \in S_\sigma \cap T_a$ where $F_\ell(k_0) = 0$) and the corresponding function $f_\ell(k_0, r)$. To be used in Section 4 we estimate the quantities $| f_\ell(k_0,r) - e^{ik_0 r} |$ and $\left| \frac{d}{dr} f_\ell(k_0,r) - ik_0 e^{ik_0 r} \right|$ $(r > R_\sigma)$:

By (3.5)

$$| f_\ell^1(k_0,r) - e^{ik_0 r} | \leq | e^{ik_0 r} | \left(\exp\left\{ \frac{1}{|k_0|} \int_{\gamma(r)} | dz | | V_1(z) | \right\} - 1 \right)$$
$$\leq (B_{V_1}(r,k_0) - 1) | e^{ik_0 r} | . \tag{3.11}$$

Using (3.1), (3.8), (3.9) and (3.11) we easily find that

$$| f_\ell(k_0,r) - e^{ik_0 r} | \leq (B_{V_1}(r,k_0) - 1) | e^{ik_0 r} |$$
$$+ | e^{ik_0 r} | B_{V_1}(r,k_0) \left(\exp\left\{ \frac{1}{|k_0|} B_{V_1}(r,k_0)^2 C_{V_2}(r) \right\} - 1 \right). \tag{3.12}$$

By differentiating (3.6) and applying (3.5) we get that

$$\left| \frac{d}{dr} f_\ell^1(k_0,r) \right| \leq | k_0 e^{ik_0 r} | B_{V_1}(r,k_0) .$$

We use the estimate in the equation (3.10) and find that

$$\left| \frac{d}{dr} f_\ell(k_0, r) - ik_0 e^{ik_0 r} \right| \leq | k_0 e^{ik_0 r} | (B_{V_1}(r, k_0) - 1) + | k_0 e^{ik_0 r} | B_{V_1}(r, k_0) \left(\exp \left\{ \frac{1}{| k_0 |} B_{V_1}(r, k_0)^2 C_{V_2}(r) \right\} - 1 \right). \quad (3.13)$$

We introduce ε_1 and ε_2 by the equations ($r > R_\sigma$)

$$f_\ell(k_0, r) = e^{ik_0 r}(1 + \varepsilon_1) \quad \text{and} \quad \frac{d}{dr} f_\ell(k_0, r) = ik_0 e^{ik_0 r}(1 + \varepsilon_2). \quad (3.14)$$

Now we can rewrite (3.12) and (3.13) as follows:

$$| \varepsilon_i | \leq B_{V_1}(r, k_0) - 1 + B_{V_1}(r, k_0) \left(\exp \left\{ \frac{1}{| k_0 |} B_{V_1}(r, k_0)^2 C_{V_2}(r) \right\} - 1 \right), \quad i = 1, 2. \quad (3.15)$$

In an analogous way we introduce for $k \in \mathbb{R} \setminus \{ 0 \}$ and $r > R_\sigma$ the quantities ε_3 and ε_4:

$$f_\ell(k, r) = e^{ikr}(1 + \varepsilon_3), \qquad \frac{d}{dr} f_\ell(k, r) = ike^{ikr}(1 + \varepsilon_4). \quad (3.16)$$

Using (2.1) it is easy to prove that

$$| \varepsilon_i | \leq \exp \left\{ \frac{1}{| k |} \int_r^\infty dr' \, | V(r') + \ell(\ell + 1)/r'^2 | \right\} - 1, \quad i = 3, 4. \quad (3.17)$$

For $R > R_\sigma$ we introduce $\varepsilon = \varepsilon(R)$ defined to be the largest of

$$B_{V_1}(R, k_0) - 1 + B_{V_1}(R, k_0) \left(\exp \left\{ \frac{1}{| k_0 |} B_{V_1}(R, k_0)^2 C_{V_2}(R) \right\} - 1 \right)$$

and
$$\exp \left\{ \frac{2^{1/2}}{\alpha} \int_R^\infty dr \, | V(r) + \ell(\ell + 1)/r^2 | \right\} - 1, \quad (3.18)$$

where (recalling the definitions of $B_{V_1}(r, k)$ and $C_{V_2}(r)$)

$$B_{V_1}(R, k_0) = \exp \left\{ \frac{1}{| k_0 |} \max \left[\sup_{r' \geq R} \int_{\gamma(r')} | dz | \, | V_1(z) |, \int_R^\infty dr' \, | V_1(r') | \right] \right\}$$

(the path γ is given in Diagram 1)

and
$$C_{V_2}(R) = \int_R^\infty dr' e^{2ar'} | V_2(r') | .$$

The « choice » $| k | = \alpha 2^{-1/2}$ is to some extent arbitrary. For k satisfying $| k | \geq \alpha 2^{-1/2}$, $| \varepsilon_i(k, R) | \leq \varepsilon(R)$ ($i = 1, \ldots, 4$) cf. (3.15) and (3.17).

