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JAKOB YNGVASON

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Invariant states on Borchers' tensor algebra

by

Jakob YNGVASON
Science Institute, University of Iceland, Reykjavik, Iceland

ABSTRACT. — A method is presented for constructing states, invariant under various groups of space-time transformations, on the test function algebra for quantum fields. These states are used to determine the intersection of the kernels resp. left kernels of all states satisfying either the locality condition and translational invariance, or the spectrum condition and Poincaré invariance.

RÉSUMÉ. — On présente une méthode pour construire des états sur l'algèbre des fonctions d'essai des champs quantiques, qui soient invariants sous l'action de divers groupes de transformations d'espace-temps. On utilise ces états pour déterminer l'intersection des noyaux (resp. des noyaux à gauche) de tous les états satisfaisant soit la condition de localité et l'invariance par translation, soit la condition spectrale et l'invariance de Poincaré.

1. INTRODUCTION

This paper continues the study of ideals and automorphisms of Borchers's tensor algebra that was begun in [*1*] [*2*]. It is motivated by the long standing question about existence and abundance of nontrivial Wightman fields in four-dimensional space-time. One aim of this study is to determine what linear conditions, besides those explicitly stated in the Wightman axioms, are implied by the nonlinear positivity condition. Perturbation theory gives rise to a host of examples of linear functionals on Borchers's algebra that satisfy all Wightman conditions, except positivity [3]. It would be a major step towards a solution of the existence problem if the functionals in these examples could be shown to belong to the linear span of the Wightman states.

Linear restrictions that follow from positivity have been known for a long time. An example is the observation in [4] that the Wightman functions are Fourier transforms of measures in each of the difference variables when they have been smeared in the others. Another example is the continuity property (« $\mathcal{N}$-continuity ») discussed in [5]. By combining these two conditions a complete characterization of the linear span of translationally invariant states on Borchers's algebra satisfying a spectrum condition was given in [1].

In the present paper we partly extend these results to Poincaré-invariant functionals. We do not obtain a complete characterization of the linear span of invariant states, but derive instead a sufficient condition for a functional to belong to this linear span (prop. 5.6). Using this, we show that such functionals are dense in the set of all invariant functionals with spectrum conditions. We also consider the intersection of the left kernels of Poincaré invariant states with spectrum condition and prove that this is equal to the spectrum ideal (thm. 5.4). Analogous results are obtained for the locality ideal and the translation group (thms. 3.1 and 3.3), extending thms. (4.5) and (4.6) in [2], which dealt with the locality ideal alone.

We now introduce some notation and discuss briefly the general mathematical context of these results. Let $\mathscr{A}$ be a locally convex algebra over $\mathbb{C}$ with a separately continuous product and a continuous, antilinear involution *. We denote the dual space by $\mathscr{A}'$ and the cone of positive functionals, i. e. functionals $T \in \mathscr{A}'$ with $T(a^*a) \geq 0$ for all a, by $\mathscr{A}^{+\prime}$. If $\mathscr{A}$ has a unit element $\mathbb{1}$, and $T \in \mathscr{A}^{+\prime}$ is normalized such that $T(\mathbb{1}) = 1$, then T is called a *state*. The kernel of $T \in \mathscr{A}'$ is denoted by K(T), the left kernel of $T \in \mathscr{A}^{+\prime}$ by L(T), and the kernel of the corresponding GNS-representation by I(T). Let $\mathscr{C}$ be a subset of $\mathscr{A}^{+\prime}$ and consider the following closed subspaces of $\mathscr{A}$:

$$K(\mathscr{C}) := \bigcap_{T \in \mathscr{C}} K(T),$$

$$L(\mathscr{C}) := \bigcap_{T \in \mathscr{C}} L(T),$$

$$I(\mathscr{C}) := \bigcap_{T \in \mathscr{C}} I(T).$$

The following properties are easily verified (cf. [2], prop. 2.1):

i) $K(\mathscr{C})$ is *-invariant, and if $\sum_{i=1}^{\infty} a_i^* a_i \in K(\mathscr{C})$, then $\overline{a_i^*\mathscr{A} + \mathscr{A}a_i} \subset K(\mathscr{C})$ for all i.

ii) $\overline{L(\mathscr{C}) + L(\mathscr{C})^*} \subset K(\mathscr{C})$. The ideal $L(\mathscr{C})$ is the largest left ideal contained in $K(\mathscr{C})$, and if $\sum_{i=1}^{\infty} a_i^* a_i \in \overline{L(\mathscr{C}) + L(\mathscr{C})^*}$ then $a_i \in L(\mathscr{C})$ for all i.

iii) $I(\mathscr{C})$ is *-invariant and contained in $L(\mathscr{C}) \cap L(\mathscr{C})^* \subset K(\mathscr{C})$. The ideal $I(\mathscr{C})$ is the largest two-sided ideal in $K(\mathscr{C})$, and if $\Sigma a_i^* a_i \in I(\mathscr{C})$, then $a_i \in I(\mathscr{C})$ for all i.

It is natural to ask whether the conditions stated in *i*)-*iii*) characterize precisely those subspaces, resp. ideals of $\mathscr{A}$ which have the form $K(\mathscr{C})$ resp. $L(\mathscr{C})$ or $I(\mathscr{C})$ for some $\mathscr{C} \subset \mathscr{A}^{+\prime}$. For Borchers's algebra $\underline{\mathscr{S}}$ (more generally for the tensor algebra, or symmetric tensor algebra, over an infinite dimensional nuclear F-space), J. Alcántara has given examples which show that this is not the case. This should be compared with the real version of Hilbert's Nullstellensatz [*6*], which states that this *does* hold for ideals in the symmetric tensor algebra over a finite dimensional space. An interesting open question is whether the same is at least true for *graded* subspaces, resp. ideals in $\underline{\mathscr{S}}$.

In this paper $\mathscr{C}$ consists of positive functionals that annihilate a given left ideal and are invariant under a group of automorphisms. Let G be a topological group and $\tau \mapsto \alpha_\tau$ be a representation of G by *-automorphisms of $\mathscr{A}$, such that $(\tau, a) \mapsto \alpha_\tau a$ is separately continuous in $\tau \in G$ and $a \in \mathscr{A}$. A functional $T \in \mathscr{A}'$ is G-invariant, i. e. $T(\alpha_\tau a) = T(a)$ for all $\tau \in G$, $a \in \mathscr{A}$, iff $T \in \mathscr{K}_G{}^\perp$, where

$$\mathscr{K}_G := \text{cl. conv. hull} \{ \alpha_\tau a - a \mid \tau \in G, a \in \mathscr{A} \} .$$

and $\mathscr{K}_G{}^\perp$ denotes the set of all $T \in \mathscr{A}'$ that vanish on $\mathscr{K}_G$.

Let $\mathscr{L}$ be a G-invariant left ideal in $\mathscr{A}$. We say that $\mathscr{L}$ and G are *well behaved with respect to positivity* if

$$K(\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp \cap \mathscr{A}^{+\prime}) = \overline{\mathscr{L} + \mathscr{L}^* + \mathscr{K}_G} \tag{1.1}$$

and

$$L(\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp \cap \mathscr{A}^{+\prime}) = \mathscr{L} . \tag{1.2}$$

One has then also

$$I(\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp \cap \mathscr{A}^{+\prime}) = \mathscr{I}_{\mathscr{L}} := \text{largest two sided ideal contained in } \mathscr{L} \cap \mathscr{L}^* . \tag{1.3}$$

In the next sections it will be shown that (1.1) and (1.2) hold if G is the translation group resp. the Poincaré group and $\mathscr{L}$ is the locality ideal resp. the spectrum ideal. In an appendix we discuss a series of examples and counterexamples which show that this is by no means a general feature of the ideals and automorphisms of Borchers's algebra.

By duality, (1.1) is equivalent to the statement that the linear span of $\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp \cap \mathscr{A}^{+\prime}$ is dense in $(\mathscr{L} + \mathscr{L}^*)^\perp \cap \mathscr{K}_G{}^\perp$. A precise characterization of this linear span is a much harder task, and theorems 3.3 and 5.4 below are only a first step in this direction.

The construction of invariant states in this paper is based on an averaging procedure applied to truncated functionals. This method has been used to solve a different problem in [*7*], and a brief account of it was presented in [*8*]. There it is explained why one has to consider the truncated functionals instead of the functionals themselves, and also why the Poincaré group is easier to deal with in connection with the spectrum ideal than the locality ideal. We refer also to [*9*] for a general discussion and further motivation of these problems. Finally, we want to mention the remarkable work of Baumgärtel and Wollenberg [*10*], who construct states satisfying parts of the Wightman axioms by entirely different means, although their method is perhaps not particulary suited for the questions dealt with here.

The notation used in this paper is the same as in [*1*] and [*2*]. Borchers's algebra $\underline{\mathscr{S}}$ is the (completed) tensor algebra over Schwartz-space $\mathscr{S}_1 = \mathscr{S}(\mathbb{R}^d)$; its elements are sequences $\underline{f} = (f_0, f_1, f_2, \ldots, f_n, 0, 0, \ldots)$ with $f_0 \in \mathbb{C}$, $f_m \in \mathscr{S}_m = \mathscr{S}(\mathbb{R}^{d.m})$. Addition and multiplication by scalars are defined componentwise, and the product is the usual tensor product for functions. The dual space $\underline{\mathscr{S}}'$ consists of sequences $T = (T_0, T_1, \ldots)$ with $T_n \in \mathscr{S}'_n$, and $T(\underline{f}) = \Sigma T_n(f_n)$. The cone of positive functionals in $\underline{\mathscr{S}}'$ is denoted by $\underline{\mathscr{S}}^{+\prime}$.

2. SOME AUXILIARY RESULTS ON THE SPLITTING OF LINEAR FUNCTIONALS

A functional $T \in \underline{\mathscr{S}}'$ is called *conditionally positive*, if $T(\underline{f}^* \otimes \underline{f}) \geq 0$ for all $\underline{f} \in \underline{\mathscr{S}}$ with $f_0 = 0$ [*11*]. We shall denote the cone of conditionally positive functionals by $\mathring{\underline{\mathscr{S}}}^{+\prime}$. Note that the components T_0 and T_1 are not involved in the definition. For this reason, conditionally positive, invariant functionals are easier to construct than invariant states. One can often simply integrate over the group. This procedure will not work for states if the group is not compact, because of the constant term $T_0 \neq 0$. In order to pass afterwards from conditionally positive functionals to positive ones we need the so-called s-product [*12*].

If $T, S \in \underline{\mathscr{S}}'$ one defines $TsS \in \underline{\mathscr{S}}'$ by

$$(TsS)_0 = T_0 S_0,$$
$$(TsS)_n(f_1 \otimes \ldots \otimes f_n) = \Sigma T_k(f_{i_1} \otimes \ldots \otimes f_{i_k}) S_l(f_{j_1} \otimes \ldots \otimes f_{j_l}),$$

where the sum ranges over all partitions of $\{1, \ldots, n\}$ into two complementary, ordered subsets $(i_1, \ldots, i_k)$, $(j_1, \ldots, j_l)$ (including the empty set for $k = 0$ or $l = 0$). With the **s**-product one can also form infinite series [12]. If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence r, then $A(T)_{|s} := \sum_{n=1}^{\infty} a_n T^n{}_{|s}$ converges in $\underline{\mathcal{S}}'$ for all $T \in \underline{\mathcal{S}}'$ with $|T_0| < r$. In particular we can define $\exp_{|s} T = \Sigma 1/n! \cdot T^n{}_{|s}$ for all T. If $T_0 \neq 0$, there is also an inverse to this:

$$\log_{|s} T = \log T_0 + \sum_{n=1}^{\infty} (-1)^{n+1}/n \cdot T_0^{-n} \cdot (T/T_0 - 1)^n{}_{|s}.$$

$T^t = \log T$ is precisely the *truncated functional* of T, and writing $T = \exp_{|s} T^t$ leads to the usual expansion into truncated functions. The following result is due to Hegerfeldt:

2.1. THEOREM ([11], thm. 2.1). — *If* $T \in \mathring{\underline{\mathcal{S}}}^{+\prime}$ *is hermitian, then* $\exp_{|s} T \in \underline{\mathcal{S}}^{+\prime}$.

