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TAKU MATSUI

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Explicit formulas for correlation functions of ground states of the 1 dimensional XY model

by

Taku Matsui
Research institute for Mathematical Sciences Kyoto university,
Kyoto, 606, Japan

ABSTRACT. — We consider the odd correlation functions of pure ground states for the 1 dimensional XY model. A reduction formula is proved. 1 and 2 point functions are computed explicitly.

RÉSUMÉ. — On considère les fonctions de corrélation impaires des états fondamentaux purs pour le modèle XY sur un réseau à 1 dimension. On démontre une formule de réduction. On calcule aussi les fonctions de corrélation à 1 point et à 2 points.

§ 1. INTRODUCTION

The structure of ground state representations of the XY model on a 1 dimensional lattice is analyzed in [*1*].

The Hamiltonian is specified by two parameters λ and γ :

$$\mathrm{H} = -\Sigma \{ (1+\gamma)\sigma_x^{(i)}\sigma_x^{(i+1)} + (1-\gamma)\sigma_y^{(i)}\sigma_y^{(i+1)} + 2\lambda\sigma_z^{(i)} \} \tag{1.1}$$

where $\sigma_\alpha^{(i)}$ ($\alpha = x, y, z$) are Pauli spin matrices located on each lattice site i.

It defines the time automorphism α_t of the C* algebra $\mathscr{A}$ generated by $\sigma_\alpha^{(i)}$ ($i \in \mathbb{Z}$ $\alpha = x, y, z$).

The time automorphism α_t has a $\mathbb{Z}_2$ symmetry i. e. it commutes with the automorphism Θ determined by

$$\Theta(\sigma_\alpha^{(i)}) = -\Theta(\sigma_\alpha^{(i)})\alpha = x, y \qquad \Theta(\sigma_z^{(i)}) = \sigma_z^{(i)}. \tag{1.2}$$

The algebra $\mathscr{A}$ is graded by Θ:

$$\mathscr{A} = \mathscr{A}_+ + \mathscr{A}_-, \qquad \mathscr{A}_\pm = \{ a \in \mathscr{A} : \Theta(a) = \pm a \}.$$

The number of pure ground states is

i) 1 if $|\lambda| \geqq 1$ or $\gamma = 0$,
ii) 2 if $|\lambda| < 1, \gamma \neq 0$ and $(\lambda, \gamma) \neq (0, \pm 1)$,
iii) ∞ if $(\lambda, \gamma) \neq (0, \pm 1)$.

In the case *ii*) pure ground states are denoted by $\varphi^{\pm}$.
They are translationally invariant but Θ symmetry is broken.

$$\varphi^{+} \circ \Theta = \varphi^{-}. \tag{1.3}$$

In the case *iii*) there are pure ground states $\varphi^{\pm}$ which are limits of $\varphi^{\pm}$ for the case *ii*). In addition, two other mutually disjoint ground state representations, with two families of soliton-like ground states, exist. (The soliton-like states are product states of the spin algebra, so correlation functions can be immediately computed.)

The purpose of this note is to give an explicit formula for correlation functions corresponding to $\varphi^{\pm}$, namely we give a reduction formula for odd correlation functions and compute the 1 point functions.

Note that even correlation functions can be immediately computed from 2 point functions as a quasifree state.

The 2 point functions are also explicitely given.

§ 2. NOTATIONS

To describe our results, we introduce the CAR elgebra and the Jordan Wigner transformation (on a two sided infinite chain).

Let Θ_{-} be an automorphism of $\mathscr{A}$ determined by

$$\Theta_{-}\left(\sigma_{\alpha}^{(j)}\right) = \varepsilon^{j} \sigma_{\alpha}^{(j)} \qquad \alpha = x, y \tag{2.1a}$$

$$\Theta_{-}\left(\sigma_{z}^{(j)}\right) = \sigma_{z}^{(j)} \tag{2.1b}$$

$$\varepsilon^{j} = \begin{cases} 1 & j \geqq 1 \\ -1 & j \leqq 0 \end{cases} \tag{2.1c}$$

$\tilde{\mathscr{A}}$ is the crossed product of $\mathscr{A}$ by this $\mathbb{Z}_2$ action ($\Theta_{-}^{2} = \text{id.}$), i. e. $\tilde{\mathscr{A}}$ is generated by $\mathscr{A}$ and a selfadjoint unitary T satisfying

$$\mathrm{TAT} = \Theta_{-}(\mathrm{A}) \qquad \mathrm{A} \in \mathscr{A}. \tag{2.2}$$

