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BERND THALLER
VOLKER ENSS

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Asymptotic observables and Coulomb scattering for the Dirac equation

by

Bernd THALLER and Volker ENSS
Institut für Mathematik I, Freie Universität Berlin, Arnimallee 2-6,
D-1000 Berlin 33, West-Germany

ABSTRACT. — We analyse the long-time behaviour of scattering states for the Dirac equation, in particular propagation properties in phase space. A major tool are the temporal asymptotics of observables like e. g. position, velocity, or the projections to positive/negative kinetic energy. Fairly general potentials of long and of short range are admitted. As a special application we give a simple proof of asymptotic completeness for the relativistic Coulomb system.

RÉSUMÉ. — On analyse le comportement à grand temps des états de diffusion pour l'équation de Dirac, et en particulier les propriétés de propagation dans l'espace de phase. Un outil essentiel est constitué par le comportement asymptotique en temps d'observables tels que la position, la vitesse, ou les projections sur les états d'énergie cinétique positive ou négative. On admet des potentiels assez généraux à longue et à courte portée. Comme application particulière, on donne une preuve simple de la complétude asymptotique pour le problème coulombien relativiste.

1. INTRODUCTION AND MAIN RESULTS

The relativistic motion of a spin-1/2 particle in an external field is described by the Dirac equation written as a Cauchy problem in $\mathscr{H} = L^2(\mathbb{R}^3)^4$, the Hilbert space of four component square integrable functions. Our aim is to investigate relativistic potential scattering for this system. We show that some observables are asymptotically constant and apply this to a proof of asymptotic completeness, where potentials of long range are included.

In nonrelativistic scattering theory, the asymptotic behaviour of certain observables, like position and velocity, can be applied as a useful ingredient in a proof of asymptotic completeness [*3*] [*4*] [*6*] [*11*] [*13*] [*19*]. As a typical result one obtains for suitable Ψ in the continuous spectral subspace of the Schrödinger operator H_s that

$$e^{iH_s t}(\boldsymbol{x}/t - \boldsymbol{p}/m)e^{-iH_s t}\Psi \to 0, \quad \text{as} \quad |t| \to \infty, \tag{1.1}$$

i. e. the average velocity $\{ \boldsymbol{x}(t) - \boldsymbol{x}(0) \}/t \approx \boldsymbol{x}(t)/t$ approaches the velocity $\boldsymbol{p}(t)/m$ at large times t. One obtains a localization in phase space which can be derived without detailed information about the interacting time evolution.

As a relativistic generalization of (1.1) one could try to replace H_s in (1.1) by the Dirac operator H and $\boldsymbol{p}/m$ by the relativistic velocity operator $c\boldsymbol{\alpha}$ (see below). Unfortunately, *that* statement is *false*, even for free particles:

The free time evolution of a particle with mass m is generated by

$$H_0 = c\boldsymbol{\alpha}.\boldsymbol{p} + \beta mc^2 \tag{1.2}$$

which is self-adjoint on the Sobolev space

$$\mathscr{D}(H_0) = W^{1,2}(\mathbb{R}^3)^4 . \tag{1.3}$$

Here $\boldsymbol{p} = -i\nabla$ acting componentwise, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are the hermitean 4×4 Dirac matrices defined by the commutation relations (A.1) given in the Appendix. In the sequel we keep a representation fixed where H_0 has this form for some $\boldsymbol{\alpha}$, β. The constant c denotes the speed of light. We choose the localization observable (« position operator ») to be in this representation the multiplication operator $\boldsymbol{x}$ acting on each component of $\Psi \in \mathscr{H}$. Then the velocity operator is given by $i[H_0, \boldsymbol{x}] = c\boldsymbol{\alpha}$. (For other localization operators see the remarks at the end of this section). Its time dependence is obtained by integrating eq. (A.2) with (1.13)

$$c\boldsymbol{\alpha}(t) \equiv e^{iH_0 t}c\boldsymbol{\alpha}e^{-iH_0 t} = c^2\boldsymbol{p}H_0^{-1} + e^{2iH_0 t}(c\boldsymbol{\alpha} - c^2\boldsymbol{p}H_0^{-1}). \tag{1.4}$$

Thus the velocity oscillates without damping (« Zitterbewegung » [*18*]) around a conserved mean value $c^2\boldsymbol{p}H_0^{-1}$ which is the velocity in *classical* relativistic kinematics.

On the other hand, integrating (1.4) yields on $\mathscr{D}(\boldsymbol{x})$

$$\boldsymbol{x}(t) = \boldsymbol{x} + c^2\boldsymbol{p}H_0^{-1}t + (e^{2iH_0 t} - \mathbb{1})(2iH_0)^{-1}(c\boldsymbol{\alpha} - c^2\boldsymbol{p}H_0^{-1}). \tag{1.5}$$

Since the last summand in (1.5) is bounded uniformly in t, we conclude for $\Psi \in \mathscr{D}(\boldsymbol{x})$, $H = H_0$ that

$$e^{iHt}\{ \boldsymbol{x}/t - c^2\boldsymbol{p}H_0^{-1} \} e^{-iHt}\Psi \to 0, \quad \text{as} \quad |t| \to \infty . \tag{1.6}$$

For particles scattered by external fields the operator $c^2\boldsymbol{p}H_0^{-1}$ will be time dependent and one does not have explicit integrals like (1.4), (1.5). But in the asymptotic region quantum particles behave quasiclassically and the fully interacting time evolution is approximated by the free one. This is made precise by showing a version of (1.6) for an extremely large class of interactions if Ψ is a scattering state. See Theorem 1.7 and the discussion following it. Beyond the straightforward generalizations of the corresponding nonrelativistic treatment [*4*] our proof requires additional results on the « Zitterbewegung », on invariant domains, and on positive/negative energy states which have no nonrelativistic analogue (Theorems 1.1, 2.1, and Corollary 1.4). Also long-range magnetic fields require the introduction of suitably adjusted observables as given in Section 3.

In the following we state the assumptions on the potential and our main results. The external field usually is described by an operator of multiplication by a hermitean 4×4-matrix valued function $V(\boldsymbol{x})$. This includes electric as well as magnetic or scalar fields and even magnetic moment interactions, etc. We assume that the interacting Dirac operator H is self-adjoint on a domain $\mathscr{D}(H)$ with the property

$$\mathscr{D}(H) \subset \mathscr{D}(|H_0|^{1/2}) \cap \mathscr{D}(|V|^{1/2}). \tag{1.7}$$

It is a sum of H_0 and V in the sense that for all $\Psi \in \mathscr{D}(H)$ and for all $\Phi \in \mathscr{D}(|H_0|^{1/2}) \cap \mathscr{D}(|V|^{1/2})$

$$(\Phi, H\Psi) = \left(|H_0|^{1/2}\Phi, H_0^{1/2}\Psi\right) + \left(|V|^{1/2}\Phi, V^{1/2}\Psi\right) \tag{1.8}$$

(where $H_0^{1/2} = |H_0|^{1/2} \operatorname{sgn}(H_0)$, etc.).

E. g. for the Coulomb potential $V(\boldsymbol{x}) = (\gamma/|\boldsymbol{x}|)\mathbb{1}$ it is well known that the minimal operator $H_0 + \gamma/|\boldsymbol{x}|$ on $C_0^\infty(\mathbb{R}^3)^4$, $|\gamma| < c$, though not essentially self-adjoint for $c\sqrt{3}/2 < |\gamma| < c$, admits a self-adjoint extension uniquely characterized by (1.7). This extension is given by (1.8) [*12*] or equivalently as the norm resolvent limit of Hamiltonians with cut-off potentials [*20*] [*9*].

In order to formulate our decay requirements we assume that one can split V into two parts

$$V = V_s + V_l \tag{1.9}$$

such that *(each element of)* the long-range potential matrix V_l is continuously differentiable and satisfies

$$V_l(\boldsymbol{x}) \to 0 \quad \text{and} \quad \boldsymbol{x}\cdot(\nabla V_l)(\boldsymbol{x}) \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{1.10}$$

The short-range part V_s contains the singularities of the potential and satisfies

$$(H_0 - z)^{-1}(1 + |\boldsymbol{x}|)V_s(H - z)^{-1} \quad \text{is compact}. \tag{1.11}$$

Note that by (1.7) and the closed graph theorem $(H_0 - z)^{1/2}(H - z)^{-1}$

is bounded. This implies with compactness of $F(|\boldsymbol{x}| < R)(H_0 - z)^{-1/2}$ the local compactness property of H. Together with decay at infinity this is equivalent to the compactness of $(H - z)^{-1} - (H_0 - z)^{-1}$. Thus (1.11) is essentially a condition on the asymptotic behaviour of V_s requiring a decay faster than $1/|\boldsymbol{x}|$ as $|\boldsymbol{x}|$ tends to infinity. The compactness of the difference of the resolvents also implies equality of the essential spectra by Weyl's Theorem:

$$\sigma_{ess}(H) = \sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, \infty). \tag{1.12}$$

We now proceed to state our main theorems. As we have seen above the kinematics of the Dirac equation does not exactly correspond to the intuition gained from classical relativity. In order to control the influence of quantum phenomena like « Zitterbewegung » we first investigate the asymptotic behaviour of the Dirac matrices $\boldsymbol{\alpha}$ and β. We define the bounded self-adjoint operators

$$\mathbf{F} := c\boldsymbol{\alpha} - c^2\boldsymbol{p}H_0^{-1} \quad \text{and} \quad G := \beta - mc^2H_0^{-1}. \tag{1.13}$$

F describes the Zitterbewegung (cf. eq. (1.4)) in a given inertial frame. It measures the difference between the actual velocity operator $c\boldsymbol{\alpha}$ and the operator corresponding to the classical velocity $\boldsymbol{v}=c^2\boldsymbol{p}/E$. G is related to a « fourth component » for **F**. It describes the difference between β and the operator corresponding to the classical expression $(1-v^2/c^2)^{1/2}=mc^2/E$. We have listed some of the algebraic properties of **F** and G in the Appendix. For any operator Q we denote the time translated one by

$$Q(t) = e^{iHt}Qe^{-iHt},$$

and P_{cont} is the projection onto the continuous spectral subspace $\mathscr{H}_{cont}$ of the Hamiltonian H.