We shall need the following elaboration of this theorem:

2.2. THEOREM. — *Suppose* $\mathcal{K}$ *is a graded, linear subspace of* $\underline{\mathcal{S}}$ (*i. e.* $f \in \mathcal{K} \Rightarrow f_n \in \mathcal{K}$ *for all n*) *satisfying the condition*

$$T \in \mathcal{K}^{\perp} \text{ implies } \exp_{|s} T \in \mathcal{K}^{\perp}. \tag{A}$$

The following are equivalent for $T \in \underline{\mathcal{S}}'$:

i) T *is a linear combination,*

$$T = (T^{(1)} - T^{(2)}) + i(T^{(3)} - T^{(4)}) \qquad \text{with} \qquad T^{(\nu)} \in \underline{\mathcal{S}}' \cap \mathcal{K}^{\perp}, \quad \nu = 1, \ldots, 4.$$

ii) T *is a linear combination of functionals*

$$T^{(\nu)} \in \mathring{\underline{\mathcal{S}}}^{+\prime} \cap \mathcal{K}^{\perp}, \qquad \nu = 1, \ldots, 4.$$

If $T_0 \neq 0$, *this is also equivalent to*

iii) T^t *is a linear combination of functionals*

$$S^{(\nu)} \in \underline{\mathcal{S}}^{+\prime} \cap \mathcal{K}^{\perp}, \qquad \nu = 1, \ldots, 4.$$

iv) T^t *is a linear combination of functionals*

$$S^{(\nu)} \in \mathring{\underline{\mathcal{S}}}^{+\prime} \cap \mathcal{K}^{\perp}, \qquad \nu = 1, \ldots, 4.$$

Moreover, if T *varies in such a way that the functionals* $T^{(\nu)}$, resp. $S^{(\nu)}$ *in one of the cases i*)-*iv*) *run through an equicontinuous set, then the functionals* $T^{(\nu)}$, *resp.* $S^{(\nu)}$ *can also in the other cases be chosen from equicontinuous sets.*

Proof. — The essential step is to show *iv*) $\Rightarrow$ *i*). This goes in exactly the same way as in prop. 2.4 in [7]. For convenience of the reader we repeat here the main points of the argument. First, using the fact that all norms on the finite dimensional space of polynomials of degree n are equivalent, one shows that for any $\lambda \in \mathbb{C}$ there are constants $C_n(\lambda)$, $n = 0, 1, 2, \ldots$, such that for any $S \in \underline{\mathcal{S}}'$ and $f_n \in \mathcal{S}_n$ one has

$$| \exp_{|s} \lambda S(f_n) | \leq C_n(\lambda) \sup_{0 \leq \varepsilon \leq 1} | \exp_{|s} \varepsilon S(f_n) | .$$

Next, every sequence can be dominated by a sequence of positive type (i. e. a sequence $\{ \alpha_\nu \}$, s. t. $(\alpha_{i+j})_{i,j} = 0, 1, \ldots$ is a positive semidefinite matrix), and by a slight refinement of this statement one argues that there is a sequence $C'_n(\lambda)$ such that for all $\underline{f} \in \underline{\mathcal{S}}$

$$| \exp_{|s} \lambda S(\underline{f}) | \leq \sup | (\exp_{|s} \varepsilon S)_{\{\alpha_n\}} (\underline{f}) | . \tag{2.1}$$

where the sup is taken over all $\varepsilon \in [0, 1]$ and all sequences $\{ \alpha_n \}$ of positive type with $| \alpha_n | \leq C'_n(\lambda)$. Here we have used the notation $T_{\{\alpha_n\}} = (\alpha_0 T_0, \alpha_1 T_1, \ldots)$ if $T \in \underline{\mathcal{S}}'$ and $\{ \alpha_n \}$ is a sequence of complex numbers. If S is hermitian and conditionally positive, the right hand side of (2.1) is a monotonous seminorm on $\underline{\mathcal{S}}$ w. r. t. the usual order defined by $\underline{\mathcal{S}}^+$. Moreover, if $S \in \mathcal{K}^\perp$ and $\mathcal{K}$ is graded and satisfies condition (A), this seminorm vanishes on $\mathcal{K}$. From Prop. 1.21 and 1.15 in [7] it follows that $\exp_{|s} \lambda S$ can be written as a linear combination of positive functionals $T^{(\nu)}$ which are continuous w. r. t. this seminorm and thus annihilate $\mathcal{K}$ also. From this it is also clear that if S runs through an equicontinuous set and λ stays bounded, the $T^{(\nu)}$ remain within an equicontinuous set. If $S \in \mathring{\underline{\mathcal{S}}}^{+\prime}$ is not hermitian, it is of the form $Q + iR$ with Q and R hermitian in $\mathring{\underline{\mathcal{S}}}^{+\prime}$, We may then use the formula $\exp \lambda S = \exp \lambda Q s \exp \lambda i R$ and the fact that the s-product of two functionals in $\underline{\mathcal{S}}^{+\prime}$ is also in $\underline{\mathcal{S}}^{+\prime}$ to obtain a decomposition for $\exp \lambda S$. The proof of *iv)* $\Rightarrow$ *i)* is completed by writing

$$\exp_{|s} T^t = (\exp_{|s} S^{(1)}) s (\exp_{|s} (-S^{(2)}) s (\exp_{|s} i S^{(3)}) s (\exp_{|s} (-iS^{(4)}),$$

noting that the s-product of two equicontinuous sets is an equicontinuous set. The other implications are simple: *i)* $\Rightarrow$ *iii)* and *ii)* $\Rightarrow$ *iii)* are obvious, and *ii)* $\Rightarrow$ *iii)* follows directly from the formula for T^t. Note that *ii)* $\Rightarrow$ *i)* holds also if $T_0 = 0$, because $T = (T+1) - 1$, and $1 := (1, 0, 0, \ldots) \in \underline{\mathcal{S}}^{+\prime}$.

In addition to theorem 2.2, we shall in section 5 make use of a few other constructions.

2.3. PROPOSITION. — *Suppose* $\mathcal{K}$ *is a graded subspace of* $\underline{\mathcal{S}}$ *and* T *a hermitian functional in* $\underline{\mathcal{S}}'$. *Define* $T_{(n)} = (0, \ldots, 0, T_n, 0, \ldots)$. *The following are equivalent:*

i) There exist functionals T^1 *and* $T^2 \in \mathcal{K}^\perp \cap \underline{\mathcal{S}}^{+\prime}$ *such that* $T = T^1 - T^2$.

ii) For all n, there exist functionals $T^{1,n}$ *and* $T^{2,n} \in \mathcal{K}^\perp \cap \underline{\mathcal{S}}^{+\prime}$, *such that*

$T_{(n)} = T^{1,n} - T^{2,n}$, *and there is a continuous seminorm* $\| \,.\, \|$ *on* $\underline{\mathcal{S}}$ *(independent of n) such that*

$$| T^{i,n}(\underline{f}) | \leq C_n \| \underline{f} \| \qquad \text{for all} \quad n, \underline{f}, \text{ and } i = 1, 2,$$

with some constants C_n.

Proof. — The implication *i)* $\Rightarrow$ *ii)* follows immediately from the fact that any sequence of real numbers, (α_ν), in particular $\alpha_\nu = S_{\nu n}$ (n fixed), is a difference of two sequences of positive type. To prove *ii)* $\Rightarrow$ *i)* choose $\lambda_n > 0$, such that

$$\sum_{n=0}^{\infty} C_n \lambda_n^{-(n-\nu)} < \infty$$

for all ν (e. g. $\lambda_n = 2^{(1+C_n)}$). We define

$$T^i = \sum_n \lambda_n^{-n}(T_0^{i,n}, \lambda_n T_1^{i,n}, \ldots, \lambda_n^\nu T_\nu^{i,n}, \ldots), \qquad i = 1, 2.$$

The sum is convergent because of the estimate for $T^{i,n}$. It is clear that $T^i \in \underline{\mathcal{S}}^{+\prime} \cap \mathcal{K}^\perp$, because every $T^{i,n}$ has been multiplied with the moment sequence $(\lambda_n^\nu)_{\nu=0,1,\ldots}$. Finally,

$$T_\nu^1 - T_\nu^2 = \sum_n \lambda_n^{\nu-n}(T_\nu^{1,n} - T_\nu^{2,n}) = \sum_n \lambda_n^{\nu-n} T_\nu \delta_{n\nu} = T_\nu \quad \text{for all} \quad \nu.$$

2.4. PROPOSITION. — *Let* T *be a positive functional on* $\underline{\mathcal{S}}$, *h a function in* $\mathcal{O}_M$ *(polynomially bounded* C^∞*-functions on* $\mathbb{R}^d$*) and* $t_2 \in \mathcal{S}'_2$ *positive, i. e.* $t_2(f^* \otimes f) \geq 0$ *for all* $f \in \mathcal{S}_1$. *Then the following functionals* T' *and* T'' *are conditionally positive*:

i) $T'_n(f_1 \otimes \ldots \otimes f_n) = T_n(h.f_1 \otimes f_2 \ldots f_{n-1} \otimes h^*.f_n)$,

ii) $T''_n(f_1 \otimes \ldots \otimes f_n) = t_2(f_1 \otimes f_n) T_{n-2}(f_2 \otimes \ldots \otimes f_{n-1})$.

Proof. — The first part is clear, because the linear mapping

$$f_1 \otimes \ldots \otimes f_n \mapsto hf_1 \otimes f_2 \otimes \ldots \otimes f_{n-1} \otimes h^* f_n \qquad (n \geq 2)$$

maps elements of the form $\underline{g}^* \otimes \underline{g}$ with $g_0 = 0$ into elements of the same form. For the second part we note that every $\underline{f} \in \underline{\mathcal{S}}$ with $f_0 = 0$ can be written

$$\underline{f} = \sum_i \underline{g}_i \otimes h_i$$

with $h_i \in \mathcal{S}_1$, and one has

$$T''(\underline{f}^* \otimes \underline{f}) = \sum_{i,j} t_2(h_i^* \otimes h_j) T(\underline{g}_i^* \otimes \underline{g}_j).$$

Now $t_2(h_i^* \otimes h_j)$ and $T(g_i^* \otimes g_j)$ are both positive semidefinite matrices. $T''(\underline{f}^* \otimes \underline{f})$ is just the trace of their product and hence nonnegative.

For the next proposition we recall the notation

$$T_{\{\alpha_n\}} = (\alpha_0 T_0, \alpha_1 T_1, \ldots)$$

if $T \in \underline{\mathcal{S}}'$ and $\{ \alpha_n \}$ is a sequence in $\mathbb{C}$.

2.5. PROPOSITION. — *There is a sequence $\{ \alpha_n \}$ of positive type, such that for all $T \in \underline{\mathcal{S}}^{+\prime}$ and $\underline{f} \in \underline{\mathcal{S}}$*

$$\sum_n T_{2n}(f_n^* \otimes f_n) \leq T_{\{\alpha_n\}}(\underline{f}^* \otimes \underline{f}).$$

Proof. — The sequence $\{ \alpha_n \}$ can be constructed by induction over n (cf. e. g. [2], p. 1 070) such that the matrix $A = (\alpha_{i+j})_{i,j=0,1,\ldots}$ dominates $I = (\delta_{ij})$ in the sense that

$$\Sigma \mid \lambda_i \mid^2 \leq \Sigma \alpha_{i+j} \bar{\lambda}_i \lambda_j$$

for all finite sequences $\lambda_i \in \mathbb{C}$. But then we have also

$$\operatorname{Tr} M \leq \operatorname{Tr} AM$$

for all positive semidefinite matrices M, in particular for $M = (T_{i+j}(f_i^* \otimes f_j))$.

2.6. PROPOSITION. — *There exist sequences $\{ \alpha_n \}$ and $\{ \beta_n \}$ such that for every conditionally positive functional T and all $\underline{f} \in \bigoplus_{n=1}^{\infty} \mathcal{S}_n$, one has*

$$\sum_n T_{2n}(f_n^* \otimes f_n) \leq (\exp_{|s} T_{\{\beta_n\}})_{\{\alpha_n\}}(\underline{f}^* \otimes \underline{f}).$$

Proof. — In view of prop. 2.5, one has to find a sequence $\{ \beta_n \}$ of positive type such that for all n

$$T_{2n}(f_n^* \otimes f_n) \leq (\exp_{|s} T_{\{\beta_n\}})(f_n^* \otimes f_n),$$

i. e.

$$0 \leq \sum_{\text{Part}} \beta_{k_1} \ldots \beta_{k_j}(T_{k_1} \otimes \ldots \otimes T_{k_j})(f_n^* \otimes f_n) + (\beta_{2n} - 1)T_{2n}(f_n^* \otimes f_n) = (\exp_{|s} T_{\{\beta_\nu^{(n)}\}})(f_n^* \otimes f_n)$$

with $\{ \beta_\nu^{(n)} \} := (\beta_1, \ldots, \beta_{2n-1}, \beta_{2n}^{-1}, \beta_{2n+1}^{(n)}, \ldots)$. One has thus to choose β_ν (and $\beta_\nu^{(n)}$ for $\nu \geq 2n + 1$) such that $\{ \beta_\nu^{(n)} \}$ is a sequence of positive type for all n. But this can be achieved by the induction method of [2], p. 1 070; in fact one has only to let $\beta_\nu^{(n)}$ grow sufficiently fast with ν.

