ANNALES DE L'I. H. P., SECTION A

M. ASOREY
P. K. MITTER

Cohomology of the Yang-Mills gauge orbit space and dimensional reduction

Annales de l'I. H. P., section A, tome 45, nº 1 (1986), p. 61-78
<http://www.numdam.org/item?id=AIHPA_1986__45_1_61_0>

Ann. Inst. Henri Poincaré,
Vol. 45, n° 1, 1986, p. 61-78. *Physique théorique*

Cohomology of the Yang-Mills gauge orbit space and dimensional reduction

by

M. ASOREY (1)
Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain

and

P. K. MITTER
Université de Paris VI, LPTHE (2), 4, place Jussieu,
Tour 16, 1er étage, 75230 Paris Cedex 05, France

ABSTRACT. — Let $\mathcal{M}$ be the gauge orbit space of pure Yang-Mills theory and $H^i(\mathcal{M}, \mathbb{Z})$, $i = 1, 2$ be its first and second cohomology groups. We give a differential geometric proof, based on a dimensional reduction procedure, that certain cohomology class on $\mathcal{M}$, obtained by a method due to Atiyah and Singer, give the non-trivial generators of $H^i(\mathcal{M}, \mathbb{Z})$ $i = 1, 2$ in cases where these groups are non-trivial. The dimensional reduction procedure is a generalization of the method used to obtain the canonical formalism from the covariant one in the temporal gauge. The relation between generating forms of $H^1(\mathcal{M}, \mathbb{Z})$, Chern-Simmons classes and global gauge anomalies in odd dimensions is also pointed out in the light of this method.

RÉSUMÉ. — Soit $\mathcal{M}$ l'espace des orbites de jauge d'une théorie de Yang-Mills pure et $H^i(\mathcal{M}, \mathbb{Z})$, $i = 1, 2$, son premier et son second groupe de cohomologie. On donne une démonstration de géométrie différentielle, basée sur un procédé de réduction dimensionnelle, du fait que certaines classes

(1) Partly supported by the CAICYT (Plan de Altas Energías).
(2) Laboratoire Associé au CNRS 280.

de cohomologie de $\mathscr{M}$, obtenues par une méthode due à Atiyah et Singer, fournissent les générateurs non triviaux de $H^i(\mathscr{M}, \mathbb{Z})$, $i = 1, 2$, dans des cas où ces groupes sont non triviaux. Le procédé de réduction dimensionnelle est une généralisation de la méthode utilisée pour obtenir le formalisme canonique à partir du formalisme covariant dans la jauge temporelle. On signale aussi, à la lumière de cette méthode, la relation entre les formes génératrices de $H^1(\mathscr{M}, \mathbb{Z})$, les classes de Chern-Simmons, et les anomalies de jauge globales.

1. INTRODUCTION

The physical relevance of cohomological properties of the Yang-Mills orbit space $\mathscr{M}$ has been recently stressed in two different contexts. First, Atiyah-Singer [*4*] (see also Stora [*13*]) have pointed out the connection between the non-abelian anomalies in even dimensions and the second cohomology group $H^2(\mathscr{M}, \mathbb{Z})$ of $\mathscr{M}$. On the other hand it has been shown that in the canonical Hamiltonian formalism, the θ-vacuum phenomenon can be expressed in terms of the generating 1-forms of the first cohomology group $H^1(\mathscr{M}, \mathbb{Z})$ [*1*] [*5*] [*15*]. In a similar way, the generating 2-forms of $H^2(\mathscr{M}, \mathbb{Z})$ turn out to be very relevant for the canonical formalism of Yang-Mills theories in $2 + 1$ dimensions with a Chern-Simmons term [*2*] [*5*] [*15*].

Let P be a smooth principal SU(N)-bundle ($N \geqslant 2$) on a compact smooth manifold V, $\mathscr{G}^0$ the group of pointed gauge transformations and $\mathscr{A}$ the space of connections in P. The corresponding orbit space $\mathscr{A}/\mathscr{G}^0$ is denoted $\mathscr{M}$. In [*4*] Atiyah and Singer introduced the principal bundle π : $(P \times \mathscr{A})/\mathscr{G}^0 \to V \times \mathscr{M}$ with SU(N) as its structure group. Then the Chern class C_r of this bundle gives an element of $H^{2r}(V \times \mathscr{M}, \mathbb{R})$. The Kunneth formula and the (known) cohomology of V lead to elements of $H^{2r}(\mathscr{M}, \mathbb{R})$, $H^{2r-1}(\mathscr{M}, \mathbb{R})$, $\ldots$, $H^{2r-\dim V}(\mathscr{M}, \mathbb{R})$. In this paper we will show, by differential geometric techniques, for $V = S^{2N-2}$ (respectively $V = S^{2N-1}$) that the cocycles in $H^2(\mathscr{M}, \mathbb{R})$ (respectively $H^1(\mathscr{M}, \mathbb{R})$) constructed by the above procedure ($r = N$) are in fact generators of those cohomology groups.

The proof proceds via a dimensional reduction technique which generalizes the standard approach to the canonical formalism in the temporal gauge $A_0 = 0$. This technique establishes 1) a relation between the points of the gauge orbit space when the initial base manifold is $V \times S^1$ with closed curves in the orbit space when the initial base manifold is V 2) a relation between closed S^2-surfaces in the orbit space corresponding to V and points of the orbit space corresponding to $V \times S^2$. The results are in agreement with those of [*4*] although the method is completely different.

Our method should be particularly palatable to physicists with a background in differential geometry.

We also point out by this method the existence of a relation between the generating 1-forms of $H^1(\mathcal{M}, \mathbb{Z})$, the global chiral anomaly and the Chern-Simmons terms in $2N - 1$ dimensions.

The paper is organized as follows. In Section 2 we calculate the number of connected components of the extended orbit space $\mathcal{M}_V^{SU(N)}$ i. e. the number of inequivalent principal fibre bundles over V with gauge group SU(N). We also point out how the generating functions of $H^0(\mathcal{M}_V^{SU(N)}, \mathbb{Z})$ can be given in terms of Chern numbers. The computation of the cohomology groups $H^1(\mathcal{M}, \mathbb{Z})$ and $H^2(\mathcal{M}, \mathbb{Z})$ is carried out in Section 3 for some manifolds V. Section 4 is devoted to the analysis of the generating 1-forms of $H^1(\mathcal{M}, \mathbb{Z})$ by means of the dimensional reduction method for $V = S^{2N-1}$ and $S^{2N-2} \times S^1$. In Section 5 the generalization of this method for two-dimensional surfaces of $\mathcal{M}$, provides a simple proof of non-exactness and the normalization of the explicit expression of the generating 2-forms of $H^2(\mathcal{M}, \mathbb{Z})$ for the case $V = S^{2N-2}$, $G = SU(N)$. Finally the application of the first cohomology group $H^1(\mathcal{M}, \mathbb{Z})$ to the study of global gauge anomalies in odd dimensional space-times is considered in Section 6.

