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A quantization for Kähler fields
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by

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ABSTRACT. — We examine a quantization of the Kähler equation in terms of inhomogeneous differential forms on a class of static space-times. Anticommutation relations between the quantized field operator and its conjugate are imposed and their relation to the classical propagator is demonstrated.

RÉSUMÉ. — On examine une quantification de l'équation de Kähler en termes de formes différentielles inhomogènes sur une classe d'espaces-temps statiques. On impose des relations d'anticommutation entre l'opérateur de champ quantique et son conjugué, et on démontre leur relation au propagateur classique.

1. INTRODUCTION

The dynamical properties of a field system described by an inhomogeneous differential form have recently been studied intensively. The field equation first written down by E. Kähler [1] may be regarded as a system of coupled equations for the components of all possible antisymmetric tensors on a manifold with a Pseudo Riemannian metric structure. In [3] the relation of this system to the Dirac equation in Minkowski space was

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clarified and a quantization effected using anticommutators for the fundamental quantum brackets.

In this article, the problem of establishing a consistent quantization in a more general space-time is dealt with. Working in a space-time with a parallel time-like Killing vector field ensures the existence of stationary state solutions which may be used to construct a basis of a multiparticle Fock Space. Although the field system consists of dynamically coupled complex p-forms we are led to a consistent quantization using anticommutators between the quantized field and its quantized conjugate field.

It is to be noted that the classical dynamical system admits a class of constrained solutions that also satisfy the Klein-Gordon or Proca field equations. These systems would be described as quantum field theories on a symmetric Fock space rather than an antisymmetric one. However, for the general case discussed in this article, the demand that the hypersurface Hamiltonian be bounded from below leads to basic anticommutation relations, and the unequal time anticommutator between the field and its conjugate field gives the propagator for a general inhomogeneous form solution of the massive Kähler equation.

In the following, we shall work on space-time a metric g whose signature is $(-, +, +, +)$. Forms will take values in the complex field.

The Clifford product associated with g between any two covectors α, β is denoted by $\vee$ and satisfies

$$\alpha \vee \beta + \beta \vee \alpha = 2g(\alpha, \beta).$$

It may be related to the exterior product $\wedge$ according to

$$\alpha \vee \beta = \alpha \wedge \beta + g(\alpha, \beta). \tag{1.1}$$

We denote by η the main involution of the Clifford algebra and by ξ the main anti-involution. For further details on Clifford analysis and its relation to the pseudo-Riemannian structure of space-time see [3].

2. THE KÄHLER EQUATION

The Kähler equation [1] [2] [3], can be written on a manifold M as

$$(\displaystyle{\not}d - m)\Phi = 0 \tag{2.1}$$

where Φ is an inhomogeneous differential form and $\displaystyle{\not}d\Phi = e^\mu \vee \nabla_{X_\mu} \Phi$ for any basis of vector fields $\{ X_\mu \}$ and its dual basis $\{ e^\mu \}$. ∇ is the pseudo-Riemannian connection of space-time. The equation (2.1) may be obtained from a variational principle with an action density 4-form

$$\mathscr{L} = S_0(\xi\eta\Phi^* \vee \displaystyle{\not}d\Phi - m\xi\eta\Phi^* \vee \Phi) * \mathbb{1} \tag{2.2}$$

In this paper, we restrict to space-times which are static with a parallel

time-like Killing vector, in which the metric tensor may be written in terms of a local coframe $\{\, e^\mu \,\}$ as

$$g = - e^0 \otimes e^0 + \sum_{a=1}^{3} e^a \otimes e^a \qquad (2.3)$$

with $e^0 = dt$. The local co-frame satisfies the properties

$$\nabla_X e^0 = 0 \qquad \forall X \in TM$$
$$\nabla_{\partial/\partial t} e^a = 0 \qquad a = 1, 2, 3 \qquad (2.4)$$

and the space-time has global topology $M = \mathbb{R} \times M_3$. M_3 is assumed to be a compact Riemannian manifold. The induced metric on M_3 will be denoted $\hat{g} = \sum_{a=1}^{3} e^a \otimes e^a$ (Indeed, if we write $g = - dt \otimes dt + \hat{g}$ for some Riemannian metric $\hat{g}$ on M_3 then we can always rewrite g in the above form with the properties (2.4). We use these properties to perform a 3 + 1 splitting of the Kähler equation, as follows.

Every general inhomogeneous differential form Φ can be written locally as

$$\Phi = \alpha + e^0 \wedge \beta \qquad (2.5)$$

with α, β satisfying $i_{\partial/\partial t}\alpha = i_{\partial/\partial t}\beta = 0$. Using the above properties, this becomes

$$\Phi = \alpha + e^0 \vee \beta . \qquad (2.6)$$

Writing $\quad \dslash = e^\mu \vee \nabla_{X_\mu} = e^0 \vee \nabla_{\partial/\partial t} + e^a \vee \nabla_{X_a} \qquad a = 1, 2, 3$

we express $\dslash\Phi$ as follows:

$$\dslash\Phi = e^0 \vee \nabla_{\partial/\partial t}\alpha + e^0 \vee \nabla_{\partial/\partial t}(e^0 \vee \beta) + e^a \vee \nabla_{X_a}\alpha + e^a \vee \nabla_{X_a}(e^0 \vee \beta)$$

$$= e^0 \vee \frac{\partial\alpha}{\partial t} + e^0 \vee e^0 \vee \nabla_{\partial/\partial t}\beta + \underline{\dslash}\alpha + e^a \vee e^0 \vee \nabla_{X_a}\beta$$

$$= e^0 \vee \frac{\partial\alpha}{\partial t} - \frac{\partial\beta}{\partial t} + \underline{\dslash}\alpha - e^0 \vee \underline{\dslash}\beta .$$

Where $\underline{\dslash} = e^a \vee \nabla_{X_a}$ $a = 1, 2, 3$ and $\dfrac{\partial\alpha}{\partial t}$ is written for $\nabla_{\frac{\partial}{\partial t}}\alpha$.

Substituting into equation (2.1) we find

$$e^0 \vee \left(\frac{\partial\alpha}{\partial t} - \underline{\dslash}\beta - m\beta \right) + \left(-\frac{\partial\beta}{\partial t} + \underline{\dslash}\alpha - m\alpha \right) = 0$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix} 0 & \underline{\dslash}+m \\ \underline{\dslash}-m & 0 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \qquad (2.7)$$

Consequently, both α and β satisfy the equation

$$\frac{\partial^2 \alpha}{\partial t^2} = (\underline{d}^2 - m^2)\alpha . \tag{2.8}$$

Writing $\underline{d}$ for the exterior derivative on M_3 for this decomposition and putting $\hat{*} d \hat{*} = \delta$ where $\hat{*}$ denotes the Hodge dual with respect to $\hat{g}$, we can write $\underline{d} = \underline{d} + \mu \delta$.