Fermion creation and anihilation operators are introduced as follows:

$$c_j^{*} = \mathrm{TS}_j\left(\sigma_x^{(j)} + i\sigma_y^{(j)}\right) \times \frac{1}{2}$$
$$c_j = \mathrm{TS}_j\left(\sigma_x^{(j)} - i\sigma_y^{(j)}\right) \times \frac{1}{2} \tag{2.3}$$

where
$$S_j = \begin{cases} \prod_{k=1}^{j-1} \sigma_z^{(k)} & j \geqq 1, \\ \mathbb{1} & j = 0, \\ \prod_{k=j}^{0} \sigma_z^{(k)} & j < 0. \end{cases} \tag{2.4}$$

They satisfy the CAR (canonical anticommutation relations).

The ground state of the XY model is described in terms of a non gauge-invariant Fock state of the CAR. Thus we use the selfdual formalism of CAR algebra:

Let $K = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ and

$$B(h) = c^*(f) + c(g) \qquad (f, g) \in K, \tag{2.5a}$$

$$c^*(f) = \Sigma c_j^* f_j \qquad c(g) = \Sigma c_j g_j \qquad \{ f_j \} \{ g_j \} \in l^2(\mathbb{Z}). \tag{2.5b}$$

Let $\Gamma\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}$ where the bar denotes complex conjugation, then $B^*(h) = B(\Gamma h)$.

$\mathscr{A}^{CAR}$ is the C* subalgebra of $\tilde{\mathscr{A}}$ generated by $B(h)$, $h \in K$.

A Fock state φ_E of $\mathscr{A}^{CAR}$ is determined by any projection E having the property $\Gamma E \Gamma = 1 - E$ via characterization

$$\varphi_E(B(h)^* B(h)) = \| Eh \|^2 . \tag{2.6}$$

Let U be a unitary operator of K satisfying $\Gamma U \Gamma = U$.

The Bogoliubov automorphism α_U is defined by

$$\alpha_U(B(h)) = B(Uh). \tag{2.7}$$

Θ_- has a natural extension to $\tilde{\mathscr{A}}$ by $\Theta_-(T) = T$ and its restriction to $\mathscr{A}^{CAR}$ is a Bogoliubov automorphism α_θ :

$$\Theta_-(B(h)) = B(\theta_- h), \tag{2.8}$$

$$\theta_-\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \theta_- f \\ \theta_- g \end{pmatrix} \qquad \text{and} \qquad (\theta_- f)_j = \varepsilon^j f_j . \tag{2.9}$$

The ground state of the XY model which is invariant under the lattice translation and Θ is given by

$$\varphi(A_+ + A_-) = \varphi_E(A_+) \qquad A_\pm \in \mathscr{A}_\pm \tag{2.10}$$

for some basis projection E (cf. § 4).

Note that $\mathscr{A}_+ \subset \mathscr{A}^{\mathrm{CAR}}$.

In the case $|\lambda| < 1$ and $\gamma \neq 0$, pure ground states $\varphi^{\pm}$ are

$$\varphi^{\pm}(\mathrm{A}_+ + \mathrm{A}_-) = \frac{1}{2}\{ \varphi_{\mathrm{E}}(\mathrm{A}_+) \pm (\Omega_{\mathrm{E}}, \mathrm{U}(\Theta_-)\Pi(\mathrm{TA}_-)\Omega_{\mathrm{E}}) \}. \tag{2.11}$$

Here Ω_{E} is the GNS cyclic vector of the Fock state φ_{E}, Π is the GNS representation and $\mathrm{U}(\Theta_-)$ is a selfadjoint unitary of the Fock space which implements θ_- i. e.

$$\mathrm{U}(\Theta_-)\Pi(\mathrm{Q}) = \Pi(\mathrm{Q})\mathrm{U}(\Theta_-), \qquad \mathrm{Q} \in \mathscr{A}^{\mathrm{CAR}}. \tag{2.12}$$

Note that $\mathrm{TA}_- \in \mathscr{A}^{\mathrm{CAR}}$ if $\mathrm{A}_- \in \mathscr{A}_-$.

§ 3. MAIN RESULTS

Our interest is in the odd correlation functions in the case $|\lambda| < 1, \gamma \neq 0$. The $2n + 1$ point functions

$$\varphi^{\pm}(\sigma_{\alpha_1}^{(j_1)} \dots \sigma_{\alpha_{2n+1}}^{(j_{2n+1})}) \qquad \alpha_k = x, y \qquad j_k \in \mathbb{Z} \tag{3.1}$$

reduce to the 1 point functions $\varphi^{\pm}(\sigma_x^{(j)})$ or $\varphi^{\pm}(\sigma_y^{(j)})$ by the reduction formula which will be proved in § 5.