THEOREM 1.1. — Let H satisfy (1.7)-(1.11).

a)
$$\underset{|t|\to\infty}{s\text{-lim}} \frac{1}{t}\int_0^t ds\mathbf{F}(s)P_{cont} = 0, \tag{1.14}$$

b)
$$\underset{|t|\to\infty}{s\text{-lim}} \frac{1}{t}\int_0^t dsG(s)P_{cont} = 0. \tag{1.15}$$

For a stronger technical statement see Proposition 2.6.

Compactness of $mc^2(H_0^{-1} - H^{-1})P_{cont}$ and Lemma 2.4 below combine with (1.15) to yield a constant limit of

$$\frac{1}{t}\int_0^t ds\beta(s)P_{cont} \to mc^2H^{-1}P_{cont}. \tag{1.16}$$

The time average of the Zitterbewegung decays like const/t for the free evolution, (1.4) and (1.5). The weaker decay (1.14) is sufficient to show that these oscillations do not contribute to the asymptotics of $\boldsymbol{x}(t)/t$.

Now we define the operator which for the purposes of relativistic kinematics plays the same role as the dilation generator in the Schrödinger case (using (A.6))

$$\mathrm{A} := \frac{c^2}{2}(\mathrm{H}_0^{-1}\boldsymbol{p}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{p}\mathrm{H}_0^{-1}) = c^2\mathrm{H}_0^{-1}\boldsymbol{p}\cdot\boldsymbol{x} + \frac{ic}{2}\mathrm{H}_0^{-1}\boldsymbol{\alpha}\cdot\mathbf{F}, \quad (1.17)$$

defined and essentially self-adjoint on $\mathscr{D}(\boldsymbol{x})$ and on $C_0^\infty(\mathbb{R}^3)^4$. A is the symmetrized inner product of the classical velocity with the position of the particle. The sign of A determines whether the averaged motion of the particle is towards the origin or away from it, and it takes into account that wave packets with negative energy move in a direction opposite to their momentum. Therefore the spectral projections of A characterize incoming and outgoing states just as the spectral projections of the dilation generator D do in the nonrelativistic situation. In the following $\mathrm{F}(\cdot)$ denotes the spectral projection of the self-adjoint operator to the part of the spectrum as indicated in the parenthesis.

THEOREM 1.2. — Let H satisfy (1.7)-(1.11). Then in the sense of strong resolvent convergence

$$\lim_{|t|\to\infty} \mathrm{A}(t)/t = c^2(\mathbb{1} - m^2c^4\mathrm{H}^{-2})\mathrm{P}_{\mathrm{cont}}. \quad (1.18)$$

(Note that classical kinematics gives $v^2 = c^2(1 - m^2c^4/\mathrm{E}^2)$).

COROLLARY 1.3. — For H as above and $\Psi \in \mathscr{H}_{\mathrm{cont}}$ we have for all $v < c$

a) $$\lim_{t\to\pm\infty} \| \mathrm{F}(\pm\mathrm{A} < v^2|t|)e^{-i\mathrm{H}t}\mathrm{F}(|\mathrm{H}| > mc^2(1 - v^2/c^2)^{-1/2})\Psi \| = 0, \quad (1.19)$$

b) $$\lim_{|t|\to\infty} \| \mathrm{F}(|\mathrm{A}| < a)e^{-i\mathrm{H}t}\Psi \| = 0 \quad \text{for all } a. \quad (1.20)$$

Proof. — We have

$$\mathrm{F}(\mathrm{A}(t)/t < v^2) \to \mathrm{F}(c^2(\mathbb{1} - m^2c^4\mathrm{H}^{-2}) < v^2)\mathrm{P}_{\mathrm{cont}}$$

strongly as $t \to \infty$ because of (1.18) and Theorem VIII 24.*b*) in [*15*]. An analogous calculation holds for $t \to -\infty$. This proves *a*). To prove *b*) we note that in the expression

$$\| \mathrm{F}(|\mathrm{A}| < a)e^{-i\mathrm{H}t}\mathrm{F}(|\mathrm{H}| < mc^2(1 - v^2/c^2)^{-1/2})\Psi \| \\ + \| \mathrm{F}(|\mathrm{A}| < a)e^{-i\mathrm{H}t}\mathrm{F}(|\mathrm{H}| > mc^2(1 - v^2/c^2)^{-1/2})\Psi \|$$

$\| \mathrm{F}(|\mathrm{H}| < mc^2(1 - v^2/c^2)^{-1/2})\Psi \|$ is arbitrarily small for small v whereas the second term vanishes as $|t| \to \infty$ by (1.19). □

We denote by $\mathrm{P}^{\pm}_{\mathrm{cont}}$ and $\mathrm{P}^{\pm}_0$ the projectors on the positive and negative energy subspaces of H and H_0 respectively, corresponding to the two parts of the spectrum indicated in eq. (1.12).

COROLLARY 1.4. — Let H be as above and $\Psi \in \mathscr{H}_{\text{cont}}$

a) $$\underset{|t|\to\infty}{w\text{-lim}}\, e^{-iHt}\Psi = 0\,, \tag{1.21}$$

b) $$\lim_{|t|\to\infty} P_0^{\pm}(t)\Psi = P_{\text{cont}}^{\pm}\Psi\,, \tag{1.22}$$

c) $$\lim_{|t|\to\infty} (c^2 \boldsymbol{p} H_0^{-1})^2(t) \quad \Psi = c^2(\mathbb{1} - m^2c^4H^{-2})\Psi\,. \tag{1.23}$$

Proof. — *a*) For any $\Phi \in \mathscr{H}$ and $\varepsilon > 0$

$$|(\Phi, e^{-iHt}\Psi)| \leq \| F(|A| > a)\Phi \| \, \| \Psi \| + \| \Phi \| \, \| F(|A| < a)e^{-iHt}\Psi \| < \varepsilon$$

if one chooses a large enough to have the first summand smaller than $\varepsilon/2$ and then takes $|t|$ large enough to get the second term smaller than $\varepsilon/2$ by (1.20).

b) Since $H^{-1} - H_0^{-1}$ is compact on $\mathscr{H}_{\text{cont}}$,

$$\| (H^{-1} - H_0^{-1})e^{-iHt}\Psi \| \to 0$$

for any $\Psi \in \mathscr{H}_{\text{cont}}$ by part *a*). But this implies $P_0^{\pm}(t)\Psi \to P_{\text{cont}}^{\pm}\Psi$ for $\Psi \in \mathscr{H}_{\text{cont}}$.

c) In view of $H_0^2 = c^2\boldsymbol{p}^2 + m^2c^4$ we have

$$\mathbb{1} - m^2c^4H^{-2} - c^2p^2H_0^{-2} = m^2c^4(H^{-2} - H_0^{-2})$$

which is also compact. Thus

$$\lim_{|t|\to\infty} \| (\mathbb{1} - m^2c^4H^{-2} - c^2p^2H_0^{-2})e^{-iHt}\Psi \| = 0\,. \qquad \square$$

Although the subspace of states with positive free energy is not conserved under the interacting time evolution, the components with positive/negative free energy of a continuum state are approximately conserved at late times by (1.22). This simple observation is very useful in applications (cf. Section 4).

THEOREM 1.5. — Let H be as above. Then in the sense of strong resolvent convergence

$$\lim_{|t|\to\infty} x^2(t)/t^2 = c^2(\mathbb{1} - m^2c^4H^{-2})P_{\text{cont}}\,. \tag{1.24}$$

COROLLARY 1.6. — Let H be as above, $v < c$, and $\Psi \in \mathscr{H}_{\text{cont}}$.

a) $$\lim_{|t|\to\infty} \| F(|\boldsymbol{x}| \lessgtr |vt|)e^{-iHt}F(|H| \gtrless mc^2(1 - v^2/c^2)^{-1/2})\Psi \| = 0, \tag{1.25}$$

b) $$\lim_{|t|\to\infty} \| F(|\boldsymbol{x}| \leq R)e^{-iHt}\Psi \| = 0 \quad \text{for all R}\,. \tag{1.26}$$

The Proof of this Corollary is analogous to that of Corollary 1.3. (1.25) says that a state is asymptotically localized in the region of space where it should be according to its energy support, because $mc^2(1 - v^2/c^2)^{-1/2}$ is just the energy of a classical relativistic particle with velocity v. The local decay property (1.26) is well known for the absolutely continuous subspace. Here we show it for all continuum states. We do not know whether our assumptions exclude a singular continuous spectrum.

From eqs. (1.18), (1.23), (1.24) and the corresponding proofs we conclude (1.6)

$$\| (\boldsymbol{x}/t - c^2\boldsymbol{p}\mathrm{H}_0^{-1})e^{-i\mathrm{H}t}\Psi \|^2$$
$$= (\Psi, \{ x^2(t)/t^2 - 2\mathrm{A}(t)/t + [c^4p^2\mathrm{H}_0^{-2}](t) \} \Psi) \to 0, \quad \text{as} \quad |t| \to \infty \qquad (1.27)$$

for suitable Ψ. To avoid domain questions we use bounded functions of the unbounded operators.