3. LOCALITY IDEAL AND TRANSLATION GROUP

In this section we construct $\mathbb{R}^d$-invariant states on $\underline{\mathcal{S}}$ that annihilate the locality ideal $\mathcal{J}_C$. The following simple facts will be used without further comment.

1) $\mathcal{K} = \mathcal{J}_C + \mathcal{K}_{\mathbb{R}^d}$ satisfies condition (A) of thm. 2.2.

2) If $\| \, . \, \|$ is a continuous seminorm on $\mathcal{S}_n$ and $g \in \mathcal{S}_n$, then

$$f \mapsto \int \| g \cdot f_a \| \, da \,, \qquad \text{with} \qquad f_a := f(x-a) \,,$$

is a continuous, translationally invariant seminorm on $\mathcal{S}_n$. If $\| \, . \, \|$ vanishes on $\mathcal{J}_C$ and g is totally symmetric this seminorm vanishes also on $\mathcal{J}_C$.

The following theorem is analogous to thm. 2.11 in [7]:

3.1. THEOREM. — *Suppose* $T \in \mathcal{J}_C^{\perp} \cap \mathcal{K}_{\mathbb{R}^d}^{\perp}$ *and the following holds either for* T *or the truncated functional* T^t:

There is a $k \in \mathbb{N}$, *and for each* $n = 1, 2,$ *a constant* C_n *and a rapidly decreasing continuous function* (1) g_n *on* $\mathbb{R}^d$ *such that*

$$| T_n^{(t)}(f) | \leq C_n \cdot \max_{|\alpha_i| \leq k} \sup_{x_1, \ldots, x_n} \left| \prod_{i=2}^{n} g_n(x_i - x_1) D_1^{\alpha_1} \ldots D_n^{\alpha_n} f(x_1, \ldots, x_n) \right|$$

for all $f \in \mathcal{S}_n$. *Then* T *can be written as a linear combination of four functionals in* $\mathcal{J}_C^{\perp} \cap K_{\mathbb{R}^d}^{\perp} \cap \underline{\mathcal{S}}^{+\prime}$.

Proof. — We shall make use of the following lemma, proved in the appendix in [7]:

There is a function $h \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\sup_{x_1, \ldots, x_n} \left| \prod_{i=2}^{n} g_n(x_i - x_1) D_1^{\alpha_1} \ldots D_n^{\alpha_n} (\overline{h^{\otimes n}}(x_1, \ldots, x_n))^{-1} \right| < \infty$$

for all $\alpha_1, \ldots, \alpha_n$, where $\overline{h^{\otimes n}}(x_1, \ldots, x_n) := \int h(x_1 + a) \ldots h(x_n + a) da$.

It follows that the functional T', defined by $T'_0 = T_0^{(t)}$, $T'_n = T_n^{(t)} . (\overline{h^{\otimes n}})^{-1}$ is continuous in the topology τ_∞ considered in [2]. By thm. 4.6 in [2] T' is thus a linear combination of functionals $R^{(\nu)} \in \mathcal{J}_C^{\perp} \cap \underline{\mathcal{S}}^{+\prime}$:

$$T' = (R^{(1)} - R^{(2)}) + i(R^{(3)} - R^{(4)}) \,.$$

(1) i. e. decreasing more rapidly at infinity than any negative power of $| x |$.

Consider now the functionals $S^{(\nu)}$, $\nu = 1, \ldots, 4$, defined by

$$S_0{}^{(\nu)} = R_0{}^{(\nu)},$$
$$S_n{}^{(\nu)}(f) = \int (R_n{}^{(\nu)} \cdot h^{\otimes n})(f_a)da\,.$$

These functionals are in $\mathscr{I}_C{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$ and they are also conditionally positive. Moreover,

$$(S_0{}^{(1)} - S_0{}^{(2)}) + i(S_0{}^{(3)} - S_0{}^{(4)}) = T_0{}^{(t)},$$

and using the invariance of T'_n, we have for $n \geq 1$

$$(S_n{}^{(1)} - S_n{}^{(2)}) + i(S_n{}^{(3)} - S_n{}^{(4)})(f) = \int (T'_n \cdot h^{\otimes n})(f_a)da = \int (T'_n \cdot h_a{}^{\otimes n})(f)da = T_n{}^{(t)}(f).$$

From theorem 2.2 it now follows that T is a linear combination of functionals in $\mathscr{I}_C{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$.

Remark. — The hypothesis of rapid decrease of T_n or $T_n{}^t$ in all directions in the difference variables is of course for from being necessary. For instance, truncated Wightman functions have only a polynomial decrease in time-like directions. It is an unsolved problem to find a complete characterization of the linear span of $\mathscr{I}_C{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$, but thm. 3.1 is sufficient for a proof of thm. 3.3.

3.2. LEMMA. — *$\mathscr{I}_C + \mathscr{K}_{\mathbb{R}^d}$ is closed.*

Proof. — Suppose $\underline{f} \in \overline{\mathscr{I}_C + \mathscr{K}_{\mathbb{R}^d}}$. Then $T(\underline{f}) = 0$ for all $T \in \mathscr{I}_C{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$. In particular, $f_0 = 0$ (take $T = (1, 0, \ldots,)$). Next let χ be a function in $\mathscr{S}$ with $\int \chi(x)dx = 1$, and define

$$\chi_n(x_1, \ldots, x_n) = \chi((x_1 + \ldots + x_n)/n)\,.$$

Write $f_n = g_n + h_n$ with

$$g_n(x_1, \ldots, x_n) = \int f(x_1 + a, \ldots, x_n + a)da \cdot \chi_n(x_1, \ldots, x_n)\,.$$

Then $\int h_n(x_1 + a, \ldots, x_n + a) = 0$, so $h_n \in \mathscr{K}_{\mathbb{R}^d}$ (cf. prop. 2.1 in [1]). Also, for any $S \in \mathscr{I}_C{}^{\perp}$ we have

$$S_n(g_n) = \int (S_n \cdot \chi_n)(f_{n,a})da = 0\,.$$

because the functional $f \mapsto \int (S_n \cdot \chi_n)(f_a)$ da belongs to $\mathscr{I}_C{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$. Hence $g_n \in \mathscr{I}_C$, so $\underline{f} \in \mathscr{I}_C + \mathscr{K}_{\mathbb{R}^d}$.

3.3. THEOREM. — *i)* $\mathrm{K}(\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{I}_{\mathrm{C}} + \mathscr{K}_{\mathbb{R}^d}$,
ii) $\mathrm{L}(\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{I}_{\mathrm{C}}$.

3.4. REMARK. — From *ii)* it follows in particular that the algebra $\underline{\mathscr{S}}/\mathscr{I}_{\mathrm{C}}$ has a faithful, translationally covariant Hilbert space representation.

Proof of 3.3. — *i)* In view of lemma 3.2 it suffices to show that the linear span of $\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$ is dense in $\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$. To prove this we use theorem 3.1. Let $\varphi \in \mathscr{S}$ and define

$$\psi_n(x_1, \ldots, x_n) = 1/n\,! \sum_{\mathrm{Perm}\,\pi} \prod_{i=2}^{n} \varphi(x_{\pi i} - x_{\pi 1}).$$

Then ψ_n is totally symmetric, translationally invariant, and belongs to $\mathscr{S}(\mathbb{R}^{d(n-1)})$ as a function of the difference variables $x_i - x_1$. If $\mathrm{T} \in \mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$, and $\mathrm{N} < \infty$, we define a functional $\mathrm{T}' \in \mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$ by

$$\mathrm{T}_n' = \mathrm{T}_n \cdot \psi_n \quad \text{for} \quad n \leq \mathrm{N}, \qquad \mathrm{T}_n' = 0 \quad \text{for} \quad n > \mathrm{N}.$$

This functional obviously satisfies the condition of theorem 3.1 and thus belongs to the linear span of $\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$. But T can be approximated weakly by such functionals if we take $\mathrm{N} \to \infty$ and $\varphi = 1$ on an increasingly large portion of $\mathbb{R}^d$.

ii) Suppose $\underline{f} \in \mathrm{L}(\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime})$, i. e. $\mathrm{T}(\underline{f}^* \otimes \underline{f}) = 0$ for all $\mathrm{T} \in \mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$. By theorem 4.5 in [*2*] we have to show that $\mathrm{T}(\underline{f}^* \otimes \underline{f}) = 0$ for all $\mathrm{T} \in \mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$. Obviously $f_0 = 0$. Moreover, if $\mathrm{T} \in \mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$ and $h \in \mathscr{S}$, we may consider the functionals

$$\mathrm{T}_h(\underline{g}) = \sum_n \mathrm{T}_n(h^{\otimes n} g_n) \text{ and}$$

$$\mathrm{S}_h(\underline{g}) = \int \mathrm{T}_h(\underline{g}_a)\,da \quad \text{for} \quad g_0 = 0, \qquad (\mathrm{S}_h)_0 = \mathrm{T}_0.$$

The functional S_h is in $\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp}$ and is moreover conditionally positive. By theorem 2.2 it is a linear combination of functionals in $\mathscr{I}_{\mathrm{C}}{}^{\perp} \cap \mathscr{K}_{\mathbb{R}^d}{}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$. Hence $\mathrm{S}_h(\underline{f}^* \otimes \underline{f}) = 0$, and since $\mathrm{T}_h(\underline{f}_a^* \otimes \underline{f}_a) \geq 0$, and $a \mapsto \mathrm{T}_h(\underline{f}_a^* \otimes \underline{f}_a)$ is continuous, we conclude that $\mathrm{T}_h(\underline{f}^* \otimes \underline{f}) = 0$. Since this holds for all h we have finally $\mathrm{T}(\underline{f}^* \otimes \underline{f}) = 0$ and thus $\underline{f} \in \mathscr{I}_{\mathrm{C}}$.

4. INTEGRATING OVER THE LORENTZ GROUP

This section consists of a few technical lemmas concerning integration over the proper, orthochronous Lorentz group $\mathrm{L}_+^\uparrow = \mathrm{SO}(1, d-1)$. It will be convenient to parametrize the Lorentz transformations by writing them as products of boosts and rotations. Let $p \in V^+$, i. e. $p \cdot p = m^2 > 0$ and $p^0 > 0$. We define the boost operators Λ_p as follows:

For
$$p = (p^0, 0, \ldots, 0, p^{d-1})$$
we put

$$(\Lambda_p)_0{}^0 = (\Lambda_p)_{d-1}{}^{d-1} = p^0/m\,,$$
$$(\Lambda_p)_{d-1}{}^0 = (\Lambda_p)_0{}^{d-1} = p^{d-1}/m, \qquad (\Lambda_p)_\nu{}^\nu = 1 \quad \text{for} \quad \nu = 1, \ldots, d-2\,,$$

and the other matrix elements zero. More generally, if $p = \mathrm{R}p'$ with p' of the form as above and $\mathrm{R} \in \mathrm{SO}(d-1)$ is a rotation, we define $\Lambda_p = \mathrm{R}\Lambda_{p'}\mathrm{R}^{-1}$. We then have $\Lambda_p \bar{p} = p$ with $\bar{p} = ((p\ p)^{1/2}, \vec{0})$, and moreover

$$\Lambda_{\lambda p} = \Lambda_p \qquad \text{if} \quad \lambda > 0\,,$$
$$\Lambda_{\mathrm{R}p} = \mathrm{R}\Lambda_p\mathrm{R}^{-1} \quad \text{if} \quad \mathrm{R} \in \mathrm{SO}(d-1)\,.$$

Note also that
$$\Lambda_p^{-1} = \Lambda_{\hat{p}}$$
with $\hat{p} := (p^0, -\vec{p})$, and
$$(\Lambda_p^{-1}q)^0 = (\Lambda_{\mathrm{R}\hat{p}}\mathrm{R}q)^0 = p \cdot q/m_p\,. \tag{4.1}$$

Every Lorentz transformation Λ can be written uniquely as

$$\Lambda = \Lambda_p \cdot \mathrm{R}$$

with a rotation R and a boost Λ_p. For each $m > 0$ the corresponding vector $p \in \mathrm{H}_m{}^+ := \{\, p \mid p \cdot p = m^2,\ p^0 > 0 \,\}$ is also uniquely determined by Λ.