Note that $\mathrm{B}(f_j)$ with $f_j \in (1 - \mathrm{E})\mathrm{K}$ can be handled by the CAR and $\Pi(\mathrm{B}(f_j))\Omega_{\mathrm{E}} = 0$.

Reduction formula.

Set

$$\mathrm{A} \equiv \mathrm{E}\theta_-(1 - \mathrm{E})\theta_-, \qquad \mathrm{D} \equiv \mathrm{A}^*(1 - \mathrm{A})^{-1}(1 - \mathrm{E} \wedge \theta_-(1 - \mathrm{E})\theta_-) \tag{3.2}$$

and

$$[f_1, f_2, f_{2k+1}]^{(k)} \equiv (\Omega_{\mathrm{E}}, \mathrm{U}(\theta_-)\Pi(\mathrm{B}(f_1) \dots \Pi(\mathrm{B}(f_{2k+1})\Omega_{\mathrm{E}}) \tag{3.3}$$

If $f_1, f_2, \dots, f_{2k+1} \in \mathrm{EK}$, then

$$[f_1, f_2 f_{2k+1}]^{(k)} = \sum_{j=2}^{2k+1} (-1)^j (\Gamma \mathrm{D} f_1, f_j)[f_2 \check{f}_j, f_{2k+1}]^{(k-1)}$$
$$+ \sum_{j=3}^{2k+1} (-1)^{j+1} (\Gamma \mathrm{D} f_2, f_j)[\mathrm{E} \wedge \theta_-(1 - \mathrm{E})\theta_- f_1, f_2 f_j]^{(k-1)}, \tag{3.4}$$

where $\check{f}_j$ denotes the absence of f_j.

We compute the 1 point functions by the cluster property of the 2 point functions, taking advantage of the translation invariance of ground states $\varphi^{\pm}$. They are computed in section 6.

2 point functions.

$$\varphi^{\pm}(\sigma_x^{(1)}\sigma_x^{(j)}) = \det X_{1,k}(\lambda, \gamma), \qquad 1 \leqq 1, k \leqq j-1 \tag{3.5a}$$

$$\varphi^{\pm}(\sigma_y^{(1)}\sigma_y^{(j)}) = \det X_{1,k}(\lambda, -\gamma), \qquad 1 \leqq 1, k \leqq j-1 \tag{3.5b}$$

$$\varphi^{\pm}(\sigma_x^{(1)}\sigma_y^{(j)}) = \varphi^{\pm}(\sigma_y^{(1)}\sigma_x^{(j)}) = 0, \qquad \text{for} \quad j > 1. \tag{3.5c}$$

$$\varphi^{\pm}(\sigma_z^{(j)}) = X_{1,0}. \tag{3.5d}$$

Here $X_{1,k}$ is a Tœplitz matrix whose entries are

$$X_{1,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\cos\theta - \lambda) - i\gamma\sin\theta}{\mu(\theta)} e^{-i(1-k-1)\theta} d\theta, \tag{3.6}$$

$$\mu(\theta) = [(\cos\theta - \lambda)^2 + \gamma^2 \sin^2\theta]^{1/2}. \tag{3.7}$$

1 point functions.

$$\varphi^{\pm}(\sigma_x^{(j)}) = \begin{cases} \pm(1-\lambda^2)^{1/8}\left[\dfrac{4\gamma}{(1+\gamma)^2}\right]^{1/4} & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma < 0. \end{cases} \tag{3.8}$$

$$\varphi^{\pm}(\sigma_y^{(j)}) = \begin{cases} 0 & \text{if} \quad \gamma > 0, \\ \pm(1-\lambda^2)^{1/8}\left[\dfrac{-4\gamma}{(1-\gamma)^2}\right]^{1/4} & \text{if} \quad \gamma < 0. \end{cases} \tag{3.9}$$

§ 4. FOCK STATES

We present here an explicit form of the basis projection E which specifies the Fock state we need.