THEOREM 1.7. — Let H be as above, $\Psi \in \mathscr{H}_{\text{cont}}$, and assume f or $1 - f$ to be the Fourier transform of an integrable function on $\mathbb{R}^3$. Then

a) $$\lim_{|t|\to\infty} \| \{ f(\boldsymbol{x}/t - c^2\boldsymbol{p}\mathrm{H}_0^{-1}) - f(0) \} e^{-i\mathrm{H}t}\Psi \| = 0, \qquad (1.28)$$

a') $$\lim_{|t|\to\infty} \| \{ f(\boldsymbol{x}/t) - f(c^2\boldsymbol{p}\mathrm{H}_0^{-1}) \} e^{-i\mathrm{H}t}\Psi \| = 0, \qquad (1.29)$$

b) $$\lim_{|t|\to\infty} \| \{ f(\boldsymbol{x}/t \mp \boldsymbol{u}(\boldsymbol{p})) - f(0) \} e^{-i\mathrm{H}t}\mathrm{P}^{\pm}_{\text{cont}}\Psi \| = 0, \qquad (1.30)$$

b') $$\lim_{|t|\to\infty} \| \{ f(\boldsymbol{x}/t) - f(\pm \boldsymbol{u}(\boldsymbol{p})) \} e^{-i\mathrm{H}t}\mathrm{P}^{\pm}_{\text{cont}}\Psi \| = 0, \qquad (1.31)$$

where $\boldsymbol{u}(\boldsymbol{p}) = c^2\boldsymbol{p}(c^2p^2 + m^2c^4)^{-1/2}$.

This theorem expresses a correlation between the localization of a scattering state at late times with its velocity (momentum), i. e. it describes propagation in phase space. Consider e. g. a smooth function f with small support around some average velocity $\boldsymbol{v}_0$. Then at late times t the component of Ψ with velocities near $\boldsymbol{v}_0$ is localized near $\boldsymbol{v}_0 t$. This does not contradict the uncertainty principle because the spread of localization increases linearly in time: $f(\boldsymbol{x}/t)$ characterizes localization in a narrow cone in a space-time picture. The possibility of this almost linear increase is the reason why such a wide class of potentials can be admitted (slow decay at infinity) and why the proof is simple. Nevertheless the results are strong enough to conclude that asymptotically the angle between the position and velocity tends to zero (π) as $t \to \infty$ ($t \to -\infty$). Scattering states have been « incoming » in the remote past and will be « outgoing » in the far future, moving away from the region of significant interactions. This information is sufficient for some applications to scattering theory.

It can be seen that *a'* (resp. *b'*) of Theorem 1.7 is an equivalent reformulation of *a* (resp. *b*). Whenever (1.22) holds, also *b* (with both signs) is equivalent to *a*. Another reformulation of Theorem 1.7 *a*) due to Sinha and Muthuramalingam [*19*] is

COROLLARY 1.8. — For H as above, $\Psi \in \mathscr{H}_{\text{cont}}$, any $\delta > 0$

$$\lim_{|t| \to \infty} \| F(|x| > \delta t) e^{iH_0 t} e^{-iHt} \Psi \| = 0 . \tag{1.32}$$

Proof. — By (1.4), (1.5) on $\mathscr{D}(|x|)$

$$e^{iH_0 t}(x/t - c^2 p H_0^{-1}) = (x/t) e^{iH_0 t} + O(t^{-1}) .$$

Let f satisfy $f(0) = 0$, $f(z) = 1$ for $|z| \geq \delta$, then (1.28) implies (1.32) □

This can be interpreted as follows. The localization of Ψ at a late time t is the same as that of some state starting from a ball of radius $\delta |t|$ ($\delta > 0$ arbitrarily small) which moves freely with the same velocity distribution as that of $\exp(-iHt)\Psi$. If the long-range forces are restricted such that a « modified free » time evolution $U(t, 0)$ exists, then $\exp(iH_0 t)$ can be replaced in (1.32) by $U(0, t)$.

As a typical application of our results we prove asymptotic completeness for fairly arbitrary short-range matrix valued potentials and a long-range part of the physically relevant Coulombic type. The precise assumptions and results are given in Theorem 4.1. More general long-range forces can be treated as well but this will not be given in the present paper [*21*].

For our discussion the precise choice of the localization operator is irrelevant. Most « position operators » discussed in the literature differ from ours only by bounded operators [*8*]. Therefore $x(t)/t$ has the same limit. In particular the Newton-Wigner operator belongs to this class. Although it gives the « correct » velocity under the free time evolution it is not straightforward to define it in the interacting case. Our simple choice avoids these problems and is adequate for the applications as seen e. g. in Section 4.

2. INVARIANT DOMAINS, PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1 and some auxiliary results. We start by showing that the domain of $|x(t)|$ is invariant and we give bounds on $|x^n(t)|\Psi$. Corresponding results in nonrelativistic quantum mechanics have been given e. g. in [*7*] [*14*].

In our case the existence of a uniform bound on velocities enables us to obtain simpler results than in the Schrödinger operator case where the velocities are unbounded. This can easily be understood on a heuristic level. Formally, the time derivative of $x(t)$

$$\frac{d}{dt} x(t) = i[H, x(t)] = c\alpha(t)$$

is bounded. In

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t ds\, c\boldsymbol{\alpha}(s) \tag{2.1}$$

the second summand is bounded for all finite t and thus $\mathscr{D}(\boldsymbol{x}(t)) = \mathscr{D}(\boldsymbol{x})$.

THEOREM 2.1. — Let H satisfy (1.7), (1.8) and let $\boldsymbol{x}$ be the multiplication operator acting componentwise. Then for all $n \in \mathbb{N}$ the domain of $|\,\boldsymbol{x}\,|^n$ is invariant with respect to t:

$$\exp(-\,i\mathrm{H}t)\mathscr{D}(|\,\boldsymbol{x}\,|^n) = \mathscr{D}(|\,\boldsymbol{x}\,|^n)\,. \tag{2.2}$$

Moreover, for all $\Psi \in \mathscr{D}(|\,\boldsymbol{x}\,|^n)$ and $n \in \mathbb{N}$

$$\|\,|\,\boldsymbol{x}\,|^n \exp(-\,i\mathrm{H}t)\Psi\,\| \leq k_n(\Psi)(1 + c\,|\,t\,|)^n \tag{2.3}$$

with a constant $k_n(\Psi)$ depending only on n and Ψ.

Proof. — Observe that the result is trivial for $n = 0$ with $k_0(\Psi) = \|\,\Psi\,\|$. We proceed inductively and assume the result to be true for $n - 1$. We first consider the regularization of $x^n := |\,\boldsymbol{x}\,|^n$

$$\mathrm{B}_\lambda \equiv \mathrm{B}_\lambda(\boldsymbol{x}) := (1 + \lambda x^n)^{-1} x^n \tag{2.4}$$

which is bounded for each $\lambda > 0$. Since also the (distributional) derivative of B_λ is bounded we can apply (a slight modification of) Proposition 10 of [*16*, Appendix to IX.4] to conclude that multiplication by B_λ is also a bounded map of $\mathrm{W}^{m,2}(\mathbb{R}^3)^4$ into itself for all $|\,m\,| \leq 1$ and $\lambda > 0$. Therefore for any $\Phi \in \mathscr{D}(\mathrm{H}) \subset \mathscr{D}(|\,\mathrm{H}_0\,|^{1/2}) = \mathrm{W}^{\frac{1}{2},2}(\mathbb{R}^3)^4$ also $\mathrm{B}_\lambda\Phi \in \mathscr{D}(|\,\mathrm{H}_0\,|^{1/2})$. Moreover, B_λ leaves $\mathscr{D}(|\,\mathrm{V}\,|^{1/2})$ invariant and we obtain

$$\mathrm{B}_\lambda\Phi \in \mathscr{D} := \mathscr{D}(|\,\mathrm{H}_0\,|^{1/2}) \cap \mathscr{D}(|\,\mathrm{V}\,|^{1/2}) \tag{2.5}$$

for all $\Phi \in \mathscr{D}(\mathrm{H})$ and all $\lambda > 0$. The time derivative of $\mathrm{B}_\lambda(\boldsymbol{x}(t))$ as a quadratic form on $\mathscr{D}(\mathrm{H}) \times \mathscr{D}(\mathrm{H})$ is given by

$$i(\mathrm{H}e^{-i\mathrm{H}t}\Phi, \mathrm{B}_\lambda(\boldsymbol{x})e^{-i\mathrm{H}t}\Psi) - i(e^{-i\mathrm{H}t}\Phi, \mathrm{B}_\lambda(\boldsymbol{x})\mathrm{H}e^{-i\mathrm{H}t}\Psi)\,. \tag{2.6}$$

It follows from (1.8) that we can calculate this commutator to obtain

$$\left(\Phi, e^{+i\mathrm{H}t} c \frac{\boldsymbol{\alpha}\cdot\boldsymbol{x}}{x} \frac{n x^{n-1}}{(1 + \lambda x^n)^2} e^{-i\mathrm{H}t}\Psi\right). \tag{2.7}$$

This expression uniquely defines a bounded linear operator on $\mathscr{H}$. Integrating from 0 to t we infer from (2.6) and (2.7)

$$\|\,\mathrm{B}_\lambda(\boldsymbol{x})e^{-i\mathrm{H}t}\Psi\,\| \leq \|\,\mathrm{B}_\lambda(\boldsymbol{x})\Psi\,\| + \int_0^{|t|} ds\, nc \left\| \frac{x^{n-1}}{(1 + \lambda x^n)^2} e^{-i\mathrm{H}s}\Psi \right\|$$

for all $\Psi \in \mathscr{H}$. For $\Psi \in \mathscr{D}(x^n) \subset \mathscr{D}(x^{n-1})$ all expressions on the r. h. s. are bounded uniformly in $\lambda \to 0$. Thus $\exp(-iHt)\Psi \in \mathscr{D}(x^n)$ and

$$\begin{aligned} \| x^n e^{-iHt}\Psi \| &\leq \| x^n \Psi \| + \int_0^{|t|} ds\, nc \| x^{n-1} e^{-iHs}\Psi \| \\ &\leq \| x^n \Psi \| + \int_0^{|t|} ds\, nc\, k_{n-1}(\Psi)(1 + c\,|\,s\,|)^{n-1} \\ &\leq \| x^n \Psi \| + \tilde{k}_n(\Psi)(1 + c\,|\,t\,|)^n \\ &\leq k_n(\Psi)(1 + c\,|\,t\,|)^n . \qquad \square \end{aligned}$$

Remarks. — Note that the assumptions of this theorem do not contain any restrictions on the behaviour of the potential at infinity. For $n = 1$ we obtain by an explicit calculation

$$\| \,|\,\boldsymbol{x}\,|\, e^{-iHt}\Psi \| \leq \| \,|\,\boldsymbol{x}\,|\,\Psi \| + c\,|\,t\,|\,\|\Psi\|$$

which implies

$$\limsup_{|t| \to \infty} \| (|\,\boldsymbol{x}\,|/t) e^{-iHt}\Psi \| \leq c \| \Psi \| . \tag{2.8}$$

This reflects the fact that irrespective of the external field the particle cannot escape faster than with the velocity of light. Another consequence of the finite propagation speed is that the self-adjointness properties of H are insensitive to the large $|\,\boldsymbol{x}\,|$ behaviour of the potential [*1*]. This also shows up in the following.