The (left- and right-) invariant Haar measure on $\mathrm{SO}(1, d-1)$ can in this parametrization be written

$$d\Lambda = m^{-(d-2)}\theta(p^0)\delta(p \cdot p - m^2)dp d\mathrm{R} \tag{4.2}$$

where $d\mathrm{R}$ is the (normalized) Haar measure on $\mathrm{SO}(d-1)$ and dp is the Lebesgue measure on $\mathbb{R}^d$. The factor $m^{-(d-2)}$ is so chosen that $d\Lambda$ is independent of m.

The eigenvalues of Λ_p are $(|\,p^0\,| - |\,\vec{p}\,|)/m_p$, 1 and $(|\,p^0\,| + |\,\vec{p}\,|)/m_p$, where $m_p := (p \cdot p)^{1/2}$. We thus have

$$\|\,\Lambda_p\,\| = (|\,p^0\,| + |\,\vec{p}\,|)/m_p \leq 2\,|\,p\,|/m_p \tag{4.3}$$

where $|\,p\,|$ is the Euclidean norm of p. Also, since the smallest eigenvalue is

$$(p^0 - |\,\vec{p}\,|)/m_p = m_p/(p^0 + |\,\vec{p}\,|) \geq m_p/2\,|\,p\,|$$

we have for all $q \in \mathbb{R}^d$
$$|\,\Lambda_p q\,| \geq m_p \cdot |\,q\,|/2\,|\,p\,| \tag{4.4}$$

If $q \in V^+$ and $m_p = m_q = m$, it is easy to see that

$$|\,\Lambda_p q\,| = |\,\Lambda_q p\,|\,.$$

(Either use the fact that $\Lambda_p\Lambda_q = \mathrm{R}\Lambda_q\Lambda_p$ with a rotation R, or use formula (4.1) noting that $|\Lambda_p q|^2 = 2((\Lambda_p q)^0)^2 - m_q^2)$.) Hence, for such q we have also

$$|\Lambda_p q| \geq m_p \cdot |p|/2|q|. \tag{4.5}$$

If $\varepsilon > 0$, we denote $\overline{V}_\varepsilon^+ = \{ p \mid p \cdot p \geq \varepsilon^2, p^0 > 0 \}$.

4.1. LEMMA. — *Let h be a C^∞-function on $\mathbb{R}^{d.n}$ satisfying*

i) $\operatorname{supp} h \subset \{ (q_1, \ldots, q_n) \mid q_1 \in \overline{V}_\varepsilon^+ \}$ *for some* $\varepsilon > 0$,

ii) $\sup_{q_1,\ldots,q_n} |(1 + (|\vec{q}_1|/m_{q_1}))^N D^\alpha h(q_1, \ldots, q_n)| < \infty$

for all N *and all multiindices* $\alpha \in \mathbb{N}^{d.n}$.

If $\| \cdot \|$ *is a continuous seminorm on* $\mathcal{S}_n$, *then*

$$f \mapsto \int_{L_+^\uparrow} \| h \cdot f_\Lambda \| d\Lambda \qquad \textit{and} \qquad f \mapsto \int_{L_+^\uparrow} \| h_\Lambda \cdot f \| d\Lambda$$

with $f_\Lambda(q_1, \ldots, q_n) := f(\Lambda q_1, \ldots, \Lambda q_n)$ *are also continuous seminorms on* $\mathcal{S}_n$.

Proof. — We consider the first integral; the second one is treated similarly. Since $\| \cdot \|$ is a continuous seminorm we have for some M and all N, using *ii*)

$$\int \| hf_\Lambda \| d\Lambda \leq \text{const.} \sum_{|\alpha| \leq M} \iint (1 + (|\vec{q}_1|/m_{q_1}))^{-N}$$

$$\times \prod_{i=1}^{n} (1 + |q_i|)^M P_\alpha(\Lambda) |D^\alpha f(\Lambda q_1, \ldots, \Lambda q_n)| dq_1 \ldots dq_n d\Lambda$$

where P_α is a polynomial in the matrix elements of Λ and the integration ranges over $q_1 \in \overline{V}_\varepsilon^+$, $q_2, \ldots, q_n \in \mathbb{R}^d$ because of *i*). We now use the formula (4.2) for the Haar measure with $m = m_{q_1} = m_1$. Since $f \in \mathcal{S}_n$ we have for all L and M a continuous seminorm $\| \cdot \|_{L,M}$ such that

$$\max_{|\alpha| \leq M} |D^\alpha f(q_1, \ldots, q_n)| \leq \| f \|_{L,M} \prod_{i=1}^{n} (1 + |q_i|)^{-L}.$$

The degree of the polynomial P_α is $\leq$ M, and because of (4.3) we have thus to show that the following integral is finite for sufficiently large N and L:

$$\int (1 + |\vec{q}_1|/m_1)^{-N} \prod_{i=1}^{n} (1 + |q_i|)^M |p|^M/m_1^M$$

$$\times \prod_{i=1}^{n} (1 + |\Lambda_p R q_i|)^{-L} m_1^{d-2} \theta(p^0) \delta(p \cdot p - m_1^2) dp dR dq_1 \ldots dq_n .$$

By (4.5) we have

$$(1 + |\Lambda_p R q_1|)^{-L} \leq (1 + m_1 \cdot |p|^{\,2} |q_1|)^{-L}$$

and by (4.4)

$$\prod_{i=2}^{n} (1 + |\Lambda_p R q_i|)^{-L} \leq (1 + \max_{2 \leq i \leq n} |q_i| \cdot m_1/2 |p|)^{-L}.$$

Hence

$$\prod_{i=1}^{n} (1 + |\Lambda_p R q_i|)^{-L} \leq \text{const.} (|q_1|/m_1)^{L} \cdot (1 + |p| + \varepsilon \max_{2 \leq i \leq n} |q_i|)^{-L}.$$

The integrals over p and $q_2, \ldots, q_n$ thus converge for sufficiently large L, and by choosing N large we make the integral over q_1 convergent also.

4.2. LEMMA. — *Let $f \in \mathcal{S}_n$ and suppose h is as in lemma 4.1. The integrals*

$$\bar{f}(q_1, \ldots, q_n) := \int_{L_+^\uparrow} f(\Lambda q_1, \ldots, \Lambda q_n) d\Lambda$$

and

$$\bar{h}(q_1, \ldots, q_n) := \int_{L_+^\uparrow} h(\Lambda q_1, \ldots, \Lambda q_n) d\Lambda$$

exist if $q_1 \in V^+$. Moreover, $h\bar{f} \in \mathcal{S}_n$ and $\bar{h}f \in \mathcal{S}_n$ and there is a sequence of compact sets $K_\nu \subset L_+^\uparrow$, such that

$$h\bar{f} = \lim_{\nu \to \infty} hf_\nu, \qquad \bar{h}f = \lim_{\nu \to \infty} h_\nu f$$

in the topology of $\mathcal{S}$, with $f_\nu := \int_{K_\nu} f_\Lambda d\Lambda$, $h_\nu := \int_{K_\nu} h_\Lambda d\Lambda$.

Proof. — The convergence of the integrals follows immediately from the preceding lemma. If $K \subset L_+^\uparrow$ is compact, one has

$$h \cdot \int_K f_\Lambda d\Lambda \in \mathcal{S}_n \qquad \text{and} \qquad \left(\int_K h_\Lambda d\Lambda \right) \cdot f \in \mathcal{S}_n .$$

Furthermore, for any continuous seminorm $\| \,.\, \|$ on $\mathcal{S}_n$, $\left\| h \int_K f_\Lambda d\Lambda \right\|$ and $\left\| \left(\int_K h_\Lambda d\Lambda \right) f \right\|$ are uniformly bounded in K by lemma 4.1. Since $\mathcal{S}_n$ is a Montel space, there is a sequence of compact sets $K_\nu \subset L_+^\uparrow$, exhausting $L_+^\uparrow$ such that hf_ν resp. $h_\nu f$ converge in $\mathcal{S}_n$. Since these sequences converge pointwise to $h\bar{f}$ resp. $\bar{h}f$, these functions are the limits of hf_ν resp. $h_\nu f$ in the topology of $\mathscr{S}_n$.

4.3. LEMMA. — *Let* $\varepsilon > 0$. *There exist* C^∞*-functions h on* $\mathbb{R}^d$ *such that*

i) $$\operatorname{supp} h \subset \overline{V^+}_{\varepsilon/2}\,,$$

ii) $h(Rp)=h(p)$ *for all* $R \in SO(d)$, *and* $h(\lambda p)=h(p)$ *for* $\lambda \geq 1$ *and* $p \in \overline{V}_\varepsilon{}^+$.

iii) $\sup |(1 + (|\vec{q}\,|/m_q))^N D^\alpha h(q)| < \infty$ *for all* α *and* N.

iv) $$\int h_\Lambda d\Lambda = 1 \quad \text{on} \quad \overline{V}_\varepsilon^+\,.$$

Proof. — Take $h_1 \in \mathscr{S}(\mathbb{R})$ with $h_1(t) > 0$ for all t and normalize it, so that

$$\int h_1(p^0)\delta(p\cdot p - 1)\theta(p^0)dp = 1\,.$$

If χ is a C^∞-function on $\mathbb{R}$ such that $\chi(m) = 1$ for $m \geq \varepsilon$, $\chi(m) = 0$ for $m \leq \varepsilon/2$, we define

$$h(p) = \chi(m_p)h_1(p^0/m_p)\,.$$

The properties *i)*-*iv)* are immediately verified.

4.4. LEMMA. — *If* g_n, $n = 1, 2, \ldots$ *are rapidly decreasing continuous function on* $[1, \infty[$, *and* $\varepsilon > 0$, *there exists a function h satisfying the conditions of lemma 4.3, such that for all n and all multiindices* $\underline{\alpha}$

$$\sup_{p,q \in \overline{V}_\varepsilon^+} |g_n(p\cdot q/m_p m_q)D^\alpha(\overline{h^{\otimes 2}}(p, q)^{-1})| < \infty$$

where

$$\overline{h^{\otimes 2}}(p, q) := \int_{L_+^\uparrow} h(\Lambda p)h(\Lambda q)d\Lambda\,.$$

Proof. — The construction is very similar to that in the appendix of [7]. First one argues as in [7] that it is sufficient to consider a single, positive function g instead of the whole sequence $\{ g_n \}$. Next we define a function f on $[1, \infty[$ by

$$f(t) = \sup_{\substack{\hat{u}\cdot v = t \\ u,v \in H_1^+}} g(u^0)g(v^0)$$

where $\hat{u} = (u^0, -\vec{u})$, $H_1{}^+ = \{ p \mid m_p = 1,\ p^0 > 0 \}$. The function f is also continuous and strongly decreasing, for $t = \hat{u}\cdot v \leq |u|\ |v| \leq 2u^0v^0$, so either u^0 or v^0 is larger than $(t/2)^{-1/2}$. It is convenient to replace f by a monotonus function:

$$M(r) := \begin{cases} \sup_{t \geq r} f(t) & \text{for} \quad r > 1 \\ f(1) & \text{for} \quad r \leq 1 \end{cases}$$

and the function h is defined by regularizing and cutting M:

$$h(p) = \chi(m_p) \int \gamma(p^0/m_p - r)M(r)dr$$

with $\gamma(s) = \exp(-(1+s^2)^{1/2})$, χ a C^∞-function with $\chi(m) = 0$ for $m \leq \varepsilon/2$, $\chi(m) = \text{const.} = k$ for $m \geq \varepsilon$. One immediately verifies that h has properties *i)*-*iv)* of lemma 4.3 if the normalization k is suitably chosen. Note also that

$$h(p) \geq \text{const.}\ f(p^0/m_p) \tag{4.6}$$

with

$$\text{const.} \geq \int_1^\infty \gamma(s)ds > 0\,,$$

and

$$|D^\alpha h(p)| \leq C_\alpha h(p) \tag{4.7}$$

with $C_\alpha < \infty$, because $|\gamma^{(n)}(s)| \leq \gamma(s)$ for all n, and all derivatives of p^0/m_p are uniformly bounded on $\bar{V}_\varepsilon^+$. Without restriction we may assume $\varepsilon \leq 1$. Because of property *ii)* we have

$$|\overline{h^{\otimes 2}}(p, q)| = \int h(\Lambda p)h(\Lambda q)d\Lambda = \int h(u)h(\Lambda_u\Lambda_p^{-1}q)\delta(u\cdot u - 1)\theta(u^0)du\,. \tag{4.8}$$