Let us introduce some notation. We consider only four kinds of manifolds $V = S^n$, $S^n \times S^1$, $S^n \times S^2$, $S^n \times S^1 \times S^1$. Let $\mathcal{A}_P$ denote the space of connections defined in any principal fibre bundle $P(V, SU(N))$. $\mathcal{G}_P$ the group of gauge transformations of P and $\mathcal{G}_P^0$ the normal subgroup of $\mathcal{G}_P$ which leaves fixed a given point of P. If we choose connections in the Sobolev class k and gauge transformations in class $k + 1$ with $k > \dfrac{\dim V}{2}$, then $\mathcal{A}_P$ is an affine Hilbert space, and $\mathcal{G}_P$ and $\mathcal{G}_P^0$ Hilbert Lie groups. The orbit space

$$\mathcal{M}_P = \mathcal{A}_P / \mathcal{G}_P^0 \tag{1.1}$$

defined by usual action of $\mathcal{G}_P$ on $\mathcal{A}_P$

$$\begin{aligned} \mathcal{A}_P \times \mathcal{G}_P &\to \mathcal{A}_P \\ (A, g) &\mapsto g^{-1}Ag + g^{-1}dg \end{aligned} \tag{1.2}$$

is a smooth separable paracompact Hilbert manifold [*1*] [*9*] [*11*].

If we restrict ourselves to the space of irreducible connections $\overline{\mathcal{A}}_P$ and $\overline{\mathcal{G}}_P$ is the quotient of $\mathcal{G}_P$ by its center $Z(\mathcal{G}_P) \approx \mathbb{Z}^N$, then the orbit space

$$\overline{\mathcal{M}}_P = \overline{\mathcal{A}}_P / \overline{\mathcal{G}}_P \tag{1.3}$$

is also a smooth separable paracompact Hilbert [*7*] [*9*] [*11*]. Moreover, $\mathcal{A}_P(\mathcal{M}, \mathcal{G}_P^0)$ and $\overline{\mathcal{A}}_P(\overline{\mathcal{M}}_P, \overline{\mathcal{G}}_P)$ are principal fibre bundles [*1*] [*7*] [*9*] [*11*].

2. THE EXTENDED ORBIT SPACE

For any principal bundle P(V, SU(N)) the orbit space $\mathscr{M}_P(\overline{\mathscr{M}}_P)$ is a connected manifold because $\mathscr{A}_P(\overline{\mathscr{A}}_P)$ is so and the $\mathscr{G}_P^0(\overline{\mathscr{G}}_P)$-action is closed [9]. Then, $H^0(\mathscr{M}_P, \mathbb{Z}) = \mathbb{Z}$. It is obvious that the orbit spaces of equivalent principal bundles are diffeomorphic. Let [P] denote the class of principal fibre bundles equivalent to P.

The extend orbit space

$$\mathscr{M}_V^{SU(N)} = \bigcup_{[P]} \mathscr{M}_{[P]} \tag{2.1}$$

is defined by the disjoint union of the orbit spaces of the different equivavalence classes of principal fibre bundles on V with the same gauge group SU(N).

It is believed that an (extended) orbit space is the physical configuration space in the Euclidean approach to Continuum Quantum Yang-Mills theory.

In general $\mathscr{M}_V^{SU(N)}$ is not connected, and the number of its connected components is the cardinal of the set $H^1(V, \underset{\sim}{SU(N)})$ of classes of principal fibre bundles on V with gauge group SU(N). Now, as is well known [12]

$$H^1(V, \underset{\sim}{SU(N)}) = [V, BSU(N)] \tag{2.2}$$

where BSU(N) is the classifying space of the Lie group SU(N) and [V, BSU(N)] is the set of classes of homotopy equivalent maps of V into BSU(N). Since the total space of the universal bundle on BSU(N) is contractible $\pi_n(BSU(N)) = \pi_{n-1}(SU(N))$. Thus,

$$\begin{aligned}
[S^n, BSU(N)] &= \pi_{n-1}(SU(N)) \\
[S^n \times S^1, BSU(N))] &= \pi_n(SU)N)) + \pi_{n-1}(SU(N)) \\
[S^n \times S^2, BSU(N))] &= \pi_{n+1}(SU(N)) + \pi_{n-1}(SU(N)) \\
[S^n \times S^1 \times S^1, BSU(N)] &= \pi_{n+1}(SU(N)) + 2\pi_n(SU(N)) + \pi_{n-1}(SU)N))
\end{aligned} \tag{2.3}$$

Let us consider some particular cases. If $V = S^{2N-1} \times S^1$, (2.3) implies that

$$H^1(S^{2N-1} \times S^1, \underset{\sim}{SU(N)}) = \mathbb{Z} \tag{2.4}$$

for $N \geqslant 2$. The corresponding classes of principal fibre bundles are classified by the Chern-number

$$C_N(P) = \int_{S^{2N-1} \times S^1} \chi_N(\overset{N}{\wedge} F(A)) \tag{2.5}$$

where $\overset{N}{\wedge}$ denotes the N-fold exterior product,

$$F(A) = dA + A \wedge A \tag{2.6}$$

is the curvature of any connection A of P, and χ_N is the normalized invariant N-polynomial of the Lie algebra of SU(N) [4]. In the same way, one gets

$$H^1(S^{2N-2}\times S^1\times S^1, \underline{SU(N)}) = H^1(S^{2N-2}\times S^2, \underline{SU(N)}) = \mathbb{Z}\oplus\mathbb{Z} \qquad (2.7)$$

for $N > 2$, which corresponds to the Chern numbers $C_N(P)$ and

$$C_{N-1}(P) = \int_{S^{2N-2}} \chi_{N-1}(\overset{N-1}{\wedge} F(A)) \qquad (2.8)$$

of $P(S^{2N-2} \times S^1 \times S^1, SU(N))$ and $P(S^{2N-2} \times S^2, SU(N))$. If $N = 2$

$$H^1(S^2 \times S^1 \times S^1, \underline{SU(2)}) = H^1(S^2 \times S^2, \underline{SU(2)}) = \mathbb{Z} \qquad (2.9)$$

and the SU(2)-principal fibre bundles on $S^2 \times S^1 \times S^1$ and $S^2 \times S^2$ are completely classified by $C_2(P)$. Since

$$H^1(S^{2N-2}, \underline{SU(N)}) = \mathbb{Z} \qquad (2.10)$$

for $N > 2$, the corresponding bundles are labeled by $C_{N-1}(P)$. Finally,

$$H^1(S^2, \underline{SU(2)}) = 0 \qquad (2.11)$$

and

$$H^1(S^{2N-1}, \underline{SU(N)}) = 0 \qquad (2.12)$$

The above results also hold for the extended orbit spaces associated with $\overline{\mathcal{M}}_P$. They are summarized in

Proposition 2.1. — The 0-cohomology groups of the extended orbit spaces on $S^{2N-1} \times S^1$, $S^{2N-2} \times S^2$, S^{2N-1}, S^{2N-2} with gauge group SU(N) ($N \geqslant 2$) are given by

i) $H^0(\mathcal{M}^{SU(N)}_{S^{2N-1}\times S^1}, \mathbb{Z}) = \mathbb{Z}^{\mathbb{Z}}$

ii) $H^0(\mathcal{M}^{SU(N)}_{S^{2N-2}\times S^1\times S^1}, \mathbb{Z}) = H^0(\mathcal{M}^{SU(N)}_{S^{2N-2}\times S^2}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\mathbb{Z}\oplus\mathbb{Z}}, & N > 2 \\ \mathbb{Z}^{\mathbb{Z}}, & N = 2 \end{cases}$

iii) $H^0(\mathcal{M}^{SU(N)}_{S^{2N-2}}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\mathbb{Z}}, & N > 2 \\ 0, & N = 2 \end{cases}$

iv) $H^0(\mathcal{M}^{SU(N)}_{S^{2N-1}}, \mathbb{Z}) = 0$

The corresponding generating functions are given by the following Chern numbers

i) $C_N(P)$ for $V = S^{2N-1} \times S^1$

ii) $C_N(P)$ and $C_{N-1}(P)$ for $V = S^{2N-2} \times S^2, S^{2N-2} \times S^1 \times S^1$ and $N > 2$
$C_2(P)$ for $S^2 \times S^2$ and $S^2 \times S^1 \times S^1$, and $N = 2$.

iii) $C_{N-1}(P)$ for $V = S^{2N-2}$ and $N > 2$.