COROLLARY 2.2. — Let H satisfy (1.7), (1.8).
Then for each $n \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{0\}$, $\operatorname{Im} z \neq 0$

a) $(H - z)^{-k}\mathscr{D}(|\,\boldsymbol{x}\,|^n)$ is a core for H and

$$(H - z)^{-k}\mathscr{D}(|\,\boldsymbol{x}\,|^n) \subset \mathscr{D}(|\,\boldsymbol{x}\,|^n), \tag{2.9}$$

b) $\mathscr{D}(|\,\boldsymbol{x}\,|^n) \cap \mathscr{D}(H)$ is a core for H.

Proof. — *a*) For $\Psi \in \mathscr{D}(|\,\boldsymbol{x}\,|^n)$ and $\operatorname{Im} z > 0$ we obtain using (2.3)

$$\begin{aligned} \| \,|\,\boldsymbol{x}\,|^n (H - z)^{-k}\Psi \| &= \text{const} \left\| \int_0^\infty dt\, e^{izt} t^{k-1} |\,\boldsymbol{x}\,|^n e^{-iHt}\Psi \right\| \\ &\leq \int_0^\infty dt\, e^{-(\operatorname{Im} z)t} t^{k-1} k_n(\Psi)(1 + ct)^n < \infty . \end{aligned}$$

This proves (2.9). $\mathscr{D} := (H - z)^{-k}\mathscr{D}(|\,\boldsymbol{x}\,|^n)$ is dense for all $k \geq 0$ because $\mathscr{D}(|\,\boldsymbol{x}\,|^n)$ is dense and $(H - z)^{-k}$ is a bounded operator with injective adjoint. Since $(H - z)\mathscr{D} = (H - z)^{-k+1}\mathscr{D}(|\,\boldsymbol{x}\,|^n)$ is dense, H is essentially self-adjoint on $\mathscr{D}$.

b) This follows from *a*) and

$$\mathscr{D} = (H - z)^{-1}\mathscr{D}(|\,\boldsymbol{x}\,|^n) \subset \mathscr{D}(|\,\boldsymbol{x}\,|^n) \cap \mathscr{D}(H) . \qquad \square$$

An immediate further consequence of Theorem 2.1 is

COROLLARY 2.3. — Let H be as above. Then for all $\Psi \in \mathscr{D}(|\boldsymbol{x}|)$

a)
$$\underset{|t|\to\infty}{w\text{-lim}} \frac{1}{t} |\boldsymbol{x}| e^{-iHt}\Psi = 0, \qquad (2.10)$$

b)
$$\| Ae^{-iHt}\Psi \| \leq k(\Psi)(1 + c|t|), \qquad (2.11)$$

where the constant $k(\Psi)$ depends only on Ψ.

Proof. — *a)* For $\Phi \in \mathscr{H}$ arbitrary and $\Phi' \in \mathscr{D}(|\boldsymbol{x}|)$

$$\left| \left(\Phi, \frac{1}{t} |\boldsymbol{x}| e^{-iHt}\Psi \right) \right| \leq \| \Phi - \Phi' \| \cdot \left\| \frac{1}{t} |\boldsymbol{x}| e^{-iHt}\Psi \right\| + \frac{1}{|t|} \| |\boldsymbol{x}| \Phi' \| \cdot \| \Psi \| .$$

The first summand can be made smaller than $\varepsilon/2$ by choosing Φ' appropriately. The second summand is smaller than $\varepsilon/2$ for $|t|$ large enough.

b)
$$\| Ae^{-iHt}\Psi \| = \left\| \left\{ c^2 H_0^{-1} \boldsymbol{p}\cdot\boldsymbol{x} + \frac{ic}{2} H_0^{-1} \boldsymbol{\alpha}\cdot\mathbf{F} \right\} e^{-iHt}\Psi \right\|$$
$$\leq \text{const}\left(\| |\boldsymbol{x}| e^{-iHt}\Psi \| + \| \Psi \| \right).$$

Now the result follows from (2.3). □

One of our main tools will be

LEMMA 2.4. — Let C be compact and H be self-adjoint. Then

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t ds\, e^{iHs} C e^{-iHs} P_{\text{cont}} \right\| = 0 \qquad (2.12)$$

where P_{cont} projects onto the continuous subspace of H. This lemma follows from Wiener's theorem (Theorem XI.114 in [*17*]). A direct proof is given in [*3*].

The time evolution of **F** is controlled by the commutator $i[\mathrm{H}, \mathbf{F}]$. We avoid domain questions by regularizing **F** with the resolvent $(\mathrm{H} - z)^{-1}$.

$$\frac{d}{dt}(\mathrm{H} - z)^{-1} \mathrm{F}_k(t) (\mathrm{H} - z)^{-1} = i[\mathrm{F}_k(t), (\mathrm{H} - z)^{-1}]. \qquad (2.13)$$

LEMMA 2.5. — Let H satisfy (1.7)-(1.11).
Then for all $z \notin \sigma(\mathrm{H})$, $k = 1, 2, 3$,

$$i[\mathrm{F}_k, (\mathrm{H} - z)^{-1}] = (\mathrm{H} - z)^{-1} 2i\mathrm{H}\, \mathrm{F}_k (\mathrm{H} - z)^{-1} + \mathrm{K}_k \qquad (2.14)$$

where K_k is compact. Similarly for G.

Proof. — We split the commutator into two parts.

$$i[\mathrm{F}_k, (\mathrm{H} - z)^{-1}] = i[\mathrm{F}_k, (\mathrm{H}_0 - z)^{-1}] + i[\mathrm{F}_k, (\mathrm{H} - z)^{-1} - (\mathrm{H}_0 - z)^{-1}].$$

The last term is compact since F_k is bounded and the difference of the resolvents is compact. Using (A.2) we obtain

$$\begin{aligned} i[F_k, (H_0 - z)^{-1}] &= (H_0 - z)^{-1} 2iH_0 F_k (H_0 - z)^{-1} \\ &= (H - z)^{-1} 2iH F_k (H - z)^{-1} \\ &- (H - z)^{-1} 2iH F_k [(H - z)^{-1} - (H_0 - z)^{-1}] \\ &- \{ (H - z)^{-1} H - (H_0 - z)^{-1} H_0 \} 2iF_k (H_0 - z)^{-1}. \end{aligned}$$

The term in braces in the last summand equals the compact

$$[(H - z)^{-1} - (H_0 - z)^{-1}]z.$$

Therefore the second and last summands are compact. □

A strengthening of Theorem 1.1 is the following

PROPOSITION 2.6. — Let H satisfy (1.7)-(1.11) and **F**, **G** be as defined in (1.13). *a)* For $k = 1, 2, 3$ there are compact operators C_k such that

$$\lim_{|t| \to \infty} \left\| \frac{1}{t} \int_0^t ds \{ F_k(s)(H - z)^{-1} - C_k(s)[\mathbb{1} - P_{cont}] \} \right\| = 0. \quad (2.15)$$

b) If we write $\mathbb{1} - P_{cont} = \sum_j P_j$ with pairwise orthogonal one-dimensional bound state projections P_j then there is for any $\varepsilon > 0$ an $N(\varepsilon)$ such that

$$\limsup_{|t| \to \infty} \left\| \frac{1}{t} \int_0^t ds F_k(s)(H - z)^{-1} \left[\mathbb{1} - \sum_{j=1}^{N(\varepsilon)} P_j \right] \right\| < \varepsilon. \quad (2.16)$$

Similarly for **G**.

Proof. — We integrate (2.14) from 0 to t and divide by t:

$$\begin{aligned} &\frac{1}{t}(H - z)^{-1} [F_k(t) - F_k](H - z)^{-1} \\ &= (H - z)^{-1} 2iH \frac{1}{t} \int_0^t ds F_k(s)(H - z)^{-1} + \frac{1}{t} \int_0^t ds K_k(s). \quad (2.17) \end{aligned}$$

It tends to zero in norm for $|t| \to \infty$ since all operators on the left hand side are bounded. By Lemma 2.4 we can replace K_k by $K_k[\mathbb{1} - P_{cont}]$.

