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On nonlinear Schrödinger equations

by

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ABSTRACT. — The Cauchy problem for the nonlinear Schrödinger equation $i\partial_t u = \Delta u - F \circ u$, $u(0) = \phi$, on space domain $\mathbb{R}^m$, $m \geqslant 1$, is solved under the assumption that F is a C^1-function (in the real sense) on $\mathbb{C}$ to itself satisfying $F(0) = 0$ and the growth condition $|F'(\zeta)| \leqslant \text{const} |\zeta|^{p-1}$, where $p < (m+2)/(m-2)$ (or any $p < \infty$ if $m \leqslant 2$). It is shown that if $\phi \in H^1$, there exists a unique local solution $u \in C(I; H^1) \cap C^1(I; H^{-1})$, $I = [0, T]$. Any solution $u \in C(I; H^1)$ with $u(0) \in H^2$ belongs to $C(I; H^2) \cap C^1(I; L^2)$. These solutions depend continuously on the initial value. They are global if $F(\zeta) = \partial H / \partial \bar{\zeta}$ with a real-valued function H such that $H(\zeta) \geqslant c |\zeta|^2$, $c > 0$.

RÉSUMÉ. — On résout le problème de Cauchy pour l'équation de Schrödinger non linéaire $i\partial_t u = \Delta u - F \circ u$, $u(0) = \phi$, dans l'espace $\mathbb{R}^m$, $m \geqq 1$, sous l'hypothèse que la fonction F est C^1 (au sens réel) de $\mathbb{C}$ dans $\mathbb{C}$, et satisfait $F(0) = 0$ et la condition de croissance $|F'(\zeta)| \leqq \text{const} |\zeta|^{p-1}$, avec $p < (m+2)/(m-2)$ (ou $p < \infty$ si $m \leqq 2$). On montre que si $\phi \in H^1$, il existe une solution locale unique $u \in C(I; H^1) \cap C^1(I; H^{-1})$, $I = [0, T]$. Toute solution $u \in C(I; H^1)$ avec $u(0) \in H^2$ appartient à $C(I; H^2) \cap C^1(I; L^2)$. Ces solutions dépendent continûment des données initiales. Elles sont globales si $F(\zeta) = \partial H / \partial \bar{\zeta}$ pour une fonction réelle H telle que $H(\zeta) \geqq c |\zeta|^2$, $c > 0$.

INTRODUCTION

This paper is concerned with the Cauchy problem for the nonlinear Schrödinger equation

$$(0.1) \qquad i\partial_t u = -\Delta u + F(u), \qquad t \geqslant 0, \qquad x \in \mathbb{R}^m, \qquad m \geqslant 1\,.$$

Here $F(u)$ is defined pointwise by $F(u)(t, x) = F(u(t, x))$ with a complex-valued function $F(\zeta)$ of a complex variable ζ. The only assumption we need for local theory is that

$$(0.2)\quad F \in C^1(\mathbb{R}^2; \mathbb{R}^2); \qquad F(0) = 0, \quad |F'(\zeta)| \leqslant M|\zeta|^{p-1} \quad \text{for} \quad |\zeta| \geqslant 1,$$

where

$$(0.3)\qquad 1<p<(m+2)/(m-2) \quad (m>2), \qquad 1<p<\infty \quad (m\leqslant 2),$$

and where F' is the Fréchet derivative of F (see Appendix for a precise definition). Note that no assumption is made on the bahavior of $F(\zeta)$ for small ζ except continuous differentiability.

As is easily seen by cutting F smoothly into two pieces, (0.2) is equivalent to

$$(0.2)'\quad \begin{aligned} &F = F_1 + F_2, \qquad F_j \in C^1(\mathbb{R}^2; \mathbb{R}^2), \qquad F_j(0) = 0, \qquad j = 1, 2, \\ &|F_1(\zeta) - F_1(\zeta')| \leqslant M|\zeta - \zeta'|, \\ &|F_2(\zeta) - F_2(\zeta')| \leqslant M|\zeta - \zeta'|(|\zeta|^{p-1} \vee |\zeta'|^{p-1}), \end{aligned}$$

where $\vee$ means supremum. In fact we find it convenient to work directly with the Lipschitz continuity $(0.2')$ for the F_j, at least in the first part of this paper.

Under the assumption $(0.2')$, we shall prove the following theorems. (For notations see below).

THEOREM 0 (uniqueness). — A solution $u \in L^\infty([0, T]; L^2 \cap L^{p+1}) \equiv X_0$ of (0.1) has a definite initial value $u(0) \in L^2$. There is at most one solution $u \in X_0$ with a given $u(0)$.

THEOREM I (local existence and uniqueness). — If $\phi \in H^1$, there is a positive number T, depending only on $\|\phi\|_{H^1}$, such that (0.1) has a unique solution $u \in C(I; H^1) \cap C^1(I; H^{-1})$, $I = [0, T]$, with $u(0) = \phi$. Moreover, $\partial u \in L^r(I; L^{p+1})$, where $r = 4(p+1)/m(p-1) > 2$.

THEOREM I′ (continuous dependence). — The map $u(0) \mapsto u$ is continuous from H^1 into $C(I; H^1)$. More precisely, let $u \in C(I; H^1)$ be a solution of (0.1) with $u(0) = \phi$ (where I need not be the one in Theorem I), and let $\phi_n \to \phi$ in H^1, $n \to \infty$. Then the solution u_n with $u_n(0) = \phi_n$ exists on the interval I if n is sufficiently large, and $u_n \to u$ in $C(I; H^1)$.

THEOREM II (regularity). — Let $u \in C(I; H^1)$ be a solution of (0.1). If $u(0) \in H^2$, then $u \in C(I; H^2) \cap C^1(I; L^2)$. Moreover, $\partial_t u \in L^r(I; L^{p+1})$.

THEOREM II′ (continuous dependence in H^2). — The map $u(0) \mapsto u$ is continuous from H^2 into $C(I; H^2)$, in the sense similar to the statement in Theorem I′.

THEOREM III (global existence). — In addition to (0.2′) assume that $F(\zeta) = \partial H(\zeta)/\partial\bar{\zeta}$, where $H(\zeta)$ is a real-valued function such that $H(0) = 0$ and $H(\zeta) \geqslant c\,|\,\zeta\,|^2$ with a real constant $c > 0$. Then the solution u in Theorem I exists for all time ($T=\infty$), and $E(t) \equiv \|\,\partial u(t)\,\|_2^2 + \int H(u(t,x))dx$ is constant in t. ($\|\,u(t)\,\|_2$ need not be conserved.)