3. COHOMOLOGY OF THE ORBIT SPACE

The real cohomology groups $H^r(\mathcal{M}_P, \mathbb{R})$ of any orbit space $\mathcal{M}_P$ can be obtained by means of Thom's theorem [3]. The Poincaré series $\mathcal{P}_t(\mathcal{M}_P)$ of $\mathcal{M}_P$ is defined by

$$\mathcal{P}_t(\mathcal{M}_P) = \sum_{r=0}^{\infty} \dim H^r(\mathcal{M}_P, \mathbb{R}) t^r . \tag{3.1}$$

Since $\mathcal{A}_P$ is contractible, $\mathcal{M}_P$ is a classifying space for $\mathcal{G}_P^0$ and

$$\mathcal{M}_P \simeq B\mathcal{G}^0 \simeq \mathrm{Map}_0^P(V, BSU(N)) \tag{3.2}$$

where $\mathrm{Map}_0^P(V, BSU(N))$ denotes the set of pointed maps which pull-back the universal bundle on B SU(N) to the principal bundle P on V. Now,

$$B\,SU(N) \approx K(\mathbb{Z}, 4) \times \ldots \times K(\mathbb{Z}, 2N) \tag{3.3}$$

over the rationals [3]. Thus,

$$\mathrm{Map}_0^P(V, B\,SU(N)) \approx \mathrm{Map}_0^P(V, K(\mathbb{Z}, 4) \times \ldots \times K(\mathbb{Z}, 2N)) \tag{3.4}$$

and according to Thom's theorem

$$\mathrm{Map}_0^P(V, K(\mathbb{Z}, r)) = \prod_{q=1}^{r-1} K(H^q(V, \mathbb{R}), r - q) \tag{3.5}$$

In special cases, using the known real cohomology of Eilenberg-MacLane spaces we obtain

i) $V = S^{2N-1}, \quad \mathcal{P}_t(\mathcal{M}_P) = 1 + t$ (3.6)

ii) $V = S^{2N-2}, \quad \mathcal{P}_t(\mathcal{M}_P) = (1 - t^2)^{-1}$ (3.7)

iii) $V = S^{2N-2} \times S^1, \quad \mathcal{P}_t(\mathcal{M}_P) = (1 - t^2)^{-1}(1 + t)(1 + t^3) \ldots (1 + t^{2N-1})$ (3.8)

PROPOSITION 3.1. — For any principal bundle P(V, SU(N)), on

i) $V = S^{2N-1}, \quad H^1(\mathcal{M}_P, \mathbb{Z}) = \mathbb{Z}, \quad H^2(\mathcal{M}_P, \mathbb{Z}) = 0$

ii) $V = S^{2N-2}, \quad H^1(\mathcal{M}_P, \mathbb{Z}) = 0 \quad H^2(\mathcal{M}_P, \mathbb{Z}) = \mathbb{Z}$

iii) $V = S^{2N-2} \times S^1, \quad H^1(\mathcal{M}_P, \mathbb{Z}) = \mathbb{Z}, \quad H^2(\mathcal{M}_P, \mathbb{Z}) = \mathbb{Z}.$

Proof. — The Poincaré series (3.6)-(3.8) imply that $H^1(\mathcal{M}_P, \mathbb{R}) = \mathbb{R}$ and $H^2(\mathcal{M}_P, \mathbb{R}) = 0$ for $V = S^{2N-1}$, $H^1(\mathcal{M}_P, \mathbb{R}) = 0$ and $H^2(\mathcal{M}_P, \mathbb{R}) = \mathbb{R}$ for $V = S^{2N-2}$, and $H^1(\mathcal{M}_P, \mathbb{R}) = \mathbb{R}$ and $H^2(\mathcal{M}_P, \mathbb{R}) = \mathbb{R}$ for $V = S^{2N-2} \times S^1$. Now the cohomology groups $H^r(\mathcal{M}_P, \mathbb{Z})$ with the integer coefficients differ from the integer submoduli of $H^r(\mathcal{M}_P, \mathbb{R})$ only by the torsion. But, in this case, since $\mathscr{M}_P$ is connected $H^1(\mathscr{M}_P, \mathbb{Z}) = \mathrm{Hom}(\pi_1(\mathscr{M}_P), \mathbb{Z})$ is torsionless because $\pi_1(\mathscr{M}_P)$ is so.

On the other hand the exact sequence of homomorphisms of abelian gauge group

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{j} U(1) \to 0 \tag{3.9}$$

induces the exact sequence of cohomology groups

$$\ldots \to H^1(\mathscr{M}_P, \mathbb{R}) \xrightarrow{j_1} H^1(\mathscr{M}_P, U(1)) \xrightarrow{\partial_1} H^2(\mathscr{M}_P, \mathbb{Z}) \xrightarrow{i_2} H^2(\mathscr{M}_P, \mathbb{R}) \to \ldots \tag{3.10}$$

Now, by the properties of cohomology groups

$$H^1(\mathscr{M}_P, \mathbb{R}) = \mathrm{Hom}(\pi_1(\mathscr{M}_P), \mathbb{R}), \quad H^1(\mathscr{M}_P, U(1)) = \mathrm{Hom}(\pi_1(\mathscr{M}_P), U(1))$$

and since $\pi_1(\mathscr{M}_P)$ is torsionless, j_1 is surjective and ∂_1 is trivial. Thus, i_2 is injective, which implies that $H^2(\mathscr{M}_P, \mathbb{Z})$ is also torsionless and coincides with the submoduli of integer elements of $H^2(\mathscr{M}_P, \mathbb{R})$. ||

Throughout we consider cohomology groups of the Čech cohomology theory. However, since $\mathscr{M}$ is a paracompact (Hausdorff) manifold all sheaf cohomology theories are isomorphic. In particular, $H^r(\mathscr{M}, \mathbb{R})$ is isomorphic to the corresponding De Rham cohomology group.

It is easy to show that proposition 3.1 also holds for $\overline{\mathscr{M}}_P$. The difference between the cohomology groups of $\mathscr{M}_P$ and $\overline{\mathscr{M}}_P$ appears only for higher orders. Since the results do not depend either on the principal fibre bundle P(V, SU(N)) hereafter we shall drop the sub-script P in the notation for $\mathscr{M}_P$, $\mathscr{A}_P$ and $\mathscr{G}^0_P$.

As remarked in [4] Chern-Weil theory for the principal bundle $(P \times \mathscr{A})/\mathscr{G}^0(V \times \mathscr{M}, SU(N))$ together with the Kunneth formula give, in particular, elements in $H^1(\mathscr{M}, \mathbb{R})$ and $H^2(\mathscr{M}, \mathbb{R})$.

Let us consider a given connection ω in $\mathscr{A}(\mathscr{M}, \mathscr{G}^0)$ and the induced connection $\tilde{\omega}$ of $(P \times \mathscr{A})/\mathscr{G}^0(V \times \mathscr{M}, SU(N))$ defined by

$$\tilde{\omega}_{\{(u,A)\}}(w) = A(\tilde{v}) + \omega(\tilde{\tau})(u)$$

for any $\{(u, A)\} \in (P \times \mathscr{A})/\mathscr{G}^0$, $w \in T_{\{(u,A)\}}((P \times \mathscr{A})/\mathscr{G}^0)$, where $\tilde{v} \in T_u(P)$ and $\tilde{\tau} \in T_A(\mathscr{A})$ are the respective components of any lift $\tilde{w} \in T_{(u,A)}(P \times \mathscr{A})$ of w.