By (1.12) zero can only be a discrete eigenvalue of H. Therefore $F(H \neq 0) 2iH(H - z)^{-1}$ is boundedly invertible (and time invariant), and $F(H = 0)$ is compact. With

$$K'_k := F(H = 0) F_k (H - z)^{-1} F(H = 0) = \frac{1}{t} \int_0^t ds K'_k(s)$$

we have

$$\left\| F(H=0)\frac{1}{t}\int_0^t ds F_k(s)(H-z)^{-1} - K_k' \right\|$$
$$= \left\| F(H=0)\, F_k\,(H-z)^{-1}\,\frac{1}{t}\int_0^t ds\, e^{-iHs} F(H \neq 0) \right\|.$$

This vanishes as $|t| \to \infty$ because

$$\left\| \frac{1}{t}\int_0^t ds\, e^{-iHs} F(H \neq 0) \right\| \leq \frac{2}{|t|} \| H^{-1} F(H \neq 0) \|.$$

We have verified (2.15) if we set

$$C_k := K_k' + \frac{i}{2}(H - z)H^{-1}F(H \neq 0)K_k.$$

(2.16) follows from

$$\lim_{N\to\infty} \left\| C_k \sum_{j>N} P_j \right\| = 0. \quad \square$$

Proof of Theorem 1.1. — The time-average of **F** is uniformly bounded and $\mathscr{D}(H) = \operatorname{Ran}(H-z)^{-1}$ is dense. Then (1.14) is implied by (2.15). Similarly for G. □

The following estimate will be needed to control the time derivative of $x^2(t)$ in the proof of Theorem 1.5.

LEMMA 2.7. — Let H be as above. Then for $\Psi \in (H-z)^{-1}\mathscr{D}(|\boldsymbol{x}|)$

$$\lim_{|t|\to\infty} \frac{1}{t^2}\int_0^t ds\, e^{iHs}(\mathbf{F}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\mathbf{F})e^{-iHs}\Psi = 0. \tag{2.18}$$

Proof. — Since $\mathbf{F}\cdot\boldsymbol{x} - \boldsymbol{x}\cdot\mathbf{F}$ is bounded (A.6) it is sufficient to estimate e. g. $F_k(s)x_k(s)$. If we insert bound state projections P_j, then for any $N < \infty$

$$\left\| \frac{1}{t^2}\int_0^t ds F_k(s) \sum_{j=1}^{N} P_j x_k(s)\Psi \right\| \leq \| F_k \| \frac{1}{t^2}\int_0^t s ds \left\| \sum_{j=1}^{N} P_j \frac{x_k(s)}{s}\Psi \right\|.$$

Compactness of $\sum_{j=1}^{N} P_j$ and Corollary 2.3 *a)* imply that the integrand vanishes as $|s| \to \infty$ and also the weighted time average as $|t| \to \infty$. Moreover

$$(H-z)^{-1}x_k(s)(H-z)\Psi - x_k(s)\Psi = (H-z)^{-1}ic\alpha_k(s)\Psi$$

is bounded uniformly in s and the error from this substitution vanishes as $|t| \to \infty$. For the following time translation invariant operator we introduce the shorthand

$$\{\dots\} := \left\{ (\mathrm{H} - z)^{-1} \left[\mathbb{1} - \sum_{j=1}^{\mathrm{N}} \mathrm{P}_j \right] \right\}.$$

The lemma is proved if we show that the following expression is small uniformly for N and $|t|$ large:

$$\begin{aligned} & \left\| \frac{1}{t^2} \int_0^t ds \mathrm{F}_k(s) \{ \dots \} x_k(s)(\mathrm{H} - z)\Psi \right\| \\ = & \left\| \frac{1}{t^2} \int_0^t d\tau \mathrm{F}_k(\tau) \{ \dots \} x_k(t)(\mathrm{H} - z)\Psi \right. \\ & \left. - \frac{1}{t^2} \int_0^t ds \int_0^s d\tau \mathrm{F}_k(\tau) \{ \dots \} c\alpha_k(s)(\mathrm{H} - z)\Psi \right\| \\ \leq & \left\| \frac{1}{t} \int_0^t d\tau \mathrm{F}_k(\tau) \{ \dots \} \right\| \cdot \left\| \frac{x_k(t)}{t} (\mathrm{H} - z)\Psi \right\| \\ & + \frac{1}{t^2} \int_0^t s\, ds \left\| \frac{1}{s} \int_0^s d\tau \mathrm{F}_k \{ \dots \} \right\| \cdot \| c\alpha_k \| \cdot \| (\mathrm{H} - z)\Psi \| . \end{aligned}$$

In the first step we have performed a partial integration. $|t|^{-1} \| x_k(t)(\mathrm{H}-z)\Psi \|$ is uniformly bounded for $|t| \geq 1$. Proposition 2.6 *b)* ensures that the other factor is small for N and $|t|$ ($|s|$, resp.) large. □

3. PROOFS OF THE MAIN THEOREMS

To study the long time behaviour of $\mathrm{A}(t)/t$ we use the freedom to replace A by some A_l as long as $[\mathrm{A}(t) - \mathrm{A}_l(t)]/t \to 0$ in a suitable sense. Our goal is to find such an A_l that its time derivative differs from the limit $c^2(\mathbb{1} - m^2c^4\mathrm{H}^{-2})$ by a relatively compact term. Then Lemma 2.4 implies convergence on $\mathscr{H}_{\mathrm{cont}}$. Unlike the simpler Schrödinger case this does not hold for A if long-range potentials are present which do not commute with $\boldsymbol{\alpha}$ or β.

For a bound state $\mathrm{H}\Psi = \mathrm{E}\Psi$ and any self-adjoint operator A, f bounded and continuous

$$\lim_{|t|\to\infty} f(\mathrm{A}(t)/t)\Psi = \lim_{|t|\to\infty} e^{i(\mathrm{H}-\mathrm{E})t} f(\mathrm{A}/t)\Psi = f(0)\Psi .$$

This verifies convergence in the strong resolvent sense

$$\lim_{|t|\to\infty} \mathrm{A}(t)/t = 0 \quad \text{on} \quad (\mathbb{1} - \mathrm{P}_{\mathrm{cont}})\mathscr{H} \tag{3.1}$$

in (1.18), also for any A_l. Similarly for (1.24)

$$\lim_{|t|\to\infty} x^2(t)/t^2 = 0 \quad \text{on} \quad (\mathbb{1} - P_{\text{cont}})\mathcal{H} . \tag{3.2}$$

For A_l we replace H_0 in the definition (1.17) of A by $H_l = H_0 + V_l$. The splitting (1.9) may be chosen such that $0 \notin \sigma(H_l)$ e. g. by adding and subtracting a smooth function of compact support. Then we define the bounded

$$\boldsymbol{v} := \frac{c^2}{2}(H_l^{-1}\boldsymbol{p} + \boldsymbol{p}H_l^{-1}), \tag{3.3}$$

and on $\mathcal{D}(|\boldsymbol{x}|)$ the symmetric

$$\begin{aligned} A_l &:= \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{v}) \\ &\equiv \boldsymbol{v}\cdot\boldsymbol{x} + i\frac{c^2}{2}\left[3H_l^{-1} - \frac{1}{2}\{H_l^{-1}c\boldsymbol{\alpha}H_l^{-1}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot H_l^{-1}c\boldsymbol{\alpha}H_l^{-1}\}\right]. \end{aligned} \tag{3.4}$$

Since $V_lH_l^{-1}$ is compact also the difference

$$c^2\boldsymbol{p}H_0^{-1} - \boldsymbol{v} = \frac{c^2}{2}(H_l^{-1}V_lH_0^{-1}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot H_0^{-1}V_lH_l^{-1}) \tag{3.5}$$

is compact.

To calculate the time derivative $i[H, A_l]$ we regularize again as in the proof of Theorem 1.1 to avoid domain questions. If zero happens to be a spectral value of H it must lie in the discrete spectrum. It is convenient to set

$$H^{-1}F(H = 0) := 0 .$$

With such a convention the compactness of $(H - z)^{-2} - (H_l - z)^{-2}$ for $z \notin \sigma(H) \cup \sigma(H_l)$ extends to the discrete spectra and in particular

$$H^{-2} - H_l^{-2} \quad \text{is compact}. \tag{3.6}$$

LEMMA 3.1. — Let H satisfy (1.7)-(1.11) and be $z \notin \sigma(H)$. Then

$$i[A_l, (H - z)^{-1}] = (H - z)^{-2}c^2(\mathbb{1} - m^2c^4H^{-2}) + K \tag{3.7}$$

where K is compact.

Proof. — A direct calculation of the quadratic form between vectors in the dense set $\mathcal{D}(H_0) \cap \mathcal{D}(|\boldsymbol{x}|)$ gives the bounded operator

$$i[H_l, A_l] = \frac{c}{2}(\boldsymbol{\alpha}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\alpha}) + K_1 + K_1^* , \tag{3.8}$$

$$K_1 = -\frac{c^2}{4}\{2H_l^{-1}\boldsymbol{x}\cdot(\nabla V_l) + (\nabla V_l)\cdot H_l^{-1}ic\boldsymbol{\alpha}H_l^{-1}\}.$$

K_1 is compact. By (3.5) the first summand in (3.8) differs only by a compact operator from (see (A.5))

$$\frac{c}{2}(\boldsymbol{x}\cdot c^2\boldsymbol{p}H_0^{-1} + c^2\boldsymbol{p}H_0^{-1}\cdot\boldsymbol{x}) = c^2(\mathbb{1} - m^2c^4H_0^{-2}).$$

This implies

$$\begin{aligned} i[A_l, (H_l - z)^{-1}] &= (H_l - z)^{-1} i[H_l, A_l](H_l - z)^{-1} \\ &= (H_l - z)^{-1}c^2(\mathbb{1} - m^2c^4H_0^{-2})(H_l - z)^{-1} + K_2 \\ &= (H_l - z)^{-2}c^2(\mathbb{1} - m^2c^4H_l^{-2}) + K_3 \end{aligned}$$

(K_2 and K_3 being compact). From (1.11) we conclude compactness of $A_l[(H - z)^{-1} - (H_l - z)^{-1}]$.