More detailed results are found in various lemmas given below.

REMARKS. — *a*) These theorems improve known results (see e. g. [*1*] [*2*]) for (0.1) in various directions, as far as the Cauchy problem is concerned. First: we have eliminated the customary assumptions such as

$$|\,F(\zeta)\,| \leqslant M\,|\,\zeta\,|^p + N\,|\,\zeta\,|^q, \qquad p > q > 1,$$

which not only severely restrict the behavior of F for small $|\,\zeta\,|$ but greatly complicate the proofs. In [*2*], Ginibre and Velo removed such a condition in the existence theorem, but not in the uniqueness and other problems.

b) We assume a minimum smoothness C^1 of $F(\zeta)$. Actually the Lipschitz continuity expressed by the inequalities in (0.2′) suffices for the existence and uniqueness theorems (even for H^2-solutions); the C^1-property is needed only in the proof of regularity and continuous dependence. (That H^2-solutions can be constructed in the special case $F(\zeta) = |\,\zeta\,|^{p-1}\zeta$ for any $m \geqslant 1$ was recently proved by Tsutsumi [*4*].)

c) Nothing is assumed in the local theory about $\arg F(\zeta)$ or related quantities; such an assumption is introduced only for global theory (Theorem III). Theorem III gives only a particular sufficient condition for global existence, which, incidentally, does not imply conservation of the L^2-norm; there are other possible conditions which we do not discuss here. The following rather queer example is allowed in Theorem III, but could not be covered by other existing theorems.

$$F(u) = (a\,|\,\mathrm{Re}\,u\,|^{p-1} + b)(\mathrm{Re}\,u) + i(c\,|\,\mathrm{Im}\,u\,|^{p-1} + d)(\mathrm{Im}\,u),$$

where $a, c \geqslant 0, b, d > 0$.

d) In the proofs we make extensive use of the space-time behavior of the free Schrödinger group $(e^{it\Delta})$, as in most papers on the subject. But we do it directly, without using any regularization procedure. In this process, the contraction property of the basic nonlinear integral operator in the norm of $L^{2,\infty} \cap L^{p+1,r}$ is essential. (Analogous property of the linear operator was used by Yajima [*6*].) In fact the proofs of existence and uniqueness theorems are rather simple; a greater part of the paper is devoted to proving regularity and continuous dependence.

NOTATION. — The following notations are used without particular comments.

$$\partial_t = \partial/\partial t, \qquad \partial = (\partial_1, \ldots, \partial_m), \qquad \partial_j = \partial/\partial x_j;$$
$$\partial/\partial\zeta = (\partial/\partial\xi - i\partial/\partial\eta)/2, \qquad \partial/\partial\bar{\zeta} = (\partial/\partial\xi + i\partial/\partial\eta)/2 \qquad (\zeta = \xi + i\eta);$$
$$\mathrm{L}^q = \mathrm{L}^q(\mathbb{R}^m), \qquad \mathrm{H}^1 = \mathrm{H}^1(\mathbb{R}^m), \quad \text{etc.}; \qquad \| \ \|_q = \mathrm{L}^q\text{-norm};$$
$$\mathrm{L}^{q,r} = \mathrm{L}^r(\mathrm{I}; \mathrm{L}^q) \quad (\mathrm{I} = [0, \mathrm{T}]); \qquad ||| \ |||_{q,r} = \mathrm{L}^r(\mathrm{I}; \mathrm{L}^q)\text{-norm};$$

$\mathrm{B_R(Z)}$ = closed ball in Z with center 0, radius R.

Other notations will be introduced according to necessity.

Finally we list several constants derived from p that are used throughout the paper.

$$\begin{aligned} &\alpha = m(1/2 - 1/(p+1)), \qquad 0 < \alpha < 1, \\ &r = 2/\alpha = 4(p+1)/m(p-1), \qquad 2 < r < \infty, \\ &r' = r/(r-1), \qquad 1 < r' < 2. \end{aligned} \tag{0.4}$$

1. THE INTEGRAL EQUATION. UNIQUENESS

We convert (0.1) into the integral equation

$$u = \mathrm{G}_0\phi - i\mathrm{GF}(u), \tag{1.1}$$

where $\phi = u(0)$ and G_0, G are linear operators given by

$$\mathrm{G}_0\phi(t) = e^{it\Delta}\phi, \tag{1.2}$$

$$\mathrm{G}v(t) = \int_0^t e^{i(t-s)\Delta}v(s)ds. \tag{1.3}$$

LEMMA 1.1. — If $u \in \mathrm{X}_0 = \mathrm{L}^\infty(\mathrm{I}; \mathrm{L}^2 \cap \mathrm{L}^{p+1})$, (0.1) and (1.1) are equivalent.

Before proving Lemma 1.1, we have to study the properties of the operators G_0, G and F. To this end we introduce the following function spaces on $\mathrm{I} = [0, \mathrm{T}]$, which are Banach spaces.

$$\begin{aligned} &\overline{\mathrm{X}} = \overline{\mathrm{X}}(\mathrm{I}) = \mathrm{C}(\mathrm{I}; \mathrm{L}^2) \cap \mathrm{L}^{p+1,r}, \qquad r = 4(p+1)/m(p-1), \\ &\mathrm{X} = \mathrm{X}(\mathrm{I}) = \mathrm{L}^{2,\infty} \cap \mathrm{L}^{p+1,r}, \\ &\mathrm{X}' = \mathrm{X}'(\mathrm{I}) = \mathrm{L}^{2,1} + \mathrm{L}^{1+1/p,r'}, \qquad r' = r/(r-1). \end{aligned} \tag{1.4}$$

$\overline{\mathrm{X}}$ is a closed subspace of X; X is the dual of X′; X′ is almost (but not precisely) the dual of $\overline{\mathrm{X}}$. The norms in these spaces are defined by

$$\begin{aligned} &|||v|||_{\mathrm{X}} = |||v|||_{2,\infty} \vee |||v|||_{p+1,r}, \\ &|||f|||_{\mathrm{X}'} = \inf\{|||g|||_{2,1} + |||h|||_{1+1/p,r'}; f = g + h\}. \end{aligned} \tag{1.5}$$

We note once for all that $\bar{X}$, X, X' and other spaces to be introduced below depend on $I = [0, T]$, although in most cases we do not indicate the T-dependence explicitly. For this reason, it is important to keep track of the T-dependence of various vector and operator norms. We note that the norms in X and X' are *homogeneous* in the sense that they are covariant under the scale transformation $t \mapsto \lambda^2 t$ and $x \mapsto \lambda x$.