An important property of $\underline{d}$ is given by the following relation, a proof of which is given in the appendix:

$$\langle \underline{d}\alpha, \beta \rangle_{M_3} = - \langle \alpha, \underline{d}\beta \rangle_{M_3} \tag{2.9}$$

for all $\alpha, \beta \in \Lambda(M_3)$ which are C^∞. We have used the following notation

$$\langle \alpha, \beta \rangle_{M_3} := \int_{M_3} S_0(\xi\alpha \vee \beta) \, \hat{*} \, \mathbb{1} . \tag{2.10}$$

We use (2.9) to investigate the spectral properties of the operator

$$H = \begin{pmatrix} 0 & (\underline{d}+m) \\ (\underline{d}-m) & 0 \end{pmatrix}$$

which enters in (2.7).

3. SPECTRAL THEORY

DEFINITION 3.1. — i) $R = - \underline{d}^2 + m^2$.
ii) $H^0 = L^2(\Lambda(M_3))$ where

$$L^2(\Lambda(M_3)) = \{ \alpha \in \Lambda(M_3) : \| \alpha \|_0^2 = \langle \alpha^*, \alpha \rangle_{M_3} < \infty \} .$$

We denote the inner product in H^0 by $(\alpha; \beta)_0$ and we have the relation

$$(\alpha; \beta)_0 = \langle \alpha^*, \beta \rangle_{M_3} = \int_{M_3} S_0(\xi\alpha^* \vee \beta) \, \hat{*} \, \mathbb{1} . \tag{3.1}$$

Equation (2.9) still holds as $\underline{d}$ is a real differential operator.

   iii) $H^l = \{ \alpha \in H^0 : \| \alpha \|_l < \infty \}$

where

$$\| \alpha \|_l = \| R^{l/2}\alpha \|_0 \qquad \text{with} \qquad l \in \mathbb{N}.$$

The inner product in H^l is denoted by $(\alpha; \beta)_l$ and one has

$$(\alpha; \beta)_l = (R^{l/2}\alpha; R^{l/2}\beta)_0 .$$

H^l is a Hilbert space with this inner product.

Remark. — $R^{l/2}$ makes sense since $\underline{d}^2$ is a negative definite operator (this

follows from (2.9)) and since $\underline{d}^2$ is self-adjoint with respect to (2.11). Thus R is a positive definite self-adjoint operator, and as such one can define its square root [6].

PROPOSITION 1. — For $\alpha \in H^2$ we have

$$(\alpha; R\alpha)_0 \geqq m^2 \| \alpha \|_0^2$$

R is an invertible operator with bounded inverse.

Proof. — We have

$$(\alpha; R\alpha)_0 = -(\alpha; \underline{d}^2\alpha)_0 + m^2(\alpha; \alpha)_0 = (\underline{d}\alpha; \underline{d}\alpha)_0 + m^2 \| \alpha \|_0^2 \geqq m^2 \| \alpha \|_0^2$$

where we have used (2.9). Therefore R is, as claimed, bounded away from zero and injective. Hence an inverse exists, which we denote by R^{-1}. By the spectral theorem, $R^{-1/2}$ exists (the positive square root of R^{-1}). Now Ran R $\subset$ Dom $R^{-1/2}$ and $R^{-1/2}$ maps Ran R to Ran $R^{1/2}$ and both these are dense in H^0. Thus $R^{-1/2}$ is densely defined and so for $\alpha \in$ Ran R we have

$$\| R^{-1/2}\alpha \|_0 \leqq \frac{1}{m} \| \alpha \|_0 .$$

As $R^{-1/2}$ maps Ran R onto Ran $R^{1/2}$ we also have

$$\| R^{-1}\alpha \|_0 \leqq \frac{1}{m^2} \| \alpha \|_0 .$$

Thus R^{-1} is bounded on a dense set, which proves that it can be extended to a bounded operator in H^0. Thus R^{-1} is the bounded inverse of R.

PROPOSITION 2. — The operators $(\underline{d} \pm m)$ map H^l onto H^{l-1} for $l \geqq 1$. They have bounded inverses $-(\underline{d} \pm m)R^{-1}$.

Proof. — First we prove the boundedness below for $(\underline{d} + m)$, the case for $(\underline{d} - m)$ being almost the same. For α in the domain of $\underline{d}$

$$\| (\underline{d} + m)\alpha \|_0^2 = (\underline{d}\alpha; \underline{d}\alpha)_0 + m \{ (\alpha; \underline{d}\alpha)_0 + (\underline{d}\alpha; \alpha)_0 \} + m^2(\alpha; \alpha)_0$$
$$= \| \underline{d}\alpha \|_0^2 + m^2 \| \alpha \|_0^2 \geqq m^2 \| \alpha \|_0^2 .$$

Thus $(\underline{d} + m)$ is injective, and hence invertible. As R^{-1} is a bounded operator commuting with $\underline{d}$ then for $\alpha \in$ Dom $\underline{d}$ we have

$$- (\underline{d} - m)R^{-1}(\underline{d} + m)\alpha = -(\underline{d} - m)(\underline{d} + m)R^{-1}\alpha = R.R^{-1}\alpha = \alpha .$$

To prove boundedness of the inverse we note, for $\alpha \in$ Dom $\underline{d}$

$$\| (\underline{d}-m)R^{-1}\alpha \|_0^2 = ((\underline{d}-m)R^{-1}\alpha; (\underline{d}-m)R^{-1}\alpha)_0$$
$$= -(R^{-1}\alpha; (\underline{d}+m)(\underline{d}-m)R^{-1}\alpha)_0 = (R^{-1}\alpha; R.R^{-1}\alpha)_0$$
$$= (R^{-1}\alpha; \alpha)_0 \leqq \| \alpha \|_0 . \| R^{-1}\alpha \|_0 \leqq \frac{1}{m^2} \| \alpha \|_0^2$$

using the Schwartz inequality and Proposition 1. Thus $(\underline{d} - m)R^{-1}$ is bounded on Dom $\underline{d}$, which is dense in H^0 as Dom $\underline{d}$ contains all C^∞ differential forms on M_3, M_3 being compact. This means that $-(\underline{d} - m)R^{-1}$ is bounded on a dense subset of H^0 and so can be extended to a bounded operator on H^0. That proves the boundedness of the inverse of $\underline{d} + m$. The same goes for $\underline{d} - m$.