Thus

$$\begin{aligned} i[A_l, (H-z)^{-1}] = {} & i[A_l, (H_l-z)^{-1}] \\ & + iA_l[(H-z)^{-1} - (H_l-z)^{-1}] - i[(H-z)^{-1} - (H_l-z)^{-1}]A_l \\ = {} & (H_l-z)^{-2}c^2(\mathbb{1} - m^2c^4H_l^{-2}) + K_4 \\ = {} & (H-z)^{-2}c^2(\mathbb{1} - m^2c^4H^{-2}) + K \end{aligned} \tag{3.9}$$

with compact operators K_4 and K. □

Proof of Theorem 1.2. — Comparison of (1.17) with (3.4) shows

$$A - A_l = (c^2\boldsymbol{p}H_0^{-1} - \boldsymbol{v})\cdot\boldsymbol{x} + B$$

with a bounded B. For any $\Psi \in \mathscr{D}(|\boldsymbol{x}|)$

$$t^{-1}\|(A - A_l)e^{-iHt}\Psi\| \to 0 \quad \text{as} \quad |t| \to \infty \tag{3.10}$$

by compactness (3.5) and Corollary 2.3 *a*).

Since the limit operator is bounded any dense set $\mathscr{D}$ is a core. Thus strong resolvent convergence (1.18) follows if we show for $\Phi \in \mathscr{D}$ that

$$\{ A_l(t)/t - c^2(\mathbb{1} - m^2c^4H^{-2})P_{\text{cont}} \}\Phi$$

is small for large t. Using (3.1) and $\mathscr{D} = (H - z)^{-2}\mathscr{D}(|\boldsymbol{x}|)$ it remains to show that

$$\{ A_l(t)/t - c^2(\mathbb{1} - m^2c^4H^{-2}) \}\Phi \tag{3.11}$$

is small if $|t|$ is large and $\|(\mathbb{1} - P_{\text{cont}})(H - z)^2\Phi\|$ is small. For $\Psi = (H - z)^2\Phi \in \mathscr{D}(|\boldsymbol{x}|)$ we integrate

$$\frac{d}{dt}(H - z)^{-1}A_l(t)(H - z)^{-1}\Psi = i[A_l(t), (H - z)^{-1}]\Psi$$

to obtain with Lemma 3.1

$$\begin{aligned} & (H - z)^{-1}\frac{1}{t}\{ A_l(t) - A_l \}(H - z)^{-1}\Psi \\ & \qquad = c^2(\mathbb{1} - m^2c^4H^{-2})(H - z)^{-2}\Psi + \frac{1}{t}\int_0^t ds K(s)\Psi. \end{aligned} \tag{3.12}$$

The last term in (3.12) is bounded by

$$\left\| \frac{1}{t} \int_0^t ds \mathrm{K}(s) \mathrm{P}_{\mathrm{cont}} \right\| + \| \mathrm{K} \| \, \| (\mathbb{1} - \mathrm{P}_{\mathrm{cont}}) \Psi \|$$

which is small for large $|t|$ by Lemma 2.4 and by choice of Ψ. Finally

$$\left\| \left[(\mathrm{H} - z)^{-1}, \frac{1}{t} \{ \mathrm{A}_l(t) - \mathrm{A}_l \} \right] \right\| \leq \frac{2}{|t|} \| [(\mathrm{H} - z)^{-1}, \mathrm{A}_l] \|$$

vanishes as $|t| \to \infty$. We have shown the desired estimate of (3.11). □

Proof of Theorem 1.5. — For $\Phi, \Psi \in (\mathrm{H} - z)^{-1} \mathscr{D}(x^2)$

$$\begin{aligned} \frac{d}{dt} (\Phi, x^2(t)\Psi) &= (\Phi, i[\mathrm{H}, x^2(t)]\Psi) \\ &= (\Phi, 2\mathrm{A}(t)\Psi) + (\Phi, e^{i\mathrm{H}t} \{ \mathbf{F} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \mathbf{F} \} e^{-i\mathrm{H}t}\Psi) . \end{aligned} \quad (3.13)$$

The integrated formula extends by continuity to arbitrary $\Phi \in \mathscr{H}$ and thus

$$\frac{x^2(t)}{t^2} \Psi = \frac{x^2}{t^2} \Psi + \frac{2}{t^2} \int_0^t ds \mathrm{A}(s) \Psi + \frac{1}{t^2} \int_0^t ds \, e^{i\mathrm{H}s} \{ \mathbf{F} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \mathbf{F} \} e^{-i\mathrm{H}s} \Psi \quad (3.14)$$

for $\Psi \in (\mathrm{H} - z)^{-1} \mathscr{D}(x^2)$. The last term in (3.14) vanishes as $|t| \to \infty$ by Lemma 2.7 and clearly the first decays as well. For the second summand we apply Theorem 1.2 and obtain

$$\begin{aligned} &\left\| \frac{2}{t^2} \int_0^t ds \mathrm{A}(s) \Psi - c^2 (\mathbb{1} - m^2 c^4 \mathrm{H}^{-2}) \mathrm{P}_{\mathrm{cont}} \Psi \right\| \\ &\leq \frac{2}{t^2} \int_0^t s \, ds \left\| \left\{ \frac{\mathrm{A}(s)}{s} - c^2 (\mathbb{1} - m^2 c^4 \mathrm{H}^{-2}) \mathrm{P}_{\mathrm{cont}} \right\} \Psi \right\| \to 0 \quad \text{as} \quad |t| \to \infty . \end{aligned}$$

This shows convergence on a dense set, i. e. on a core for the bounded limit operator. Strong resolvent convergence (1.24) follows. □

Proof of Theorem 1.7. — To simplify notation we write

$$\lambda(\boldsymbol{p}) := (c^2 p^2 + m^2 c^4)^{1/2} = \mathrm{H}_0 \mathrm{P}_0^+ - \mathrm{H}_0 \mathrm{P}_0^- = |\mathrm{H}_0| \quad (3.15)$$

and

$$\boldsymbol{u}(\boldsymbol{p}) := (\nabla_p \lambda)(\boldsymbol{p}) = \pm c^2 \boldsymbol{p} \mathrm{H}_0^{-1} \mathrm{P}_0^{\pm} = c^2 \boldsymbol{p} / \lambda(\boldsymbol{p}) . \quad (3.16)$$

It is clear from Corollary 1.4 *b*) and eq. (3.16) that

$$\{ f(\boldsymbol{x}/t) - f(c^2 \boldsymbol{p} \mathrm{H}_0^{-1}) \} e^{-i\mathrm{H}t} \mathrm{P}_{\mathrm{cont}}^{\pm} \Psi$$

is asymptotically equal to

$$\{ f(\boldsymbol{x}/t) - f(\pm \boldsymbol{u}(\boldsymbol{p})) \} e^{-i\mathrm{H}t} \mathrm{P}_{\mathrm{cont}}^{\pm} \Psi .$$

Thus *b'*) implies *a'*). In the following we assume

$$\Psi = \mathrm{P}_{\mathrm{cont}}^{+} \Psi$$

(the same arguments apply to negative energy states).

Then it remains to prove convergence of

$$\| \{ f(x/t) - f(u(p)) \} e^{-iHt}\Psi \| .$$

The statements for f trivially imply those for $(\mathbb{1} - f)$. If we replace Ψ by $\Phi \in (H - z)^{-2}\mathscr{D}(x^2)$ with $\| \Phi - \Psi \| < \varepsilon$ then the error is bounded by $2 \| f \| \varepsilon$. We will use later that by Corollary 1.4 *b)*

$$\| (\mathbb{1} - P_{\text{cont}})\Phi \| < \varepsilon \quad \text{and} \quad \| P_0^- e^{-iHt}\Phi \| \to 0 . \tag{3.17}$$

Let $\hat{f} \in L^1$ then

$$\| \{ f(x/t) - f(u(p)) \} e^{-iHt}\Phi \| \\ \leq (2\pi)^{-3/2} \int d^3q \, | \hat{f}(q) | \, \| \{ e^{iq\cdot x/t} - e^{iq\cdot u(p)} \} e^{-iHt}\Phi \| . \tag{3.18}$$

By the Lebesgue dominated convergence theorem the integral vanishes if the norm vanishes as $| t | \to \infty$ for any fixed q.

For an application in Section 4 we note that it would be sufficient to have $| \widehat{\nabla f} | = | q\hat{f} | \in L^1_{\text{loc}}$ because the norm has at least a first order zero at $q = 0$. The following estimates cover this case as well.