$K = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ can be identified with $L^2(S^1) \oplus L^2(S^1)$ by the Fourier transform. Then E is a multiplication operator

$$E(\theta) = 1/2 \left\{ 1 + \frac{1}{\mu(\theta)} \begin{bmatrix} \cos\theta - \lambda & -i\gamma\sin\theta \\ i\gamma\sin\theta & -(\cos\theta - \lambda) \end{bmatrix} \right\} \tag{4.1}$$

($\theta \in \mathbb{R} \bmod 2\pi$ denotes a point on S^1 and the 2×2 matrix refers to the 2 direct summands.)

LEMMA 1. — If $|\lambda| < 1$ and $\gamma \neq 0$, then

$$\dim \theta_- E \theta_- \wedge (1 - E) = \dim E \wedge \theta_-(1 - E)\theta_- = 1. \tag{4.2}$$

Proof. — Let $v = 2^{-1/2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then

$$vEv^* = \frac{1}{2}\left\{ 1 + \begin{bmatrix} 0 & Y^* \\ Y & 0 \end{bmatrix} \right\}, \tag{4.3}$$

$$Y = \frac{(\cos\theta - \lambda) - i\gamma\sin\theta}{\mu(\theta)}. \tag{4.4}$$

If $h \in ((1 - E) \wedge \theta_- E\theta_-)K$, $vh = \begin{pmatrix} f \\ g \end{pmatrix}$ satisfies the following equations:

$$f + Y^*g = 0, \qquad f - \theta_- Y^* \theta_- g = 0 \tag{4.5}$$

or equivalently

$$\{ Y^* + \theta_- Y^* \theta_- \} g = 0, \tag{4.6a}$$

$$\{ Y + \theta_- Y \theta_- \} f = 0. \tag{4.6b}$$

The first equation of (4.5) is equivalent due to $YY^* = 1$. If (4.6*b*) is satisfied, the $g = Yf$ automatically satisfies both equations of (4.5) due to $Y^*Y = 1$ and $\theta_-^2 = 1$.

Therefore we have only to solve (4.6*b*).

Let p and q be mutually orthogonal projections such that $p - q = \theta_-$. The equation (4.6*b*) reduces to

$$pYpf = qYqf = 0. \tag{4.7}$$

The operator X is the product of a shift operator U or $U^*((Uf)_j = f_{j+1}$ for $f = \{ f_j \} \in l^2(\mathbb{Z}))$ and a unitary W,

$$Y = \begin{cases} WU = UW & \text{if } \gamma > 0, \\ WU^* = U^*W & \text{if } \gamma < 0, \end{cases} \tag{4.8}$$

where W is a multiplication operator of the function

$$\frac{(\cos\theta - \lambda) - i\gamma\sin\theta}{\mu(\theta)} e^{i(\operatorname{sgn}\gamma)\theta}, \tag{4.9}$$

which has the winding number zero.

By the definition of U, p, and q,

$$U^*p = pU^*p, \qquad Uq = qUq. \tag{4.10}$$

Thus we are solving the following equations.

(case 1) $\gamma > 0$

$$(pUpWq)f = (qWqUq)f = 0. \tag{4.11}$$

(case 2) $\gamma < 0$

$$(pWpU^*p)f = (qU^*qWq)f = 0. \tag{4.12}$$

By the theory of Tœplitz operators (cf. [*3*]), qWq qW^*q, pWp, and pW^*p

are invertible as operators of $ql^2(\mathbb{Z})$ or $pl^2(\mathbb{Z})$. If $\gamma > 0$, we obtain from (4.11)

$$(pWp)f \in \ker pUp = \mathbb{C}e_1, \tag{4.13}$$

$$(qUq)f = 0 \text{ i. e. } qf = 0, \tag{4.14}$$

where $(e_i)_j = \delta_{ij}$. By (4.13) and (4.14), we have

$$f = c(pWp)^{-1}e_1 \qquad \text{(where } c \text{ is a constant.)} \tag{4.15}$$

Thus $\dim \theta_- E \theta_- \wedge (1 - E) = 1$.

The case $\gamma < 0$ is proved by the same way. Q. E. D.

Remark. — The proof of this lemma is suggested to the author by Professor D. Evans.