4. THE MATHEMATICAL RESULTS

Throughout this section we fix an ℓ-wave resonance $k_0 = \alpha - i\beta$. We find certain bounds in terms of k_0, t and ε (defined by (3.18)). ε and $\frac{\beta}{\alpha}$ are considered to be small compared with 1. ε_i ($i = 1, \ldots, 4$) defined by (3.14) and (3.16) are utilized.

LEMMA 4.1. — For $R > R_\sigma$ we have that ($\varepsilon_i = \varepsilon_i(R)$, $i = 1, 2$)

$$\| f_R \|^2 = \int_0^R | f_\ell(k_0, r)|^2 dr = \frac{| f_\ell(k_0, R)|^2}{2\beta} \alpha^{-1} \operatorname{Im}\left\{ \frac{d}{dr} f_\ell(k_0, R)/ f_\ell(k_0, R) \right\}$$
$$= \frac{e^{2\beta R}}{2\beta}\left(1 + \operatorname{Re}\left\{ \frac{k_0}{\alpha}(\varepsilon_2 + \bar{\varepsilon}_1 + \varepsilon_2\bar{\varepsilon}_1) \right\}\right).$$

Proof. — We integrate (2.6) and use that $f_\ell(k_0, r)$ is regular at $r = 0$.

LEMMA 4.2. — We consider $k > 0$ and $R > R_\sigma$. Let

$$(\varepsilon_i = \varepsilon_i(k, R)) \; h(\varepsilon_i) = h(\varepsilon_1, \ldots, \varepsilon_4) = (1+\varepsilon_4)(1+\varepsilon_3)^{-1} - (1+\varepsilon_2)(1+\varepsilon_1)^{-1}$$

and

$$x(k, R) = \frac{1}{2}\{(-1)^\ell S_\ell(k) f_\ell(k, R) - f_\ell(-k, R)\} f_\ell(k, R)$$
$$= \frac{1}{2}\{(-1)^\ell S_\ell(k) e^{2ikR}(1+\varepsilon_3)^2 - (1+\varepsilon_3)(1+\varepsilon_3(-k, R))\}.$$

Then

$$\langle \psi_\ell^+(k, \cdot), f_R \rangle := \int_0^\infty \overline{\Psi_\ell^+(k, r)} f_R(r) dr$$
$$= \frac{-k}{k^2 - k_0^2} e^{i\ell\pi/2} S_\ell(-k) f_\ell(k_0, R)/ f_\ell(k, R)$$
$$\left[1 + \frac{x(k, R)}{ik}\left\{ \frac{d}{dr} f_\ell(k, R)/ f_\ell(k, R) - \frac{d}{dr} f_\ell(k_0, R)/ f_\ell(k_0, R) \right\}\right]$$
$$= \frac{-k}{k^2 - k_0^2} e^{i\ell\pi/2} S_\ell(-k) e^{i(k_0 - k)R}(1+\varepsilon_1)(1+\varepsilon_3)^{-1}\{1 + h(\varepsilon_i) x(k, R)\}$$
$$\left[1 + \frac{k - k_0}{k}(1+\varepsilon_2)(1+\varepsilon_1)^{-1} x(k, R)\{1 + h(\varepsilon_i) x(k, R)\}^{-1}\right].$$

Proof. — By integrating (2.7) and using the known boundary condition at $r = 0$ we obtain

$$\int_0^R \varphi_\ell(k, r) f_\ell(k_0, r) dr = \frac{-1}{k^2 - k_0^2}\left(- \varphi_\ell(k, \mathrm{R}) \frac{d}{dr} f_\ell(k_0, \mathrm{R}) + \frac{d}{dr} \varphi_\ell(k, \mathrm{R}) f_\ell(k_0, \mathrm{R}) \right). \quad (4.1)$$