By (4.1), (4.4), (4.5) and (4.6) we have

$$\begin{aligned} h(\Lambda_u\Lambda_p^{-1}q) &= h(\Lambda_u\Lambda_p^{-1}q/m_q) \geq f((\Lambda_u\Lambda_p^{-1}q/m_q)^0) \\ &= f(\hat{u}\cdot\Lambda_p^{-1}q/m_q) \geq g(u^0)g((\Lambda_p^{-1}q/m_q)^0) = g(u^0)g(p\cdot q/m_p\cdot m_q)\,, \end{aligned}$$

so the integral (4.8) is

$$\geq \left(\int h(u)g(u^0)\delta(u\cdot u - 1)\theta(u^0)du\right)g(p\cdot q/m_p\cdot m_q)\,.$$

Now we consider the derivatives. Using (4.3) we have

$$\begin{aligned} |D_p^{\alpha_1}D_q^{\alpha_2}\overline{h^{\otimes 2}}(p, q)| &= \left|\int P_{\alpha_1,\alpha_2}(\Lambda)h(\Lambda p)h(\Lambda q)d\Lambda\right| \\ &\leq \text{const.}\int(1+|\vec{u}|)^N h(u)h(\Lambda_u\Lambda_p^{-1}q/m_q)\delta(u\cdot u - 1)\theta(u^0)du \end{aligned}$$

with a Polynomial P_{α_1,α_2} of degree $N = |\alpha_1| + |\alpha_2|$. We split the integral into two parts. In the first part we integrate over $u^0 \leq (\Lambda_p^{-1}q/m_q)^0 = p\cdot q/m_pm_q$. This part can be estimated by

$$\text{const.}\,(1 + |p\cdot q/m_pm_q|)^N\overline{h^{\otimes 2}}(p, q)\,.$$

In the second part we integrate over $u^0 > p\cdot q/m_p\cdot m_q$. Since $h(u)$ is for $u \in H_1^+$ a monotonously decreasing function of u^0 (denoted again by h), this part is estimated by

$$h(p\cdot q/m_p\cdot m_q)\int(1 + |\vec{u}|)^N h(\Lambda_u\Lambda_p^{-1}q/m_q)\delta(u\cdot u - 1)\theta(u^0)du\,.$$

Because h is rapidly decreasing and because of (4.3) and (4.4), the integral can here be estimated by

$$\text{const.}\ (1 + | p\cdot q/m_p m_q |)^{N'} .$$

The next step is to show

$$h(p\cdot q/m_p\cdot m_q) \leq \text{const.}\ \overline{h^{\otimes 2}}(p, q) . \tag{4.9}$$

We use again monotionicity of h:

$$\overline{h^{\otimes 2}}(p, q) \geq \int_{u^0 \leq p\cdot q/m_p m_q} h(u)h(\Lambda_u\Lambda_p{}^{-1}q)\delta(u\cdot u - 1)\theta(u^0)du$$

$$\geq h(pq/m_p m_q) \cdot \int_{u^0 \leq p\cdot q/m_p m_q} h(\Lambda_u\Lambda_p{}^{-1}q)\delta(u\cdot u - 1)\theta(u^0)du .$$

If $p\cdot q/m_p\cdot m_q \geq 2$, the integral is bounded below by a constant > 0, so (4.9) holds for such p, q. On the other hand, $\overline{h^{\otimes 2}}(p, q)$ is a strictly positive function of $p\cdot q/m_p\cdot m_q$ for m_p, $m_q > \varepsilon$, so the estimate (4.9) holds also for $p\cdot q/m_p m_q \leq 2$.

In summary, we have shown that

$$| g(p\cdot q/m_p m_q)\overline{h^{\otimes 2}}(p, q)^{-1} | \leq \text{const.},$$

and

$$| D^\alpha \overline{h^{\otimes 2}}(p, q)\cdot \overline{h^{\otimes 2}}(p, q)^{-1} | \leq P_\alpha(p\cdot q/m_p m_q)$$

with a polynomial P_α. Using this, one shows by induction over the degree of the differential operator in the same way as in the appendix in [7] that

$$| g(p\cdot q/m_p m_q)D^\alpha(\overline{h^{\otimes 2}}(p, q))^{-1} | \leq P'_\alpha(p\cdot q/m_p m_q) \tag{4.10}$$

with another Polynomial P'_α. Finally, if g is rapidly decreasing, so is $| g |^{1/2}$. Replacing g by $| g |^{1/2}$ in (4.10) we obtain

$$| g(p\cdot q/m_p m_q)D^\alpha(\overline{h^{\otimes 2}}(p, q)^{-1}) | \leq \text{const.}\ (\alpha)$$

as desired.

5. SPECTRUM CONDITION AND POINCARÉ GROUP

The *spectrum ideal* $\mathscr{L}_S$, corresponding to a closed set $S \subset \mathbb{R}^d$ with $0 \in S$, is the set of all $\underline{f} \in \underline{\mathscr{S}}$ such that $f_0 = 0$ and $f_n(p_1, \ldots, p_n) = 0$ if $p_k + \ldots + p_n \in S$ for $k = 1, \ldots, n$, $n \geq 1$.

According to [1] there is an abundance of states on $\underline{\mathscr{S}}$ that annihilate $\mathscr{L}_S$ and are invariant under the translation group. We now consider Lorentz invariant sets S and want to obtain similar results with the Poincaré group replacing the translation group.

The method used here requires that S is an additive subset of $\mathbb{R}^d$ and that S is contained in the light cone $\overline{V}$; S is then either in $\overline{V}^+$ or $\overline{V}^-$ and we may choose $\overline{V}^+$ by convention. For some results we shall also require S to have a lower mass gap. We note first some simple consequences of these restrictions on S:

5.1. PROPOSITION. — *If* S *is additive, then* $\mathcal{L}_S$ *satisfies condition* (A) *of theorem (2.2) i. e.* $T \in \mathcal{L}_S{}^{\perp}$ *implies* $\exp_{|s} T \in \mathcal{L}_S{}^{\perp}$.

Proof. — Suppose $T \in \mathcal{L}_S{}^{\perp}$ and $f \in \mathcal{S}_n \cap \mathcal{L}_S$. We want to show that

$$\int f(p_1, \ldots, p_n) T_{n_1}(p_{i_{1,1}}, \ldots, p_{i_{1,n_1}}) \ldots T_{n_k}(p_{i_{k,1}}, \ldots, p_{i_{k,n_k}}) dp_1 \ldots dp_n = 0 \tag{5.1}$$

for any partition of $\{ 1, \ldots, n \}$ into (ordered) subsets $I_j = \{ i_{j,1}, \ldots, i_{j,n_j} \}$. For any $v = 1, \ldots, n$ we can write

$$\{ v, \ldots, n \} = \bigcup_j (\{ v, \ldots, n \} \cap I_j)$$

and by additivity of the spectrum we have that

$$\sum_{k \in \{v, \ldots, n\} \cap I_j} p_k \in S \quad \text{for all } j \text{ implies } p_v + \ldots + p_n \in S .$$

Using lemma (*a*), p. 417 in [*1*] and the hypothesis $T \in \mathcal{L}_S{}^{\perp}$ and $f \in \mathcal{L}_S$, we conclude that (5.1) holds, so $\exp_{|s} T \in \mathcal{L}_S{}^{\perp}$.

5.2. PROPOSITION. — *If* S *is Lorentz invariant and additive, then* $\mathcal{L}_S$ contains no right ideal except $\{ 0 \}$.

Proof. — The essential property of S which follows from the hypotheses is that for any $p \in \mathbb{R}^d$ there is a $p' \in S$ such that $p + p' \in S$. We want to show that for every $\underline{f} \neq 0$, there is a $\underline{g} \in \underline{\mathcal{S}}$ such that $\underline{f} \otimes \underline{g} \notin \mathcal{L}_S$. The elements of $\mathcal{L}_S$ have all vanishing zero component, so we may assume that $f_0 = 0$. Suppose that $f_n(\overline{p}_1, \ldots, \overline{p}_n) \neq 0$ for some $n \geq 1$ and some $\overline{p}_1, \ldots, \overline{p}_n \in \mathbb{R}^d$. Choose $\overline{p}'_i \in S$ such that $\overline{p}_i + \overline{p}'_i \in S$ for $i = 1, \ldots, n$, and pick a $g_n \in \mathcal{S}_n$ such that $g_n(\overline{p}'_n, \ldots, \overline{p}'_1) \neq 0$, and hence

$$(f_n \otimes g_n)(\overline{p}_1, \ldots, \overline{p}_n, \overline{p}'_n, \ldots, \overline{p}'_1) \neq 0 .$$

From the additivity of the spectrum we have $\overline{p}'_k + \ldots + \overline{p}'_1 \in S$, and also $\overline{p}_k + \ldots + \overline{p}_n + \overline{p}'_n + \ldots + \overline{p}'_1 \in S$ for $k = 1, \ldots, n$. Hence $\underline{f} \otimes \underline{g} \notin \mathcal{L}_S$ if $\underline{g} := (0, \ldots, 0, g_n, 0, \ldots)$.

In the following we denote the proper, orthochronous Lorentz group by $\mathcal{P}_+^{\uparrow}$. Space- and time inversion can be included in an obvious way, and all results of this section hold also for the full Poincaré group $\mathcal{P}$ replacing $\mathcal{P}_+^{\uparrow}$.

5.3. PROPOSITION. — *Let* S *be Lorentz-invariant and* $S \subset \{0\} \cup \overline{V}_\varepsilon^+$ *for some* $\varepsilon > 0$.

Then

$$\overline{\mathscr{L}_S + \mathscr{L}_S^* + \mathscr{K}_{\mathscr{P}_+^\uparrow}} = \mathscr{L}_S + \mathscr{K}_{\mathbb{R}^d} + \mathscr{K}_{L_+^\uparrow},$$

and $f \in \mathscr{L}_S + \mathscr{K}_{\mathbb{R}^d} + \mathscr{K}_{L_+^\uparrow}$ *iff the conditions*

a) $$f_0 = 0$$

b) $$f_n(0, \ldots, 0) = 0$$

c) $$\int_{L_+^\uparrow} f_n(\Lambda p_1, \ldots, \Lambda p_n) d\Lambda = 0$$

hold whenever $p_1 + \ldots + p_n = 0$, $p_\nu + \ldots + p_n \in S$ *for* $\nu = 2, \ldots, n$, *and at least one* $p_\nu \neq 0$.

Proof. — Since $\mathscr{P}_+^\uparrow$ contains the translation group $\mathbb{R}^d$ we have

$$\mathscr{L}_S + \mathscr{L}_S^* + \mathscr{K}_{\mathscr{P}_+^\uparrow} = \mathscr{L}_S + \mathscr{K}_{\mathscr{P}_+^\uparrow}$$

by prop. 2.6 *iii)* in [*1*]. Every $f \in \mathscr{L}_S + \mathscr{K}_{\mathscr{P}_+^\uparrow}$ satisfies *a)*, and since the functionals $\delta(p_1) \ldots \delta(p_n)$ and $\int \delta(p_1 - \Lambda\bar{p}_1) \ldots \delta(p_n - \Lambda\bar{p}_n) d\Lambda$ belong to $\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp$ if $\bar{p}_1 + \ldots + \bar{p}_n = 0$, $\bar{p}_\nu + \ldots + \bar{p}_n \in S$, (not all $\bar{p}_\nu = 0$), it is also clear that *b)* and *c)* hold for all $f \in \mathscr{L}_S + \mathscr{K}_{\mathscr{P}_+^\uparrow}$.