We show first that uniformly for q in bounded sets

$$\frac{1}{| q |} \| e^{-iq\cdot u(p)} e^{iq\cdot x/t} - e^{iq\cdot[x/t - u(p)]} \| \to 0 \tag{3.19}$$

as $| t | \to \infty$. Then the formula corresponding to (3.18) for (1.30) is covered as well and *b)* implies *a)*. Note that x and p are operator valued vectors. By the canonical commutation relations

$$e^{-iq\cdot u(p)} e^{iq\cdot x/t} = e^{iq\cdot x/t} e^{-iq\cdot u(p+q/t)} \tag{3.20}$$

and

$$e^{iq\cdot[x/t - u(p)]} = e^{-i\lambda(p)t} e^{iq\cdot x/t} e^{i\lambda(p)t} = e^{iq\cdot x/t} e^{it[\lambda(p) - \lambda(p+q/t)]} . \tag{3.21}$$

Since the factors on the right commute we estimate the difference of the exponents.

$$| - t[\lambda(p) - \lambda(p + q/t)] - q\cdot u(p + q/t) | \\ = \left| \int_0^1 ds \, q\cdot \{ u(p + sq/t) - u(p + q/t) \} \right| \leq \frac{q^2}{| t |} \| \nabla u \| . \tag{3.22}$$

It vanishes uniformly in p as $| t | \to \infty$ for bounded q. Also the exponentials are asymptotically equal and (3.19) follows. It remains to estimate for bounded q

$$\frac{1}{q^2} \| \{ \exp iq\cdot[x/t - u(p)] - \mathbb{1} \} e^{-iHt}\Phi \|^2 \\ = \frac{1}{q^2} \left\| \int_0^1 ds \exp \{ isq\cdot[x/t - u(p)] \} \, q\cdot[x/t - u(p)] e^{-iHt}\Phi \right\|^2 \\ \leq \| [x/t - u(p)] e^{-iHt}\Phi \|^2 .$$

Writing

$$u(p) = c^2 p \mathrm{H}_0^{-1} - 2c^2 p \mathrm{H}_0^{-1} \mathrm{P}_0^-$$

we can estimate this by

$$\left\| \left\{ \frac{x^2(t)}{t^2} - 2\frac{\mathrm{A}(t)}{t} + c^2(\mathbb{1} - m^2c^4\mathrm{H}^{-2})\mathrm{P}_{\mathrm{cont}} \right\} \Phi \right\|$$
$$+ m^2c^6 \left\| (\mathrm{H}^{-2} - \mathrm{H}_0^{-2})\mathrm{P}_{\mathrm{cont}} e^{-i\mathrm{H}t}\Phi \right\|$$
$$+ \left\| c^2(\mathbb{1} - m^2c^4\mathrm{H}_0^{-2}) \right\| \cdot \left\| (\mathbb{1} - \mathrm{P}_{\mathrm{cont}})\Phi \right\| + 2 \left\| c^2 p \mathrm{H}_0^{-1}\mathrm{P}_0^- e^{-i\mathrm{H}t}\Phi \right\|^2 .$$

The first term vanishes as $|t| \to \infty$ by Theorems 1.2 and 1.5, the second by compactness and Corollary 1.4 *a*). The rest is small uniformly in t by (3.17). □

4. APPLICATION TO ASYMPTOTIC COMPLETENESS

We want to use the information obtained from asymptotic observables for a particularly simple proof of asymptotic completeness in the case of relativistic Coulomb scattering. Another proof using geometric methods has been given recently by Muthuramalingam [*10*] for a more general equation. Let the short-range part V_s of the potential be H_0-bounded with decay

$$\| \mathrm{V}_s f(\mathrm{H}_0)\mathrm{F}(|x| > \mathrm{R}) \| \in \mathrm{L}^1(\mathbb{R}_+, d\mathrm{R}) \tag{4.1}$$

for each $f \in \mathrm{C}_0^\infty(\mathbb{R})$. This includes Coulomb type singularities at least for $|\gamma| < c$ since $\gamma/|x|$ has relative bound $2|\gamma|/c$ with respect to H_0. The long range part V_l should be of purely electrostatic type, i. e.

$$\mathrm{V}_l(x) = \Phi_l(x)\mathbb{1} , \tag{4.2}$$

with 4×4 unit matrix $\mathbb{1}$, and the differentiable Φ_l satisfies

$$\Phi_l(x) = \gamma/|x|, \quad \gamma \in \mathbb{R}, \quad |x| \geq \mathrm{R}_0 > 0 . \tag{4.3}$$

For other long range potentials cf. the remark at the end of this section. In particular these assumptions imply (1.7)-(1.11). We will approximate V_l by bounded functions V_R with

$$\mathrm{V}_\mathrm{R}(x) = \varphi_\mathrm{R}(x)\cdot\gamma/|x| \quad \text{for} \quad \mathrm{R} > 0 , \tag{4.4}$$

where φ_R is a smooth cut-off function

$$\varphi_\mathrm{R}(x) = \varphi(|x|/\mathrm{R}) \quad \text{with} \quad \varphi \in \mathrm{C}^\infty(\mathbb{R}_+),$$

$$\varphi(r) = \begin{cases} 0 & \text{for} \quad r \leq \dfrac{1}{2} \\ 1 & \text{for} \quad r \geq 1 . \end{cases}$$

Then for $R \geq R_0$, u, $s > 0$

$$\{ V_l(x) - V_R(x) \} F(|x| \geq R) = 0, \tag{4.5}$$

$$V_{u \cdot s}(x) = \frac{1}{s} V_u(x/s), \tag{4.6}$$

$$|q| \hat{V}_u(q) \quad \text{and} \quad q^2 \hat{V}_u(q) \in L^1(\mathbb{R}^3). \tag{4.7}$$

The last property follows from $\nabla V_u \in L^2$, i. e. $|q| \hat{V}_u(q) \in L^1_{loc}$ for any $u > 0$, and $[\widehat{(-\Delta)^n V_u}](q) = q^{2n} \hat{V}_u(q)$ is bounded for $n \geq 1$.

As suggested by the work of Dollard and Velo [2] we define the modified free time evolution by

$$U(t + \tau, \tau) := U^+(t + \tau, \tau) P_0^+ + U^-(t + \tau, \tau) P_0^-,$$

$$U^\pm(t + \tau, \tau) := e^{\mp i\lambda(p)t} \exp\left\{ - i \int_\tau^{t+\tau} ds\, V_l(\pm u(p)s) \right\}, \tag{4.8}$$

with $P_0^\pm$ as defined in Section 1 and λ, u as in (3.15), (3.16). The following theorem states strong asymptotic completeness of the modified wave operators for arbitrary T

$$\Omega_\mp(T) := \underset{t \to \pm\infty}{s\text{-lim}}\, e^{iHt} U(t + T, T). \tag{4.9}$$

THEOREM 4.1. — Let $H = H_0 + V_s + V_l$ satisfy (4.1)-(4.3). Then for any $\Psi \in \mathcal{H}_{cont}$

$$\lim_{\tau \to \infty} \sup_{t \geq 0} \| \{ e^{-iHt} - U(t + \tau, \tau) \} e^{-iH\tau} \Psi \| = 0. \tag{4.10}$$

Proof. — We assume $\Psi \in \mathcal{H}^+_{cont} := P^+_{cont} \mathcal{H}_{cont}$, the proof for $\Psi \in \mathcal{H}^-_{cont}$ is essentially the same. Then, by Corollary 1.4 *b*, we only have to prove (4.10) with $U(t + \tau, \tau)$ replaced by $U^+(t + \tau, \tau)$. It is sufficient to do this for Ψ in a dense set $\mathcal{D} \subset \mathcal{H}^+_{cont}$. Our strategy is similar to Section 5 of [4]. We choose $\mathcal{D}$ to be the set of states with compact energy support away from thresholds, i. e. for $\Psi \in \mathcal{D}$ there are constants a, b, with $mc^2 < a < b < \infty$, such that

$$\Psi = F(a < H < b)\Psi \in \mathcal{H}^+_{cont}. \tag{4.11}$$

Define $u_a := c(1 - m^2c^4/a^2)^{1/2}$, u_b similar. We conclude from Corollary 1.4 *c* that

$$\lim_{|\tau| \to \infty} \| \{ \mathbb{1} - F(u_a^2 < u^2(p) < u_b^2) \} e^{-iH\tau} F(a < H < b)\Psi \| = 0. \tag{4.12}$$

For a fixed $\Psi \in \mathcal{D}$ choose $g \in C_0^\infty(\mathbb{R}_+)$ with support contained in the interval $(4u^2, c^2)$, $u = u_a/4 > 0$, such that $g(\cdot) = 1$ on $[u_a^2, u_b^2]$, $|g| \leq 1$. Then we obtain for any $\delta > 0$

$$\| \{ \mathbb{1} - P_0^+ g(u^2(p)) F(|x| < \delta\tau) \} e^{i\lambda(p)\tau} e^{-iH\tau} \Psi \|$$
$$\leq \| F(|x| > \delta\tau) e^{i\lambda(p)\tau} e^{-iH\tau} \Psi \| + \| \{ \mathbb{1} - P_0^+ g(u^2(p)) \} e^{-iH\tau} \Psi \|. \tag{4.13}$$

By Corollaries 1.8 and 1.4 *b* the first summand tends to zero, as $|\tau| \to \infty$. The second summand vanishes in the limit $|\tau| \to \infty$ by Corollary 1.4 *b* and (4.12).

Thus we can approximate the scattering state $\exp(-iH\tau)\Psi$ in (4.10) by

$$\Phi(\tau) := e^{-i\lambda(p)\tau} P_0^+ g(u^2(p)) \, F(|x| < \delta\tau) e^{i\lambda(p)\tau} e^{-iH\tau}\Psi. \tag{4.14}$$

It is sufficient to consider

$$\sup_{t>0} \| \{ e^{-iHt} - U^+(t+\tau, \tau) \} \Phi(\tau) \| \leq \int_0^\infty dt \, \| \{ V_s(x) + V_l(x) - V_l(u(p)(t+\tau)) \} U^+(t+\tau,\tau)\Phi(\tau) \|, \tag{4.15}$$

where we have applied the Cook estimate. Regularizing V_s by inserting some $f \in C_0^\infty(\mathbb{R})$ with $f(H_0)g(u^2) = P_0^+ g(u^2)$, the short-range part of (4.15) can be estimated by

$$\int_0^\infty dt \, \| V_s f(H_0) \| \cdot \| F(|x| < R) U^+(t+\tau,\tau) e^{-i\lambda(p)\tau} g(u^2) F(|x| < \delta\tau) \| + \int_0^\infty dt \, \| V_s f(H_0) F(|x| > R) \|. \tag{4.16}$$

If we choose $R = (t+\tau)u$ and $\delta = u/2$, then both summands in (4.16) vanish as $\tau \to \infty$, the first one because of Lemma 4.2 below, and the second one by (4.1).

