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# The canonical structure of the supersymmetric non-linear $\sigma$ model in the constrained Hamiltonian formalism

by

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**ABSTRACT.** — The canonical structure of a supersymmetric non-linear  $\sigma$ -model is investigated in the constraint Hamiltonian formalism due to Dirac. The model is invariant under global  $O(N)$  rotations and local  $O(p)$  transformations. The Dirac brackets between canonical variables are computed in axial gauge. It is shown that the model admits an infinite set of conserved non-local currents at the classical level.

**RÉSUMÉ.** — On étudie la structure canonique d'un modèle  $\sigma$  non linéaire supersymétrique dans le formalisme Hamiltonien avec contraintes dû à Dirac. Le modèle est invariant par rotations globales de  $O(N)$  et transformations locales de  $O(p)$ . On calcule les crochets de Dirac des variables canoniques dans la jauge axiale. On montre que le modèle admet, au niveau classique, une famille infinie de courants conservés non locaux.

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## 1. INTRODUCTION

The purpose of this article is to present our investigations of the canonical structure of a supersymmetric chiral model in  $1 + 1$  dimensions. This note is in sequel to our earlier work [1] on generalized non-linear

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$\sigma$ -model where the scalar fields were defined over the Grassmann manifold  $O(N)/O(N - p) \times O(p)$ .

The chiral models [2] in  $1 + 1$  dimensions are known to possess striking similarities with the non-Abelian gauge theories in  $3 + 1$  dimensions and they have played an important role to simulate the characteristics of the physical theories in four dimensions. Some of the common attributes of the two models are: *i*) scale invariance, *ii*) asymptotic freedom, *iii*) existence of a topological charge, and *iv*) the non-perturbative particle spectrum. Furthermore, the chiral models are known to admit an infinite sequence of non-local conserved currents [3] and hence they are completely integrable systems at the classical level. The conservation laws survive quantization in specific cases and such models are known to have exact factorizable S-matrix. There have been attempts to study the algebraic structure of the non-local charges and to reveal the hidden symmetries associated with the algebra of these charges. Moreover, the non-linear  $\sigma$ -models and their supersymmetric generalizations have proved quite useful to study the mechanism of dynamical symmetry breaking [4] and they have provided an attractive basis for construction of composite models [5].

Recently, the non-linear  $\sigma$ -models have played an important role in the context of the string theories. These models together with the topological term (Wess-Zumino term) are known to have a non-trivial fixed point at critical dimensions and the resulting theory is conformally invariant.

The chiral models and their supersymmetric versions [6] are described by singular Lagrangians. Therefore, the conventional method of canonical quantization is not adequate in order to quantize these models. We adopt the constrained Hamiltonian approach due to Dirac to investigate the canonical structure of the model. There have already been some attempts [7] in this direction and our aim is to present a more systematic analysis of the constraint structure of the model. The model under consideration possesses global symmetries as well as local non-Abelian gauge symmetries in addition to the global supersymmetries. The gauge fields are known to be composite fields at the tree level and they acquire dynamical degrees of freedom as a result of quantum fluctuations.

We determine all the constraints of the system: primary and secondary and then proceed as follows. The primary Dirac brackets are obtained in order to eliminate all bosonic constraints (those constraints which involve bose fields and bilinear in fermionic fields). Then we use the secondary Dirac brackets to exhaust the fermionic constraints. At this stage all second class constraints are eliminated and we are only left with a set of first class constraints. Next we fix the gauge and thus eliminate all the constraints. It is worth while to mention here that the supercharge obtained from the supercurrent does not give the proper transformation properties of the fermionic fields if we compute the naive commutator of the super-

charge with fermions. The appropriate transformation property is recovered only if we compute the necessary Dirac brackets and then the proper commutation relation is applied. The model admits an infinite sequence of non-local conserved currents. We prove the existence of an infinite sequence of non-local conserved currents.

The paper is organized as follows. In Section 2 we present the model and outline the constrained Hamiltonian formalism. The analysis of the structure of the constraints is contained in Section 3. The infinite set of non-local conserved currents are constructed in Section 4. In Section 5 we present our conclusions. There are two appendices listing useful formulas and conventions followed.

## 2. THE MODEL

We are seeking supersymmetric generalizations of the Lagrangian density

$$L = \frac{1}{2} \text{Tr} [(D_\mu \phi(x))^T D^\mu \phi(x)] \tag{1}$$

where  $\phi_i^A(x)$  are set of scalar fields defined over the Grassmann manifold  $O(N)/O(N - p) \times O(p)$  and they are subjected to the constrains

$$\phi_i^A(x)\phi_j^A(x) = \delta_{ij} \quad A=1 \dots N \quad i, j=1 \dots p, \quad p < N \tag{2}$$

The covariant derivative is defined as

$$D_\mu^{ij} = \partial_\mu \delta^{ij} - A_\mu^{ij}(x). \tag{3}$$

The scalar fields belong to the fundamental representation of  $O(N)$ ;  $A = 1, \dots, N$ , and that of  $O(p)$ ,  $i = 1, \dots, p$ . The gauge fields  $A_\mu^a$  belong to the adjoint representation of  $O(p)$ . We use the convention  $A_\mu^{ij} = A_\mu^a (T_a)^{ij}$ ,  $T_a$  are the generators of  $O(p)$  rotations; consequently  $A_\mu^{ij} = -A_\mu^{ji}$ . The Lagrangian density is invariant under global  $O(N)$  rotations and is also invariant under local  $O(p)$  gauge transformations.