We use (2.2) in the following calculation:

$$\begin{aligned} &\left(-\varphi_\ell(k, \mathrm{R}) \frac{d}{dr} f_\ell(k_0, \mathrm{R}) + \frac{d}{dr} \varphi_\ell(k, \mathrm{R}) f_\ell(k_0, \mathrm{R}) \right) \\ &= f_\ell(k_0, \mathrm{R})/f_\ell(k, \mathrm{R}) \Big[-\varphi_\ell(k, \mathrm{R}) \frac{d}{dr} f_\ell(k, \mathrm{R}) + \frac{d}{dr} \varphi_\ell(k, \mathrm{R}) f_\ell(k, \mathrm{R}) \\ &+ \left\{ \frac{d}{dr} f_\ell(k, \mathrm{R})/f_\ell(k, \mathrm{R}) - \frac{d}{dr} f_\ell(k_0, \mathrm{R})/f_\ell(k_0, \mathrm{R}) \right\} \varphi_\ell(k, \mathrm{R}) f_\ell(k, \mathrm{R}) \Big] \\ &= f_\ell(k_0, \mathrm{R})/f_\ell(k, \mathrm{R}) \Big[1 + \Big\{ \frac{d}{dr} f_\ell(k, \mathrm{R})/f_\ell(k, \mathrm{R}) \\ &- \frac{d}{dr} f_\ell(k_0, \mathrm{R})/f_\ell(k_0, \mathrm{R}) \Big\} \frac{x(k, \mathrm{R})}{ik} \Big] \mathrm{F}_\ell(k). \quad (4.2) \end{aligned}$$

Now the Lemma easily follows from (2.4), (4.1) and (4.2).

LEMMA 4.3. — Let $\mathrm{R} > \mathrm{R}_\sigma$ be given. Introducing P_δ where $\delta = \alpha 2^{-1/2}$ (defined by (2.5)) we have the estimate ($\varepsilon = \varepsilon(\mathrm{R})$)

$$\| (\mathrm{I} - \mathrm{P}_\delta) f_\mathrm{R} \|^2 \leq \| f_\mathrm{R} \|^2 (e^{4\frac{\beta}{\alpha}} - 1) \left\{ 1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha} \pi \right\}^2 \left\{ 1 + 2\left(1 + \frac{|k_0|}{\alpha}\right)\varepsilon + \mathrm{O}_1(\varepsilon^2) \right\}.$$

REMARK 4.4. — The term $\mathrm{O}_1(\varepsilon^2)$ can be given explicitly, see (4.9).

Proof. — Let $d > 0$ be given. We define functions φ_R, ψ_R and g_R as follows:

$$\varphi_\mathrm{R}(r) = \begin{cases} 1 & , \quad r < \mathrm{R} \\ \frac{1}{2} + \frac{1}{2} \cos\left(\frac{r - \mathrm{R}}{d} \pi \right), & \mathrm{R} \leq r \leq \mathrm{R} + d \\ 0 & , \quad \mathrm{R} + d < r, \end{cases}$$

$$\psi_\mathrm{R} = \left(-\frac{d^2}{dr^2} \varphi_\mathrm{R} \right) f_\ell(k_0, \cdot) - 2\left(\frac{d}{dr} \varphi_\mathrm{R} \right) \frac{d}{dr} f_\ell(k_0, \cdot)$$

and

$$g_\mathrm{R} = \varphi_\mathrm{R} f_\ell(k_0, \cdot).$$

It is easily verified that $g_\mathrm{R} \in \mathrm{D}(\mathrm{H}^\ell)$ and that $(\mathrm{H}^\ell - k_0^2) g_\mathrm{R} = \Psi_\mathrm{R}$.

Hence

$$g_R = (H' - k_0^2)^{-1}\Psi_R .$$

An application of the spectral theorem now provides the identity

$$\int_{(-\infty,\delta^2]} dE'_\lambda g_R = \int_{(-\infty,\delta^2]} (\lambda - k_0^2)^{-1} dE'_\lambda \psi_R . \tag{4.3}$$

From (4.3) we obtain, applying the spectral theorem again, the following inequality

$$\| (I - P_\delta) f_R \| \le \| f_R - g_R \| + 2\alpha^{-2} \| \psi_R \| . \tag{4.4}$$

To estimate the right-hand side we observe that

$$f_R(r) - g_R(r) = e^{ik_0 r}(1 + \varepsilon_1(r))\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{r - R}{d}\pi\right)\right)\chi_{(R,R+d)}(r) \tag{4.5}$$

and

$$\psi_R(r) = e^{ik_0 r}\left\{\frac{1}{2}\left(\frac{\pi}{d}\right)^2 (1 + \varepsilon_1(r))\cos\left(\frac{r - R}{d}\pi\right) + i\frac{\pi}{d}k_0(1 + \varepsilon_2(r))\sin\left(\frac{r - R}{d}\pi\right)\right\}\chi_{(R,R+d)}(r) . \tag{4.6}$$

Using (4.4) together with the expressions (4.5) and (4.6) we now find that

$$\|(I - P_\delta) f_R\|^2 \le \int_R^{R+d} e^{2\beta r} dr (1 + \varepsilon(R))^2 \left\{1 + 2\alpha^{-2}\left(\frac{1}{2}\left(\frac{\pi}{d}\right)^2 + \frac{\pi}{d}|k_0|\right)\right\}^2 . \tag{4.7}$$