Let us note that if $\alpha \in H^{l+1}$, $R^{1/2}\alpha \in H^l$. This is easily seen from the equalities

$$\| \alpha \|_{l+1} = \| R^{\frac{l+1}{2}} \alpha \|_0 = \| R^{l/2} . R^{1/2}\alpha \|_0 = \| R^{1/2}\alpha \|_l .$$

PROPOSITION 3. — 1) The operators $A_\pm = (\pm \underline{d} + m)R^{-\frac{1}{2}}$ are bounded in H^l for $l \geqslant 0$

2) $A_+A_- = A_-A_+ = 1$.

3) $A_\pm{}^* = A_\mp$.

4) $(\underline{d} \pm m)$ maps H^{l+1} isometrically onto H^l for $l \geqslant 0$.

Proof.

1) If $\alpha \in H^l$ then $R^{-\frac{1}{2}}\alpha \in H^{l+1}$, and since $R^{\frac{1}{2}}\beta \in H^l$ for $\beta \in H^{l+1}$ it follows that $R^{-\frac{1}{2}}$ maps H^l onto H^{l+1}. Now Proposition 2 shows that $(\pm\underline{d}+m)$ maps H^{l+1} onto H^l for $l \geqslant 0$, so $A_\pm$ maps H^l onto H^l.

Therefore the graph of $A_\pm$ is $H^l \times H^l$, which is a closed space in $H^l \times H^l$ (trivially !). Hence, by the closed graph theorem (Theorem 3.2 of [6]) $A_\pm$ is bounded.

2) For any α we have

$$\begin{aligned}
A_+A_-\alpha &= (\underline{d} + m)R^{-\frac{1}{2}}(-\underline{d} + m)R^{-\frac{1}{2}}\alpha \\
&= (\underline{d} + m)(-\underline{d} + m)R^{-1}\alpha \\
&= (-\underline{d}^2 + m^2)R^{-1}\alpha \\
&= R.R^{-1}\alpha \\
&= \alpha .
\end{aligned}$$

The same argument works for $A_-A_+ = 1$.

3) Let $\alpha \in \mathrm{Dom}\,\overline{d}$ and $\beta \in H^0$. Then we have

$$\begin{aligned}
(A_+\alpha; \beta)_0 &= ((\underline{d} + m)R^{-\frac{1}{2}}\alpha; \beta)_0 \\
&= (R^{-\frac{1}{2}}(\underline{d} + m)\alpha; \beta)_0 \\
&= ((\underline{d} + m)\alpha; R^{-\frac{1}{2}}\beta)_0
\end{aligned}$$

as $R^{-\frac{1}{2}}$ is self adjoint and commutes with $\underline{d}$ on Dom $\underline{d}$. Hence we have

$$(A_+\alpha; \beta)_0 = (\alpha; A_-\beta)_0$$

and as Dom $\underline{d}$ is dense in H^0, it follows that $A_+{}^* = A_-$. The same proof gives $A_-{}^* = A_+$.

4) (1), (2) and (3) above show that A_+ is a unitary operator and, noting that if $\alpha \in H^l$, $l \geqslant 1$, then $(\underline{d} + m)\alpha = A_+ R^{\frac{1}{2}}\alpha$. We obtain

$$\| \underline{d} + m)\alpha \|_l = \| R^{\frac{1}{2}}\alpha \|_l = \| \alpha \|_{l+1}$$

using the unitary of A_+. Therefore we conclude that $(\underline{d} + m)$ maps H^{l+1} onto H^l isometrically.

We now turn to investigate the spectrum of H in the spaces $H^l \times H^l$, $l \in \mathbb{N}$. On $H^l \times H^l$ is defined the scalar product

$$\{ \Phi; \Psi)_{l,l} := (\alpha; \alpha')_l + (\beta; \beta')_l$$

where $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\Psi = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$. Defining addition and scalar multiplication in the usual way, we turn $H^l \times H^l$ into a Hilbert space, H^l being such a space.

PROPOSITION 4. — i) For $l \geqq 1$ H is a toplinear isomorphism of $H^l \times H^l$ onto $H^{l-1} \times H^{l-1}$

ii) $\{ H\Phi; \Psi \}_{l,l} = \{ \Phi; H\Psi \}_{l,l}$ for $l \geqq 1$
iii) iH^{-1} is a compact, self-adjoint operator in $H^l \times H^l$.

Proof. — i) This follows from Proposition 2 and 3. The inverse of H is a bounded operator. Indeed,

$$H^{-1} = \begin{pmatrix} 0 - (\underline{d} + m)R^{-1} \\ -- (\underline{d} - m)R^{-1}0 \end{pmatrix}$$

ii) follows directly using (2.9)
iii) H^{-1} is skew adjoint from ii) and so iH^{-1} is self adjoint. It maps $H^l \times H^l$ onto $H^{l+1} \times H^{l+1}$. As M_3 is a compact manifold the inclusion map inc: $H^{l+1} \times H^{l+1} \to H^l \times H^l$ is a compact map ([5], Theorem 2.6.3). Thus we have that $iH^{-1}: H^l \times H^l \to H^l \times H^l$ is the product of a bounded operator and a compact one. This ensures ([6], Theorem VI.12) that iH^{-1} is compact.

$\frac{1}{i}H$ is self-adjoint with domain $H^1 \times H^1$. Thus it has a complete set of eigenfunctions in $H^0 \times H^0$. The operator iH^{-1} has a discrete spectrum which is bounded, and each non-zero eigenvalue has finite multiplicity 0 being the only possible limit point ([6], Theorem VI.15). Furthermore, iH^{-1} has a complete orthonormal basis $\{ u_p \} \in H^0 \times H^0$, $p \in \mathbb{N}$

with $iH^{-1}u_p = \lambda_p u_p$ and $\lambda_p \to 0$ as $p \to \infty$. It then follows that $\frac{1}{i} H u_p = l_p u_p$ with $l_p \to \infty$ as $p \to \infty$, hence the spectrum of $\frac{1}{i} H$ is discrete and has no finite limit point. As H is a real operator and the eigenvalues l_p are real, it follows that u_p^*, the complex conjugate of u_p, corresponds to the eigenvalue $-l_p$. Therefore $\frac{1}{i} H$ has a symmetric spectrum. The u_p are all C^∞ on M_3 as they are in the kernel of an elliptic operator, $R^2 - l_p^2$ [5].