We intend to investigate the supersymmetric version of the Lagrangian density (1). Let us introduce a superfield

$$\Sigma_i^A(x, \theta) = \phi_i^A(x) + \bar{\theta} \psi_i^A(x) + \frac{1}{2} \bar{\theta} \theta F_i^A(x). \tag{4}$$

Here  $\theta$  is the extra anticommuting co-ordinate and it is a two-component real spinor (see Appendix A for notations and the conventions followed for the  $\gamma$ -matrices). The superfield  $\Sigma_i^A(x, \theta)$  is a generalization of the one introduced for the  $O(N)$  non-linear  $\sigma$ -model. The superfield required to satisfy the following constraints

$$\Sigma_i^A(x, \theta) \Sigma_j^A(x, \theta) = \delta_{ij} \tag{5}$$

and consequently it imposes following constraints among the component fields:

$$\phi_i^\Lambda(x)\phi_j^\Lambda(x) = \delta_{ij} \quad (6)$$

$$\phi_i^\Lambda(x)\psi_j^\Lambda(x) = \phi_i^\Lambda(x)\bar{\psi}_i^\Lambda(x) = 0 \quad (7)$$

$$\phi_i^\Lambda(x)F_j^\Lambda(x) + F_i^\Lambda(x)\phi_j^\Lambda(x) = \frac{1}{2}\bar{\psi}_i^\Lambda(x)\psi_j^\Lambda(x). \quad (8)$$

The matter field  $\psi_i^\Lambda(x)$  is a two-component real Majorana spinor;  $\bar{\psi}_i^\Lambda = \psi_i^{+\Lambda}\gamma^0$  is pure imaginary in our convention. The supersymmetry invariant action can be constructed in a straightforward manner [8]

$$S_1 = -\frac{1}{2}\text{Tr} \int d^2x d\theta_1 d\theta_2 [\Sigma(x, \theta)\bar{\nabla}\nabla\Sigma(x, \theta)] \quad (9)$$

and the supercovariant derivative is defined as

$$\nabla_\alpha^{ij} = D_\alpha^{ij} + V_\alpha^{ij}, \quad i, j = 1 \dots p \quad (10)$$

with

$$D_\alpha^{ij} = \left( \frac{\partial}{\partial\theta_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu \right) \delta^{ij}. \quad (11)$$

The spinor superfield has following components in the Wess-Zumino gauge [8]

$$V_\alpha^{ij} = i(\gamma^\mu\theta)_\alpha A_\mu^{ij} + (\gamma_5\theta)_\alpha S^{ij} + \frac{1}{2}i\bar{\theta}\theta\chi_\alpha^{ij} \quad (12)$$

and

$$V_\alpha^{ij} = -V_\alpha^{ji}. \quad (13)$$

Notice that there is no term corresponding to the kinetic energy of the « gauge fields » and consequently their equations of motion are merely constraint equations. We shall eliminate the auxiliary fields  $S$ ,  $\chi$  and  $F$  from the action. Integrating over the Grassmann variables and eliminating the auxiliary fields we get

$$S_1 = \frac{1}{2} \int d^2x \text{Tr} \left[ (D^\mu\phi)^T(D_\mu\phi) + i\bar{\psi}\not{D}\psi + \frac{1}{4}(\bar{\psi}\psi)^2 - \frac{1}{4}(\bar{\psi}\gamma_5\psi)^2 \right] \quad (14)$$

with the constraints

$$\phi_i^\Lambda(x)\phi_j^\Lambda(x) = \delta_{ij} \quad (15)$$

$$\phi_i^\Lambda(x)\psi_j^\Lambda(x) = \phi_i^\Lambda(x)\bar{\psi}_j^\Lambda(x) = 0 \quad (16)$$

and the covariant derivative is

$$D_\mu = \partial_\mu - A_\mu, \quad A_\mu^\top = -A_\mu. \quad (17)$$

We may mention here in passing that we have set the coupling constant (which is a dimensionless parameter in two dimensions) to unity. This choice is adopted to simplify the computation of all Poisson brackets;

otherwise we have to carry this parameter in all our calculations. Note that term  $(\bar{\psi}_i^\Lambda \gamma_5 \psi_j^\Lambda)^{2/4}$  in Eq. (14) vanishes due to the fact that  $\psi_i^\Lambda$  are Majorana spinors. It arises from the elimination of the field  $S^{ij}(x)$  occurring as a component of  $V_\alpha$  in (12).

The fields transform as follows under supersymmetry transformations:

$$\delta\phi_i^\Lambda(x) = \bar{\varepsilon}\psi_i^\Lambda(x) \tag{18}$$

$$\delta\psi_i^\Lambda(x) = i\rlap{\not{D}}^{ij}\phi_j^\Lambda(x)\varepsilon + \frac{1}{2}\bar{\psi}_i^\Lambda(x)\psi_i^\Lambda(x)\phi_i^\Lambda(x)\varepsilon. \tag{19}$$

Here  $\varepsilon$  is a constant Majorana spinor, and the supercurrent is given by

$$J_\alpha^\mu(x) = \rlap{\not{D}}^{ij}\phi_i^\Lambda(x)\gamma^\mu\psi_{i\alpha}^\Lambda(x). \tag{20}$$

The action (14) is invariant under following symmetry transformations: (a) global  $O(N)$ , (b) local  $O(p)$  gauge rotations, and (c) global supersymmetry given by (18) and (19).

It is more convenient to introduce the modified Lagrangian density in order to implement the constraints (15) and (16):

$$\begin{aligned} L = \frac{1}{2} \text{Tr} [ & (D^\mu\phi)^\text{T}(D_\mu\phi) + i\bar{\psi}\rlap{\not{D}}\psi + \frac{1}{4}(\bar{\psi}\psi)^2 - \frac{1}{4}(\bar{\psi}\gamma_5\psi)^2 ] \\ & - \lambda_{ij}(x)(\phi_i^\Lambda\phi_j^\Lambda - \delta_{ij}) - \bar{\chi}_{ij}(x)\phi_i^\Lambda\psi_j^\Lambda \end{aligned} \tag{21}$$

where  $\lambda_{ij}(x)$  and  $\bar{\chi}_{ij}(x)$  are the two space-time dependent Lagrangian multiplier fields introduced in order to implement the required constraints.  $\chi_{ij}(x)$  is a Majorana spinor.