Then, the Chern-Weil theory and Kunneth formula lead to forms Ω^1 and Ω^2 on $\mathscr{M}_P$ defined by

$$\Omega^1_{[A]}(\eta) = N \int_V \chi_N(\eta^h \overset{N-1}{\wedge} F(A)) \quad \textit{magnetic 1-form} \tag{3.11}$$

$$\begin{aligned} \Omega^2_{[A]}(\eta, \tau) = 1/2N(N-1) \int_V \chi_N(\eta^h \wedge \tau^h \overset{N-2}{\wedge} F(A)) - \\ - N \int_V \chi_N(\Theta(\eta^h, \tau^h) \overset{N-1}{\wedge} F(A)) \quad \textit{ultramonopole 2-form} \end{aligned} \tag{3.12}$$

for any $[A] \in \mathcal{M}$, τ, $\eta \in T_{[A]}\mathcal{M}$, where Θ is the curvature of ω and η^h, τ^h are the horizontal lifts of η and τ with respect to ω.

REMARK 3.2. — The form Ω^1 was already considered in [1] for the analysis of θ-vacua in the canonical formalism of Yang-Mills theories. In the same way Ω^2 appears in the study of the canonical formalism of effective Yang-Mills theories in 3 dimensions with a Chern-Simmons term in the Lagrangian [2].

We shall now show that Ω^1 and Ω^2 are generators of the cohomology groups $H^1(\mathcal{M}, \mathbb{Z})$, $H^2(\mathcal{M}, \mathbb{Z})$ when $V = S^{2N-1}$ and $V = S^{2N-2}$, respectively.

4. MAGNETIC 1-FORMS

We have shown in the preceding section that for $V = S^{2N-2} \times S^1$ or S^{2N-1} and $G = SU(N)$, the first cohomology group $H^1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. We shall show that the magnetic form Ω^1 (3.11) is in the generating class of this cohomology group [1].

PROPOSITION 4.1. — The magnetic form Ω^1 is a closed 1-form of $\mathcal{M}$.

Proof. — Let $\tilde{\Omega}^1 = \pi^*\Omega^1$ be the pull-back by the projection map $\pi : \mathcal{A} \to \mathcal{M}$. It is sufficient to show that $\tilde{\Omega}^1$ is closed. Once a connection A_0 in $\mathcal{A}$ is chosen, $\mathcal{A} \approx \Gamma(\text{ad}\, P \otimes \Lambda^1(V))$. Let $\{\tau_\alpha\}$ be any Hilbert basis of $\Gamma(\text{ad}\, P \otimes \Lambda^1(V))$. Then

$$\tilde{\Omega}^1_A(\tau_\alpha) = N \int_V \chi_N(\tau_\alpha \overset{N-1}{\wedge} F(A)) \tag{4.1}$$

Since for any $t \in \mathbb{R}$

$$F(A + t\tau_\beta) - F(A) = t d_A \tau_\beta + \frac{t^2}{2} [\tau_\beta, \tau_\beta] \tag{4.2}$$

we have

$$\delta F(A)/\delta\tau_\beta = d_A \tau_\beta \,. \tag{4.3}$$

Thus, the exterior differential of the 1-form $\tilde{\Omega}^1$ is given by

$$2\delta\tilde{\Omega}^1_A(\tau_{\alpha_1}, \tau_{\alpha_2}) = \delta\tilde{\Omega}^1(\alpha_2)/\delta\tau_{\alpha_1} - \delta\tilde{\Omega}^1(\tau_{\alpha_1})/\delta\tau_{\alpha_2} =$$
$$= N(N-1) \int_V \chi_N(\tau_{\alpha_2} \wedge d_A\tau_{\alpha_1} \overset{N-2}{\wedge} F(A)) - N(N-1) \int_V \chi_N(\tau_{\alpha_1} \wedge d_A\tau_{\alpha_2} \overset{N-2}{\wedge} F(A)) \tag{4.4}$$

Now, since for any α_i, $i = 1, \ldots, N$, Lie SU(N)-valued tensorial m_i-forms defined in P(V, SU(N))

$$d\chi_N(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_N) = \sum_{i=1}^{N} (-1)^{m_1 + \ldots + m_{i-1}} \chi_N(\alpha_1 \wedge \ldots \wedge d\alpha_i \wedge \ldots \wedge \alpha_N) \tag{4.5}$$

and

$$0 = \sum_{i=1}^{N} (-1)^{m_1+\dots m_{i-1}} \chi_N(\alpha_1 \wedge \alpha_2 \wedge \dots \wedge [A, \alpha_i] \wedge \dots \wedge \alpha_N), \quad (4.6)$$

$$d\chi_N(\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_N) = \sum_{i=1}^{N} (-1)^{m_1+\dots+m_{i-1}} \chi_N(\alpha_1 \wedge \dots \wedge d_A\alpha_i \wedge \dots \wedge \alpha_N). \quad (4.7)$$

Then,

$$\begin{aligned} 2\delta\tilde{\Omega}^1_A(\tau_{\alpha_1}, \tau_{\alpha_2}) = &- N(N-1)\int_V d\chi_N(\tau_2 \wedge \tau_1 \overset{N-2}{\wedge} F(A)) + \\ &+ N(N-1)(N-2)\int_V \chi_N(\tau_{\alpha_2} \wedge \tau_{\alpha_1} \wedge d_A F(A) \overset{N-3}{\wedge} F(A)) \end{aligned} \quad (4.8)$$

which vanishes because of the Bianchi identity $d_A F(A) = 0$ and the Stokes theorem. This means that $\tilde{\Omega}^1$ is closed. In the same way it can be shown that $\tilde{\Omega}^1$ is horizontal with respect to any connection of $\mathscr{A}(\mathscr{M}, \mathscr{G}^0)$. ||

PROPOSITION 4.2. — The magnetic form Ω^1 is not exact, and it belongs to the generating cohomology class of $H^1(\mathscr{M}, \mathbb{Z}) = \mathbb{Z}$.

Proof. — We have shown in Section 2 that the equivalence classes of principal fibre bundles $P(S^{2N-1} \times S^1, SU(N))$ are labeled by the Chern numbers $C_N(P)$. In the same way it can be shown that any principal fibre bundle on S^{2N-1} with gauge group SU(N) is trivial. Let $P_t(S^{2N-1} \times \{t\}, SU(N))$, $t \in [0, 2\pi]$ be the principal fibre bundles obtained by the restriction of a bundle P with $C_N(P) = n$ to $\pi_P^{-1}(S^{2N-1} \times \{t\})$. For any $t \in [0, 2\pi]$, P_t is a trivial bundle. The orbit space of connections of $P_0(S^{2N-1} \times \{0\}, SU(N))$ is diffeomorphic to $\mathscr{M}$.

Let B be a connection of P such that its restriction to $\pi_P^{-1}(\{(1, 0, \dots, 0)\} \times S^1)$ is trivial, and

$$\tau_t^n : P_0 \to P_t$$

be the fibre bundle isomorphism established by the parallel transport with respect to B along the curves

$$C_{x,t} : [0, t] \to S^{2N-1} \times S^1$$

such that $C_{x,t}(s) = (x, s)$, $0 \leqslant s \leqslant t$. Let $B(t)$ denote the connection of P_0 obtained by pull-back through τ_t^n of the restriction of B to P_t. Since $P_0 = P_{2\pi}$, $\tau_{2\pi}^n$ is an automorphism of P_0, which implies that $\alpha_n(t) = [B(t)]$ is a closed (differentiable) curve in $\mathscr{M}$. Let $A(t)$ be the horizontal lift of α_n with respect to any connection ω of $\mathscr{A}(\mathscr{M}, \mathscr{G}^0)$ with $A(0) = B(0)$.