If we write $u_p = \begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix}$ then an easy calculation shows that:

$$\begin{align}
(\underline{d} + m)\beta_p &= il_p\alpha_p, & (\underline{d} + m)\beta_p^* &= -il_p\alpha_p^* \\
(\underline{d} - m)\alpha_p &= il_p\beta_p, & (\underline{d} - m)\alpha_p^* &= -il_p\beta_p^*
\end{align} \tag{3.2}$$

These relations will be useful in section 6 and in Proposition 5.

Now we are able to decompose solutions of (2.7).

If we assume that for each t, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in H^0 \times H^0$, then we can make the expansion:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \Sigma_p \lambda_p(t) u_p + \mu_p(t) u_p^*$$

where $p \in N$, u_p belongs to the positive eigenvalues l_p and u_p^* belongs to negative eigenvalues $-l_p$. Substituting into (2.7) we obtain the general solution

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \Sigma_p \lambda_p e^{il_p t} u_p + \mu_p^* e^{-il_p t} u_p^* \tag{3.3}$$

where λ_p, μ_p are constants. The corresponding Kähler field is

$$\Phi = \Sigma_p \lambda_p e^{il_p t} \phi_p + \mu_p^* e^{-il_p t} \phi_p^* \tag{3.4}$$

where $\phi_p = \alpha_p + e^0 \vee \beta_p$

PROPOSITION 5. — ϕ_p satisfies the property

$$(\underline{d} - m)\phi_p = -il_p e^0 \vee \phi_p.$$

Proof.

$$\begin{align}
(\underline{d} - m)\phi_p &= (\underline{d} - m)\alpha_p + (\underline{d} - m)(e^0 \vee \beta_p) \\
&= (\underline{d} - m)\alpha_p - e^0 \vee (\underline{d} + m)\beta_p \text{ (using (2.4))} \\
&= il_p\beta_p - il_p e^0 \vee \alpha_p \text{ (using (3.2))} \\
&= -il_p e^0 \vee (\alpha_p + e^0 \vee \beta_p) \text{ (using } e^0 \vee e^0 = -1) \\
&= -il_p e^0 \vee \phi_p.
\end{align}$$

In formulating the quantum conditions on the Kähler field, it is useful to introduce the inhomogeneous form

$$\Pi = e^0 \vee \eta \Phi^* \qquad (3.5)$$

which we shall refer to as the conjugate field to Φ. Expanding Φ as $\alpha + e^0 \vee \beta$ and using the identity $e^0 \vee \Phi - \eta \Phi \vee e^0 = -2i_{\partial/\partial t}\Phi$ gives

$$\Pi = (\alpha^* - e^0 \vee \beta^*) \vee e^0.$$

Writing $\pi_p = \alpha_p - e^0 \vee \beta_p$ and $\pi_p^* = \alpha_p^* - e^0 \vee \beta_p^*$ we obtain by combining (2.14) and (2.15) the expansion

$$\Pi = \Sigma_p \lambda_p^* e^{-il_p t} \pi_p^* \vee e^0 + \mu_p e^{il_p t} \pi_p \vee e^0. \qquad (3.6)$$

The property $(\underline{d} + m)\pi_p = il_p e^0 \vee \pi_p$ follows along the lines of Proposition 4.

4. QUANTIZATION RULES

To quantize the Kähler system, we seek a quantum algebra that enables the construction of a multi-particle Fock space to be established, consistent with a Hamiltonian with a spectrum bounded below. To this end we investigate the commutation relations between the Fock space operators associated with the eigenvalues l_p.

We define the classical energy density, given by equation II.30 in [3] as

$$\mathscr{H} = 2 \operatorname{Re} S_0(\xi \eta \Phi^* \vee e^0 \vee \mathscr{L}_{\partial/\partial t}\Phi) \qquad (4.1)$$

where $\mathscr{L}_{\partial/\partial t}$ is the Lie derivative with respect to the vector field $\dfrac{\partial}{\partial t}$. Using the orthonormality properties of $\begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix}$ we find the classical energy

$$\int \mathscr{H} \hat{*} \mathbb{1} = \Sigma_p l_p \{ \lambda_p^* \lambda_p - \mu_p \mu_p^* \}. \qquad (4.2)$$

As mentioned before, the eigenvalues l_p are positive.
This motivates us to adopt for the quantum field

$$\hat{\Phi} = \Sigma_p \{ C_p e^{il_p t} \phi_p + B_p^* e^{-il_p t} \phi_p^* \} \qquad (4.3)$$

in a basis of Fock space operators C_p, C_p^*, B_p, B_p^* satisfying the rules

$$\begin{aligned} \{ C_p^*, C_q \}_+ &= \delta_{pq} \\ \{ B_p^*, B_q \}_+ &= \delta_{pq} \end{aligned} \qquad (4.4)$$

with all other anticommutators zero. Then we may adopt as quantum Hamiltonian

$$\hat{H} = \Sigma_p l_p \left\{ N_p(C_p) + N_p(B_p) \right\}$$

where the number operators $N_p(C_p) = C_p^* C_p$, $N_p(B_p) = B_p^* B_p$ have eigenvalues zero or one, and the multiparticle Fock space is constructed from a ground state Ω defined by

$$C_p \Omega = B_p \Omega = 0 .$$

5. FIELD ANTICOMMUTATORS

We now evaluate the field anticommutators between

$$\hat{\Phi} = \Sigma_p C_p \Phi_p + B_p^* \Phi_p^* \tag{5.1}$$

and

$$\hat{\Pi} = \Sigma_p C_p^* \Pi_p^* \vee e^0 + B_p \Pi_p \vee e^0 \tag{5.2}$$

where $\Phi_p = \phi_p e^{il_p t}$, $\Pi_p = \pi_p e^{il_p t}$ and we define $\hat{\Pi}_1$ by

$$\hat{\Pi} = \hat{\Pi}_1 \vee e^0 .$$

If f and h denote C^∞ inhomogeneous differential forms on M with compact support, we define

$$\Phi_p(f) := \int_M S_0(\xi \Phi_p \vee f) * \mathbb{1} . \tag{5.3}$$

The operators $\hat{\Phi}(f)$, $\hat{\Pi}_1(h)$ on Fock space are then given by

$$\begin{aligned}
\hat{\Phi}(f) &:= \Sigma_p C_p \Phi_p(f) + B_p^* \Phi_p^*(f) \\
\hat{\Pi}_1(h) &:= \Sigma_p C_p^* \Pi_p^*(h) + B_p \Pi_p(h) .
\end{aligned} \tag{5.4}$$