Now we are in a position to investigate the canonical structure of the model described by the Lagrangian density (21). The canonical momenta are

$$\pi_i^\Lambda(x) \equiv \frac{\partial L}{\partial\dot{\phi}_i^\Lambda(x)} = D_0^{ij}\phi_j^\Lambda(x) = \dot{\phi}_i^\Lambda(x) - A_0^{ij}(x)\phi_j^\Lambda(x) \tag{22}$$

$$\pi_{ij}^\lambda(x) \equiv \frac{\partial L}{\partial\dot{\lambda}_{ij}(x)} \approx 0$$

$$\pi_\mu^{ij}(x) \equiv \frac{\partial L}{\partial\dot{A}_\mu^{ij}(x)} \approx 0 \tag{24}$$

$$\pi_\chi^{ij}(x) \equiv \frac{\partial L}{\partial\dot{\chi}^{ij}(x)} \approx 0 \tag{25}$$

$$p_{i\alpha}^{\psi^\Lambda} \equiv \frac{\partial L}{\partial\dot{\psi}_{i\alpha}^\Lambda(x)} = -i\psi_{i\alpha}^{+\Lambda}(x). \tag{26}$$

Here  $\simeq$  means the weak equality. Since  $\psi_i^A(x)$  is a real Majorana spinor, it follows from (26) that

$$p_{i\alpha}^{\psi^A}(x) + \frac{i}{2} \psi_{i\alpha}^A(x) \approx 0. \quad (27)$$

This is a set of second class constraints and we can eliminate these constraints at this stage (for definition of second class constraints see below) and use the canonical brackets between  $\psi$ 's now on. Furthermore, we adopt the following equal time Poisson bracket relations among canonically conjugate variables

$$[\phi_i^A(x), \pi_j^B(y)] = \delta_{ij} \delta^{AB} \delta(x - y) \quad (28)$$

$$[A_\mu^{ij}(x), \pi_\nu^{km}(y)] = \delta_{\mu\nu} \delta_{ik} \delta_{jm} \delta(x - y) \quad (29)$$

$$[\lambda_{ij}(x), \pi_\lambda^{km}(y)] = \delta_{ik} \delta_{jm} \delta(x - y) \quad (30)$$

$$[\bar{\chi}^{ij}(x), \pi_\chi^{km}(y)] = \delta_{ik} \delta_{jm} \delta(x - y) \quad (31)$$

$$[\psi_{i\alpha}^A(x), \psi_{j\beta}^B(y)] = -i \delta^{AB} \delta_{ij} \delta_{\alpha\beta} \delta(x - y). \quad (32)$$

The canonical Hamiltonian density is

$$\begin{aligned} \mathcal{H}_c = & \frac{1}{2} \pi_i^A(x) \pi_i^A(x) + \frac{1}{2} \pi_i^A(x) A_\mu^{ij}(x) \phi_j^A(x) + \frac{1}{2} (\mathbf{D}_1 \phi^A(x))_i (\mathbf{D}_1 \phi^A(x))_i \\ & + \frac{i}{2} \bar{\psi}_i^A(x) \not{D}_1 \psi_i^A(x) + \frac{i}{2} \bar{\psi}_i^A(x) A_\mu^{ij}(x) \psi_j^A(x) - \frac{1}{4} (\bar{\psi}_i^A(x) \psi_j^A(x))^2 \\ & + \frac{1}{4} (\bar{\psi}_i^A(x) \gamma_5 \psi_j^A(x))^2 + \lambda_{ij}(x) (\phi_i^A(x) \phi_j^A(x) - \delta_{ij}) + \bar{\chi}_{ij}(x) \phi_i^A(x) \Psi_j^A \end{aligned} \quad (33)$$

### 3. THE ANALYSIS OF CONSTRAINTS

#### 3.1. The constraint Hamiltonian formalism.

Let us briefly recapitulate the constraint Hamiltonian formalism due to Dirac [9]. The canonical momenta corresponding to the fields  $\lambda$ ,  $\chi$  and  $A_\mu$  are constraint equations (23) (25). Thus the total Hamiltonian is the sum of the canonical Hamiltonian and a linear combination of these constraints, called the primary constraints. If we want that these constraints should hold good for all times then the Poisson bracket of each constraint with the total Hamiltonian must vanish. As a consequence we generate new constraints in general. They are known as the secondary constraints. If we further demand that the secondary constraints should hold good for all times then they should have vanishing Poisson brackets with the total Hamiltonian. This in turn generates more constraints. We continue the process until no new constraints are generated. Now the set of all constraints are classified as follows. A set of constraints whose Poisson bracket with

any other member of the set gives a linear combination of the constraints of the set is called first class. The Poisson bracket of a first class constraint with the total Hamiltonian gives a linear combination of first class constraints. The set of constraints whose mutual Poisson brackets are constants, independent of the phase space variables, is called second class, i. e., if  $\{\chi_\alpha\}$  are second class, then  $\det [\chi_\alpha, \chi_\beta]_{\text{PB}} = 0$ . The total Hamiltonian is

$$H_T = \int dx \mathcal{H}_c + \int dx (\sigma_{mn}^\mu(x) \pi_\mu^{mn}(x) + \rho_{ij}(x) \pi_\lambda^{ij}(x) + \eta_{ij}(x) \pi_\lambda^{ij}(x)) \quad (34)$$

Here  $\sigma_{mn}^\mu(x)$ ,  $\rho_{ij}(x)$  and  $\eta_{ij}(x)$  are analogues of the Lagrange multipliers. We may remark here that the Hamiltonian is a function of anticommuting Grassmann variables and one has to follow an appropriate rule for the functional differentiations; our convention is explained in Appendix A.