We now want to show that every f with *a)-c)* lies in $\mathscr{L}_S + \mathscr{K}_{\mathbb{R}^d} + \mathscr{K}_{L_+^\uparrow}$. It is convenient to introduce the coordinates $q_\nu = p_{n+1-\nu} + \ldots + p_n$, $\nu = 1, \ldots, n$, and write $f_n \equiv f$ as a function of the q_ν. By means of a C^∞-function χ with $\chi(m) = 1$ for $m \geq \varepsilon$, $\chi(m) = 0$ for $m \leq \varepsilon/2$, we split f into terms: $f = f_{(0)} + \ldots + f_{(n)}$ with $f_{(0)}(0, q_2, \ldots, q_n) = 0$, supp $f_{(1)} \cap (S \times \ldots \times S) = \{0\}$ and supp $f_{(\nu)} \cap (S \times \ldots \times S) \subset \{q_1 = 0, q_\nu \in \overline{V}^+\}$ for $\nu = 2, \ldots, n$. Then $f_{(0)} \in \mathscr{K}_{\mathbb{R}^d}$ by prop. 2.1 in [*1*], and $f_{(1)} \in \mathscr{L}_S$ because $f(0, \ldots, 0) = 0$. For $\nu = 2, \ldots, n$ we write

$$f_{(\nu)}(q_1, \ldots, q_n) = h(q_\nu) f_{(\nu)}(q_1, \ldots, q_n) + g_{(\nu)}(q_1, \ldots, q_n),$$

where h is a function with properties as in lemma 4.3. Since

$$\text{supp } f_{(\nu)} \cap (S \times \ldots \times S) \subset \{q_1 = 0, q_\nu \in \overline{V}_\varepsilon^+\},$$

and

$$\int f(0, \Lambda q_2, \ldots, \Lambda q_n) d\Lambda = 0 \qquad \text{on} \qquad S \times \ldots \times S,$$

the first term belongs to $\mathscr{L}_S$. Finally, if $T \in \mathscr{K}_{L_+^\uparrow}^\perp$ we have

$$T(g_{(\nu)}) = T(g_{(\nu)}\bar{h}(q_\nu)) = T(\bar{g}_{(\nu)}h) = 0,$$

where the bar denotes integration over $L_+^\uparrow$. Hence $g_{(\nu)} \in \mathscr{K}_{L_+^\uparrow}$, and the proof is complete.

5.4. THEOREM. — *Suppose* S *is Lorentz-invariant, additive, and* $S \subset \{0\} \cup \overline{V}_\varepsilon^+$ *for some* $\varepsilon > 0$.

Then

i) $$K(\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{L}_S + \mathscr{K}_{\mathbb{R}^d} + \mathscr{K}_{L_+^\uparrow},$$

ii) $$L(\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{L}_S,$$

iii) $$I(\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \{0\}.$$

5.5. REMARK. — Since $\mathscr{L}_{\overline{V}^+} = \bigcap_{\varepsilon>0} \mathscr{L}_{S_\varepsilon}$ with $S_\varepsilon = \{0\} \cup \overline{V}_\varepsilon^+$, it follows that *ii)* and *iii)* hold also for $S = \overline{V}^+$.

The proof of 5.4 is similar to that of theorem 3.3. It makes use of a partial characterization of the linear span of $\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp \cap \underline{\mathscr{S}}^{+\prime}$.

The functionals in this space must in any case satisfy an estimate

$$|T(\underline{f}^* \otimes \underline{g})| \leq \| f \| \, \| g \| \tag{5.2}$$

with a $\mathscr{P}_+^\uparrow$-invariant Hilbert-seminorm $\| \, . \, \|$ vanishing on $\mathscr{L}_S$. In the case of the translation group, such a condition is also sufficient to characterize the linear span of the invariant states. It is not known whether this is true for the Poincaré group. Instead we shall have to make use of a somewhat elaborate sufficient condition which means essentially that in addition to (5.2), the matrix elements $T(f \otimes \alpha_\Lambda \underline{g})$ decrease rapidly in Λ for all $\underline{f}$, $\underline{g}$ with $f_0 = g_0 = 0$.

We shall use the following notation [1]: If $p_1, \ldots, p_n \in \mathbb{R}^d$, ν, $\mu \in \mathbb{N}$, $\nu + \mu = n$, and $f \in S_\nu$, $g \in S_\mu$, we write

$$q'_i = p_1 + \ldots + p_i, \qquad i = 1, \ldots, \nu$$

$$q_j = p_{n+1-j} + \ldots + p_n, \qquad j = 1, \ldots, \mu$$

$$f_{\nu 0}(q'_1, \ldots, q'_\nu) = f(p_1, \ldots, p_\nu)$$

$$g_{0\mu}(q_\mu, \ldots, q_1) = g(p_{\nu+1}, \ldots, p_n)$$

5.6. PROPOSITION. — *Suppose* S *is Lorentz invariant, additive,* $S \subset \{0\} \cup \overline{V}_\varepsilon^+$ *for some* $\varepsilon > 0$, *and* S *contains at most a finite numer of isolated mass shells.*

The following condition is sufficient for a functional $T \in \underline{\mathscr{S}}'$ *to belong to the linear span of* $\mathscr{L}_S^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}^\perp \cap \underline{\mathscr{S}}^{+\prime}$:

T_n *has support in* $\{ -p_1 \in \overline{V}_\varepsilon^+, p_n \in \overline{V}_\varepsilon^+ \}$ *for all* n *and there are constants* k_n

and rapidly decreasing continuous functions φ_n *on* $\mathbb{R}_+$ *such that the following holds for all* $f \in \mathcal{S}_\nu$, $g \in \mathcal{S}_\mu$, $\nu + \mu = n$:

$$|T_n(f \otimes g)| \leq \sup_{\substack{-q_1', q_1 \in V_\varepsilon^+ \\ -q_2', \ldots, q_2 \in S}} \max_{\substack{|\alpha_\nu| \leq k_\nu \\ |\alpha_\mu| \leq k_\mu}} \varphi_n(-q'_1 \cdot q_1/(m_{q_1}' \cdot m_{q_1})) \times \prod_{i=1}^{\nu} (1 + |q_i'|)^{k_\nu} \int_{i=1}^{\nu-1} D_i^{\alpha_\nu} f_{\nu 0}(q_1', \ldots, q_\nu') \times \prod_{j=1}^{\mu} (1 + |q_j|)^{k_\mu} \prod_{j=1}^{\mu-1} D_j^{\alpha_\mu} g_{0\mu}(q_\mu, \ldots, q_1).$$

Proof. — By lemma 4.4, the condition implies that there is a function h, such that the functional T', defined by

$$T_n'(p_1, \ldots, p_n) = \overline{(h^{\otimes 2}(-p_1, p_n))}^{-1} T_n(p_1, \ldots, p_n)$$

satisfies the criterium of theorem 5.1 in [1]. (The finite number of isolated mass shells together with $L_+^\uparrow$ invariance guarantees that S is regular in the sense of [1]). T' can thus be decomposed into $\mathbb{R}^d$-invariant, positive functionals in $\mathcal{L}_S^\perp$. If R is such a functional, it follows from lemma 4.2 that

$$\overline{R}_h(p_1, \ldots, p_n) = \int R(\Lambda p_1, \ldots, \Lambda p_n) h(-p_1) h(p_n) d\Lambda$$

is well defined, conditionally positive and $\mathcal{P}_+^\uparrow$-invariant. Since T is $\mathcal{P}_+^\uparrow$-invariant, it follows in the same way as in the proof of thm. 3.1 that T is a linear combination of conditionally positive functionals in $\mathcal{L}_S^\perp \cap \mathcal{K}_{\mathcal{P}_+^\uparrow}$. The assertion thus follows from theorem 2.2.

Proportion 5.6 is not quite sufficient for proving theorem 5.4 because of the requirement that supp T_n should be contained in $\{ -p_1 \in V_\varepsilon^+, p_n \in V_\varepsilon^+ \}$.

The next proposition takes care of this.

5.7. PROPOSITION. — *Suppose* T *belongs to the linear span of*

$$\mathcal{L}_S^\perp \cap \mathcal{K}_{\mathcal{P}_+^\uparrow}^\perp \cap \underline{\mathcal{S}}^{+\prime}.$$

Then this is also true for the functional T' *with*

$$T_n'(p_1, \ldots, p_n) = \delta(p_1) \ldots \delta(p_{i_n}) T_{n-i_n-j_n}(p_{i_n+1}, \ldots, p_{n-j_n}) \times \delta(p_{n-j_n+1}) \ldots \delta(p_n)$$

for any choice of $i_n, j_n \in \mathbb{N} \cup \{0\}$, $i_n + j_n \leq n$.

Proof. — Since T can be decomposed into functionals in $\mathcal{L}_S^\perp \cap \mathcal{K}_{\mathcal{P}_+^\uparrow}^\perp \cap \underline{\mathcal{S}}^{+\prime}$, we have

$$|T(\underline{f}^* \otimes \underline{g})|^2 \leq R(\underline{f}^* \otimes \underline{f}) \cdot R(\underline{g}^* \otimes \underline{g})$$

for some $R \in \mathcal{L}_S^\perp \cap \mathcal{K}_{\mathcal{P}_+^\uparrow}^\perp \cap \underline{\mathcal{S}}^{+\prime}$.

Consider now for fixed n the case $i_n = 1$, $j_n = 0$. If $f_\nu \in \mathscr{S}_\nu$, $g_\mu \in \mathscr{S}_\mu$, $\nu + \mu = n$, $\nu \geq 1$, we have

$$|\mathrm{T}_n'(f_\nu^* \otimes \mathrm{S}_\mu)| \leq (\delta \otimes \mathrm{R}_{2\nu-1} \otimes \delta)(f_\nu^* \otimes f_\nu)\cdot \mathrm{R}_{2\mu-1}(g_\mu^* \otimes g_\mu).$$

It follows by prop. 2.6 that the hermitean functionals

$$\left(0, \ldots, 0, \frac{1}{2}(\mathrm{T}_n' + \mathrm{T}_n'^*), 0, \ldots\right)$$

and

$$\left(0, \ldots, 0, \frac{1}{2i}(\mathrm{T}_n' - \mathrm{T}_n'^*), 0, \ldots\right)$$

are dominated on $\underline{\mathscr{S}}^+$ by the positive functional

$$\mathrm{R} + (\exp_{|s}(\delta \otimes (\mathrm{R} - 1) \otimes \delta)_{\{\beta_n\}})_{\{\alpha_n\}}$$

for some sequences $\{ \alpha_n \}$ and $\{ \beta_n \}$ of positive type, so these functionals are decomposable. The same argument holds for $i_n = 0$, $j_n = 1$. By induction over i_n and j_n we conclude, using propositions 2.4 *ii*) and 2.6 repeatedly, that the functional $\mathrm{T}_{(n)}' = (0, \ldots, 0, \mathrm{T}_n', 0, \ldots)$ is decomposable for each n. Moreover, since the m-point functions occurring in these decompositions are constructed from T_m', δ and R_k with $k \leq m$, it is clear that these functionals are all continuous w. r. t. an n-independent seminorm on $\underline{\mathscr{S}}$. By prop. 2.5 it follows that T′ is decomposable.

Proof of theorem 5.4. — *i*) Suppose $\underline{f} \notin \mathscr{L}_\mathrm{S} + \mathscr{K}_{\mathbb{R}^d} + \mathscr{K}_{\mathrm{L}_+^\uparrow}$. By prop. 5.3 there are three possibilities:

a) $f_0 \neq 0$, in which case $\mathrm{T}(\underline{f}) \neq 0$ with $\mathrm{T} = (1, 0, \ldots) \in \mathscr{L}_\mathrm{S}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}$;

b) $f_n(0, \ldots, 0) \neq 0$ for some n, i. e. $\mathrm{T}(\underline{f}) \neq 0$ with $\mathrm{T} = (0, \ldots, 0, \delta^{\otimes n}, 0 \ldots)$. This T belongs to lin. span $\mathscr{L}_\mathrm{S}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}$ by prop. 5.7;

c)

$$\int_{\mathrm{L}_+^\uparrow} f(\Lambda\bar{p}_1, \ldots, \Lambda\bar{p}_n) d\Lambda \neq 0$$

for some n and $\bar{p}_1, \ldots, \bar{p}_n \in \mathbb{R}^d$, $\bar{p}_1 + \ldots + \bar{p}_n = 0$, $\bar{q}_\nu = \bar{p}_\nu + \ldots + \bar{p}_n \in \mathrm{S}$, not all $\bar{q}_\nu = 0$. Then $\mathrm{T}(\underline{f}) \neq 0$ for $\mathrm{T} = (0, \ldots, 0, \mathrm{T}_n, 0, \ldots)$ with

$$\mathrm{T}_n(p_1, \ldots, p_n) = \delta(p_1 + \ldots + p_n)\delta(p_1) \ldots \delta(p_k) \times \int_{\mathrm{L}_+^\uparrow} \delta(p_{k+1} - \Lambda\bar{p}_{k+1}) \ldots \delta(p_{n-j} - \Lambda\bar{p}_{n-j}) d\Lambda \times \delta(p_{n-j+1}) \ldots \delta(p_n),$$

and $\bar{p}_{n-j} \neq 0$ for some j. This functional is in the linear span of

$$\mathscr{L}_\mathrm{S}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}$$

by prop. 5.6 and prop. 5.7. (The condition of prop. 5.6 that S has only a finite number of isolated mass shells is insignificant here: We can consider instead of S the smallest, $L_+^\uparrow$-invariant and additive subset of S containing all the $\bar{q}_\nu$.) Altogether we thus have

$$\underline{f} \notin K(\mathscr{L}_S{}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime})$$

so

$$K(\mathscr{L}_S{}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) \subset \mathscr{L}_S + \mathscr{K}_{L_+^\uparrow} + K_{\mathbb{R}^d}.$$

The other inclusion is trivial.

ii) Consider the functionals

$$T_{(2n)} = (0, \ldots, 0, T_{2n}, 0, \ldots)$$

with either

$$T_{2n} = \delta^{\otimes 2n}$$

or

$$T_{2n}(q'_1, \ldots, q'_n, q_n, \ldots, q_1) = \prod_{i=1}^{k} \delta(q'_i + q_i) \prod_{j=1}^{k} \delta(q_j) \int \prod_{j=k+1}^{n} \delta(q_j - \Lambda\bar{q}_j) d\Lambda$$

with $\bar{q}_{k+1}, \ldots, \bar{q}_n \in S$, $\bar{q}_{k+1} \neq 0$.