Using the anticommutation rules (4.4) together with (5.4) gives an algebra satisfying

$$\begin{aligned}
[\hat{\Phi}(f), \hat{\Pi}_1(h)]_+ &= \Sigma_p \left\{ \Phi_p(f) \Pi_p^*(h) + \Phi_p^*(f) \Pi_p(h) \right\} \\
[\hat{\Phi}f), \hat{\Phi}(h)]_+ &= 0 \\
[\hat{\Pi}_1(f), \hat{\Pi}_1(h)]_+ &= 0 .
\end{aligned} \tag{5.5}$$

The right-hand side of the first equation in (5.5) defines the action of a bi-distribution kernel F on $f \otimes h$, an action which we denote by $F(f \otimes h)$ with

$$F = \Sigma_p \Phi_p \otimes \Pi_p^* + \Phi_p^* \otimes \Pi_p \tag{5.6}$$

which can also be written as

$$F = \Sigma_p e^{il_p t'} \phi_p \otimes e^{-il_p t} \pi_p + e^{-il_p t'} \phi_p \otimes e^{il_p t} \pi_p \qquad (5.7)$$

Thus F links up two space-like hypersurfaces which we label $\Sigma_{t'} = \{ t' \} \times M_3$ and $\Sigma_t = \{ t \} \times M_3$. In the next section we show that F is the propagator of the Kähler equation: it solves the Cauchy problem relative to the hypersurface Σ_t. When $t' = t$, F reduces to F_0 given by

$$F_0 = \Sigma_p \phi_p \otimes \pi_p^* + \phi_p^* \otimes \pi_p \qquad (5.8)$$

which is a realization of the Dirac distribution on Σ_t relative to the inner product (2.10).

In terms of the bidistribution F the quantum conditions may be summarised in terms of operator-valued inhomogeneous form distributions satisfying the following algebra:

$$[\hat{\Phi}, \hat{\Pi}]_+ = F \vee (\mathbb{1} \otimes e^0)$$
$$[\hat{\Phi}, \hat{\Phi}]_+ = [\hat{\Pi}, \hat{\Pi}]_+ = 0 \qquad (5.9)$$

where $\hat{\Pi} = \hat{\Pi} \vee e^0$ and the space-time Clifford algebra commutes with the algebra of quantum operators.

6. THE CAUCHY PROBLEM FOR THE KÄHLER EQUATION

In Appendix 2, we prove the validity of the equation

$$\Psi(\phi) = - \int_\Sigma \{ (d_\phi G, \Psi)_1 - (_\phi G, d\Psi)_1 \} \qquad (6.1)$$

which solves the Cauchy problem for the equation

$$(d^2 - m^2)\Psi = 0, \qquad \Psi \in \Lambda(M) \qquad (6.2)$$

(for the notation and symbols, see the Appendix).

Let us now impose the condition that Ψ also satisfy $d\Psi = m\Psi$. Substituting this into (6.1) gives

$$\Psi(\phi) = \int_\Sigma ((d - m)_\phi G, \Psi)_1 . \qquad (6.3)$$

This formula holds for all globally hyperbolic space-times and is the solution of the Cauchy problem of (2.1). Thus we can say that

$$- \mathbb{1} \otimes (d - m)G = (d + m) \otimes \mathbb{1} G$$

solves the Cauchy problem for (2.1) relative to the space-like hypersurface Σ and the inner product $\langle, \rangle_\Sigma^{(1)}$ defined by

$$\langle \Phi, \Psi \rangle_\Sigma^{(1)} := \int_\Sigma (\Phi, \Psi)_1 . \qquad (6.4)$$

This then justifies the remark made by the authors at the end of their article [9]. (6.1) may also be written

$$\Psi(\phi) = -\{\langle (\displaystyle{\not{d}} - m)_\phi G, \Psi \rangle_\Sigma^{(1)} - \langle {}_\phi G, (\displaystyle{\not{d}} - m)\Psi \rangle_\Sigma^{(1)} \} \qquad (6.1\,a)$$

Let us now return to the Cauchy problem of (2.1) in a static space-time $R \times M_3$ where M_3 is a compact Riemannian manifold, with the metric given in § 2. We consider the problem relative to the hypersurfaces defined by $t = $ constant and denote each by Σ_t.

LEMMA 1. — With notation Φ_p, Π_p of (5.1) and (5.2) we have

(1) $(\displaystyle{\not{d}} - m)\Phi_p = 0$

(2) $(\displaystyle{\not{d}} - m)\Phi_p^{-t} = -2ilpe^0 \vee \Phi_p^{-t}$

(3) $(\displaystyle{\not{d}} - m)\Pi_p = 2ilpe^0 \vee \Pi_p$

(4) $(\displaystyle{\not{d}} - m)\Pi_p^{-t} = 0$

(5) $(\displaystyle{\not{d}}^2 - m^2)\Psi = 0$ for $\Psi = \Phi_p^{\pm t}, \Pi_p^{\pm t}$.

Here we have written $\Pi_p^{\pm t} = e^{\pm ilpt}\pi_p$, $\Phi_p^{\pm t} = e^{\pm ilpt}\phi_p$.
The conjugate results hold for the complex conjugates $\Pi_p^{\pm t*}$, $\Phi_p^{\pm t*}$.

Proof. — One uses the decomposition

$$\displaystyle{\not{d}} - m = dt \vee \nabla_{\partial/\partial t} + (\underline{\not{d}} - m)$$

and the Propositions 5 and 6 of § 3.

LEMMA 2. — The general classical solution of $(\displaystyle{\not{d}}^2 - m^2)\Psi = 0$ can be written in the form

$$\Psi = \Sigma_p \{\lambda_p \Phi_p^t + \mu_p \Phi_p^{t*}\} + \Sigma_p \{\sigma_p \Phi_p^{-t} + \tau_p \Phi_p^{-t*}\} \qquad (6.5)$$

with $\lambda_p, \mu_p, \sigma_p, \tau_p \in \mathbb{Z}$.

Proof. — One uses the basis $\{\phi_p, \phi_p^*\}$ to find equations for time dependent coefficients and the above form follows from the resulting differential equations.

The first sum of (6.5) is recognised as a solution to the Kähler equation (2.1). Lemma 1 (2) shows that the second sum is not a Kähler solution.

THEOREM 1. — The kernel

$$G = i\Sigma_p \frac{1}{2lp}\{\Phi_p^t \otimes \Pi_p^{t*} - \Phi_p^{t*} \otimes \Pi_p^t\}$$

$$- i\Sigma_p \frac{1}{2lp}\{\Phi_p^{-t} \otimes \Pi_p^{-t*} - \Phi_p^{-t*} \otimes \Pi_p^{-t}\}$$

satisfies (6.1a) for solutions given by (6.2).