By Lemma 4.1

$$\frac{e^{2\beta R}}{2\beta} \le \| f_R \|^2 \left\{1 - \frac{|k_0|}{\alpha}(2\varepsilon + \varepsilon^2)\right\}^{-1} . \tag{4.8}$$

Calculating the integral on the right-hand side of (4.7) we finally obtain from (4.7) and (4.8) that

$$\|(I - P_\delta) f_R\|^2 \le \| f_R \|^2 (e^{2\beta d} - 1)\left\{1 + 2\alpha^{-2}\left(\frac{1}{2}\left(\frac{\pi}{d}\right)^2 + \frac{\pi}{d}|k_0|\right)\right\}^2 (1 + \varepsilon)^2 \left\{1 - \frac{|k_0|}{\alpha}(2\varepsilon + \varepsilon^2)\right\}^{-1} .$$

Hence, taking $d = 2\alpha^{-1}$,

$$\begin{aligned} &\|(I - P_\delta) f_R\|^2 \\ &\le \| f_R \|^2 (e^{4\frac{\beta}{\alpha}} - 1)\left\{1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi\right\}^2 (1 + \varepsilon)^2 \left\{1 - \frac{|k_0|}{\alpha}(2\varepsilon + \varepsilon^2)\right\}^{-1} \\ &= \| f_R \|^2 (e^{4\frac{\beta}{\alpha}} - 1)\left\{1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi\right\}^2 \left\{1 + 2\left(1 + \frac{|k_0|}{\alpha}\right)\varepsilon + O_1(\varepsilon^2)\right\} . \end{aligned} \tag{4.9}$$

The proof of the Lemma is complete.

We fix $R_1 > R_\sigma$ in the remaining part of this Section. By ε and ε_i ($i = 1, \ldots, 4$) we shall from now on always understand $\varepsilon(R_1)$ and $\varepsilon_i(k, R_1)$, respectively.

LEMMA 4.5. — Introducing $R_2 = R_2(t) = 2\alpha t + R_1$, $t \geq 0$, and

$$J(t)=\left(\frac{\beta}{\alpha}\right)^{1/4}\left\{2^{-1/2}3^{-1}(t\Gamma)^{1/2}\left(1+\left(\frac{\beta}{\alpha}\right)^{1/4}\right)^2\left(1+t\Gamma\frac{3}{32}\frac{\beta}{\alpha}\right)+\left(\frac{\beta}{\alpha}\right)^{1/4}\right\}^{1/2}$$

we have the estimate (P_δ given as in Lemma 4.3)

$$\| P_\delta(e^{-itH^\ell}f_{R_1} - e^{-itk_0^2}f_{R_2}) \|^2$$
$$\leq \| f_{R_1} \|^2 \frac{16}{\pi}\left\{ J(t) + \left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\left[3\pi^{1/2} + \left(6 + \frac{|k_0|}{\alpha}\right)J(t) + \left(4 + \frac{|k_0|}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{1/2}\right] + O_2(\varepsilon^2) + J(t)O_3(\varepsilon^2)\right\}^2.$$

REMARK 4.6. — The terms $O_2(\varepsilon^2)$ and $O_3(\varepsilon^2)$ can be given explicitly by adding bounds at (4.14), (4.15), (4.16) and (4.18).

Proof. — We let $\varepsilon_i' = \varepsilon_i(k, R_2)$, $i = 1, \ldots, 4$.

By Lemma 4.2 we have for $k > \delta$ that

$$y(k,t) := \langle \Psi_\ell^+(k,\cdot), e^{-itk^2}f_{R_1} \rangle - \langle \Psi_\ell^+(k,\cdot), e^{-itk_0^2}f_{R_2} \rangle$$
$$= \frac{-k}{k^2-k_0^2}e^{i\ell\pi/2}S_\ell(-k)\left\{ e^{i(k_0-k)R_1}\frac{1+\varepsilon_1}{1+\varepsilon_3}\{1+h(\varepsilon_i)x(k,R_1)\}e^{-itk^2}\right.$$
$$\left[1+\frac{k-k_0}{k}(1+\varepsilon_2)(1+\varepsilon_1)^{-1}x(k,R_1)\{1+h(\varepsilon_i)x(k,R_1)\}^{-1}\right]$$
$$- e^{i(k_0-k)R_2}\frac{1+\varepsilon_1'}{1+\varepsilon_3'}\{1+h(\varepsilon_i')x(k,R_2)\}e^{-itk_0^2}$$
$$\left.\left[1+\frac{k-k_0}{k}(1+\varepsilon_2')(1+\varepsilon_1')^{-1}x(k,R_2)\{1+h(\varepsilon_i')x(k,R_2)\}^{-1}\right]\right\}$$
$$= \frac{-k}{k^2-k_0^2}e^{i\ell\pi/2}S_\ell(-k)\{a+b+c+d\},$$