The following lemma is basic throughout the paper.

LEMMA 1.2. — G_0 is a bounded linear operator on L^2 into $\bar{X}$. G is a bounded linear operator on X' into $\bar{X}$. The associated norms are independent of T; their supremum is denoted by γ (γ is computable in terms of the Sobolev constants.)

The lemma means that the following operator norms are finite and independent of T.

a) $G_0 : L^2 \to L^{2,\infty}$ (isometric), *b)* $G_0 : L^2 \to L^{p+1,r}$;
c) $G : L^{2,1} \to L^{2,\infty}$, *d)* $G : L^{2,1} \to L^{p+1,r}$,
e) $G : L^{1+1/p,r'} \to L^{2,\infty}$. *f)* $G : L^{1+1/p,r'} \to L^{p+1,r}$.

This is a summary of various estimates proved by many authors during the last decade, although apparently they have not been assembled in this form except in [*6*] (in a slightly different form). Actually these results are true for $T = \infty$, and are proved mostly in that case in the literature, but a finite T makes no difference. *a)* is trivial; *b) e)*, and *f)* are now well known. *d)* is (essentially) the dual of *e)*. *c)* is self-dual, and is simply a superposition of *a)*. The reader may find a more complete proof (essentially) in [*6*].

Next we study the properties of the nonlinear operator F. To this end we use the auxiliary function space X_0 (already introduced in Theorem 0 and Lemma 1.1).

$$X_0 = X_0(I) = L^{2,\infty} \cap L^{p+1,\infty} \subset X, \tag{1.6}$$
$$|||v|||_{X_0} = |||v|||_{2,\infty} \vee |||v|||_{p+1,\infty}.$$

This norm for X_0 is *not* homogeneous in the sense given above. For this reason, it is worth noting that the injection of X_0 into X is uniformly bounded for $T \leqslant 1$, say, though not for all T.

LEMMA 1.3. — F maps X_0 into X' continuously and boundedly (sends bounded sets into bounded sets). For $v, w \in X_0$, we have

$$|||F_1(v)-F_1(w)|||_{2,1} \leqslant MT\,|||v-w|||_{2,\infty}, \tag{1.7}$$
$$|||F_2(v)-F_2(w)|||_{1+1/p,r'} \leqslant MT^{1-\alpha}(|||v|||_{X_0}^{p-1} \vee |||w|||_{X_0}^{p-1})\,|||v-w|||_{p+1,r},$$

where $\alpha = m(1/2-1/(p+1)) < 1$. On each ball $B_R(X_0)$, F is uniformly Lipschitz continuous from X into X'; the Lipschitz constant is arbitrarily small if R is fixed and T is made small.

COROLLARY. — GF maps X_0 into X continuously and boundedly. On each ball $B_R(X_0)$, GF is a contraction map in the metric of X if T is sufficiently small. (Note that the spaces such as X, X_0 depend on T, but we keep R fixed when T is varied.)

Proof. — Using (0.2′) and the elementary formula

$$\| f^{p-1}g \|_{1+1/p} \leqslant \| f \|_{p+1}^{p-1} \| g \|_{p+1}, \qquad f \geqslant 0, \tag{1.8}$$

we obtain by a simple computation

$$||| F_1(v) - F_1(w) |||_{2,\infty} \leqslant M ||| v - w |||_{2,\infty},$$
$$||| F_2(v) - F_2(w) |||_{1+1/p,r} \leqslant M(||| v |||_{p+1,\infty}^{p-1} \vee ||| w |||_{p+1,\infty}^{p-1}) ||| v - w |||_{p+1,r}.$$

The estimates (1.7) then follow by

$$||| f |||_{2,1} \leqslant T ||| f |||_{2,\infty}, \qquad ||| f |||_{1+1/p,r'} \leqslant T^{1-\alpha} ||| f |||_{1+1/p,r},$$

where $1 - \alpha = 1/r' - 1/r$ (see (0.4)). In view of the definition of X and X′, we have proved Lemma 1.3. The Corollary follows by combining Lemmas 1.2 and 1.3.

Proof of Lemma 1.1. — Let $u \in X_0$. Then we have $\Delta u \in L^\infty(I; H^{-2})$. Moreover, $F(u) \in X' \subset L^\infty(I; H^{-1})$ by Lemma 1.3. Hence $-\Delta u + F(u) \in L^\infty(I; H^{-2})$. If u satisfies (0.1), it follows that $\partial_t u \in L^\infty(I; H^{-2})$, hence $u \in C(I; H^{-2})$, so that $u(0) = \phi \in H^{-2}$ exists. Actually we have $\phi \in L^2$ because $u \in L^{2,\infty}$. Since $\{ e^{it\Delta} \}$ is a continuous unitary group on H^{-2}, we obtain (1.1). The converse is proved in the same way working in H^{-2}.

Proof of Theorem 0. — As usual we may assume that T is sufficiently small. Let $u, v \in X_0$ be two solutions of (0.1) with $u(0) = v(0) = \phi \in L^2$. By Lemma 1.1, they are solutions of (1.1). Choose R so large that $u, v \in B_R(X_0)$. Since $u - v = i(GF(u) - GF(v))$ and since GF is a contraction on $B_R(X_0)$ in X-metric if T is made sufficiently small (by Lemma 1.3, Corollary), we must have $u = v$. (We may add a trivial remark that the norm such as $||| u |||_{X_0}$ for a given function u does not increase when T is decreased.)

REMARK 1.4. — The proof of Theorem 0 does not prove existence of a solution in X_0, since GF need not map X_0 into itself.

2. PROOF OF THEOREM I

Here we need the following spaces.

$$\begin{aligned} \overline{Y} &= \overline{Y}(I) = \{ v \in \overline{X}; \partial v \in \overline{X} \} \subset C(I; H^1), \\ Y &= Y(I) = \{ v \in X; \partial v \in X \} \subset L^\infty(I; H^1), \\ Y' &= Y'(I) = \{ v \in X'; \partial v \in X' \}, \end{aligned} \tag{2.1}$$

with the norms

(2.2) $|||v|||_Y = |||v|||_X \vee |||\partial v|||_X, \qquad |||f|||_{Y'} = |||f|||_{X'} \vee |||\partial f|||_{X'}$.