In the long-range part of (4.15) we approximate V_l by V_R as defined in eq. (4.4), with $R = u(t+\tau)$, $u\tau > R_0$. Note that on the range of $g(u^2)$ we have $V_l(u(p)(t+\tau)) = V_R(u(p)(t+\tau))$.

$$\begin{aligned} &\int_0^\infty dt \, \| \{ V_l(x) - V_l(u(p)(t+\tau)) \} U^+(t+\tau,\tau)\Phi(\tau) \| \\ &\leq \int_0^\infty dt \, \| \{ V_R(x) - V_R(u(p)(t+\tau)) \} U^+(t+\tau,\tau)\Phi(\tau) \| \\ &+ \int_0^\infty dt \, \| \{ V_l(x) - V_R(x) \} U^+(t+\tau,\tau)\Phi(\tau) \|. \end{aligned} \tag{4.17}$$

The last summand vanishes asymptotically by (4.5) and Lemma 4.2 below. The first summand on the right hand side of (4.17) equals by (4.6)

$$\int_0^\infty dt \frac{1}{t+\tau} \left\| \left\{ V_u\left(\frac{x}{t+\tau}\right) - V_u(u(p)) \right\} U^+(t+\tau,\tau)\Phi(\tau) \right\|. \tag{4.18}$$

The norm is bounded by

$$\left\| \left\{ V_u\left(\frac{x}{t+\tau} + u(p)\right) - V_u(u(p)) \right\} e^{i\lambda(p)(t+\tau)} U^+(t+\tau, \tau)\Phi(\tau) \right\|$$

$$\leq (2\pi)^{-3/2} \int d^3q \, |\hat{V}_u(q)| \, |q| \times \frac{1}{|q|} \| \{ e^{iq\cdot[u(p)+x/(t+\tau)]} - e^{iq\cdot u(p)} \} e^{i\lambda(p)(t+\tau)} U^+ \Phi(\tau) \| .$$

Analogous calculations as (3.20)-(3.22) show

$$\| (e^{-iq\cdot u(p)} e^{iq\cdot[u(p)+x/(t+\tau)]} - \mathbb{1}) - (e^{iq\cdot x/(t+\tau)} - \mathbb{1}) \| \leq q^2 \| \nabla u \| /(t+\tau) .$$

Thus

$$\frac{1}{|q|} \| \dots \| \leq |q| \, \| \nabla u \| /(t+\tau) + \left\| \frac{x}{t+\tau} e^{i\lambda(p)(t+\tau)} U^+(t+\tau, \tau)\Phi(\tau) \right\| .$$

With (4.7) the first summand gives a contribution to (4.18) which vanishes as $\tau \to \infty$. The norm of the second is bounded by

$$\frac{1}{t+\tau} \left\| x \exp\left\{ -i \int_\tau^{\tau+t} ds \, V_l(u(p)s) \right\} P_0^+ g(u^2(p)) \, F(|x| < \delta\tau) \right\|$$

$$\leq \frac{\delta\tau}{t+\tau} + \frac{1}{t+\tau} \| [x, P_0^+ g(u^2(p))] \|$$

$$+ \frac{1}{t+\tau} \int_\tau^{\tau+t} s \, ds \, \| (\nabla V_l)(u(p)s) \, g(u^2(p)) \| \cdot \| \nabla u \| .$$

Only vectors $|u(p)| > 2u$ are in the support of $g(u^2(p))$. Therefore the gradient-term decays like s^{-2} for $u\tau \geq R_0$. The contribution from these terms to (4.18) is bounded by

$$\text{const} \int_0^\infty dt \frac{1}{(t+\tau)^2} \left\{ \delta\tau + 1 + \ln \frac{t+\tau}{\tau} \right\} \leq 2\delta$$

for large τ. Since δ may be chosen arbitrarily small we have verified (4.10). □

LEMMA 4.2. — Let $U^\pm$, g, u be as above. Then

$$\| F(|x| < (t+\tau)u) U^\pm(t+\tau, \tau) e^{-i\lambda(p)\tau} g(u^2(p)) F(|x| < \tau u/2) \|$$
$$\leq C_n (1+t+\tau)^{-n} \quad \text{for all } n \text{ and } 0 \leq \tau, t . \quad (4.19)$$

We indicate only the proof because it is a standard propagation property of (modified) free time evolutions. If $K(t, \tau; x - y)$ denotes the kernel of $U^\pm(t+\tau, \tau) \exp\{-i\lambda(p)\tau\} g(u^2(p))$ then it is sufficient to show its rapid decay in $t+\tau$ uniformly on $|x-y| \leq 3u(t+\tau)/2$.

$$K(t, \tau; z) = \int d^3p \, e^{i(t+\tau)\varphi(p)} g(u^2(p)),$$

$$\varphi(p) := z \cdot p/(t+\tau) - \lambda(p) - \frac{1}{t+\tau} \int_\tau^{t+\tau} ds V_l(u(p)s) .$$

The gradient of the phase functions

$$z/(t+\tau) - \boldsymbol{u}(\boldsymbol{p}) - \frac{1}{t+\tau}\int_{\tau}^{t+\tau} s\,ds\,\nabla \mathrm{V}_l(\boldsymbol{u}(\boldsymbol{p})s)\,\nabla \boldsymbol{u}(\boldsymbol{p})$$

has strictly positive modulus for $t + \tau$ large since $|\,\boldsymbol{u}(\boldsymbol{p})\,| > 2u$ in supp g and $|\,z\,|/(t+\tau) \leq 3u/2$. Then standard « non-stationary phase » estimates as e. g. Theorem XI.14 in [*17*] yield rapid decay. For an explicit calculation in the closely analogous Schrödinger case see e. g. [*5*].

REMARK 4.3. — In this section we have given the completeness proof for a simple case which should cover all interactions of physical relevance like Coulomb- and Yukawa-potentials as well as multipole forces from extended objects, etc. Rotational symmetry never entered into our proof and the convenient scaling relation (4.6) can be omitted if e. g. the rescaled functions satisfy (4.7) uniformly for large s. A potential of the form

$$\mathrm{V}_l(r,\theta,\varphi) = f(\theta,\varphi)\frac{1}{r}\,[1 + \sin^2(\ln r)] \quad \text{for} \quad r > \mathrm{R} > 0$$

would be permitted here. For suitable potentials of electrostatic type with slower decay the methods of Sections 4, 5 of [*6*] can also be applied to the Dirac equation. If, however, the long-range potentials at different points do not commute, which may be the case for magnetic fields, then the modification (4.8) no longer has that simple form and a more subtle analysis would be required.

Note added in proof. After completion of this paper and of [*21*] we recieved the preprint [*22*] of Muthumaralingam and Sinha treating similar problems.

APPENDIX

For the convenience of the reader we list some commutation formulas used in the text as well as interesting properties of the operators **F** and G defined in Section 1.

$$\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{lk}\mathbb{1}\,, \qquad \alpha_k\beta + \beta\alpha_k = 0, \quad \beta^2 = \mathbb{1}\,, \tag{A.1}$$

$$i[\mathrm{H}_0, c\boldsymbol{\alpha}] = i[\mathrm{H}_0, \mathbf{F}] = 2i\mathrm{H}_0\mathbf{F}\,, \tag{A.2}$$

$$i[\mathrm{H}_0, \beta] = i[\mathrm{H}_0, \mathrm{G}] = 2i\mathrm{H}_0\mathrm{G}\,, \tag{A.3}$$

$$i[\mathrm{H}_0, \boldsymbol{x}] = c\boldsymbol{\alpha}, \quad i[\mathrm{H}_0, \boldsymbol{p}] = 0\,, \tag{A.4}$$

$$i[\mathrm{H}_0, \mathrm{A}] = \frac{c^2}{2}(\mathrm{H}_0^{-1}c\boldsymbol{\alpha}\cdot\boldsymbol{p} + c\boldsymbol{\alpha}\cdot\boldsymbol{p}\mathrm{H}_0^{-1}) = c^4p^2\mathrm{H}_0^{-2} = c^2(\mathbb{1} - m^2c^4\mathrm{H}_0^{-2})\,, \tag{A.5}$$

$$\mathbf{F}\cdot\boldsymbol{x} - \boldsymbol{x}\cdot\mathbf{F} = c^2(\boldsymbol{x}\cdot\boldsymbol{p}\mathrm{H}_0^{-1} - \mathrm{H}_0^{-1}\boldsymbol{p}\cdot\boldsymbol{x}) = ic\mathrm{H}_0^{-1}\boldsymbol{\alpha}\cdot\mathbf{F}\,, \tag{A.6}$$

$$\boldsymbol{x} = \frac{i}{2}\mathbf{F}\mathrm{H}_0^{-1} + \mathrm{P}_0^+\boldsymbol{x}\mathrm{P}_0^+ + \mathrm{P}_0^-\boldsymbol{x}\mathrm{P}_0^- \quad \text{on} \quad \mathscr{D}(x)\,, \tag{A.7}$$

$$\mathrm{P}_0^{\pm}\mathbf{F} = \mathbf{F}\mathrm{P}_0^{\mp}, \quad \mathrm{P}_0^{\pm}\mathrm{G} = \mathrm{G}\mathrm{P}_0^{\mp}\,, \tag{A.8}$$

$$\mathrm{H}_0\mathbf{F} + \mathbf{F}\mathrm{H}_0 = 0, \quad \mathrm{H}_0\mathrm{G} + \mathrm{G}\mathrm{H}_0 = 0\,, \tag{A.9}$$

$$\mathbf{F}\cdot\boldsymbol{p} + mc^2\mathrm{G} = 0\,, \tag{A.10}$$

$$\frac{1}{c}\boldsymbol{\alpha}\cdot\mathbf{F} + \beta\mathrm{G} = 3\,, \tag{A.11}$$

$$\mathrm{G}^2 = \mathbb{1} - m^2c^4\mathrm{H}_0^{-2} = 3 - \mathbf{F}^2/c^2\,. \tag{A.12}$$

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