Proof. — From Lemma 1

$$\mathbb{1} \otimes (\displaystyle{\not{d}} - m)G = \Sigma_p \Phi_p^t \otimes e^0 \vee \Pi_p^{t*} + \Phi_p^{t*} \otimes e^0 \vee \Pi_p^t.$$

Using (6.2) we obtain

$$(đ - m)\Psi = i\Sigma_p 2 l_p \{ \sigma_p e^0 \vee \Phi_p - \tau_p e^0 \vee \Phi_p \} .$$

A straightforward calculation using (6.1 a) gives us the result, taking into account that

$$\langle \Phi, \Psi \rangle_{\Sigma_t}^{(1)} = \int_{M_3} S_0(\xi\Phi \vee e^0 \vee \Psi) \hat{*} \mathbb{1} \qquad (6.6)$$

where $\hat{*} \mathbb{1} = i_{\partial/\partial t} * \mathbb{1}$ is the 3-form measure on Σ_t.

If we now combine (6.3) with (6.4), we obtain

$$\Psi(\phi) = - \int_{\{t\} \times M_3} S_0(\xi \{ (đ - m)_\phi G \} \vee e^0 \vee \Psi) \hat{*} \mathbb{1}$$

from which it follows that

$$\Psi(\phi) = - \int_{\{t\} \times M_3} S_0(\xi \{ e^0 \vee (đ - m)_\phi G \} \vee \Psi) \hat{*} \mathbb{1}$$

$$= - \langle e^0 \vee (đ - m)_\phi G, \Psi \rangle_{M_3}$$

in the notation of (2.11). Thus $- \mathbb{1} \otimes e^0 \vee (đ - m)G$ is the propagator relative to Σ_t and the inner product $\langle , \rangle_{M_3}$, for the Kähler equation.

COROLLARY. — In the static space-time $\mathbb{R} \times M_3$ with our given metric, and M_3 a compact Riemannian manifold, we have the result

$$- \mathbb{1} \otimes e^0 \vee (đ - m)G = F .$$

Proof. — Using the form $\mathbb{1} \otimes (d - m)G$ used to prove Theorem 1, it follows that

$$- \mathbb{1} \otimes e^0 \vee (đ - m)G = \Sigma_p \Phi_p^{t'} \otimes \Pi_p^{t*} + \Phi_p^{t'*} \otimes \Pi_p^t = F .$$

Theorem I and its Corollary establish the connection between the anti-commutators (5.9) and the propagator for the classical solution of the Kähler equation.

CONCLUSION

A quantization of the Kähler equation based on anticommutation relations between basic field variables has been established in a class of static space-times with a parallel timelike Killing vector. The field system has been described in terms of differential forms without recourse to any decomposition into differentiable ideals of the space-time Clifford Algebra. It is an open question whether the system always admits spinorial solutions in such space-times. The existence of such a quantization scheme for this system may require a closer scrutiny of the interrelation between the statistics of field quanta and the tensorial properties of the underlying field theory.

APPENDIX I

Here we prove formula (8) of Appendix II, and obtain (2.9). Let $\{ X_a \}$ be any space-time frame and $\{ e^b \}$ its dual basis. We write i_a for i_{X_a} and i^a for $i_{\tilde{e}^a}$ where $\tilde{e}^a$ is the metric dual vector field to the covector e^a. We have the relationship

$$i_a = g_{ab} i^b$$

where g_{ab} are the components of the metric tensor:

$$g = g_{ab} e^a \otimes e^b .$$

The relation (1.1) implies that if $* \mathbb{1}$ is the volume form of the manifold then

$$e^a \vee * \mathbb{1} = i^a * \mathbb{1} .$$

LEMMA 1. — $\qquad d2^a * \mathbb{1} = - \omega^a{}_c \wedge i^c * \mathbb{1} .$

Proof. — $\qquad di^a * \mathbb{1} = e^b \wedge \nabla_{X_b}(e^a \vee * \mathbb{1})$

$$= - e^b \wedge \omega^a{}_c(X_b)(e^c \vee * \mathbb{1})$$

$$= - \omega^a{}_c \wedge i^c * \mathbb{1} .$$

LEMMA 2. — $\qquad di_a * \mathbb{1} = \omega^c{}_a \wedge i_c * \mathbb{1} .$

Proof. — $\qquad di_a * \mathbb{1} = d(g_{ab} i^b * \mathbb{1})$

$$= dg_{ab} \wedge i^b * \mathbb{1} + g_{ab} d(i^b * \mathbb{1})$$

$$= (\omega_{ab} + \omega_{ba}) \wedge i^b * \mathbb{1} - g_{ab} \omega^b{}_r \wedge i^r * \mathbb{1}$$

(using Lemma 1 and the result

$$dg_{ab} = \omega_{ab} + \omega_{ba}$$

$$= \omega_{ba} \wedge i^b * \mathbb{1}$$

$$= \omega^b{}_a \wedge i_b * \mathbb{1}).$$

We use the above result in proving Appendix II (8). First let us note

$$S_0(\xi\Phi \vee d\Psi) = S_0(\xi\Phi \vee e^a \vee \nabla_{X_a}\Psi)$$

$$= \nabla_{X_a} . S_0(\xi\Phi \vee e^a \vee \Psi) - S_0(\xi\nabla_{X_a}\Phi \vee e^a \vee \Psi)$$

$$- S_0(\xi\Phi \vee \nabla_{X_a}e^a \vee \Psi)$$

$$= - S_0(\xi(e^a \vee \nabla_{X_a}\Phi) \vee \Psi) + \nabla_{X_a}S_0(\xi\Phi \vee e^a \vee \Psi)$$

$$+ \omega^a{}_b(X_a)S_0(\xi\Phi \vee e^b \vee \Psi) .$$

Therefore

$$S_0(\xi\Phi \vee d\Psi) + S_0(\xi d\Phi \vee \Psi) = \nabla_{X_a}S_0(\xi\Phi \vee e^a \vee \Psi) + \omega^a{}_b(X_a)S_0(\xi\Phi \vee e^b \vee \Psi).$$

Multiplying both sides by $* \mathbb{1}$ gives

$$(d\Phi, \Psi) + (\Phi, d\Psi) = \nabla_{X_a}S_0(\xi\Phi \vee e^a \vee \Psi) * \mathbb{1} + S_0(\xi\Phi \vee e^b \vee \Psi)\omega^a{}_b(X_a) * \mathbb{1} \quad (A.1)$$