§ 5. PROOF OF THE REDUCTION FORMULA

Suppose $f_1, f_2, f_{2k+1} \in EK$. Due to $\Pi(B(f))\Omega = \Pi(B(Ef))\Omega$,

$$\begin{aligned}
[f_1, \ldots, f_{2k+1}]^{(k)} &= (\Omega, U(\theta_-)\Pi(B(f_1) \ldots B(f_{2k+1}))\Omega) \\
&= (\Pi(B(E\Gamma\theta_- f_1))\Omega, U(\theta_-)(B(f_2) \ldots)\Omega) \\
&= (\Omega, U(\theta_-)\Pi(B(\theta_-(1-E)\theta_- f_1)B(f_2) \ldots)\Omega) \\
&= (\Omega, U(\theta_-)\Pi(B(f_2) \ldots B(f_{2k+1})B(E\theta_-(1-E)\theta_- f_1))\Omega) \\
&\quad + \sum_{j=2}^{2k+1} (-1)^j(\Gamma\theta_-(1-E)\theta_- f_1, Ef_j)[f_2, \ldots, \check{f}_j \ldots f_{2k+1}]^{(k-1)} \\
&= (\Omega, U(\theta_-)\Pi(B(Af_1)B(f_2) \ldots B(f_{2k+1})\Omega) \\
&\quad + \sum_{j=2}^{2k+1} (-1)^j(\Gamma A^* f_1, f_j)[f_2, \check{f}_j f_{2k+1}]^{(k-1)}
\end{aligned} \tag{5.1}$$

Since $A = E\theta_-(1 - E)\theta_-$ is a compact operator in the trace class by [*1*], A restricted to the A invariant subspace $(1 - E \wedge \theta_-(1 - E)\theta_-)K$ has a norm strictly less than 1 and

$$st\text{-}\lim_{k=\infty} A^k = E \wedge \theta_-(1 - E)\theta_- . \tag{5.2}$$

On the other hand, the repeated use of (5.1) leads to

$$\begin{aligned}
[f_1, f_{2k+1}]^{(k)} &= [A^l f_1, f_2, f_{2k+1}]^{(k)} \\
&\quad + \sum_{j=2}^{2k+1} (-1)^j(\Gamma A^*(1 + A + A^2 + A^{l-1})f_1, f_j)[f_2, \check{f}_j f_{2k+1}]^{(k-1)}
\end{aligned} \tag{5.3}$$

By splitting $f_1 = \hat{f}_1 + (E \wedge \theta_-(1 - E)\theta_-)f_1$ and applying (5.2) for

$$\hat{f}_1 = (1 - E \wedge \theta_-(1 - E)\theta_-)f_1, \tag{5.4}$$

we obtain

$$[f_1, f_2, f_{2k+1}]^{(k)} = [(E \wedge \theta_-(1 - E)\theta_- f_1, f_2, f_{2k+1}]^{(k)} + \sum_{j=2}^{2k+1} (-1)^j (\Gamma D f_1, f_j)[f_2, \check{f}_j f_{2k+1}]^{(k-1)}. \quad (5.5)$$

We now use the total antisymmetry of $[f_1, f_{2k+1}]^{(k)}$ (due to the CAR) to write the first term of (5.5) as

$$- [f_2, (E \wedge \theta_-(1 - E)\theta_-)f_1, f_3, f_{2k+1}]^{(k)}. \quad (5.6)$$

By the lemma proved above and the CAR (i. e. $B(Eh)^2 = 0$ for any h),

$$B(E \wedge \theta_-(1 - E)\theta_- f_1)B(E \wedge \theta_-(1 - E)\theta_- f_2) = 0. \quad (5.7)$$

So if we apply (5.3) again for (5.6) and substitute the resulting expression into (5.4), we obtain the reduction formula (3.4).

Note that $(\Gamma D f_2, E \wedge \theta_-(1 - E)\theta_- f_1) = 0$ due to

$$\Gamma(E \wedge \theta_-(1 - E)\theta_-)\Gamma = (1 - E) \wedge \theta_- E\theta_- \quad \text{and} \quad \theta_- E\theta_- A^* = 0.$$

§ 6. COMPUTATION OF 1 POINT FUNCTIONS

i) 2 point functions.

The 2 point functions are directly computed from the quasifree property of φ_E and

$$\varphi^{\pm}(B(f_1)B(f_2)) = \varphi_E(B(f_1)B(f_2)) = (\Gamma f_1, Ef_2) \quad (6.1)$$

Let $e_j \in l^2(\mathbb{Z})$, $(e_i)_j = \delta_{ij}$ and

$$h_j = \begin{pmatrix} e_j \\ e_j \end{pmatrix} \qquad f_j = \frac{1}{i}\begin{pmatrix} e_j \\ -e_j \end{pmatrix}. \quad (6.2)$$