### 3.2. The constraints and the Dirac brackets.

We list below all the constraints associated with the model described by the Lagrangian (21)

$$F_1 \equiv \pi_\mu^{km}(x) \simeq 0 \quad (35)$$

$$F_2 \equiv \pi_k^\Lambda(x) \phi_m^\Lambda(x) - \pi_m^\Lambda(x) \phi_k^\Lambda(x) + i \bar{\psi}_m^\Lambda(x) \gamma^0 \psi_k^\Lambda(x) \simeq 0 \quad (36)$$

$$\Lambda_1 \equiv \pi_1^{km}(x) \simeq 0 \quad (37)$$

$$\Lambda_2 \equiv \pi_\lambda^{ij}(x) \simeq 0 \quad (38)$$

$$\Lambda_3 \equiv \phi_k^\Lambda(x) D_1^{mi} \phi_i^\Lambda(x) - \frac{i}{2} \bar{\psi}_k^\Lambda(x) \gamma^1 \psi_m^\Lambda(x) \simeq 0 \quad (39)$$

$$\Lambda_4 \equiv \phi_i^\Lambda(x) \phi_j^\Lambda(x) - \delta_{ij} \simeq 0 \quad (40)$$

$$\Lambda_5 \equiv \pi_k^\Lambda(x) \phi_m^\Lambda(x) + \pi_m^\Lambda(x) \phi_k^\Lambda(x) \quad (41)$$

$$\Lambda_6 \equiv 2\lambda_{km}(x) - D_1^{kl} \phi_l^\Lambda(x) D_1^{mn} \phi_n^\Lambda(x) - \pi_k^\Lambda(x) \pi_m^\Lambda(x) \simeq 0 \quad (42)$$

$$\Delta_1 \equiv \pi_{x1}^{ij}(x) \simeq 0 \quad (43)$$

$$\Delta_2 \equiv \pi_{x2}^{ij}(x) \simeq 0 \quad (44)$$

$$\Delta_3 \equiv \phi_i^\Lambda(x) \psi_{j1}^\Lambda(x) \simeq 0 \quad (45)$$

$$\Delta_4 \equiv \phi_i^\Lambda(x) \psi_{j2}^\Lambda(x) \simeq 0 \quad (46)$$

$$\Delta_5 \equiv \bar{\chi}_1^i(x) - i \phi_i^\Lambda(x) (\gamma^0 \mathcal{D}^j)_{1\alpha} \psi_{i\alpha}^\Lambda(x) \simeq 0 \quad (47)$$

$$\Delta_6 \equiv \bar{\chi}_2^j(x) - i \phi_i^\Lambda(x) (\gamma^0 \mathcal{D}^j)_{2\alpha} \psi_{i\alpha}^\Lambda(x) \simeq 0. \quad (48)$$

The constraints  $F_1$  and  $F_2$  are first class and the rest are all second class. The constraints  $\Lambda_1, \dots, \Lambda_6$  given by Eqs. (37)-(42) are bosonic ones and  $\Delta_1, \dots, \Delta_6$  are the fermionic ones. All the constraints are  $p \times p$  matrices and we have suppressed the indices for the notational simplicity. Note that  $F_2$  are the generators of the  $O(p)$  gauge transformations.



Now we are in a position to compute all the necessary Dirac brackets. We eliminate all the second class constraints in a two-step process since we are dealing with a large number of constraints. This is a very convenient procedure. The primary Dirac bracket between two dynamical variables A and B is given by

$$\{A, B\}' = [A, B]_{\text{PB}} - \int \sum_{I, J=1}^6 dudv [A, \Lambda_I^{ij}(u)]_{\text{PB}} C_{IJ}^{-1ijkl} [\Lambda_J^{kl}(v), B]_{\text{PB}} \quad (49)$$

where  $[\ , ]_{\text{PB}}$  is the canonical Poisson bracket and  $C_{IJ}^{-1ijkl}$  is the inverse of the non-singular matrix  $C_{IJ}$  obtained by taking a canonical Poisson bracket between the pair of bosonic second class constraints  $\Lambda_I$  and  $\Lambda_J$ ,  $I, J = 1, \dots, 6$

$$[\Lambda_I^{ij}(u), \Lambda_J^{kl}(v)] = C_{IJ}^{ijkl}(u, v). \quad (50)$$

Some of the useful brackets are (equal-time)

$$\{\phi_i^A(x), \phi_j^B(y)\}' = 0 \quad (51)$$

$$\{\phi_i^A(x), \pi_j^B(y)\}' = [\delta^{AB} \delta_{ij} - (\phi_j^A(x) \phi_i^B(x) + \phi_n^A(x) \phi_n^B(x) \delta_{ij})] \delta(x - y) \quad (52)$$

$$\begin{aligned} \{\pi_i^A(x), \pi_j^B(y)\}' &= [\phi_n^A(x) \pi_n^B(x) \delta_{ij} + \phi_j^A(x) \pi_i^B(x) \\ &\quad - \pi_n^A(x) \phi_n^B(x) \delta_{ij} - \pi_j^A(x) \phi_i^B(x)] \delta(x - y) \end{aligned} \quad (53)$$

$$\{\psi_{i\alpha}^A(x), \psi_{j\beta}^B(y)\}' = -i \delta^{AB} \delta_{ij} \delta_{\alpha\beta} \delta(x - y). \quad (54)$$

Having computed the primary Dirac brackets, now we can eliminate the fermionic second class constraints and introduce the secondary Dirac brackets. If A and B are two dynamical variables, the secondary Dirac bracket between A and B is defined as

$$\{A, B\}_{s'} = \{A, B\}' - \int dw dz \sum_{\alpha, \beta=1}^6 \{A, \Delta_{\alpha}^{ij}(u)\}' D_{\alpha\beta}^{-1ijkl}(w, z) \times \{\Delta_{\beta}^{kl}(z), B\}' \quad (55)$$