We know from the proof of Lemma 1 that

$$\omega^a{}_b(X_a) * \mathbb{1} = \omega^a{}_b \wedge i_a * \mathbb{1}$$

Annales de l'Institut Henri Poincaré - Physique théorique

taking the first term of the right hand side of (A.1) we find

$$\nabla_{X_a} S_0(\xi\Phi \vee e^a \vee \Psi) * \mathbb{1} = (i_a d S_0(\xi\Phi \vee e^a \vee \Psi)) * \mathbb{1}$$
$$= i_a(d S_0(\xi\Phi \vee e^a \vee \Psi) \wedge * \mathbb{1}) + d S_0(\xi\Phi \vee e^a \vee \Psi) \wedge i_a * \mathbb{1}$$
$$= d(S_0(\xi\Phi \vee e^a \vee \Psi)i_a * \mathbb{1}) - S_0(\xi\Phi \vee e^a \vee \Psi)di_a * \mathbb{1}$$
$$= d(\Phi, \Psi)_1 - S_0(\xi\Phi \vee e^a \vee \Psi)di_a * \mathbb{1}.$$

Consequently we find

$$(\overset{\leftrightarrow}{d}\Phi, \Psi) + (\Phi, \overset{\leftrightarrow}{d}\Psi) = d(\Phi, \Psi)_1 + S_0(\xi\Phi \vee e^a \vee \Psi)\{\omega^b{}_a \wedge i_b * \mathbb{1} - di_a * \mathbb{1}\}$$
$$= d(\Phi, \Psi)_1 \quad \text{by Lemma 2}.$$

This proves (8) of Appendix II.

Proving (2.9) is now easy. Because of the properties (2.4) $\hat{*}$ obeys the equation

$$\nabla_X \hat{*} \mathbb{1} = 0$$

for all $X \in TM_3$. The above results then apply, and we find

$$S_0(\xi\underline{d}\alpha \vee \beta) \hat{*} 1 + S_0(\xi\alpha \vee \underline{d}\beta) \hat{*} \mathbb{1} = \underline{d}(\alpha, \beta)_1$$

from which we then obtain

$$\langle \underline{d}\alpha, \beta \rangle + \langle \alpha, \underline{d}\beta \rangle = \int_{M_3} \underline{d}(\alpha, \beta)_1 = 0$$

as M_3 is compact. Hence we obtain (2.9).

APPENDIX II

In this appendix, we adapt the work of Lichnerowicz [7] on the Cauchy problem of the equation

$$(\vec{d}^2 - m^2)\Psi = 0, \qquad \Psi \in \Lambda(M) \tag{1}$$

to solve the Cauchy problem for the Kähler equation

$$(\vec{d} - m)\Phi = 0.$$

DEFINITION. — For Φ, Ψ inhomogeneous differential forms on $M = \mathbb{R} \times M_3$ we define

$$(\Phi, \Psi) := \int_0 (\xi\Phi \vee \Psi) * \mathbb{1}$$

$$\langle \Phi, \Psi \rangle := \int_M (\Phi, \Psi)$$

$$(\Phi, \Psi)_1 := \int_0 (\xi\Phi \vee e^a \vee \Psi) i_{X_a} * \mathbb{1}$$

Here M is any globally hyperbolic space-time and $\{ e^a \}$, $\{ X_b \}$ any naturally dual coframe and frame respectively.

We denote by Λ_Ω the Clifford bundle over $\Omega \subseteq M$ and if $\Omega = M$ we write $\Lambda_M = \Lambda$. A section ϕ in Λ has compact support if there is a compact set $K \subset M$ with ϕ equal to the zero section in Λ_Ω for all Ω such that $\Omega \cap K = \phi$. $\mathscr{D}(\Lambda)$ is the space of all C^∞ sections in Λ of compact support. A sequence $\phi_n \in \mathscr{D}(\Lambda)$ converges to zero if in any local basis the components of ϕ_n, and the components of all its covariant derivatives of any finite order converge to zero in the topology of compact convergence, as $n \to \infty$. We call $\mathscr{D}(\Lambda)$ the space of inhomogeneous test forms. The inhomogeneous distributions $\mathscr{D}'(\Lambda)$ are linear functionals on $\mathscr{D}(\Lambda)$. We shall denote such duality between $T \in \mathscr{D}'(\Lambda)$ and $\phi \in \mathscr{D}(\Lambda)$ by the pairing $T(\phi)$ i. e.

$$T : \mathscr{D}(\Lambda) \to \mathbb{C}, \qquad \phi \mapsto T(\phi).$$

For the locally integrable distributions T,

$$T(\phi) = \langle T, \phi \rangle \qquad \forall \phi \in \mathscr{D}(\Lambda).$$

T has support in Ω if $T(\phi) = 0$ for all ϕ whose support is disjoint from Ω.

A bi-inhomogeneous kernel, or just kernel, is an element of $\mathscr{D}'(\Lambda \times \Lambda)$, the dual of $\mathscr{D}(\Lambda \times \Lambda)$ the latter being defined as the space of sections of $\Lambda \times \Lambda$ with compact support in $M \times M$. $\mathscr{D}(\Lambda) \otimes \mathscr{D}(\Lambda)$ is a subset of $\mathscr{D}(\Lambda \times \Lambda)$. If $E \in \mathscr{D}'(\Lambda \times \Lambda)$ then for $\phi, \psi \in \mathscr{D}(\Lambda)$ we know that $\phi \otimes \psi \to E(\phi \otimes \psi)$ is a functional from $\mathscr{D}(\Lambda) \otimes \mathscr{D}(\Lambda)$ to $\mathbb{C}$. For each fixed $\phi \in \mathscr{D}(\Lambda)$, $\psi \mapsto E(\phi \otimes \psi)$ defines a distribution in $\mathscr{D}'(\Lambda)$ which we denote by $_\phi E$. Thus

$$_\phi E(\psi) = E(\phi \otimes \psi). \tag{2}$$

Similarly we define $E_\psi \in \mathscr{D}'(\Lambda)$ by

$$E_\psi(\phi) := E(\phi \otimes \psi) \tag{3}$$

for each fixed $\psi \in \mathscr{D}(\Lambda)$.