Since $A(t)$ and $B(t)$ are two lifts of the same closed curve α_n of $\mathscr{M}$ there always exists a (differentiable) curve $C(t)$ in $\mathscr{A}$ such that $C(0) = A(2\pi)$ and

$C(2\pi) = B(2\pi)$ which lies completely in the fibre of $\mathscr{A}(\mathscr{M}, \mathscr{G}^0)$ over $[A(0)]$.
Since $\tilde{\Omega}^1$ is horizontal and $C(t)$ is a vertical vector we have

$$\tilde{\Omega}^1_A(\dot{C}(t)) = \int_{S^{2N-1}} \chi_N(\dot{C}(t) \overset{N-1}{\wedge} F(A)) = 0 \tag{4.9}$$

for any $t \in [0, 2\pi]$ and

$$\begin{aligned}\int_{\alpha_n} \Omega^1 &= \int_{A[t]} \tilde{\Omega}^1 \\ &= N\int_0^{2\pi} dt \int_{S^{2N-1}} \chi_N(\dot{A}(t) \overset{N-1}{\wedge} F(A(t))) = N\int_0^{2\pi} dt \int_{S^{2N-1}} (\dot{B}(t) \overset{N-1}{\wedge} F(A(t))) - \\ &- N\int_0^{2\pi} dt \int_{S^{2N-1}} \chi_N(\dot{C}(t) \overset{N-1}{\wedge} F(A(t))) = \int_{S^{2N-1}\times S^1} \chi_N(\overset{N}{\wedge} F(B)) = n\end{aligned} \tag{4.10}$$

(see Fig. 1).

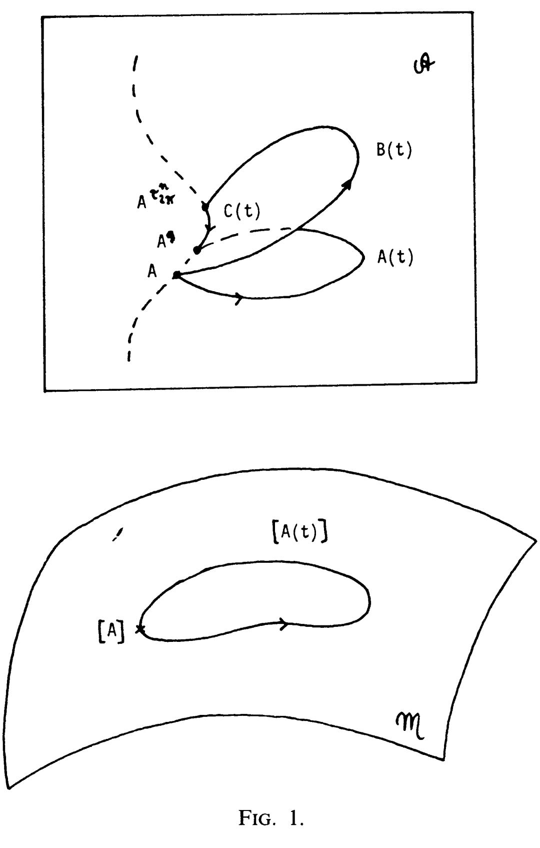

FIG. 1.

This proves that Ω^1 is not exact. Moreover since $\int_{[A(t)]} \Omega^1$ along any closed curve $[A(t)]$ constructed in such a way is an integer, and for any integer $n \in \mathbb{Z}$ there exists a curve $[A(t)]$ closed in $\mathcal{M}$ such that $\int_{[A(t)]} \Omega^1 = n$, we can conclude that Ω^1 is in the generating cohomology class of $H^1(\mathcal{M}, \mathbb{Z})$.

If $V = S^{2N-2} \times S^1$, the principal fibre bundles over $V \times S^1$ are classified by the Chern numbers $C_N(P)$ and $C_{N-1}(P)$ and those over V by $C_{N-1}(P)$. In such a case the proof proceeds in a similar way by choosing a bundle over $V \times S^1$ whose Chern number $C_{N-1}(P)$ is equal to that of the bundle over V where the connections of $\mathcal{A}$ are defined. The results for the orbit space $\overline{\mathcal{M}}$ are obtained in a similar way.

We have also shown:

COROLLARY 4.3. — The closed curve $\alpha_1 : [0, 2\pi] \to \mathcal{M}$ corresponding to $n=1$ defined in the above proof is in the generating homology class of $H_1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$ and in the first homotopy class of

$$\pi_1(\mathcal{M}) = \pi_0(\mathcal{G}^0) = \pi_{2N-1}(SU(N)) = \mathbb{Z}.$$

5. ULTRAMONOPOLE 2-FORMS

In Section 3 we have shown that for $V = S^{2N-2}$ and $G = SU(N)$, $H^2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. We shall show that the ultramonopole 2-form Ω^2 (3.12) on the orbit space $\mathcal{M}$, is in the generating class of $H^2(\mathcal{M}, \mathbb{Z})$.

PROPOSITION 5.1. — Ω^2 is a closed 2-form of $\mathcal{M}$.

Proof. — It is sufficient to show that the pull-back

$$\tilde{\Omega}^2 = \pi^* \Omega^2$$

of Ω^2 by the projection map $\pi : \mathcal{A} \to \mathcal{M}$ is closed. Let $\{ \tau_\alpha \}_{\alpha \in \mathbb{N}}$ be a Hilbert basis of $\mathcal{A}$. Then, for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$,

$$6 \delta \tilde{\Omega}^2_A(\tau_{\alpha_1}, \tau_{\alpha_2}, \tau_{\alpha_3}) = \varepsilon^{ijk} \delta / \delta \tau_{\alpha_i} \tilde{\Omega}^2_A(\tau_{\alpha_j}, \tau_{\alpha_k}) \tag{5.1}$$

because

$$[\delta / \delta \tau_{\alpha_i}, \delta / \delta \tau_{\alpha_j}] = 0 \tag{5.2}$$

Now, because of the gauge invariance of the right hand side of (3.12)

$$\delta/\delta\tau_\alpha\tilde{\Omega}^2_{\mathrm{A}}(\tau_\beta,\tau_\gamma)=1/2\mathrm{N}(\mathrm{N}-1)\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\delta/\delta\tau^h_\alpha(\tau^h_\beta)\wedge\tau^h_\gamma\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big)+$$
$$+1/2\mathrm{N}(\mathrm{N}-1)\int\chi_{\mathrm{N}}\big(\tau^h_\beta\wedge\delta/\delta\tau^h_\alpha(\tau^h_\gamma)\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big)+$$
$$+\mathrm{N}/2(\mathrm{N}-1)(\mathrm{N}-2)\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\tau^h_\beta\wedge\tau^h_\gamma\wedge d_{\mathrm{A}}\tau^h_\alpha\overset{\mathrm{N}-3}{\wedge}\mathrm{F(A)}\big)-$$
$$-\mathrm{N}\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\delta/\delta\tau^h_\alpha\Theta(\tau^h_\beta,\tau^h_\gamma)\overset{\mathrm{N}-1}{\wedge}\mathrm{F(A)}\big)-\mathrm{N}(\mathrm{N}-1)\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\Theta(\tau^h_\beta,\tau^h_\gamma)\wedge d_{\mathrm{A}}\tau^h_\alpha\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big) \tag{5.3}$$