Here  $\{, \}'$  are the primary Dirac brackets. The secondary Dirac bracket takes the following form when written in terms of Poisson brackets

$$\begin{aligned} \{A, B\}_{s'} &= [A, B] - \sum_{I, J=1}^6 \int dudv [A, \Lambda_{IJ}^i(u)] C_{IJ}^{-1ijkl}(u, v) [\Lambda_J^{kl}(v), B] \\ &\quad - \left[ \sum_{\alpha, \beta=1}^6 \int dw dz ([A, \Delta_{\alpha}^{ij}(w)] \right. \\ &\quad - \int dudv \sum_{I, J=1}^6 [A, \Lambda_I^{pq}(u)] C_{IJ}^{-1pqrs}(u, v) [\Lambda_J^{rs}(v), \Delta_{\alpha}^{ij}(w)] D_{\alpha\beta}^{-1ijkl}(w, z) ([\Delta_{\beta}^{kl}(z), B] \\ &\quad \left. - \int \sum_{I', J'=1}^6 du' dv' x [\Delta_{\beta}^{kl}(w), \Lambda_{I'}^{p'q'}(u')] C_{I'J'}^{-1p'q'r's'}(u', v') [\Lambda_{J'}^{r's'}(v'), B] \right) \end{aligned} \quad (56)$$

The object  $D_{\alpha\beta}^{-1\,ijkl}$  is the inverse of the non-singular matrix defined as follows

$$[\Delta_{\alpha}^{ij}(w), \Delta_{\beta}^{kl}(z)]_{PB} = D_{\alpha\beta}^{ijkl}(w, z) \tag{57}$$

It is now clear that the two-step process avoids the problem associated with inversion of large matrices. In our case we have to invert two  $6 \times 6$  matrices. If we wanted to eliminate all the second-class constraints in one step, then we will have to deal with a  $12 \times 12$  matrix and invert it. The relevant secondary Dirac brackets are

$$\{ \phi_i^A(x), \phi_j^B(y) \}_s = 0 \tag{58}$$

$$\{ \phi_i^A(x), \pi_j^B(y) \}_s = [\delta^{AB}\delta_{ij} - (\phi_j^A(x)\phi_i^B(x) + \phi_n^A(x)\phi_n^B(x)\delta_{ij})]\delta(x-y) \tag{59}$$

$$\{ \pi_i^A(x), \pi_j^B(y) \}_s = [\phi_n^A(x)\pi_n^B(x)\delta_{ij} + \phi_j^A(x)\pi_i^B(x) - \pi_n^A(x)\phi_n^B(x)\delta_{ij} - \pi_j^A(x)\phi_i^B(x) + 2i\psi_n^A(x)\psi_n^B(x)\delta_{ij}]\delta(x-y) \tag{60}$$

$$\{ \phi_i^A(x), \psi_j^B(y) \}_s = 0 \tag{61}$$

$$\{ \pi_i^A(x), \psi_j^B(y) \}_s = -\psi_j^A(x)\phi_i^B(x)\delta(x-y) \tag{62}$$

$$\{ \psi_{i\alpha}^A(x), \psi_{j\beta}^B(y) \}_s = [-i\delta^{AB}\delta_{ij}\delta_{\alpha\beta} - i\phi_n^A(x)\phi_n^B(x)\delta_{ij}\delta_{\alpha\beta}]\delta(x-y). \tag{63}$$

Notice that there are extra terms in the right-hand side of Eqs. (59) (60) (62) and (63) which is not present in the naive Poisson bracket relations. This is the manifestation of the constraints present in the system.

### 3.3. Gauge fixing and elimination of the first-class constraints.

We obtained the relevant secondary Dirac brackets between canonical variables. However, we have to eliminate the first-class constraints. We choose the gauge  $A_1 = 0$ . Then

$$\chi_1^{km} = \pi_k^A(x)\phi_m^A(x) - \pi_m^A(x)\phi_k^A(x) + i\bar{\psi}_m^A(x)\gamma^0\psi_k^A(x) \simeq 0 \tag{64}$$

$$\chi_2^{ij} = \phi_j^A(x)\partial_1\phi_i^A(x) - \frac{i}{2}\bar{\psi}_j^A(x)\gamma^1\psi_i^A(x) \simeq 0 \tag{65}$$

are a set of second-class constraints and we can eliminate them. The Dirac bracket now is

$$\{ A, B \}_{DB} = \{ A, B \}_s - \sum_{I,J=1}^6 \int dudv \{ A, \chi_I^j(u) \}_s G_{IJ}^{-1\,ijkl}(u, v) \times \{ \chi_J^k(v), B \}_s \tag{66}$$

where  $A$  and  $B$  are any two-dynamical variables and  $G^{-1}$  is the inverse of the matrix  $G$  obtained from mutual Poisson brackets between  $\chi_1$  and  $\chi_2$ .

Some of the useful canonical Dirac brackets are

$$\{ \phi_i^A(x), \phi_n^B(y) \}_{\text{DB}} = 0 \quad (67)$$

$$\begin{aligned} \{ \phi_i^A(x), \pi_n^B(y) \}_{\text{DB}} = & [\delta^{AB} \delta_{in} - (\phi_n^A(x) \phi_i^B(x) + \phi_j^A(x) \phi_j^B(x) \delta_{in})] \delta(x-y) \\ & - \varepsilon(x-y) [\partial_1 \phi_i^B(x) \phi_n^A(x) - \phi_n^C(x) \partial_1 \phi_i^C(x) \phi_j^B(x) \phi_j^A(x) \\ & - \phi_j^C(x) \partial_1 \phi_i^C(x) \phi_j^B(x) \phi_n^A(x) - \partial_1 \phi_j^B(x) \phi_n^A(x) \phi_j^A(x) \delta_{in} \\ & + \partial_1 \phi_j^C(x) \phi_n^C(x) \phi_i^B(x) \phi_j^A(x) + \partial_1 \phi_j^C(x) \phi_k^C(x) \phi_k^B(x) \phi_j^A(x) \delta_{in}] \end{aligned} \quad (68)$$

$$\begin{aligned} \{ \psi_{i\alpha}^A(x), \psi_{n\beta}^B(y) \}_{\text{DB}} = & [i \delta^{AB} \delta_{in} \delta_{\alpha\beta} - i \phi_j^A(x) \phi_j^B(x) \delta_{in} \delta_{\alpha\beta}] \delta(x-y) \\ & + i \varepsilon(x-y) [\psi_{i\alpha}^A(x) \psi_{i\beta}^B(x) \delta_{in} - \psi_{n\alpha}^A(x) \gamma_5 \psi_{i\beta}^B(x)]. \end{aligned} \quad (69)$$

We may remark here that a similar procedure could have been carried out for any other gauge choice. Furthermore, now the theory can be quantized either in the canonical approach or in the path integral approach. The former corresponds to the prescription that all Dirac brackets go over to quantum commutation relations with an appropriate factor of  $i$ . On the other hand, the path integral quantization is implemented using the standard prescription [10] once all the constraints have been deduced.