Obviously we have $\overline{Y} \subset \overline{X}$, $Y \subset X$, $Y' \subset X'$, and $Y \subset X_0$ by the Sobolev embedding theorem. These spaces are not homogeneous, unlike X and X′, but it is trivial that the injection maps just mentioned are independent of T:

(2.3) $|||v|||_X \leqslant |||v|||_Y, \qquad |||f|||_{X'} \leqslant |||f|||_{Y'}, \qquad |||v|||_{X_0} \leqslant c\,|||v|||_Y$,

where c depends only on m and p. Moreover, we have

LEMMA 2.1. — G_0 is bounded on H^1 to $\overline{Y}$. G is bounded on Y′ to $\overline{Y}$. The associated norms γ are equal to those in Lemma 1.2 and are independent of T.

Proof. — Since G_0 and G commute with ∂, the lemma follows immediately from Lemma 1.2.

LEMMA 2.2. — F maps Y into Y′ boundedly, with

(2.4) $$|||F(v)|||_{Y'} \leqslant M(T + cT^{1-\alpha}|||v|||_Y^{p-1})\,|||v|||_Y.$$

Proof. — Since $Y \subset X_0 \subset X$, $v \in Y$ implies $F(v) \in X'$ by Lemma 1.3. To see that $\partial F(v) \in X'$ too, let τ_h denote translation by $h \in \mathbb{R}^m$. Then it follows from (1.7) that

(2.5) $$\begin{aligned} |||(\tau_h - 1)F_1(v)|||_{2,1} &= |||F_1(\tau_h v) - F_1(v)|||_{2,1} \leqslant MT\,|||(\tau_h - 1)v|||_{2,\infty}, \\ |||(\tau_h - 1)F_2(v)|||_{1+1/p,r'} &= |||F_2(\tau_h v) - F_2(v)|||_{1+1/p,r'} \\ &\leqslant MT^{1-\alpha}|||v|||_{X_0}^{p-1}\,|||(\tau_h - 1)v|||_{p+1,r}; \end{aligned}$$

note that the function spaces considered are translation invariant in x-variable. Dividing by $|h|$ and letting $|h| \to 0$, we see that $\tau_h - 1$ may be replaced by ∂, since L^2, L^{p+1} and $L^{1+1/p}$ are reflexive. Recalling the definition of X and X′, we thus obtain

(2.6) $$|||\partial F(v)|||_{X'} \leqslant M(T + T^{1-\alpha}|||v|||_{X_0}^{p-1})\,|||\partial v|||_X.$$

This proves that $F(v) \in Y'$; the estimate (2.4) follows by $|||v|||_{X_0} \leqslant c\,|||v|||_Y$.

LEMMA 2.3. — Let $\phi \in H^1$. Choose a real number R such that $R > R' \equiv \gamma(\|\phi\|_2 \vee \|\partial\phi\|_2)$ (for γ see Lemmas 1.2 and 2.1). If T is sufficiently small, the map $\Phi(v) = G_0\phi - iGF(v)$ sends $B_R(Y)$ into itself, and it is a contraction in the X-norm.

Proof. — We have $G_0\phi \in Y$ by Lemma 2.1, with

(2.7) $$|||G_0\phi|||_Y \leqslant \gamma(\|\phi\|_2 \vee \|\partial\phi\|_2) = R'.$$

Set $R'' = R - R' > 0$. Lemmas 2.1 and 2.2 show that GF sends $B_R(Y)$ into $B_{R''}(Y)$ if T is so small that

$$(2.8) \qquad \gamma M(T + cT^{1-\alpha}R^{p-1}) \leqslant R''/R \equiv \delta < 1 .$$

Thus Φ maps $B_R(T)$ into itself.

On the other hand, Lemmas 1.2 and 1.3 show that

$$|||\, GF(v) - GF(w) \,|||_X \leqslant \gamma M(T + cT^{1-\alpha}R^{p-1}) \,|||\, v - w \,|||_X$$

for v, $w \in B_R(T)$. It follows by (2.8) that Φ is a contraction in X-metric.

We can now complete the proof of Theorem I. It is easy to see that $B_R(Y)$ with the X-metric is a complete metric space. Application of the contraction map theorem thus gives a fixed point $u \in Y$ for Φ. Then we have $u = \Phi u \in \overline{Y} \subset C(I; H^1)$ by Lemma 2.1. In view of Lemma 1.1, this completes the proof of Theorem I.

3. PROOF OF THEOREM II, PART 1

In this section we prove part of Theorem II, by constructing *local* H^2-solutions for initial values in H^2. The full regularity will be proved in next section.

LEMMA 3.1. — F maps H^2 into L^2 continuously and boundedly.

Proof. — This is obvious for F_1. For F_2, we note the following estimate, which is also used later.

$$(3.1) \qquad \begin{aligned} &\|\, f^{p-1}g \,\|_2 \leqslant \|\, f \,\|_{2p}^{p-1} \,\|\, g \,\|_{2p}, \qquad f, g \in L^{2p}, \\ &\|\, g \,\|_{2p} \leqslant c \,\|\, g \,\|_{H^k}, \qquad k = m(p-1)/2p < 2 . \end{aligned}$$

In view of (0.2′), this shows that F_2 maps $H^2 \subset H^k \subset L^{2p}$ continuously into L^2.

We now introduce another triplet of function spaces

$$(3.2) \qquad \begin{aligned} &\overline{Z} = \overline{Z}(I) = \{\, v \in \overline{X}; \partial_t v \in \overline{X}, \Delta v \in C(I; L^2 \,\} \subset \overline{Y}, \\ &Z = Z(I) = \{\, v \in X; \partial_t v \in X, \Delta v \in L^{2,\infty} \,\} \subset Y, \\ &Z' = Z'(I) = \{\, f \in L^{2,\infty}; \partial_t f \in X' \,\} \subset X', \\ &|||\, v \,|||_Z = |||\, v \,|||_X \vee |||\, \partial_t v \,|||_X \vee |||\, \Delta v \,|||_{2,\infty}, \\ &|||\, f \,|||_{Z'} = |||\, f \,|||_{2,\infty} \vee |||\, \partial_t f \,|||_{X'} . \end{aligned}$$

Again $\overline{Z}$ is a closed subspace of Z. $v \in Z$ may also be characterized by $v \in L^\infty(I; H^2)$ and $\partial_t v \in X$. But we find it convenient to use the norm given above. It is easy to see that the injections $Z \subset Y$ and $Z' \subset X'$ are bounded uniformly in $T \leqslant 1$.