Now suppose L is an operator $L: \mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda)$. Then we can, as is usual, extend L to act on $\mathcal{D}'(\Lambda)$ as follows: if $T \in \mathcal{D}'(\Lambda)$ then $LT \in \mathcal{D}'(\Lambda)$ is defined by

$$(LT)(\phi) := T(L^*\phi) \qquad \forall \phi \in \mathcal{S}(\Lambda) \tag{4}$$

where L^* is the adjoint of L in the inner product $\langle, \rangle$. Similarly we can use L to define operators $\mathbb{1} \otimes L$ and $L \otimes \mathbb{1}$ in $\mathcal{D}'(\Lambda \times \Lambda)$ by demanding

$$\{ (L \otimes \mathbb{1})E \} (\phi \otimes \psi) = E(L^*\phi \otimes \psi) \tag{5}$$
$$\{ (\mathbb{1} \otimes L)E \} (\phi \otimes \psi) = E(\phi \otimes L^*\psi)$$

for all $\phi, \psi \in \mathcal{D}(\Lambda)$. We then obtain the following using (2), (3), (4), (5):

$$\{ (\mathbb{1} \otimes L)E \} (\phi \otimes \psi) = E(\phi \otimes L^*\psi) = {}_\phi E(L^*\psi) = (L_\phi E)(\psi)$$
$$\{ (L \otimes \mathbb{1})E \} (\phi \otimes \psi) = E(L^*\phi \otimes \psi) = F_\psi(L^*\phi) = (LE_\psi)(\phi)$$

The Dirac kernel $D \in \mathcal{S}'(\Lambda \times \Lambda)$ is defined through the formula

$$D(\phi \otimes \psi) = \langle \phi, \psi \rangle \qquad \forall \phi, \psi \in \mathcal{D}(\Lambda) \tag{6}$$

it follows that

$${}_\phi D(\psi) = \langle \phi, \psi \rangle \quad \text{and} \quad D_\psi(\phi) = \langle \phi, \psi \rangle$$

so that we may identify ${}_\phi D$ with ϕ and D_ψ with ψ, if ϕ and ψ are locally integrable.

Given a differential operator $L: \mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda)$ we define its Green's kernel as the bidistribution $E \in \mathcal{D}'(\Lambda \times \Lambda)$ such that

$$(L \otimes \mathbb{1})E = (\mathbb{1} \otimes L)E = D \tag{7}$$

the equalities being understood in the sense of distributions. We consider the case $L = \tilde{d}^2 - m^2$. Now $\tilde{d}$ is skew-adjoint with respect to $\langle, \rangle$. Indeed, we have the following two results

$$d(\Phi, \Psi)_1 = (\tilde{d}\Phi, \Psi) + (\Phi, \tilde{d}\Psi) \tag{8}$$

$$(d^2\Phi, \Psi) - (\Phi, \tilde{d}^2\Psi) = d \{ (\tilde{d}\Phi, \Psi)_1 - (\Phi, \tilde{d}\Psi)_1 \} \tag{9}$$

(9) follows from (8). For a proof of (8) (which gives a proof of (2.10) see the Appendix I. (8) implies

$$\langle \tilde{d}\Phi, \Psi \rangle = - \langle \Phi, \tilde{d}\Psi \rangle$$

and this implies that $L^* = \tilde{d}^2 - m^2$. It is known [7] [8] [9] that in this case (7) gives two kernels $E^\pm$ such that ${}_\phi E^+$ has support in the future of supp ϕ and ${}_\phi E^-$ has support in the past of supp ϕ, and $E^\pm$ are regular in the sense of Schwartz [10] so that, in particular, ${}_\phi E^\pm$ are C^∞ inhomogeneous forms.

Now let us define, for $\phi \in \mathcal{S}(\Lambda)$ and Ψ an inhomogeneous form,

$${}_\phi A_\psi^\pm := (\tilde{d}_\phi E^\pm, \Psi)_1 - ({}_\phi E^\pm, \tilde{d}\Psi)_1 . \tag{10}$$

Using (6.8) it follows that

$$d_\phi A_\psi^\pm = (\tilde{d}^2 {}_\phi E^\pm, \Psi) - ({}_\phi E^\pm, \tilde{d}^2\Psi) .$$

If we further impose that Ψ satisfies (1) then we have

$$d_\phi A_\psi^\pm = ({}_\phi D, \Psi) . \tag{11}$$

Taking ψ to be a solution of (1) with support in $\Omega \subset M$ and supposing that Ψ vanishes on the boundary of Ω, we divide Ω into two parts Ω_1, Ω_2 whose boundaries share a common space-like hypersurface Σ. Then Ψ can be written as

$$\Psi = \Psi_1 + \Psi_2$$

with supp $\Psi_1 \subset \Omega_1 \cup \Sigma$, supp $\Psi_2 \subset \Omega_2 \cup \Sigma$. We consider Ψ as a distributional solution. Using (11),

$$\int_{\Omega_1 \cup \Sigma} d_\phi A_\Psi^- = \int_{\Omega_1 \cup \Sigma} (_\phi D, \Psi) = \langle \phi, \Psi_1 \rangle = \Psi_1(\phi) = \int_\Sigma {}_\phi A_\Psi^- \tag{12}$$

the last equality being obtained by Stoke's Theorem. Similarly

$$\int_{\Omega_2 \Sigma} d_\phi A_\Psi^+ = \Psi_2(\phi) = - \int_\Sigma {}_\phi A_\Psi^+ . \tag{13}$$

Combining (12) and (13) gives

$$\Psi_1(\phi) = \Psi_2(\phi) = \Psi(\phi) = - \int_\Sigma (_\phi A_\Psi^+ - {}_\phi A_\Psi^-). \tag{14}$$

Writing $E^+ - E^- = G$, and using (14) and (10) gives us the result

$$\Psi(\phi) = - \int_\Sigma \{ d_\phi G, \Psi)_1 - (_\phi G, d\Psi)_1 \} \tag{15}$$

Equation (15) represents the solution of the Cauchy problem of (1) relative to the space-like hypersurface Σ. This reformulates the Cauchy problem as treated by Lichnerowicz in [7] for (1). It can be shown that if Ψ is a 0-form then (15) reduces to

$$\Psi(\phi) = - \int_\Sigma \{ (\nabla_{\lambda\phi} G)\Psi - {}_\phi G(\nabla_\lambda \Psi) \} \, d\Sigma$$

where $d\Sigma$ represents the volume form on Σ and ∇_λ is the covariant derivative with respect to the unit time-like vector λ which is normal to the space-lime hypersurface Σ. Further, in static space-times with a metric of the form $g = - dt \otimes dt + \hat{g}$ where $\hat{g}$ is a Riemannian metric independent of t, one can show that (15) reduces to

$$\Psi(\phi) = - \int_\Sigma \{ S_0(\xi \nabla_{\lambda\phi} G \vee \Psi) - S_0(\xi_\phi G \vee \nabla_\lambda \Psi) \} \, d\Sigma$$

with the notation as before. These are the formulae given by Lichnerowicz in [7] for the solution of the Cauchy problem. For general space-times and forms of degree higher than zero, it is not always possible to recover the Lichnerowicz form for the solution of the Cauchy problem.

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