#### 4. THE NON-LOCAL CONSERVED CURRENTS

In this section we shall show the existence of an infinite sequence of non-local conserved currents in the model. We note that the general method laid down by Brézin et al. [3] for the chiral models needs modification. The chiral model is described by the Lagrangian density

$$L = \frac{1}{2} \text{Tr} \int d^2x \partial^\mu g^{-1}(x) \partial_\mu g(x) \quad (70)$$

where  $g(x)$  belongs to some compact Lie group  $G$  in a matrix representation. When  $g(x)$  varies over the whole group  $G$  the action is invariant under the global transformation of  $G \otimes G$ . The equations of motion are

$$\partial^\mu A_\mu(x) = 0 \quad (71)$$

with  $A_\mu = g^{-1}(x) \partial_\mu g(x)$  and the covariant derivative  $D_\mu = \partial_\mu + A_\mu(x)$  satisfies the following requirements:

$$[D_\mu, D_\nu] = 0, \quad \text{and} \quad [\partial_\mu, D_\nu] = 0. \quad (72)$$

For example, in the case of  $O(N)$  non-linear  $\sigma$  model  $A_\mu^{ij} = 2n^i(x) \overleftrightarrow{\partial}_\mu n^j(x)$  and the fields satisfy the constraint  $n^i(x) n^i(x) = 1$ . We may recall that (71) and (72) guarantee the existence of an infinite set of non-local conserved currents. We demonstrate in what follows that the above conditions are not satisfied in our model.

The Noether current corresponding to the global O(N) rotations in our model is

$$J_\mu^{AB} = \mathcal{A}_\mu^{AB} + 2\mathbb{B}_\mu^{AB} \tag{73}$$

where

$$\mathcal{A}_\mu^{AB} = 2\phi_i^A(x)\overleftrightarrow{D}_\mu^{ij}\phi_j^B(x) \tag{74}$$

$$\mathbb{B}_\mu^{AB} = -i\bar{\psi}_i^A(x)\gamma_\mu\psi_i^B(x). \tag{75}$$

The equations of motion are

$$(D^\mu D_\mu)^{ij}\phi_j^A - \lambda_{ij}\phi_j^A - \bar{\chi}_{ij}\psi_j^A = 0 \tag{76}$$

$$i\overleftrightarrow{D}^{ij}\psi_j^A - \chi_{ij}\phi_j^A - \frac{1}{2}\psi_j^A. \tag{77}$$

We can easily eliminate the Lagrangian multipliers  $\lambda_{ij}$  and  $\bar{\chi}_{ij}$  by appropriately multiplying  $\phi_j^A$  and  $\psi_i^A$  and using the constraints (15) and (16). Furthermore, we find that the relations

$$\partial^\mu \mathcal{A}_\mu^{AB} = -\mathbb{B}_\mu^{AC}\mathcal{A}^{\mu CB} + \mathcal{A}^{\mu AC}\mathbb{B}_\mu^{CB} \tag{78}$$

$$\partial^\mu \mathbb{B}_\mu^{AB} = -\frac{1}{2}\mathcal{A}^{\mu AC}\mathbb{B}_\mu^{CB} + \frac{1}{2}\mathbb{B}_\mu^{AC}\mathcal{A}^{\mu CB} \tag{79}$$

hold good on-shell, i. e., when the fields  $\phi_i^A(x)$  and  $\psi_i^A(x)$  satisfy the equations of motion and  $J_\mu^{AB}$  is conserved although  $\mathcal{A}_\mu$  and  $\mathbb{B}_\mu$  are not conserved separately. If we define a covariant derivative  $D_\mu^{AB} = \partial_\mu\delta^{AB} + J_\mu^{AB}$ , then it is not curl free:

$$[D_\mu, D_\nu]^{AB} \neq 0 \tag{80}$$

as is evident from the following relations satisfied on-shell:

$$\varepsilon^{\mu\nu}\partial_\mu\mathcal{A}_\nu^{AB} = -\varepsilon^{\mu\nu}\mathcal{A}_\mu^{AC}\mathcal{A}_\nu^{CB} \tag{81}$$

$$\varepsilon^{\mu\nu}\partial_\mu\mathbb{B}_\nu = -\varepsilon^{\mu\nu}\mathbb{B}_\mu^{AC}\mathbb{B}_\nu^{CB} - \frac{1}{2}(\mathcal{A}_\mu^{AC}\mathbb{B}_\nu^{CB} + \mathbb{B}_\mu^{AC}\mathcal{A}_\nu^{CB}). \tag{82}$$

We construct a covariant derivative from the linear combination  $\mathcal{A}_\mu$  and  $\mathbb{B}_\mu$  and their duals in such a way that the corresponding curvature tensor vanishes [11] [12]. Thus  $\Delta_\mu$  is such that

$$[\Delta_\mu(x, t, \lambda), \Delta_\nu(x, t, \lambda)]^{AB} = 0 \tag{83}$$

where

$$\Delta_\mu^{AB} = \partial_\mu\delta^{AB} + K_\mu^{AB}(x, t, \lambda) \tag{84}$$

$$K_\mu^{AB}(x, t, \lambda) = -\frac{\lambda}{1-\lambda^2}\left(\lambda\mathcal{A}_\mu + \frac{4\lambda}{1-\lambda^2}\mathbb{B}_\mu + \varepsilon_{\mu\nu}\mathcal{A}^\nu + 2\frac{(1+\lambda^2)}{1-\lambda^2}\varepsilon_{\mu\nu}\mathbb{B}^\nu\right)^{AB} \tag{85}$$