LEMMA 3.2. — G_0 is a bounded operator on H^2 into $\overline{Z}$. G is a bounded operator on Z' into $\overline{Z}$. The associated norms are uniformly bounded in $T \leqslant 1$. More precisely we have

$$\| G_0 \phi \|_Z \leqslant \gamma(\| \phi \|_2 \vee \| \Delta\phi \|_2) = \gamma \| \phi \|_{H^2}, \tag{3.3}$$

$$\| Gf \|_Z \leqslant (2\gamma + 1) \| f \|_{Z'}, \qquad T \leqslant 1. \tag{3.4}$$

Proof. — In the proof we may replace $\overline{Z}$ by Z, since the operators considered map smooth functions into $\overline{Z}$, so that the desired results follow by the standard approximation procedure.

Lemma 1.2 gives

$$\| G_0 \phi \|_X \leqslant \gamma \| \phi \|_2, \quad \| \Delta G_0 \phi \|_{2,\infty} \leqslant \| \Delta G_0 \phi \|_X = \| \partial_t G_0 \phi \|_X \leqslant \gamma \| \Delta\phi \|_2 .$$

Thus G_0 maps H^2 into Z, with the norm given by (3.3).

Again, it is obvious that $f \in Z' \subset X'$ implies $Gf \in X$, with

$$\| Gf \|_X \leqslant \gamma \| f \|_{X'} \leqslant \gamma \| f \|_{2,1} \leqslant \gamma T \| f \|_{2,\infty} \leqslant \gamma T \| f \|_{Z'} .$$

Next we note that

$$\partial_t Gf = G\partial_t f + G_0 f(0) . \tag{3.5}$$

This formula is justified because $f \in Z'$ implies $\partial_t f \in X' \subset L^1(I; H^{-1})$, hence $f \in C(I; H^{-1})$. In particular $f(0) \in H^{-1}$ exists, and in fact $f(0) \in L^2$ because $f \in L^{2,\infty}$. It follows from Lemma 1.2 that $\partial_t Gf \in X$, with

$$\| \partial_t Gf \|_X \leqslant \gamma \| \partial_t f \|_{X'} + \gamma \| f(0) \|_2 \leqslant 2\gamma \| f \|_{Z'} .$$

Similarly we have

$$\Delta Gf = i(- \partial_t Gf + f), \tag{3.6}$$

again by $f \in C(I; H^{-1})$. Hence $\Delta Gf \in L^{2,\infty}$, with

$$\| \Delta Gf \|_{2,\infty} \leqslant \| \partial_t Gf \|_{2,\infty} + \| f \|_{2,\infty} \leqslant (2\gamma + 1) \| f \|_{Z'} .$$

Collecting the results, we have $Gf \in Z$ with the estimate (3.4).

LEMMA 3.3. — $v \in Z$ implies

$$v \in \operatorname{Lip}(I; L^2) \cap C^\theta(I; H^k) \subset \operatorname{Lip}(I; L^2) \cap C^\theta(I; L^{2p}), \tag{3.7}$$
$$k = m(p - 1)/2p < 2, \qquad \theta = 1 - k/2 > 0,$$
$$\| v(t) - v(s) \|_{2p} \leqslant c \,| t - s |^\theta \| v \|_Z,$$

$$F(v) \in C^\theta(I; L^2), \tag{3.8}$$
$$\| F(v(t)) - F(v(s)) \|_2 \leqslant cM(| t - s | \| v \|_Z + | t - s |^\theta \| v \|_Z^p) .$$

Proof. — $v \in Z$ implies $\partial_t v \in X \in L^{2,\infty}$, hence $v \in \operatorname{Lip}(I; L^2)$. Since $v \in L^\infty(I; H^2)$ too, this proves (3.7) by interpolation. Regarding (3.8), it is obvious that $F_1(v) \in \operatorname{Lip}(I; L^2)$ too. On the other hand, (3.1) and (3.7) show that $F_2(v) \in C^\theta(I; L^2)$ with Hölder coefficient $\| v \|_{2p,\infty}^p$, which is dominated by $c\,|||\,v\,|||_Z^p$ because $L^{2p} \supset H^k \supset H^2$. Thus we obtain (3.8) for $F = F_1 + F_2$.

LEMMA 3.4. — F maps Z into Z′ boundedly, with

$$(3.9) \qquad |||\,F(v) - F(v(0))\,|||_{Z'} \leqslant cM(T\,|||\,v\,|||_Z + T^{1-\alpha}\,|||\,v\,|||_Z^p),$$

where $F(v(0))$ is regarded as a constant function in t. (Thus the image of a bounded set of Z′ may not be close to the origin when T is small, but the diameter of the image is small if, for example, all v have a common value $v(0)$.)

Proof. — Let $v \in Z$. We first note that

$$(3.10) \quad |||\,\partial_t(F(v) - F(v(0))\,|||_{X'} \leqslant cM(T + T^{1-\alpha}\,|||\,v\,|||_{X_0}^{p-1})\,|||\,\partial_t v\,|||_X .$$

This is an analog of (2.6), and could be proved in the same way by estimating the difference quotients in t (instead of in x) and going to the limit $|\,h\,| \to 0$, if X′ we reflexive. To avoid the difficulty that the L^1-component of X′ is not reflexive, we may first replace this L^1-norm by $L^{1+\varepsilon}$-norm, thereby strengthening the space X′. It is easy to see that the estimates for the difference quotients are still valid for this modified norm, provided the first factor T is replaced by $T^{1/(1+\varepsilon)}$. Since the modified space is reflexive, we can replace the difference quotient by the derivative. Afterwards we may let $\varepsilon \to 0$ to recover (3.10).

On the other hand, (3.8) gives

$$|||\,F(v) - F(v(0))\,|||_{2,\infty} \leqslant cM(T\,|||\,v\,|||_Z + T^{\theta}\,|||\,v\,|||_Z^p).$$

Since $\theta > 1 - \alpha$, as can easily be verified, we arrive at the required result (3.9) (note that $|||\,v\,|||_{X_0} \leqslant c\,|||\,v\,|||_Z$).

LEMMA 3.5. — Let $\phi \in H^2$, so that $F(\phi) \in L^2$ by Lemma 3.1. Define the set

$$E = \{\, v \in B_R(Z)\,;\, v(0) = \phi \,\},$$

where R is a real number such that $R > \gamma(\|\,\phi\,\|_2 \vee \|\,\Delta\phi\,\|_2) + (2\gamma+1)\|\,F(\phi)\,\|_2$. E is not empty ($G_0\phi \in E$). If T is sufficiently small, $\Phi(v) = G_0\phi - iGF(v)$ maps E into itself.