It is easy to show that

$$6\delta\tilde{\Omega}^2_{\mathrm{A}}(\tau_{\alpha_1},\tau_{\alpha_2},\tau_{\alpha_3})=-\mathrm{N}(\mathrm{N}-1)\varepsilon^{ijk}\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(d_{\mathrm{A}}\Theta(\tau_{\alpha_i},\tau_{\alpha_j})\wedge\tau^h_{\alpha_k}\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big)+$$
$$+1/2\mathrm{N}(\mathrm{N}-1)(\mathrm{N}-2)\varepsilon^{ijk}\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\tau^h_{\alpha_i}\wedge\tau^h_{\alpha_j}\wedge d_{\mathrm{A}}\tau^h_{\alpha_k}\overset{\mathrm{N}-3}{\wedge}\mathrm{F(A)}\big)-$$
$$-\mathrm{N}(\mathrm{N}-1)\varepsilon^{ijk}\int_{\mathrm{V}}\chi_{\mathrm{N}}\big(\Theta(\tau_{\alpha_i},\tau_{\alpha_j})\wedge d_{\mathrm{A}}\tau_{\alpha_k}\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big) \tag{5.4}$$

when the symmetry properties of χ_{N} and the Bianchi identity of ω have been taken into account.

Now, by using (4.7) and the Bianchi identity $d_{\mathrm{A}}\mathrm{F(A)} = 0$, we obtain

$$6\delta\tilde{\Omega}^2_{\mathrm{A}}(\tau_{\alpha_1},\tau_{\alpha_2},\tau_{\alpha_3})=-\mathrm{N}(\mathrm{N}-1)\varepsilon^{ijk}\int_{\mathrm{V}}d\chi_{\mathrm{N}}\big(\Theta(\tau_{\alpha_i},\tau_{\alpha_j})\wedge\tau_{\alpha_k}\overset{\mathrm{N}-2}{\wedge}\mathrm{F(A)}\big)+$$
$$+1/2(\mathrm{N}-1)(\mathrm{N}-2)\varepsilon^{ijk}\int_{\mathrm{V}}d\chi_{\mathrm{N}}\big(\tau^h_{\alpha_i}\wedge\tau^h_{\alpha_j}\wedge\tau^h_{\alpha_k}\overset{\mathrm{N}-3}{\wedge}\mathrm{F(A)}\big), \tag{5.5}$$

which vanishes because of the Stokes theorem. ||

PROPOSITION 5.2. — The ultramonopole 2-form Ω^2 is not exact and is a generator of the second cohomology group $\mathrm{H}^2(\mathcal{M},\mathbb{Z}) = \mathbb{Z}$.

Proof. — Let $\sigma : \mathrm{S}^2 \to \mathcal{M}$ be a differentiable map in the generating class of the second homotopy group of $\mathcal{M}$, $\pi_2(\mathcal{M}) = \pi_1(\mathcal{G}^0) = \pi_{2\mathrm{N}-1}(\mathrm{SU(N)}) = \mathbb{Z}$ Let $\alpha : (0, 2\pi)^2 \to \mathrm{S}^2 - \{(0, 0, 1)\}$ be a parametrization of the S^2 sphere without its north pole, defined by composition of the $1/z$ coordinates (s, t) of the open disc $\mathring{\mathrm{D}}_2 = \{ x\in\mathbb{R}^2, \|x\| < 1 \} \approx \mathbb{R}^2$ with the inverse of the stereographic map. This map α can be extended to the natural map

$$\bar{\alpha} : \mathrm{S}^1 \times \mathrm{S}^1 \to \mathrm{S}^1 \wedge \mathrm{S}^1 \simeq \mathrm{S}^2 .$$

Let A be a given connection in the fibre over $[\mathrm{A}] = \sigma(0, 0, 1)$. We define $\mathrm{A}(s, t_0)$ as the horizontal lift with respect to the curve $\sigma_0\bar{\alpha}(s, t_0)$ of $\mathcal{M}$ such that $\mathrm{A}(0, t_0) = \mathrm{A}$ (see Fig. 2).

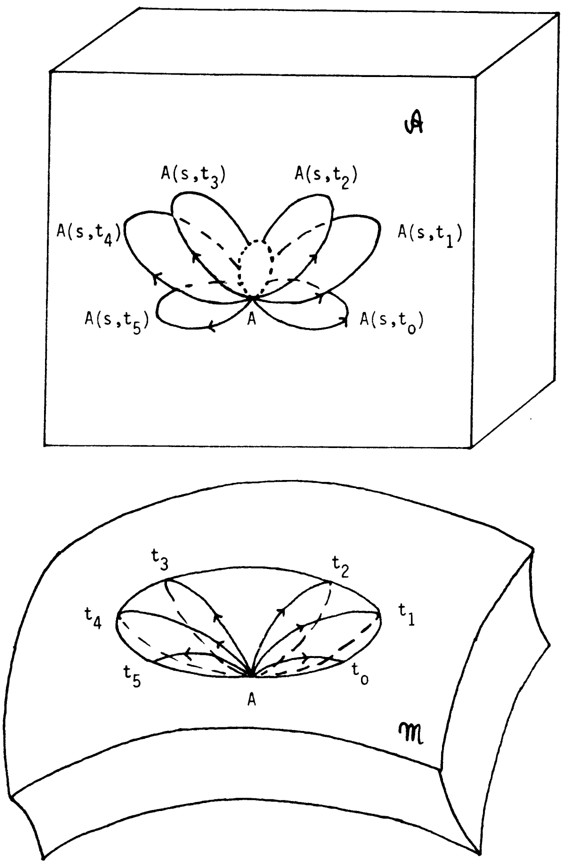

FIG. 2.

Let $\tau(t)$ be the elements of $\mathscr{G}^0$ defined by

$$A(2\pi, t) = A(0, t)^{\tau(t)}. \tag{5.6}$$

The closed curve τ in $\mathscr{G}^0$ belongs to the generating class of $\pi_1(\mathscr{G}^0)$ because of the exact sequence of homotopy groups

$$0 \simeq \pi_2(\mathscr{A}) \rightarrow \pi_2(\mathscr{M}) \rightarrow \pi_1(\mathscr{G}^0) \rightarrow \pi_1(\mathscr{A}) \simeq 0 \tag{5.7}$$

Let us consider a principal fibre bundle $P_{2N}(S^{2N-2} \times S^2, SU(N))$ such that

$$\begin{aligned} C_{N-1}(P_{2N}) &= C_{N-1}(P) \\ C_N(P_{2N}) &= 1 \end{aligned} \tag{5.8}$$

and let B be any connection of P_{2N}. Recall that P is the principal bundle where the connections of $\mathscr{A}$ are defined. Parallel transport with respect to B along the curves $\overline{\alpha}(s, t_0)$ induces a family $\phi(s, t)$ of isomorphisms of fibre bundles

$$\phi(s, t) : P^{s,t}(S^{2N-2} \times \{(s, t)\}, SU(N)) \rightarrow P^{(0,0)}(S^{2N-2} \times \{0, 0)\}, SU(N)) \tag{5.9}$$

where $P^{s,t}(S^{2N-2} \times \{ s, t \}, SU(N))$ is the family of principal fibre bundles obtained by the restriction of $P_{2N}(S^{2N-2} \times S^2, SU(N))$.

Because of (5.8), $P^{(0,0)} \approx P$ and the closed curve $\phi(2\pi, t)$ of $\mathscr{G}^0$ belongs to the generating class of $\pi_1(\mathscr{G}^0)$ too. Let us choose a connection B_0 in P_{2N} such that the corresponding maps $\phi(2\pi, t)$ coincide with $\tau(t)$. The existence of such a connection is obvious.