These functionals belong to the linear span of $\mathscr{L}_S{}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}$ by prop. 5.7. Hence $T_{(2n)}(\underline{f}^* \otimes \underline{f}) = 0$ for all $\underline{f} \in L(\mathscr{L}_S{}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime})$. This implies that the highest component f_n of $\underline{f}$ belongs to $\mathscr{L}_S$. Now prop. 2.5 implies that $L(\mathscr{L}_S{}^\perp \cap \mathscr{K}_{\mathscr{P}_+^\uparrow}{}^\perp \cap \underline{\mathscr{S}}^{+\prime})$ is graded, so we can repeat the argument and finally obtain $\underline{f} \in \mathscr{L}_S$.

iii) Is an immediate consequence of *ii*) and prop. 5.2.

APPENDIX

In this appendix we discuss some examples and counterexamples, that illustrate the preceding results in a more general setting.

We use the notation explained in the introduction and consider first the case that $\mathscr{A}$ is a C*-algebra. Here a strong version of (1.1) is valid:

A.1. Proposition. — *Let $\mathscr{A}$ be a C*-algebra, G a group acting an $\mathscr{A}$ by automorphisms and $\mathscr{L}$ a left ideal in $\mathscr{A}$. Then*

$$\text{lin. span. } \mathscr{K}_G{}^{\perp} \cap \mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime} = \mathscr{K}_G \cap \mathscr{L}^{\perp} \cap \mathscr{L}^{*\perp}$$

and hence (by duality)

$$K(\mathscr{K}_G{}^{\perp} \cap \mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}) = \overline{\mathscr{K}_G + \mathscr{L} + \mathscr{L}^*}.$$

Proof. — If T is G-invariant and annihilates $\mathscr{L}$ and $\mathscr{L}^*$, the same holds for its hermitian and antihermitian part. We may thus suppose that T is hermitian. The Jordan decomposition

$$T = T_1 - T_2$$

is uniquely determined by the conditions

$$T_i \in \mathscr{A}^{+\prime}, \quad i = 1, 2 \qquad \text{and} \qquad \| T \| = \| T_1 \| + \| T_2 \| \tag{*}$$

cf. e. g. [*13*], p. 120 and 140. Since automorphisms preserve norms, the functionals T_i are therefore G-invariant. To show that $T_i \in \mathscr{L}^{\perp}$, we pass to the universal enveloping W*-algebra $\mathscr{A}''$. The weak closure $\mathscr{L}^{00}$ of $\mathscr{L}$ in $\mathscr{A}''$ is generated by a hermitian projector e. T has a natural extension to $\mathscr{A}''$ by continuity, and since T annihilates $\mathscr{L}$ we have

$$T(a) = T((1 - e)\cdot a \cdot (1 - e))$$

for all $a \in \mathscr{A}$. Considering

$$\hat{T}_i(a) := T_i((1 - e)\cdot a \cdot (1 - e)) \tag{**}$$

we have $\hat{T}_i \in \mathscr{A}^{+\prime}$, $\| \hat{T}_i \| \leq \| T_i \|$, and by (*) and (**)

$$\| T_1 \| + \| T_2 \| = \| T \| \leq \| \hat{T}_1 \| + \| \hat{T}_2 \| \leq \| T_1 \| + \| T_2 \|$$

so $\| T \| = \| \hat{T}_1 \| + \| \hat{T}_2 \|$. By uniqueness of the Jordan decomposition we thus have $T_i = \hat{T}_i \in \mathscr{L}^{\perp}$.

As next we consider left kernels. If G is a group of automorphisms, we denote by $\mathscr{L}_G$ the largest left ideal contained in $\mathscr{K}_G$.

A.2. Proposition. — *Let $\mathscr{A}$ be a C*-algebra and $\mathscr{L}$ a closed left ideal in $\mathscr{A}$. Then $L(\mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}) = \mathscr{L}$. If $\mathscr{A}$ is a W*-algebra, G a group of automorphisms of $\mathscr{A}$ and $\mathscr{L}$ is G-invariant, then*

$$L(\mathscr{L}^{\perp} \cap \mathscr{K}_G{}^{\perp} \cap \mathscr{A}^{+\prime}) = \mathscr{L} + \mathscr{L}_G.$$

Proof. — The first part is standard: Let e be the projector in $\mathscr{A}''$ generating the weak closure, $\mathscr{L}^{00}$, of $\mathscr{L}$. Then $T'(a) := T((1 - e)\cdot a \cdot (1 - e))$ belongs to $\mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}$ for all $T \in \mathscr{A}^{+\prime}$, and

$$\bigcap_{T \in \mathscr{A}^{+\prime}} L(T) = \{ a \mid (1 - e)\cdot a \cdot (1 - e) = 0 \} = \mathscr{L}^{00} \cap \mathscr{A} = \mathscr{L}.$$

For the second part one notes that the generating projector e for $\mathscr{L}$ is G-invariant, if $\mathscr{L}$

is G-invariant. Hence T' is G-invariant, if T is. If $a \in L(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime})$ we thus have, using prop. A.1,

$$a(1 - e) \in \mathscr{L}_G, \quad \text{so} \quad a = ae + a(1 - e) \in \mathscr{L} + \mathscr{L}_G.$$

A.3. REMARKS. — If $\mathscr{A}$ is a C*-algebra, but not necessarily a W*-algebra, the same argument as in the proof above shows that

$$L(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = (\mathscr{L}^{00} + \tilde{\mathscr{L}}_G) \cap \mathscr{A}$$

where $\tilde{\mathscr{L}}_G$ is the intersection of the left kernels *in* $\mathscr{A}''$ of G-invariant functionals $T \in \mathscr{A}^{+\prime}$. If $\tilde{\mathscr{L}}_G = \mathscr{L}_G{}^{00}$, it follows from this that

$$L(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = \mathscr{L} + \mathscr{L}_G.$$

It is, however, not clear if this holds in general for a C*-algebra (except if $\mathscr{L} = \{ 0 \}$, when it follows by A.1 and the definition of $\mathscr{L}_G$).

ii) If $\mathscr{A}$ is the C*-algebra of all bounded, continuous functions on G, and α is the natural action of G on $\mathscr{A}$, then $\mathscr{K}_G^{\perp} \neq \{ 0 \}$ iff G is an amenable group. If this is not the case, one has therefore

$$\mathscr{K}_G = \mathscr{L}_G = \mathscr{A}$$

iii) Another example with $\mathscr{L}_G \neq \{ 0 \}$, but G amenable, is the following: $\mathscr{A}$ is the C*-algebra of bounded, continuous functions f on $\mathbb{R}$, such that $\lim_{x \to \infty} f(x)$ exists. (G, α) is the natural action of $\mathbb{R}$ on $\mathscr{A}$. Here $\mathscr{K}_G^{\perp}$ is one-dimensional and spanned by the functional $f \mapsto \lim_{x \to \infty} f(x)$. In this example

$$\mathscr{A} \neq \mathscr{K}_G = \mathscr{L}_G \neq \{ 0 \}.$$

For *compact* groups G, but arbitrary $\mathscr{A}$ one has

A.4. PROPOSITION. — *Let $\mathscr{A}$ be a locally convex, topological *-algebra, and suppose the states on $\mathscr{A}$ separate points. Let G be a compact group of automorphisms of $\mathscr{A}$. Then*

i)
$$K(\mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = \mathscr{K}_G$$
and
$$L(\mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = \{ 0 \}.$$

ii) *If $\mathscr{L}$ is a G-invariant left ideal, and $\mathscr{A}$ is complete, then*

$$K(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = \mathscr{K}_G + K(\mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime})$$
and
$$L(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}) = L(\mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}).$$

Proof. — Since *i*) follows from *ii*) with $\mathscr{L} = \{ 0 \}$, we consider the latter statement. Suppose $a \in K(\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime})$ and write

$$a = b + c \quad \text{with} \quad b = \int_G \alpha_\tau a d\tau,$$

where $d\tau$ denotes the normalized Haar measure on G. Since $\mathscr{A}$ is complete, $b \in \mathscr{A}$. Moreover, if $T \in \mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}$, then the functional

$$T' := \int_G T \circ \alpha_\tau d\tau$$

is in $\mathscr{L}^{\perp} \cap \mathscr{K}_G^{\perp} \cap \mathscr{A}^{+\prime}$, so

$$0 = T'(a) = T(b),$$

and thus

$$b \in K(\mathscr{L}^{\perp} \cap \mathscr{A}^{+\prime}).$$

But

$$c = a - b \in K(\mathscr{K}_G^{\perp}) = \mathscr{K}_G,$$

so

$$a \in \mathscr{K}_G + K(\mathscr{L}^\perp \cap \mathscr{A}^{+\prime}).$$

Furthermore, by G-invariance of $\mathscr{L}$, we have

$$T' \in \mathscr{L}^\perp \cap \mathscr{K}_G^\perp \cap \mathscr{A}^{+\prime} \qquad \text{for all} \qquad T \in \mathscr{L}^\perp \cap \mathscr{A}^{+\prime}.$$

Hence, if $d \in L(\mathscr{L}^\perp \cap \mathscr{K}_G^\perp \cap \mathscr{A}^{+\prime})$ then

$$T'(d^*d) = \int T((\alpha_\tau d)^*(\alpha_\tau d))d\tau = 0.$$

Since $\tau \mapsto T((\alpha_\tau d)^*(\alpha_\tau d)) \geq 0$ is continuous, it follows that $T(d^*d) = 0$, and thus $d \in L(\mathscr{L}^\perp \cap \mathscr{A}^{+\prime})$.

We have now given an account of the « good cases » ($\mathscr{A}$ = C*-algebra, and/or G compact), where some variant of (1.1) and (1.2) holds quite generally. In the remainder of the appendix we collect examples, which demonstrate some of the « bad cases » that may occur in Borchers's algebra $\underline{\mathscr{S}}$.

In the first example, we show that $\mathscr{K}_G^\perp$ may be zero, even for a single automorphism (i. e. for the amenable group $G \cong \mathbb{Z}$).

A.5. **Example.** — Let h be a hermitian functional in $\mathscr{S}_1'$, $h \neq 0$. Define an automorphism $\alpha: \underline{\mathscr{S}} \mapsto \underline{\mathscr{S}}$ by

$$\alpha(\mathbb{1}) = \mathbb{1}$$
$$\alpha(f) = f + h(f)\cdot\mathbb{1}, \qquad f \in \mathscr{S}_1$$

and canonical extension to the tensor algebra $\underline{\mathscr{S}}$. Suppose $T(-T_0, T_1, \dots) \in \underline{\mathscr{S}}'$ is invariant under α. Then $T_1(f) = T_1(f) + T_0\cdot h(f)$ for all f and thus $T_0 = 0$. In the same way $T_2(f_1 \otimes f_2) = T_2(f_1 \otimes f_2) + T_1(f_1)\cdot h(f_2) + T_1(f_2)\cdot h(f_1)$, so $T_1 = 0$, and by induction we get $T_n = 0$ for all n.

If α is a graded automorphism, i. e. $\alpha\mathscr{S}_n = \mathscr{S}_n$ for all n, then at least the trivial functionals $(T_0, 0, 0, \dots)$ are invariant. The next two examples show: 1. that there may be no others, and 2. that $K(\mathscr{K}_G^\perp \cap \underline{\mathscr{S}}^{+\prime}) \neq \mathscr{K}_G$ is possible, even for graded automorphisms.