Proof. — According to the estimates obtained above, we have

$$|||\,\Phi(v)\,|||_Z \leqslant \gamma(\|\,\phi\,\|_2 \vee \|\,\Delta\phi\,\|_2) + (2\gamma+1)(\,|||\,F(\phi)\,|||_{Z'} + |||\,F(v) - F(\phi)\,|||_{Z'}$$

where $F(v(0)) = F(\phi) \in Z'$ is regarded as a constant function. As such, it has Z′-norm $\|\,F(\phi)\,\|_2$. According to Lemma 3.4, therefore, $|||\,\Phi(v)\,|||_Z$ does not exceed R in Z-norm if T is sufficiently small. This proves the lemma; note that $\Phi(v)(0) = \phi$.

LEMMA 3.6. — Given $\phi \in H^2$, there is $T > 0$, depending only on $\|\,\phi\,\|_{H^2}$, for which there is a unique solution $u \in C(I; H^2)$, $I = [0, T]$, of (0.1) with $u(0) = \phi$. Moreover, $\partial_t u \in L^{2,\infty} \cap L^r(I; L^{p+1})$.

Proof. — Again Φ is a contraction in the X-metric if T is sufficiently small. Noting that E is complete in the X-metric, we conclude that there is a fixed point $u \in E$ for the map Φ. Then $u \in \overline{Z}$ since G_0 and G map into Z, so that u has the properties stated in the lemma.

4. PROOF OF THEOREM II, PART 2. LINEAR EQUATIONS

It remains to show that an H^1-solution with $u(0) \in H^2$ is an H^2-solution on the whole interval of existence. In view of local existence of H^2-solutions proved in Lemma 3.6, it suffices to prove

LEMMA 4.1. — Let $u \in C(I; H^1)$ be a solution of (0.1) for $I = [0, T]$, and let $u \in C(I^0; H^2)$ for $I^0 = [0, T)$ (semi-open interval). Then we have $u \in C(I; H^2)$.

For the proof we need some preparations. First we recall that an H^1-solution satisfies

$$u \in L^{2,\infty} \cap L^{p+1,\infty} \quad \text{(on I)}. \tag{4.1}$$

If we introduce the derivative $F' = F_1' + F_2'$ (see Appendix), Lemma A.2 of Appendix thus gives

$$F_1'(u) \in L^{\infty,\infty}, \qquad F_2'(u) \in L^{(p+1)/(p-1),\infty} \quad \text{(on I)}. \tag{4.2}$$

Next we consider the differential equation satisfied by $v = \partial_t u$. A formal differentiation of (0.1) gives

$$i\partial_t v = -\Delta v + F'(u).v \quad \text{for} \quad t \in I^0; \tag{4.3}$$

see Appendix for the notation $F'(u).v$. (4.3) is justified by Lemma A.3 of Appendix. Indeed, since $\partial_t u \in L^{2,\infty} \cap L^{p+1,r}$ on I^0 (because u is an H^2-solution), Lemma A.3 holds for $F = F_1$ and F_2, with $q = 1$ and $q = p$, respectively, by virtue of (4.2).

We now regard (4.3) as a real-linear Schrödinger equation for the *unknown* v, with u given as above. For this equation, we have

LEMMA 4.2. — Given $\psi \in L^2$, there is a unique solution $v \in \overline{X}(I) = C(I; L^2) \cap L^r(I; L^{p+1})$ of (4.3) with $v(0) = \psi$.

The proof is parallel to that of Theorem I, and it would suffice to give only a few remarks. (Regarding *linear* Schrödinger equations with a time-dependent potential, cf. [*3*] [*6*] for example):

a) Equation (4.3) makes sense for $v \in \overline{\mathrm{X}}$, which implies

$$\mathrm{F}'(u).v \in \mathrm{L}^{2,\infty} + \mathrm{L}^{1+1/p,r} \subset \mathrm{L}^1(\mathrm{I};\mathrm{H}^{-1}) \qquad \text{by} \quad (4.2).$$

This allows one to convert (4.3) into the corresponding integral equation (cf. the proof of Lemma 1.1).

b) Construction of the solution is simpler than in Theorem I since we can apply the contraction map theorem within the space X without using X_0 or Y, due to the strong property (4.2) of the coefficient, although the solution obtained is weaker. Indeed, (4.2) has the same effect as the assumption $u \in \mathrm{X}_0$ used in Lemma 1.3. Cf. also Remark 1.4.

Using Lemma 4.2, it is now easy to complete the proof of Lemma 4.1 and hence of Theorem II. Construct the solution of (4.3) with

$$v(0) = \partial_t u(0) = i(\Delta\phi - \mathrm{F}(\phi)) \in \mathrm{L}^2.$$

Since $\partial_t u$ satisfies the same equation with the same initial condition, we have $\partial_t u = v$ by uniqueness; note that both $\partial_t u$ and v are in

$$\mathrm{C}(\mathrm{I}^0;\mathrm{L}^2) \cap \mathrm{L}^r_{\mathrm{loc}}(\mathrm{I}^0;\mathrm{L}^{p+1}).$$

Since $v(t) \in \mathrm{L}^2$ is continuous up to $t = \mathrm{T}$, the same must be true of $\partial_t u$.

It follows that $\| \Delta u - \mathrm{F}(u) \|_2 \leqslant \mathrm{K} = \text{const.}$ for $t \in \mathrm{I}$. Hence

$$\| \Delta u - \mathrm{F}_2(u) \|_2 \leqslant \mathrm{K}$$

(with K denoting different constants), and

$$\begin{aligned} \| \Delta u \|_2^2 &\leqslant \mathrm{K}^2 + 2\mathrm{Re}\,\langle \Delta u, \mathrm{F}_2(u) \rangle \leqslant \mathrm{K}^2 - 2\mathrm{Re}\,\langle \partial_j u, \mathrm{F}_2'(u).\partial_j u \rangle \\ &\leqslant \mathrm{K}^2 + 2\,\| \partial u \|_{p+1}^2\, \| \mathrm{F}_2'(u) \|_{(p+1)/(p-1)} \leqslant \mathrm{K}^2 + \mathrm{K}\,\| \partial u \|_{\mathrm{H}^\alpha}^2 \end{aligned}$$

because $\| \mathrm{F}_2'(u) \|_{(p+1)/(p-1)} \leqslant \mathrm{K}\,\| u \|_{p+1}^{p-1} \leqslant \mathrm{K}$ is known. Since $\alpha < 1$ and $\| u \|_2 \leqslant \mathrm{K}$, it follows that $\| u \|_{\mathrm{H}^2}$ is bounded up to $t = \mathrm{T}$, completing the proof.