where

$$a=e^{-itk^2}e^{i(k_0-k)R_1}\frac{k-k_0}{k}\frac{1+\varepsilon_2}{1+\varepsilon_3}x(k,R_1),$$

$$b=-e^{-itk_0^2}e^{i(k_0-k)R_2}\frac{k-k_0}{k}\frac{1+\varepsilon_2'}{1+\varepsilon_3'}x(k,R_2),$$

$$c=-e^{-itk_0^2}e^{i(k_0-k)R_2}\left[\frac{1+\varepsilon_1'}{1+\varepsilon_3'}\{1+h(\varepsilon_i')x(k,R_2)\}-\frac{1+\varepsilon_1}{1+\varepsilon_3}\{1+h(\varepsilon_i)x(k,R_1)\}\right]$$

and

$$d=\frac{1+\varepsilon_1}{1+\varepsilon_3}\{1+h(\varepsilon_i)x(k,\mathrm{R}_1)\}e^{-itk^2}e^{i(k_0-k)\mathrm{R}_1}(1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}).$$

We will utilize the estimate

$$\|\mathrm{P}_\delta(e^{-it\mathrm{H}'}f_{\mathrm{R}_1}-e^{-itk_0^2}f_{\mathrm{R}_2})\|^2=\frac{2}{\pi}\int_\delta^\infty|y(k,t)|^2dk$$
$$\leq\frac{2}{\pi}\left(\left\{\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|a|^2\right\}^{1/2}+\ldots+\left\{\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|d|^2\right\}^{1/2}\right)^2. \tag{4.10}$$

For this purpose the following two inequalities and one equation are useful:

$$|h(\varepsilon_i)|=\left|\frac{(1+\varepsilon_4)(1+\varepsilon_1)-(1+\varepsilon_2)(1+\varepsilon_3)}{(1+\varepsilon_3)(1+\varepsilon_1)}\right|\leq\frac{2\varepsilon(2+\varepsilon)}{(1-\varepsilon)^2}, \tag{4.11}$$

$$|x(k,\mathrm{R})|\leq(1+\varepsilon)^2 \qquad \text{(cf. (2.3))}, \tag{4.12}$$

and

$$\int_0^\infty\frac{k^2}{|k^2-k_0^2|^2}dk=\frac{\pi}{4\beta} \qquad \text{(apply the Cauchy theorem)}. \tag{4.13}$$

The terms a and b: Using (4.8) and (4.12) we easily obtain

$$\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|a|^2\leq\|f_{\mathrm{R}_1}\|^2(1-\varepsilon)^{-2}\left\{1-\frac{|k_0|}{\alpha}(2\varepsilon+\varepsilon^2)\right\}^{-1}2\frac{\beta}{\alpha}(1+\varepsilon)^6$$
$$=\|f_{\mathrm{R}_1}\|^2\frac{2\beta}{\alpha}\left\{1+\left(8+2\frac{|k_0|}{\alpha}\right)\varepsilon+\mathrm{O}(\varepsilon^2)\right\}. \tag{4.14}$$

The same estimate holds for $\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|}|b|^2$. (4.15)

The term c: By using (4.8), (4.11), (4.12) and (4.13) we find

$$\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|c^2|$$
$$\leq\|f_{\mathrm{R}_1}\|^2\left\{1-\frac{|k_0|}{\alpha}(2\varepsilon+\varepsilon^2)\right\}^{-1}\frac{\pi}{2}[(2(1-\varepsilon)^{-1}(1+\varepsilon)^3+1)2\varepsilon(2+\varepsilon)(1-\varepsilon)^{-2}]^2$$
$$=\|f_{\mathrm{R}_1}\|^2 72\pi\varepsilon^2\{1+\mathrm{O}(\varepsilon)\}. \tag{4.16}$$

The term d: By inserting $k^2-k_0^2=2\alpha(k-\alpha)+(k-\alpha)^2+\beta^2+i\Gamma/2$ and $(k_0-k)2\alpha=-2\alpha(k-\alpha)-i\Gamma/2$ we find that

$$|1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}|^2=4\sin^2\left(\frac{t}{2}|k-k_0|^2\right).$$