Let $g_{t,t'} : P^{s,t} \to P^{s,t'}$ be the isomorphism of fibre bundles given by

$$g_{t,t'}(u) = \phi^{-1}(s, t')\phi(s, t)(u) \tag{5.10}$$

for any $u \in P^{s,t}$.

Then, we define a new connection B_1 in P_{2N} such that for any $v_u \in T_uP_{2N}$ with $\pi(u) = (x, \overline{\alpha}(s, t))$

$$\begin{aligned} B_1(v_u) = {} & [\phi^*(s, t)A(s, t)]\left(v_u^V + v_u^{2N-2}\right) - \tilde{v}^t B_0(\partial_{t'} g_{tt'}(u)\,|_{t=t'}) \\ & + \tilde{v}^t \omega(\partial_t A(s, t))(\phi(s, t)(u)) \end{aligned} \tag{5.11}$$

where v_u^{2N-2} and v_u^V are the components of v_u defined as follows. By definition the connection B_0 induces a splitting of T_uP_{2N}, $v_u = v_u^h + v_u^V$, and $v_u^h = v_u^{2N-2} + v_u^2$ also splits because of the product structure of the base manifold $S^{2N-2} \times S^2$. Let v^t be the t-component of $\pi_*(v_u^2)$ in the splitting of $T_{\alpha(s,t)}S^2$ induced by the coordinates (s, t) when $s,\ t \neq 0, 2\pi$. In the case $s, t = 0,\ 2\pi$, $v^t = \pi_*(v_u^2)$. $\tilde{v}^t$ denotes the real number given by $\tilde{v}^t\partial_t = v^t$. It is easy to verify that B_1 as defined above is a global continuous connection in P_{2N}.

Let $\eta : S^{2N-2} - \{(0, \dots, 0, 1)\} \to P^{(0,0)} \approx P$ be a maximal cross-section of P. This section can be extended to a maximal cross-section $\overline{\eta} : (S^{2N-2} - \{(0, \dots, 0, 1)\}) \times (S^2 - \{(0, 0, 1)\}) \to P_{2N}$ of P_{2N} by parallel transport with respect to B_1 along the curves $\overline{\alpha}(s, t_0)$. The definition of $\overline{\eta}$ is given by:

$$\overline{\eta}(x, \alpha(s, t)) = \phi^{-1}(s, t)(\eta(x)) \tag{5.12}$$

for any $(x, \alpha(s, t)) \in (S^{2N-2} - \{(0, \dots, 0, 1)\}) \times (S^2 - \{(0, 0, 1)\})$. In this gauge $B_{1,x} = \eta^*A(s, t)$, $B_{1,s} = 0$ and $B_{1,t} = \omega(\partial_t A(s, t)) \circ \eta$ by construction of $\overline{\eta}$ and B_1.

Since $C_N(P_{2N}) = 1$,

$$\begin{aligned} 1 = {} & \int_{S^{2N-2} \times S^2} \chi_N(F(B) \wedge \overset{N}{\dots} \wedge F(B)) = \\ = {} & N \int_0^{2\pi} ds \int_0^{2\pi} dt \int_{S^{2N-2}} \chi_N(\partial_s \omega(\partial_t(A(s, t)) \overset{N-1}{\wedge} F(A(s, t))) \circ \eta - \\ & - N(N-1) \int_0^{2\pi} ds \int_0^{2\pi} dt \int_{S^{2N-2}} \chi_N(\partial_s A(s, t) \wedge \partial_t A(s, t) - \\ & - d_{A(s,t)}\omega(\partial_t A(s, t) \overset{N-2}{\wedge} F(A(s, t))) \circ \eta . \end{aligned} \tag{5.13}$$

Now, since

$$\partial_t A(s, t) - d_{A(s,t)}\omega(\partial_t A(s, t)) \tag{5.14}$$

and $\partial_s A(s, t)$ are horizontal with respect to ω,

$$\partial_s \omega(\partial_t A(s, t)) = 2\Theta(\partial_s A(s, t), \partial_t A(s, t)) \tag{5.15}$$

Thus, the r. h. s. of (5.13) equals to

$$-2\int_0^{2\pi} ds \int_0^{2\pi} dt \tilde{\Omega}^2(\partial_s(\sigma_0 \alpha)(s, t), \partial_t(\sigma_0 \alpha)(s, t)) = -\int_\sigma \Omega^2 \tag{5.16}$$

and

$$\int_\sigma \Omega^2 = -1 \tag{5.17}$$

which shows that Ω^2 is not exact and belongs to the generating class of $H^2(\mathcal{M}, \mathbb{Z})$.

REMARK 5.3. — The pull-back of the restriction of the connection B_1 to $P_{2N}^{s,t}(V, SU(N))$ through the map $\phi^{-1}(s, t)$ is by definition $A(s, t)$. In this way the S^2-surface σ of $\mathcal{M}$ is naturally associated to B_1. Notice that $\phi(s, t_0)$ can be itself defined in terms of B_1. It is the map induced by parallel transport with respect to the connection B_1 along the curves $\bar{\alpha}(s, t_0)$. Hence, this dimensional reduction method leads to a correspondence between maps of S^2 into $\mathcal{M}$ and connections of P_{2N}. This correspondence can also be derived from the Narasimhan-Ramanan theorem [*8*], which shows the surjective character of the map

$$K : \mathrm{Maps}_0^P(V, BSU(N)) \to \mathcal{M} \tag{5.18}$$

defined by pull-back of the universal connection $\mathbb{A}_0$ of the universal bundle $E(BSU(N), SU(N))$, i. e. $K(\varphi) = [\varphi^* \mathbb{A}_0]$. Let $\mathcal{M}_{S^2 \times V, P}^{SU(N)}$ denote the subset of the extended orbit space defined by the orbits of connections of fibre bundles $P_{2N}(S^2 \times V, SU(N))$ whose Chern class $C_{2N}(P_{2N})$ is equal to $C_N(P)$, and $\mathrm{Maps}_0^P(S^2 \times V, BSU(N))$ the set of pointed maps $\varphi : S^2 \times V \to BSU(N)$ such that for any $a \in S^2$ the map $\varphi_a : V \longrightarrow BSU(N)$ defined by

$$\varphi_a(x) = \varphi(a, x) \tag{5.19}$$

induces the bundle P by pull-back from E. Then, if we define the correspondence

$$K_c : \mathcal{M}_{S^2 \times V, P}^{SU(N)} \to \mathrm{Maps}(S^2, \mathcal{M}) \tag{5.20}$$

by

$$K_c([\varphi^* \mathbb{A}_0])(a) = [\varphi_a^* \mathbb{A}_0] \tag{5.21}$$

for any $[\varphi^* \mathbb{A}_0] \in \mathcal{M}_{S^2 \times V, P}^{SU(N)}$, K_c does coincide with the correspondence obtained by means of the dimensional reduction method. ||

6. APPLICATION TO GAUGE ANOMALIES

We have already mentioned in the Introduction the relevance of the magnetic and ultramonopole forms Ω^1 and Ω^2, respectively, for the Hamiltonian formulation of the quantum theory of Yang-Mills fields in 3 + 1 dimensions with a θ-vacuum term in the Lagrangian and 2 + 1 dimensions with a Chern-Simmons one [*1*] [*2*] [*5*] [*15*].

On the other hand Atiyah-Singer [*4*] and Stora [*13*] have shown the existence of a relation between the ultramonopole form Ω^2 and the non-abelian gauge anomalies in S^{2N-2}. We shall show the existence of a similar relationship between the magnetic form Ω^1 and the global anomalies in S^{2N-1}.