5. PROOF OF THEOREMS I′ AND II′

We start with Theorem I′. As usual, it suffices to prove continuous dependence on a sufficiently small interval I; then it can be extended step by step to the whole interval. Thus we can apply the contraction map argument given in section 2, to construct the u_n within a fixed ball $\mathrm{B_R(Y)}$ and with a uniform contraction factor $\delta < 1$, since the ϕ_n are bounded in H^1.

Since $\mathrm{G}_0\phi_n \to \mathrm{G}_0\phi$ in X by Lemma 1.2, it follows that $u_n \to u$ in $\mathrm{X} \subset \mathrm{L}^{2,\infty}$. Since $u_n \in \mathrm{B_R(Y)}$ implies that the $u_n(t)$ are uniformly bounded in H^1, we have $u_n \to u$ in $\mathrm{C}(\mathrm{I};\mathrm{H}^k)$ for any $k < 1$. Since $\mathrm{H}^k \subset \mathrm{L}^{p+1}$ for some $k < 1$, we have

$$(5.1) \qquad u_n \to u \quad \text{in} \quad \mathrm{L}^{2,\infty} \cap \mathrm{L}^{p+1,\infty} = \mathrm{X}_0.$$

On the other hand, $\partial_j u$ and $\partial_j u_n$, $j = 1, 2, \ldots, m$, satisfy the real-linear integral equations

$$\partial_j u = G_0 \partial_j \phi - iGF'(u).\partial_j u, \qquad (5.2)$$
$$\partial_j u_n = G_0 \partial_j \phi_n - iGF'(u_n).\partial_j u_n.$$

These are again justified by Lemma A.3 in Appendix, since $\partial_j u \in X$ by $u \in Y$, etc. (5.2) implies

$$\partial_j(u_n - u) = G_0 \partial_j(\phi_n - \phi) - iGF'(u_n).\partial_j(u_n - u) - iG(F'(u_n) - F'(u)).\partial_j u. \qquad (5.3)$$

LEMMA 5.1. — If T is sufficiently small, $GF'(u_n)$ is a real-linear contraction operator on X, uniformly in n.

This was essentially proved in Lemma 4.2.

LEMMA 5.2. — $GF'(u_n).v \to GF'(u).v$ in X for each $v \in X$.

Proof. — Since G is bounded on X′ into X, it suffices to show that $F'(u_n).v \to F'(u).v$ in X′. In view of (5.1) and the definition of X and X′, the lemma is proved if we show that

(5.4) $F_1'(u_n).v \to F_1'(u).v$ in $L^{2,2}$ if $u_n \to u$ in $L^{2,\infty}$ and $v \in L^{2,2}$;

(5.5) $F_2'(u_n).v \to F_2'(u).v$ in $L^{1+1/p,r}$ if $u_n \to u$ in $L^{p+1,\infty}$ and $v \in L^{p+1,r}$.

These assertions follow from Lemma A.2 in Appendix.

Proof of Theorem I′. — Using Lemmas 5.1, 5.2 and $\partial_j u \in X$, we conclude from (5.3) that $\partial_j(u_n - u) \to 0$ in X. Since $u_n - u \to 0$ in X is known, we have proved Theorem I′.

Proof of Theorem II′. — We indicate only the main difference from the proof of Theorem I′. Again we may assume that T is sufficiently small.

The solutions u_n can be constructed by the contraction map theorem within $B_R(Z)$ with the metric of X, with $u_n(0) = \phi_n \to \phi$ in H^2. Thus $u_n \to u$ in $L^{2,\infty}$ while the $u_n(t)$ are uniformly bounded in H^2. Hence we have

$$u_n \to u \quad \text{in} \quad C(I; H^k) \qquad \text{for any } k < 2. \qquad (5.6)$$

This again implies (5.1). Then we note that $\partial_t u$ and $\partial_t u_n$ satisfy the integral equations (5.2), just as $\partial_j u$ and $\partial_j u_n$ do, with the difference that the free terms $G_0\phi$ and $G_0\phi_n$ in (5.2) should be replaced, respectively, by $G_0\psi$ and $G_0\psi_n$, where

$$\psi = i(\Delta\phi - F(\phi)), \qquad \psi_n = i(\Delta\phi_n - F(\phi_n)).$$

Since $\psi_n \to \psi$ by Lemma 3.1, the same proof given above leads to the conclusion that $\partial_t u_n \to \partial_t u$ in $C(I; L^2)$. Using the differential equation (0.1) for u and u_n, we then deduce that

$$\Delta u_n - F(u_n) \to \Delta u - F(u). \qquad (5.7)$$

But (5.6) implies that $F(u_n) \to F(u)$ in $L^{2,\infty}$; the proof is the same as in Lemma 3.1. It follows that $\Delta u_n \to \Delta u$ in $L^{2,\infty}$. With (5.6), this proves the desired result that $u_n \to u$ in $C(I; H^2)$.

6. PROOF OF THEOREM III

First we sketch the proof that if $u \in C(I; H^2)$ is a solution of (0.1), then

$$E(t) = \text{const.} \tag{6.1}$$

We start by noting that the function H given in Theorem III is uniquely determined by F, belongs to $C^2(\mathbb{R}^2; \mathbb{R})$, and $= O(|\zeta|^{p+1})$ for large ζ. Moreover, we may decompose H into sum $H_1 + H_2$ in such a way that both H_j are real-valued and

$$F_j = \partial H_j/\partial\bar{\zeta}, \qquad j = 1, 2. \tag{6.2}$$

This can be achieved by first decomposing H into the desired form by applying cut-off functions and then identifying F_j with $\partial H_j/\partial\bar{\zeta}$.

LEMMA 6.1. — If $u \in C(I; H^2)$ is a solution of (0.1), then

$$\partial_t H_j(u) = H'_j(u) . \partial_t u = 2 \operatorname{Re} F_j(u) \partial_t \bar{u} \in L^{1,r}, \qquad j = 1, 2. \tag{6.3}$$

Proof. — In view of (6.2), the first equality follows by Lemma A.3, since $H_j(u)$ satisfies (A.5) with $s = 1$ for $j = 1$ and $s = p$ for $j = 2$, while $\partial_t u \in L^{2,\infty} \cap L^{p+1,r}$; recall that $u \in Z$. The second equality is true since H_j is real-valued; recall the definition (A.1) of H' (Appendix).