Now we fix C, D > 0, define $E=\max\{0,\alpha-C\}$ and $F=\alpha+\min\{C,D\}$, and proceed, using that $\sin^2 x \leq \min\{x^2,1\}$ for all $x \geq 0$, as follows:

$$\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}|^2$$
$$\leq \int_0^E dk\frac{k^2}{|k^2-\alpha^2|^2}4+\int_E^F dk\frac{k^2}{|k^2-k_0^2|^2}t^2|k-k_0|^4$$
$$+\int_F^{\alpha+D} dk\frac{k^2}{|k^2-\alpha^2|^2}4+\int_{\alpha+D}^\infty dk\frac{k^2}{|k^2-\alpha^2|^2}4$$
$$\leq\left(\frac{\alpha+D}{2\alpha+D}\right)^2\left\{4\int_0^E dk\frac{1}{(k-\alpha)^2}+4\int_F^{\alpha+D} dk\frac{1}{(k-\alpha)^2}+t^2\int_E^F dk((k-\alpha)^2+\beta^2)\right\}$$
$$+4\int_{\alpha+D}^\infty dk\frac{1}{(k-\alpha)^2}$$
$$\leq\left(\frac{\alpha+D}{2\alpha+D}\right)^2\left\{\frac{4}{C}+\frac{4}{C}+t^2\frac{2}{3}C^2+t^2\beta^2 2C\right\}+\frac{4}{D}$$
$$\leq(1+D/\alpha)^2\{2/C+t^2C^3/6+t^2\beta^2C/2\}+4/D.$$

We take $C=\left(\frac{2}{t}\right)^{1/2}$ and $D=\alpha^{3/4}\beta^{1/4}$ and obtain

$$\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}|^2$$
$$\leq 2^{5/2}3^{-1}t^{1/2}\left(1+\left(\frac{\beta}{\alpha}\right)^{1/4}\right)^2\left(1+\frac{3}{32}t\Gamma\frac{\beta}{\alpha}\right)+4\alpha^{-3/4}\beta^{-1/4}. \quad (4.17)$$

To estimate $\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|d|^2$ suitably we utilize (4.8), (4.11), (4.12) and (4.17) and find that

$$\int_\delta^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|d|^2$$
$$\leq\|f_{R_1}\|^2\left\{1-\frac{|k_0|}{\alpha}(2\varepsilon+\varepsilon^2)\right\}^{-1}2\beta(1+\varepsilon)^2(1-\varepsilon)^{-2}\left\{1+2\varepsilon\frac{2+\varepsilon}{(1-\varepsilon)^2}(1+\varepsilon)^2\right\}^2$$
$$\left\{2^{5/2}3^{-1}t^{1/2}\left(1+\left(\frac{\beta}{\alpha}\right)^{1/4}\right)^2\left(1+\frac{3}{32}t\Gamma\frac{\beta}{\alpha}\right)+4\alpha^{-3/4}\beta^{-1/4}\right\}$$
$$=\|f_{R_1}\|^2 8J(t)^2(1+\varepsilon)^2(1-\varepsilon)^{-2}\left\{1-\frac{|k_0|}{\alpha}(2\varepsilon+\varepsilon^2)\right\}^{-1}\left\{1+2\varepsilon\frac{2+\varepsilon}{(1-\varepsilon)^2}(1+\varepsilon)^2\right\}^2$$
$$=\|f_{R_1}\|^2 8J(t)^2\left\{1+\varepsilon\left(12+2\frac{|k_0|}{\alpha}\right)+O(\varepsilon^2)\right\}. \quad (4.18)$$

Combining (4.10), (4.14), (4.15), (4.16) and (4.18) we finally obtain

$$\| P_\delta(e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2}) \|^2 \leq \| f_{R_1} \|^2 \frac{16}{\pi} \left\{ J(t) + \left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\left[3\pi^{1/2} + \left(6 + \frac{|k_0|}{\alpha}\right) J(t) + \left(4 + \frac{|k_0|}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{1/2} \right] + O_2(\varepsilon^2) + J(t) O_3(\varepsilon^2) \right\}^2 .$$

The Lemma is proved.

Our main result is as follows.

THEOREM 4.7. — Let $R_2 = R_2(t) = 2\alpha t + R_1$, $t \geq 0$. Then

$$\| e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2} \|^2 \leq \| f_{R_1} \|^2 K(t, \varepsilon),$$

where ($\varepsilon = \varepsilon(R_1)$, and $J(t)$ given as in Lemma 4.5)

$$K(t, \varepsilon) = \frac{16}{\pi} \left\{ J(t) + \left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\left[3\pi^{1/2} + \left(6 + \frac{|k_0|}{\alpha}\right) J(t) + \left(4 + \frac{|k_0|}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{1/2} \right] + O_2(\varepsilon^2) + J(t) O_3(\varepsilon^2) \right\}^2 + (e^{4\frac{\beta}{\alpha}} - 1) 4 \left\{ 1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi \right\}^2 \left\{ 1 + \varepsilon 2 \frac{|k_0|}{\alpha} + O_4(\varepsilon^2) \right\} \left\{ 1 + \varepsilon 2 \left(1 + \frac{|k_0|}{\alpha}\right) + O_1(\varepsilon^2) \right\}.$$