Since $\mathscr{A}$ is contractible, $\tilde{\Omega}^1$ is not only a closed 1-form but also an exact form. Let A_0 be a fixed connection of $\mathscr{A}$, and define $A_t = A_0 + t(A - A_0)$ for any $A \in \mathscr{A}$ and $0 \leqslant t \leqslant 1$.

PROPOSITION 6.1. — Any real function Z defined on $\mathscr{A}$ verifying that $\partial Z = \tilde{\Omega}^1$ is equal to the Chern-Simmons function

$$W_{A_0}(A) = N \int_0^1 dt \int_{S^{2N-1}} \chi_N((A - A_0) \wedge F(A_t) \wedge \overset{N-1}{\ldots} \wedge F(A))$$

modulo a constant. Moreover, the restriction of W_{A_0} to the fibre $[A_0]$ of $\mathscr{A}$ is a non trivial element of $H^0([A_0], \mathbb{Z}) = \mathbb{Z}$.

Proof. — The first statement is a consequence of the Poincaré lemma. In fact, for any vector τ of the tangent space $T_A\mathscr{A}$

$$\begin{aligned} \delta W_{A_0}(A)/\delta\tau &= N \int_0^1 dt \int_{S^{2N-1}} \chi_N(\tau \overset{N-1}{\wedge} F(A_t)) + \\ &\quad + N(N-1) \int_0^1 tdt \int_{S^{2N-1}} \chi_N((A - A_0) \wedge d_A\tau \overset{N-2}{\wedge} F(A_t)) \\ &= N \int_0^1 dtd/dt \int_{S^{2N-1}} \chi_N(t\tau \overset{N-1}{\wedge} F(A_t)) = \tilde{\Omega}^1_A(\tau) \end{aligned} \tag{6.1}$$

Thus, $\delta(W_{A_0}(A) - Z(A)) = 0$ and $W_{A_0}(A) = \alpha(A) + C$ because $\mathscr{A}$ is connected. Since $\tilde{\Omega}^1$ is horizontal with respect to any connection ω of $\mathscr{A}(\mathscr{M}, \mathscr{G}^0)$, the restriction of $W_{A_0}(A)$ to $[A]$ is locally constant i. e. $\delta W_A(A)/\delta\tau = \tilde{\Omega}^1(\tau) = 0$ for any $\tau = d_A\phi$. Since $W_{A_0}(A_0) = 0$, W_{A_0} vanishes in the connected component of $[A_0]$, which contains A_0. For any other connection A in the fibre $[A_0]$,

$$W_{A_0}(A) = \int_{[A_t]} \Omega^1 = n \in \mathbb{Z} \tag{6.2}$$

because $[A_t]$ is a closed curve of $\mathscr{M}$ and $\Omega^1 \in H^1(\mathscr{M}, \mathbb{Z})$ by Proposition 4.2. ||

Let E be the vector bundle associated to the fundamental representation of SU(N), S the spin bundle over S^{2N-1} and $\not{D}_A : C^\infty(S \otimes E) \to C^\infty(S \otimes E)$ the corresponding Dirac operator.

PROPOSITION 6.2 [*6*] [*10*]. — The ζ-function renormalized determinant of the Dirac operator $\not{D}_A$ is given by

$$\det \not{D}_A = e^{i\pi W_{A_0}(A) + \phi(A)} \tag{6.3}$$

where ϕ is a gauge invariant function.

For completness we give a proof:

Proof. — It is generalization of the Witten analysis for the Z_2-anomaly in four dimensions [*14*]. In this case proposition 3.1 implies that $H^2(\mathscr{M}, \mathbb{Z}) = 0$. Thus, the non-abelian local gauge anomaly vanishes [*4*], which implies that the restriction of $\det \not{D}_A$ to $[A]$ is locally constant.

However, it is not globally constant. In fact, the spectral flow of $\det \not{D}_{A_t}$ along a family of connections A_t is given by the index of the Dirac operator D_B^+ over $S^{2N-1} \times S^1$, B being the connection associated to A_t in the proof of proposition 4.2 [*6*] [*10*]. Now by the index theorem

$$\operatorname{Ind} D_B^+ = \int_{S^{2N-1} \times S^1} \chi_N(\overset{N}{\wedge} F(B)) = \int_{[A_t]} \Omega^1 = W_{A_0}(A) \tag{6.4}$$

Therefore, the number of eigenvalues of $D_{A_0}^+$ which change their sign when A_t goes from A_0 to A is $W_{A_0}(A)$. Thus, the ζ-function renormalized determinant

$$\det \not{D}_A = (-1)^{W_{A_0}(A)} \det D_{A_0} = \det D_{A_0} e^{i\pi W_{A_0}(A)} \tag{6.5}$$

For any other fibre $[A]$ of $\mathscr{A}$ we choose a connection $A_1 \in [A]$ and define $\phi(A_1) = -i\pi W_{A_0}(A_1) + \log \det \not{D}_{A_1}$. Then, the derivation of (6.3) for any connection in $[A_1]$ can be obtained in the same way as (6.5) by substituing A_1 for A_0. ||

Finally we remark that, by means of the dimensional reduction methods used in the proofs Propositions 4.2, 5.2, 6.1 and 6.2, similar results can be obtained for the first and second cohomology groups of the orbits spaces associated to different space-time manifolds such as T^{2N-1}, T^{2N-2}, $S^{2N-m} \times T^{m-1}$, etc.

ACKNOWLEDGMENT

We thank R. Stora for interesting conversations on anomalies and cohomology. P. K. M. thanks M. S. Narasimhan for a helpful conversation.

REFERENCES

[1] M. ASOREY, P. K. MITTER, CERN preprint TH 3424, 1982.
[2] M. ASOREY, P. K. MITTER, *Phys. Lett.*, t. **153 B**, 1985, p. 147.
[3] M. ATIYAH, R. BOTT, *Phil. Trans. R. Soc.* (London), t. **308 A**, 1982, p. 523.
[4] M. ATIYAH, I. M. SINGER, *Proc. Natl. Acad. Sci.*, t. **81**, 1984, p. 2597.
[5] R. JACKIW, in: *E. Fradkin Festchrift*, ed. A. Hilger, Bristol, 1985 (to be published) and in *Proceedings of the N. Bohr Centenial Symposium*, North Holland (to be published), MIT preprint CTP 1254.
[6] J. LOTT, *Phys. Lett.*, t. **145 B**, 1984, p. 179.
[7] P. K. MITTER, C. M. VIALLET, *Commun. Math. Phys.*, t. **79**, 1981, p. 457.
[8] M. S. NARASIMHAN, S. RAMANAN, *Amer. J. Math.*, t. **83**, 1961, p. 563.
[9] M. S. NARASIMHAN, T. R. RAMADAS, *Commun. Math. Phys.*, t. **67**, 1979, p. 21.
[10] A. N. REDLICH, *Phys. Rev. Lett.*, t. **52**, 1984, p. 18.
[11] I. M. SINGER, *Commun. Math. Phys.*, t. **60**, 1978, p. 7.
[12] N. STEENROD, *Topology of Fibre Bundles*, Princeton, 1951.
[13] R. STORA, in: *Progress in Gauge Field Theory*, eds. G't Hooft *et al.*, Plenum, N. Y., 1984.
[14] E. WITTEN *Phys. Lett.*, t. **117 B**, 1982, p. 324.
[15] V. S. WU, A. ZEE, University of Washington preprint 40048-38P 4, 1984.

(Manuscrit reçu le 26 novembre 1985)