We have explicitly exhibited the space-time dependence in the currents. Here  $\lambda$  is a non-zero real parameter. In the inverse scattering approach to this problem,  $K_\mu$  are identified as the Lax-pair and  $\lambda$  is the spectral para-

meter [13]. Note that (83) is only satisfied on-shell. Furthermore, the lack of curvature implies the integrability condition that there exists a function  $\eta(x, t, \lambda)$  satisfying

$$\Delta_\mu \eta(x, t, \lambda) = 0. \tag{86}$$

We can expand  $\eta(x, t, \lambda)$  in a power series in  $\lambda$  as follows:

$$\eta(x, t, \lambda) = \sum_{n=0}^{\infty} \lambda^n \eta^{(n-1)}(x, t). \tag{87}$$

We define  $F_\mu^{(n+1)}(x, t) = \varepsilon_{\mu\nu} \partial^\nu \eta^{(n+1)}(x, t)$  and substitute (87) in (86). If we collect the coefficients of the various powers of  $\lambda$  we get the following relation:

$$\begin{aligned} F_\mu^{(n+1)} &\equiv \varepsilon_{\mu\nu} \partial^\nu \eta^{(n+1)}(x, t) \\ &= (\partial_\mu + F_\mu^{(v)}(x, t)) \eta^{(n)}(x, t) + 2\mathbb{B}_\mu(x) [\eta^{(n-2)}(x, t) + \eta^{(4-2)}(x, t) \dots] \\ &\quad + 2\varepsilon_{\mu\nu} \mathbb{B}^\nu(x, t) [\eta^{(n-1)}(x, t) + \eta^{(n-3)}(x, t) + \dots] \end{aligned} \tag{88}$$

where

$$\eta^{(n)}(x, t) = \begin{cases} 0 & n < -1 \\ 1 & n = -1 \\ \int_{-\infty}^x d\xi F_0^{(n)}(\xi, t) & n \geq 0. \end{cases}$$

We have suppressed here all  $O(N)$  indices. Note that  $F_\mu^{(n)}$  is manifestly conserved. The conservation of the non-local charges

$$Q^{(n)} = \int_{-\infty}^{+\infty} F_0^{(n)}(\xi, t) d\xi \tag{89}$$

can be proved in the straightforward way [12].

### 5. DISCUSSIONS

We have presented a detailed analysis of the constraints in a super-symmetric non-linear  $\sigma$ -model. The Lagrangian (21) is singular and it is necessary to identify all the constraints of the model before we obtain all canonical commutation relations. If we compute the naive canonical Poisson brackets to quantize the theory then we are invariably led to incorrect results. This fact is illustrated through the following example.

The supersymmetry transformation properties of the fields  $\phi_i^A(x)$  and  $\psi_i^A(x)$  are given by (18) and (19). The supercharge is given by

$$Q_\alpha = \int dx \mathcal{D}^{ij} \phi_j^A(x, t) \gamma^0 \psi_i^A(x, t). \quad (90)$$

If we compute the transformation properties of  $\psi_i^A(x, t)$  under  $Q_\alpha$  using naive commutation relations then we do not get Eq. (19). However, the identification  $[\cdot, \cdot]_{\text{QB}} \rightarrow i \{\cdot, \cdot\}_{\text{DB}}$ , where QB stands for quantum bracket, gives the correct transformation property for  $\psi_i^A$

$$\delta \psi_i^A(x, t) = i [\bar{Q}_\alpha, \psi_i^A(x)]. \quad (91)$$

Similarly, we must always compute the Dirac brackets and then go over to the commutation (or anticommutation) relations whenever dynamical quantities are involved. We may point out that the naive anticommutator of supercharges does not reproduce the correct Hamiltonian.

It is interesting to note that non-linear  $\sigma$  models appear in the context of string theories [14] and the constraint Hamiltonian technique plays a useful role in studying the canonical structure of some of these models.

In conclusion, we have presented a systematic analysis of the canonical structure of the supersymmetric non-linear  $\sigma$ -model and have computed the canonical Dirac brackets in  $A_1 = 0$  gauge. The model admits an infinite sequence of non-local conserved currents. It will be interesting to investigate the algebraic structure [15] of these charges and to study the quantum conservation laws.

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## APPENDIX A

In this Appendix we present our notations, conventions and some useful formulas. The metric  $g^{\mu\nu}$  is chosen to be  $g_{\mu\nu}^{00} = -g^{11} = 1$  and the antisymmetric symbol  $\varepsilon^{\mu\nu}$  is such that  $\varepsilon^{01} = \varepsilon_{10} = -\varepsilon^{10} = 1$ . The  $\gamma$ -matrices are

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^0\gamma^1 = \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A1})$$

The charge conjugation matrix  $C$  has the following properties

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T, \quad C = -C^T. \quad (\text{A2})$$

In our case, the choice is  $C = \gamma^0 = C^{-1}$ . The Majorana spinor satisfies the following constraint

$$\bar{\psi} = \psi^+ \gamma^0 = \psi^T C \quad (\text{A3})$$

$\psi$  is taken to be real in our convention and consequently  $\bar{\psi}$  is pure imaginary. If  $\psi$  and  $\chi$  are two Majorana spinors then the bilinears constructed from these two spinors satisfy the following relations:

$$\bar{\psi}\chi = \bar{\chi}\psi, \quad \bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi, \quad \bar{\psi}\gamma_5\chi = -\bar{\chi}\gamma_5\psi. \quad (\text{A4})$$

Some of the useful relations among  $\gamma$ -matrices are

$$\gamma^\mu\gamma^\nu = g^{\mu\nu}1 + \varepsilon^{\mu\nu}\gamma_5, \quad \gamma^\mu = \gamma_5\varepsilon^{\mu\nu}\gamma_\nu. \quad (\text{A5})$$