REMARK 4.8. — The terms $O_1(\varepsilon^2)$, $O_2(\varepsilon^2)$, $O_3(\varepsilon^2)$ and $O_4(\varepsilon^2)$ can be given explicitly, cf. Remarks 4.4, 4.6 and the proof of Theorem 4.7 to be given below.

Proof. — We use that

$$\| e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2} \|^2 = \| P_\delta(e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2}) \|^2 + \| (I - P_\delta)(e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2}) \|^2 ,$$

where $\delta = \alpha 2^{-1/2}$.

The first term is estimated as in Lemma 4.5, the second as follows:

$$\| (I - P_\delta)(e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2}) \|^2 \leq 2(\| (I - P_\delta) e^{-itH^\ell} f_{R_1} \|^2 + \| (I - P_\delta) e^{-itk_0^2} f_{R_2} \|^2)$$
$$\leq 2(\| f_{R_1} \|^2 + e^{-\Gamma t} \| f_{R_2} \|^2)(e^{4\frac{\beta}{\alpha}} - 1) \left\{ 1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi \right\}^2 \left\{ 1 + 2\left(1 + \frac{|k_0|}{\alpha}\right)\varepsilon + O_1(\varepsilon^2) \right\}$$

(Lemma 4.3 ; we remark that the bound in Lemma 4.3 is monotone increasing in the variable ε)

$$\leq 2\| f_{R_1} \|^2 \left(\left\{ 1 + \frac{|k_0|}{\alpha}(2\varepsilon + \varepsilon^2) \right\} \left\{ 1 - \frac{|k_0|}{\alpha}(2\varepsilon + \varepsilon^2) \right\}^{-1} + 1 \right)$$
$$(e^{4\frac{\beta}{\alpha}} - 1) \left\{ 1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi \right\}^2 \left\{ 1 + 2\left(1 + \frac{|k_0|}{\alpha}\right)\varepsilon + O_1(\varepsilon^2) \right\}$$

(Lemma 4.1 and (4.8))

$$= \| f_{R_1} \|^2 4(e^{4\frac{\beta}{\alpha}} - 1) \left\{ 1 + \varepsilon 2 \frac{|k_0|}{\alpha} + O_4(\varepsilon^2) \right\}$$
$$\left\{ 1 + \frac{\pi^2}{4} + \frac{|k_0|}{\alpha}\pi \right\}^2 \left\{ 1 + 2\left(1 + \frac{|k_0|}{\alpha}\right)\varepsilon + O_1(\varepsilon^2) \right\}.$$

We have finished the proof.

Theorem 4.7 almost immediately implies the following Corollaries 4.9 and 4.10, which concern two different « measures of decay ».

COROLLARY 4.9. — For all $t \geq 0$

$$|(f_{R_1}, e^{-itH^\ell} f_{R_1})|^2 \| f_{R_1} \|^{-4} = e^{-\Gamma t} | 1 + x(t, \varepsilon) |^2 ,$$

where $$| x(t, \varepsilon) | \leq e^{\Gamma t/2} K(t, \varepsilon)^{1/2} .$$

Proof.

$$(f_{R_1}, e^{-itH^\ell} f_{R_1}) = (f_{R_1}, e^{-itk_0^2} f_{R_2}) + (f_{R_1}, e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2})$$
$$= e^{-itk_0^2} \| f_{R_1} \|^2 \{ 1 + e^{itk_0^2} \| f_{R_1} \|^{-2} (f_{R_1}, e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2}) \} .$$

Cauchy-Schwarz' inequality and Theorem 4.7 now complete the proof.

COROLLARY 4.10. — For all $t \geq 0$

$$(e^{-itH^\ell} f_{R_1}, \chi_{(0,R_1)} e^{-itH^\ell} f_{R_1}) \| f_{R_1} \|^{-2} = e^{-\Gamma t}(1 + y(t, \varepsilon)) ,$$

where $$| y(t, \varepsilon) | \leq e^{\Gamma t} K(t, \varepsilon) + 2e^{\Gamma t/2} K(t, \varepsilon)^{1/2} .$$

Proof.

$$(e^{-itH^\ell} f_{R_1}, \chi_{(0,R_1)} e^{-itH^\ell} f_{R_1}) = e^{-\Gamma t}(f_{R_2}, \chi_{(0,R_1)} f_{R_2})$$
$$+ (\{ e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2} \}, \chi_{(0,R_1)} \{ e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2} \})$$
$$+ 2\,\mathrm{Re}(\{ e^{-itH^\ell} f_{R_1} - e^{-itk_0^2} f_{R_2} \}, \chi_{(0,R_1)} e^{-itk_0^2} f_{R_2}) .$$

The first term is equal to $e^{-\Gamma t} \| f_{R_1} \|^2$. As before we can now complete the proof using Cauchy-Schwarz' inequality and Theorem 4.7.