A.6. **Example.** — Define $(\underline{\alpha f})_n = \lambda^n f_n$ with $\lambda \in \mathbb{R}$, $\lambda \neq 0$ or 1. Then $T_n = 0$ if $n \geq 1$, for any invariant $T \in \underline{\mathscr{S}}'$.

A.7. **Example.** — Let $\{ e_i \}_{i=1,2,\dots}$ be a Schauder basis for $\mathscr{S}_1$ (e. g. the Hermite functions). Suppose $\lambda \neq 0$ or 1 and define a graded automorphism α on $\underline{\mathscr{S}}$ by

$$\alpha e_j = \begin{cases} \lambda e_j & \text{if } j \text{ is even} \\ \lambda^{-1} e_j & \text{if } j \text{ is odd}. \end{cases}$$

Clearly, there are no invariant, continuous seminorms on $\mathscr{S}_1$ except 0. Hence there are no nontrivial, invariant states T on $\underline{\mathscr{S}}$, for otherwise $f \mapsto T_2(f \otimes f)^{1/2}$ would be such a seminorm. On the other hand, there are many invariant functionals in $\underline{\mathscr{S}}'$. In fact, T is invariant iff $T_n(e_{i_1} \otimes \dots \otimes e_{i_n}) = 0$ for all $(i_1, \dots, i_n)$ such that the number of odd indices is not the same as the number of even indices. This example shows also that the topology defined by invariant Hilbert seminorms on $\underline{\mathscr{S}}$ need not be the same as the topology defined by invariant states, in contrast to the case of the translation group (cf. [*1*]).

In examples A.5-A.7 it is the group action on $\underline{\mathscr{S}}$ that does not merge with positivity. As next we consider examples of « bad » ideals in $\underline{\mathscr{S}}$.

It is clear that the condition $K(\mathscr{I}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{I}$ does not hold for all two sided, *-invariant ideals in $\underline{\mathscr{S}}$.

Counterexamples are e. g. the ideals

$$\mathscr{I} = \overset{\prime}{\underset{n=N}{\oplus}} \mathscr{S}_n$$

with $N \geq 2$. We now give an example of a left ideal $\mathscr{L}$ such that

$$K(\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{L} + \mathscr{L}^*$$

but

$$L(\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) \neq \mathscr{L} .$$

A.8. Example. — Let $\mathscr{V} \subset \mathscr{S}_1$ be a subspace, such that $\mathscr{V} \neq \mathscr{S}_1$, but $\mathscr{V} + \mathscr{V}^* = \mathscr{S}_1$. For example, one might take

$$\mathscr{V} = \{ f \mid \operatorname{supp} \tilde{f} \subset \{ (p_1, \ldots, p_d), \quad p_1 > -1 \} \},$$

where $\tilde{f}$ denotes the Fourier transform of f. Define

$$\mathscr{L} = \mathscr{V} \oplus (\overset{\infty}{\underset{n=2}{\oplus}} \mathscr{S}_n) .$$

Then $\mathscr{L} + \mathscr{L}^* = \overset{\infty}{\underset{n=1}{\oplus}} \mathscr{S}_n$, so $\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime} = \{ (T_0, 0, \ldots) \mid T_0 \geq 0 \}$ and

$$L(\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) = \overset{\infty}{\underset{n=1}{\oplus}} \mathscr{S}_n \neq \mathscr{L} .$$

It is not known if there exist left ideals $\mathscr{L}$ in $\underline{\mathscr{S}}$ with $L(\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{L}$ but $K(\mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}) \neq \mathscr{L} + \mathscr{L}^*$.

The next remark about ideals is that the sum of two well-behaved ideals need not be well-behaved. The following example is based on the fact that the energy momentum spectrum cannot be arbitrary in a local quantum field theory (a related example is due to Borchers [*9*]).

A.9. Example. — Let $\mathscr{I}$ be the locality ideal and $\mathscr{L} = \mathscr{L}_S$ a spectrum ideal (cf. [*1*]) with

$$S = \{ 0 \} \cup \{ p \mid p \cdot p = m^2, p^0 > 0 \}, m^2 > 0.$$

Both $\mathscr{I}$ and $\mathscr{L}$ are well-behaved w. r. t. positivity in the sense that (1.1) and (1.2) hold [*1*] [*2*]. Also, $\mathscr{I}^{\perp} \cap \mathscr{L}_S^{\perp}$ contains at least the functionals $(T_0, T_1, T_2, 0, 0, \ldots)$ with $T_1 = \text{const.}$ and T_2 proportional to the two-point function of the free field of mass m. Hence

$$\overline{\mathscr{I} + \mathscr{L}_S} \neq \overset{\infty}{\underset{n=1}{\oplus}} \mathscr{S}_n$$

and also

$$\overline{\mathscr{I} + \mathscr{L}_S + \mathscr{L}_S^*} \neq \overset{\infty}{\underset{n=1}{\oplus}} \mathscr{S}_n .$$

On the other hand, $\mathscr{I}^{\perp} \cap \mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$ consists only of the trivial functionals $(T_0, 0, 0, \ldots)$. The proof of this is a slight modification of the usual argument for the additivity of the spectrum (2) [*14*]. By prop. 2.6 *iii*) in [*1*] every $T \in \mathscr{I}^{\perp} \cap \mathscr{L}^{\perp} \cap \underline{\mathscr{S}}^{+\prime}$ is translationally invariant. Denote by $\Phi(f)$ the corresponding field operators, $U(a)$ the representation of the translation group, Ω the invariant, cyclic vector and E_0 the projector on the space of all invariant vectors. If $f \in \underline{\mathscr{S}}$ we have by locality

$$\lim_{\substack{a \to \infty \\ a \text{ space-like}}} \| \Phi(\underline{f}) U(a) \Phi(\underline{f}) \Omega \|^2 = \| E_0 \Phi(\underline{f}^* \otimes \underline{f}) \Omega \|^2 = \| \Phi(\underline{f}) \Omega \|^2 + \| E_0{}' \Phi(\underline{f}^* \otimes \underline{f}) \Omega \|$$

with $E_0{}'$ = projector on the invariant vectors orthogonal to Ω. From this we conclude,

(2) The modification is necessary, because one cannot in general appeal to the cluster property [*15*].

that if p belongs to the spectrum of U, then so does $2p$. Since S does not meet this requirement, except if $p = 0$, we conclude that

$$\Phi(\underline{f}) = 0 \quad \text{for} \quad \underline{f} \in \overset{\prime}{\underset{n=1}{\oplus}} \mathscr{S}_n .$$

Hence

$$\overset{\prime}{\underset{n=1}{\oplus}} \mathscr{S}_n = \mathrm{K}(\mathscr{I}^\perp \cap \mathscr{L}^\perp \cap \underline{\mathscr{S}}^{+\prime}) \neq \overline{\mathscr{I} + \mathscr{L} + \mathscr{L}^*}$$

and

$$\overset{\infty}{\underset{n=1}{\oplus}} \mathscr{S}_n = \mathrm{L}(\mathscr{I}^\perp \cap \mathscr{L}^\perp \cap \underline{\mathscr{S}}^{+\prime}) \neq \overline{\mathscr{I} + \mathscr{L}} .$$

The final examples concern the combination of invariance under automorphism groups and annihilation of ideals. We shall need the following lemma.

A.10. LEMMA. — *Let $\mathscr{I}$ be the two-sided ideal in $\underline{\mathscr{S}}$ generated by all anticommutators $f \otimes g + g \otimes f$ with $f, g \in \mathscr{S}_1$ having space-like separated supports. If $\mathrm{T} \in \underline{\mathscr{S}}^+ \cap \mathscr{I}^\perp$ is invariant under $\mathbb{R}^d$, then the n-point distributions T_n vanish for n odd.*

Proof. — Let $\Phi(f)$ denote the GNS-representation defined by T and let E_0 be the projector on the translationally invariant vectors in the GNS Hilbert-space. The operators $\mathrm{E}_0\Phi(f)\mathrm{E}_0$ are well defined and can be multiplied freely, because $\mathbb{R}^d$ is an amenable group (cf. [*16*]). Moreover, $\mathrm{E}_0\Phi(f_n)\mathrm{E}_0$ anticommutes with $\mathrm{E}_0\Phi(g_m)\mathrm{E}_0$, if $f_n \in \mathscr{S}_n$, $g_m \in \mathscr{S}_m$ and both n and m are odd. It follows that $\mathrm{E}_0\Phi(f_n)\mathrm{E}_0 = 0$ for n odd, so $\mathrm{T}_n = 0$ for n odd.

A.11. EXAMPLE. — Let $\mathscr{I}$ be the ideal generated by anticommutators as in lemma A.10 and $\mathrm{G} = \mathbb{R}^d$. We claim that

$$\mathrm{K}(\mathscr{I}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{I} ,$$

$$\mathrm{K}(\mathscr{K}_\mathrm{G}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{K}_\mathrm{G},$$

but

$$\mathrm{K}(\mathscr{I}^\perp \cap \mathscr{K}_\mathrm{G}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) \neq \mathscr{I} + \mathscr{K}_\mathrm{G}$$

In fact, the last statement follows from lemma A.10, because the functional

$$\mathrm{T}(\underline{f}) = \int f_1(x)dx$$

belongs to $\mathscr{I}^\perp \cap \mathscr{K}_\mathrm{G}{}^\perp$, and cannot be approximated by functionals with vanishing 1-component. The first equality can be proven in a similar way as the corresponding statement for the locality ideal [*2*], thm. 5.5. The only change from the proof in [*2*] is the occurrence of minus signs in formula 3.10 in [*2*], but with this modification the proof remains valid. (Note that $\mathscr{I}$ is not generated by *all* anticommutators; in particular $f^2 \notin \mathscr{I}$ unless $f = 0$.) Finally the equation $\mathrm{K}(\mathscr{K}_{\mathbb{R}^d}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{K}_{\mathbb{R}^d}$ was proven in [*1*], thm. 5.3.

A.12. EXAMPLE. — We give an example of a group G and a G-invariant left ideal $\mathscr{L}$, such that

$$\mathrm{L}(\mathscr{K}_\mathrm{G}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \{0\}$$

$$\mathrm{L}(\mathscr{L}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \mathscr{L}$$

but

$$\mathrm{L}(\mathscr{L}^\perp \cap \mathscr{K}_\mathrm{G}{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) \neq \mathscr{L} .$$

This example is based on the spin-statistics theorem. We take for G the orthochronous, proper Lorentz group $\mathrm{L}_+^\uparrow$ with its natural action on $\underline{\mathscr{S}}$. The left ideal $\mathscr{L}$ is defined as follows: Let Ψ_α be a massive, free spinor field, and define a hermitian field Φ by

$$\Phi(x) = \Psi_1(x) + \Psi_1(x)^* .$$

Put $\Phi_\Lambda(x) = \Phi(\Lambda x)$ for $\Lambda \in L_+^\uparrow$, and let T_Λ be the corresponding positive functional on $\underline{\mathscr{S}}$. Then we define

$$\mathscr{L} = \bigcap_\Lambda L(T_\Lambda).$$

$\mathscr{L}$ is G-invariant and satisfies, by definition, the condition $\mathscr{L} = L(\mathscr{L}^\perp \cap \underline{\mathscr{S}}^{+\prime})$. Moreover, $\mathscr{L}$ contains both the spectrum ideal $\mathscr{L}_S$, with S = spectrum for a massive, free field, and the two-sided ideal generated by anticommutators as in lemma A.10. Since $S \cap -S = \{0\}$, every hermitian functional annihilating $\mathscr{L}$ is translationally invariant. A positive functional in $\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp$ thus corresponds to a Poincaré covariant scalar field with the wrong commutation relations at space-like distances. Hence, by the spin-statistics theorem,

$$L(\mathscr{L}^\perp \cap \mathscr{K}_G{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \bigoplus_{n=1}^{\infty} \mathscr{S}_n \neq \mathscr{L}.$$

To show that $L(\mathscr{K}_G{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \{0\}$, we use the results of section 2. Repeated use of prop. 2.4 with

$$t_2(f \otimes g) = \int f(x)g(x)dx$$

shows that there is a Lorentz-invariant, positive functional T such that

$$T(\underline{f}^* \otimes \underline{f}) \geq \sum_n \int |f_n(x_1, \dots, x_n)|^2 dx_1 \dots dx_n,$$

i. e. $L(T) = \{0\}$, so

$$L(\mathscr{K}_G{}^\perp \cap \underline{\mathscr{S}}^{+\prime}) = \{0\}.$$

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