Lemma 6.1 shows that $\partial_t H(u) \in L^{1,r}$, hence $H(u) \in C(I; L^1)$. It follows that

$$\partial_t \langle H(u(t)), 1 \rangle = 2 \langle F(u(t)), \partial_t u(t) \rangle, \tag{6.4}$$

where $\langle\, , \rangle$ denotes the duality on $\mathbb{R}^m$. On the other hand, the property $u \in Z$ justifies the calculation

$$\partial_t \| \partial u(t) \|_2^2 = -2 \operatorname{Re} \langle \Delta u(t), \partial_t u(t) \rangle. \tag{6.5}$$

Adding these results yields

$$\partial_t E(t) = 2 \operatorname{Re} \langle -\Delta u(t) + F(u(t)), \partial_t u(t) \rangle = 0,$$

where (0.1) is used in the final stage.

Next we extend (6.1) to a solution $u \in C(I; H^1)$. For this we have only to approximate $u(0) = \phi \in H^1$ by a sequence $\phi_n \in H^2$ in the H^1-norm, and construct the corresponding solutions u_n with $u_n(0) = \phi_n$. Theorem I′ shows that the $u_n \in C(I; H^1)$ exist for sufficiently large n and $u_n \to u$. Theorem II shows that they belong to $C(I; H^2)$. Thus they satisfy (6.1).

Since the map $u(t) \mapsto E(t)$ is continuous from H^1 to $\mathbb{R}$, we have established (6.1) for u.

According to the assumption of Theorem III, (6.1) implies that $\| u(t) \|_{H^1} \leqslant K$ (another constant depending only on $\| u(0) \|_{H^1}$). In view of Theorem I, a standard argument shows that u can be continued to a global solution, with the estimates preserved.

APPENDIX

Here we collect some lemmas on the continuity properties of the « Nemyckii operator » $u \mapsto F(u)$ and its derivative.

Let $F: \mathbb{C} \to \mathbb{C}$ be continuously differentiable as a map on $\mathbb{R}^2$ into itself. The derivative $F'(\zeta)$ is defined as a real-linear operator acting on $\omega \in \mathbb{C}$ by

$$F'(\zeta).\omega = \lim_{\varepsilon\to 0} \varepsilon^{-1}(F(\zeta + \varepsilon\omega) - F(\zeta)) = (\partial F/\partial\zeta)\omega + (\partial F/\partial\bar{\zeta})\bar{\omega}. \tag{A.1}$$

We shall identify $F'(\zeta)$ with the pair $(\partial F/\partial\zeta, \partial F/\partial\bar{\zeta})$, and define its norm by

$$| F'(\zeta) | \equiv | \partial F/\partial\zeta | + | \partial F/\partial\bar{\zeta} |. \tag{A.2}$$

F' is continuous on $\mathbb{C}$ into $\mathbb{C}^2$, with

$$| F'(\zeta).\omega | \leqslant | F'(\zeta) | \, | \omega |. \tag{A.3}$$

For a complex-valued function u on $\mathbb{R}^m$ or $I \times \mathbb{R}^m$, $F(u) = F \circ u$ and $F'(u) = F' \circ u$ are defined pointwise in the usual way.

From now on we assume in addition that

$$| F(\zeta) | \leqslant M | \zeta |^{s+1}, \qquad | F'(\zeta) | \leqslant M | \zeta |^{s}, \tag{A.4}$$

where $s \geqslant 0$. The functions F_1, F_2 in the text are examples of (A.4), with $s = 0$ and $p - 1$.

LEMMA A.1. — Let $q \geqslant s + 1$. $\phi \mapsto F(\phi)$ is bounded and continuous on L^q into $L^{q/(s+1)}$. $(\phi, \psi) \mapsto F'(\phi).\psi$ is bounded and continuous on $L^q \times L^q$ into $L^{q/(s+1)}$.

These are special cases of Nemyckii's theorems (see e. g. Vainberg [5]). The proof is easy in our case since $F(\phi)$ does not depend on x explicitly.

LEMMA A.2. — Let $q \geqslant s + 1$. Let $u_n, u \in L^{q,\infty}$ and let $u_n(t) \to u(t)$ in L^q, boundedly for a. e. $t \in I$. Let $v_n \to v$ in $L^{q,r}$, where $1 \leqslant r < \infty$. Then $F'(u_n).v_n \to F'(u).v$ in $L^{q/(s+1),r}$.

This follows partly by L^r-convergence and partly by dominated convergence in t, applied to the integral expressing the $L^{q/(s+1),r}$-norm in question, by going over to a subsequence if necessary. Note that the limit does not depend on the subsequence.

LEMMA A.3. — Let $q \geqslant s + 1$. Let $u \in L^{q,\infty}$ and $\partial_j u \in L^{q,r}$, where j is any one of $0, 1, 2, \ldots, m$, with $\partial_0 = \partial_t$. Then

$$\partial_j F(u) = F'(u).\partial_j u \in L^{q/(s+1),r}. \tag{A.5}$$

Proof. — First we note that (A.5) makes sense. Indeed, the first member exists because $F(u) \in L^{q/(s+1),\infty}$, while the second member belongs to $L^{q/(s+1),r}$, by Lemma A.1. Thus it suffices to show that the two members are equal as distributions.

To this end we apply the Friedrichs mollifier to u, v to construct approximating sequences, u_n, v_n, $n = 1, 2, \ldots$. Then the u_n are bounded in $L^{q,\infty}$ and tend to u in $L^{q,q}$, while $v_n \to v$ in $L^{q,r}$. Going over to a subsequence, we may assume that $u_n(t) \to u(t)$ in L^q for a. e. t. Then we obtain $F'(u_n).v_n \to F'(u).v$ by Lemma A.2. On the other hand, $F(u_n) \to F(u)$ in $L^{q/(s+1),q/(s+1)}$ by a variant of Lemma A.1 (replace the space function ϕ with the space-time function u). Hence $\partial_j F'(u_n) \to \partial_j F(u)$ as distributions.

Since the equality (A.5) is obviously true for smooth functions u_n, v_n, we obtain (A.5) in the limit $n \to \infty$.

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