5. APPLICATION TO α-DECAY

The usual simplified model describing α-decay concerns only s-waves. For higher angular momentum numbers ($\ell \geq 1$) tunneling is expected to be very slow due to the $\ell(\ell + 1)/r^2$-barrier.

Within the framework of the α-decay model we now present a proof of the validity of the exponential law for some time-interval.

We let R_1 be the radius of detection, and assume k_0 is a resonance and that f_{R_1} is the α-particle state at time $t = 0$. The probability P_t that the α-particle is detected during the time-interval $(0, t)$ is calculated using Corollary 4.10 ($y(t, \varepsilon)$ is given there):

$$P_t = 1 - e^{-\Gamma t}(1 + y(t, \varepsilon)). \tag{5.1}$$

If for some « large » time-interval $(0, t_0)$, $| y(t, \varepsilon) |$ is « small » compared with 1, then (5.1) is precisely the law of exponential decay.

The data in the first two rows in the following Table have been taken from [11].

TABLE 1.

	RaC′	RaA	Ur
Lifetime Γ^{-1}	$4{,}4.10^{-8}$ mi.	4,4 mi.	$4{,}4.10^{15}$ mi.
Speed	$1{,}92.10^{9}$ cm/s	$1{,}69.10^{9}$ cm/s	$1{,}4.10^{9}$ cm/s
$2\beta R_1$, $R_1 = 1$ m	2.10^{-2}	2.10^{-10}	3.10^{-25}
Γ/E	3.10^{-17}	4.10^{-25}	6.10^{-40}

In the evaluation of $| y(t, \varepsilon) |$ we can use $\frac{1}{4}\Gamma/E$ instead of the quantity $\frac{\beta}{\alpha}$.

Also we remark that if $\varepsilon = \varepsilon(R_1) < 10^{-1}\left(\frac{\beta}{\alpha}\right)^{1/4}$ and $\frac{\alpha}{\beta} > t\Gamma > 1$, $K(t, \varepsilon)$ is given (approximately) by

$$K(t, \varepsilon) \simeq 2^{7/2}(3\pi)^{-1}(t\Gamma)^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2}. \tag{5.2}$$

Using (5.2) we easily find that $| y(t, \varepsilon) |$ is smaller than 0,2 or 0,01 for $t \in (0, t_0)$, where $t_0\Gamma$ are given as follows:

TABLE 2.

	RaC′	RaA	Ur
$\mid y(t, \varepsilon) \mid < 0{,}2$ for $t_0\Gamma =$	13	22	39
$\mid y(t, \varepsilon) \mid < 0{,}01$ for $t_0\Gamma =$	7	16	33

To illustrate the mathematical results in the case $\ell \geq 1$ we will now make the physically probably wrong hypothesis that the data in Table 1 represent ℓ-wave resonances for $\ell \geq 1$. We detect at a distance of $R_1 = 1$ m and assume that $V(r) = 0$ for $r > R_1$. Because of this assumption we can calculate $\varepsilon = \varepsilon(R_1)$. The following estimate holds true:

$$\varepsilon < \frac{2}{\alpha} \int_{R_1}^{\infty} \ell(\ell + 1)/r^2 dr = 2\ell(\ell + 1)\frac{\beta}{\alpha}(\beta R_1)^{-1} . \qquad (5.3)$$

Using (5.3) and data from Table 1 we find that $\varepsilon < 2\ell(\ell + 1)10^{-15}$. From Table 1 we can now conclude that $\varepsilon < 10^{-1}\left(\frac{\beta}{\alpha}\right)^{1/4}$, if $\ell < 70$ is assumed. Hence in this case, (5.2) and the statements of Table 2 hold true. If ℓ is large (> 70) the ε-dependence of $K(t, \varepsilon)$ is visible (or predominant: $K(t, \varepsilon) \simeq 144\, \varepsilon^2$), and the numbers in Table 2 must be replaced by smaller numbers. We remark that in the case « ℓ large » the error estimate [*20*, Theorem 4.7 together with (6.2)] is typically much better than the one of Theorem 4.7.

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