The spinor trilinears satisfy the following relations:

$$\begin{pmatrix} \psi(\bar{\chi}\phi) \\ \gamma_\mu\psi(\bar{\chi}\gamma^\mu\phi) \\ \gamma_5\psi(\bar{\chi}\gamma_5\phi) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 \\ -2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \phi(\bar{\chi}\psi) \\ \gamma_\mu\phi(\bar{\chi}\gamma^\mu\psi) \\ \gamma_5\phi(\bar{\chi}\gamma_5\psi) \end{pmatrix} \quad (\text{A6})$$

*Poisson Brackets*

We follow the convention [16] described here in defining Poisson brackets. Let the Lagrangian density be a function of even variables  $\phi^i(x)$  and odd variables  $\psi^a(x)$  where  $i$  and  $a$  denote the degrees of freedom of the fields.  $L = L(\phi^i, \partial_\mu\phi^i, \psi^a, \partial_\mu\psi^a)$ . The canonical momenta are defined as follows:

$$\pi^i(x) \equiv \frac{\partial L}{\partial \dot{\phi}^i(x)} \quad (\text{A7})$$

$$\theta^a(x) \equiv \frac{\partial L}{\partial \dot{\psi}^a(x)} \quad (\text{A8})$$

The canonical Hamiltonian density is

$$\mathcal{H} = \dot{\phi}^i(x)\pi^i(x) + \dot{\psi}^a(x)\theta^a(x) - L. \quad (\text{A9})$$

Let  $A[\phi, \psi]$  and  $B[\phi, \psi]$  be two dynamical variables which are functionals of  $\phi$  and  $\psi$  and they are even. Then

$$[A, B]_{\text{PB}} = \int dx \left[ \left( \frac{\partial B}{\partial \pi^i(x)} \frac{\partial A}{\partial \phi^i(x)} - \frac{\partial B}{\partial \phi^i(x)} \frac{\partial A}{\partial \pi^i(x)} \right) - \left( \frac{\partial B}{\partial \theta^a(x)} \frac{\partial A}{\partial \psi^a(x)} + \frac{\partial B}{\partial \psi^a(x)} \frac{\partial A}{\partial \theta^a(x)} \right) \right] \quad (\text{A10})$$

If  $O$  and  $A$  are two odd and even dynamical variables, respectively, then their Poisson bracket is:

$$[O, A]_{\text{PB}} = \int dx \left[ \left( \frac{\partial O}{\partial \phi^i(x)} \frac{\partial A}{\partial \pi^i(x)} - \frac{\partial A}{\partial \phi^i(x)} \frac{\partial O}{\partial \pi^i(x)} \right) - \left( \frac{\partial O}{\partial \psi^a(x)} \frac{\partial A}{\partial \theta^a(x)} + \frac{\partial A}{\partial \psi^a(x)} \frac{\partial O}{\partial \theta^a(x)} \right) \right] \quad (\text{A11})$$

Finally, the Poisson bracket between two odd variables is given by

$$[O_1, O_2]_{\text{PB}} = \int dx \left[ \left( \frac{\partial O_1}{\partial \phi^i(x)} \frac{\partial O_2}{\partial \pi^i(x)} + \frac{\partial O_2}{\partial \phi^i(x)} \frac{\partial O_1}{\partial \pi^i(x)} \right) - \left( \frac{\partial O_1}{\partial \psi^a(x)} \frac{\partial O_2}{\partial \theta^a(x)} + \frac{\partial O_2}{\partial \psi^a(x)} \frac{\partial O_1}{\partial \theta^a(x)} \right) \right] \quad (\text{A12})$$

The Dirac brackets are defined in the appropriate manner.



## APPENDIX B

We present the matrices useful for computation of various Dirac brackets:

$$C^{-1} = \begin{bmatrix} 0 & 0 & -\delta_{ik}\delta_{mj} & 0 & 0 & 0 \\ 0 & 0 & C_{23}^{-1ikmj} & C_{24}^{-1ikmj} & \delta^{ik}\delta^{mj} & C_{26}^{-1ikmj} \\ \delta_{ik}\delta_{mj} & -C_{23}^{-1ikmj} & 0 & 0 & 0 & 0 \\ 0 & -C_{24}^{-1ikmj} & 0 & 0 & \frac{1}{2}(\delta_{mi}\delta_{kj} \\ & & & & + \delta_{ki}\delta_{mj}) & 0 \\ 0 & -\delta_{ik}\delta_{mj} & 0 & \frac{1}{2}(\delta_{mi}\delta_{kj} \\ & & & + \delta_{ki}\delta_{mj}) & 0 & 0 \\ 0 & -C_{26}^{-1ikmj} & 0 & 0 & 0 & 0 \end{bmatrix} \delta(u-v) \quad (B1)$$

The matrix D has a rather simple form

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & -\delta_{ik}\delta_{jm} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_{ik}\delta_{jm} \\ 0 & 0 & -i\delta_{ik}\delta_{jm} & 0 & D_{35}^{ikjm} & 0 \\ 0 & 0 & 0 & -i\delta_{ik}\delta_{jm} & 0 & D_{46}^{ikjm} \\ -\delta_{ik}\delta_{jm} & 0 & D_{35}^{ikjm} & 0 & D_{55}^{ikjm} & 0 \\ 0 & -\delta_{ik}\delta_{jm} & 0 & D_{46}^{ikjm} & 0 & D_{66}^{ikjm} \end{bmatrix} \delta(u-v) \quad (B2)$$

Finally, the matrix G has the following form in  $A_1 = 0$  gauge:

$$G = \begin{bmatrix} 0 & 2(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{kj}) \\ -2(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{kj}) & 0 \end{bmatrix} \partial_x \delta(x-y) \quad (B3)$$

We may remark here that it is easy to show that the constraints (i)  $\phi_i^A \phi_j^A = \delta_{ij}$ , and (ii)  $\psi_i^A \phi_j^A = 0$  are supersymmetry invariant.

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