Then,

$$(f_j, Ef_k) = (h_j, Eh_k) = \delta_{jk}. \quad (6.3a)$$

$$(f_k, Eh_1) = \frac{1}{-i2\pi} \int d\theta e^{i(1-k)\theta} \frac{(\cos\theta - \lambda) - i\gamma \sin\theta}{\mu(\theta)}. \quad (6.3b)$$

In terms of $B(h_j)$ and $B(f_j)$, $\sigma_x^{(1)}\sigma_x^{(j)}$, $\sigma_x^{(1)}\sigma_y^{(j)}$ and $\sigma_y^{(1)}\sigma_y^{(j)}$ are expressed as follows for the case $j > 1$.

$$\sigma_x^{(j)} = TS_j B(h_j) \qquad \sigma_y^{(j)} = TS_j B(f_j) \quad (6.4)$$

$$\sigma_x^{(1)}\sigma_x^{(j)} = (-i)^{j-1} B(f_1)\{B(h_2)B(f_2) \dots B(h_{j-1})B(f_{j-1})\} B(h_j) \quad (6.5a)$$

$$\sigma_y^{(1)}\sigma_y^{(j)} = -1 \times (-i)^{j-1} B(h_1)\{B(h_2)B(f_2) \dots B(h_{j-1})B(f_{j-1})\} B(f_j) \quad (6.5b)$$

$$\sigma_x^{(1)}\sigma_y^{(j)} = (-i)^{j-1} B(f_1)\{B(h_1)B(f_1) \dots B(h_{j-1})B(f_{j-1})\} B(f_j). \quad (6.5c)$$

(By $\Gamma h_j = h_j$ $\Gamma f_j = f_j$, $B(h_j)^2 = 1$ $\{B(h_j), B(f_k)\} = 0$.)

Next let us recall

$$\varphi_{\mathrm{E}}(\mathrm{B}(g_1)\ldots\mathrm{B}(g_n)) = \sum_p \operatorname{sgn} p \prod_{j=1}^{\frac{n}{2}} (\Gamma g_{p(2j-1)}, \mathrm{E} g_{p(2j)}), \tag{6.6}$$

where the summation is taken for all the permutations satisfying

$$p(2j-1) < p(2j) \quad \text{and} \quad p(1) < p(3) < p(5)\ldots.$$

(6.5*c*) and (6.6) imply $\varphi_{\mathrm{E}}(\sigma_x^{(1)}\sigma_y^{(j)}) = 0$. This is because the number of f_k appearing in B( ) of (6.5*c*) is j while that of h_k is $j-2$, so that in each summand of (6.6) $(f_k, \mathrm{E}f_m)$ $k \neq m$ always appears as a factor and (6.6) is zero due to (6.3*a*).

Next let us compute $\varphi_{\mathrm{E}}(\sigma_x^{(1)}\sigma_x^{(j)}) = \varphi^{\pm}(\sigma_x^{(1)}\sigma_x^{(j)})$.

By (6.5*a*), (6.6) and the CAR,

$$\begin{aligned}\varphi_{\mathrm{E}}(\sigma_x^{(1)}\sigma_x^{(j)}) &= (-i)^{j-1}(-1)^{(j-2)(j-1)/2}\varphi_{\mathrm{E}}(\mathrm{B}(f_1)\ldots\mathrm{B}(f_{j-1})\mathrm{B}(h_2)\ldots\mathrm{B}(h_j)) \\ &= \sum_p (-i)^{j-1}(-1)^{(j-2)(j-1)/2} \operatorname{sgn} p \prod_k (g_{p(2k-1)}, \mathrm{E} g_{p(2k)}),\end{aligned} \tag{6.7}$$

where $g_l = f_l$ or h_{l-j+2}.

(6.3*a*) implies $g_{p(2k-1)} = f_k$ and $g_{p(2k)} = h_{k'}$ for some k' and for all $k = 1, 2, 3, j-1$. So let q be the permutation defined by $q(k) = p(2k)$, then

$$(-1)^{(j-2)(j-1)} \operatorname{sgn} q = \operatorname{sgn} p. \tag{6.8}$$

The last line of (6.7) is equal to

$$(-i)^{j-1} \sum_q \operatorname{sgn} q \prod_{l=1}^{j-1} (f, \mathrm{E} h_{q(l)+1}), \tag{6.9}$$

where the sum is taken over all the permutations of $\{1, 2, j-1\}$. Then by (6.3*b*) and $\Gamma f_j = f_j$,

$$(6.9) = \sum_q \operatorname{sgn} q \prod_l \mathrm{X}_{l,q(l)} = \det(\mathrm{X}_{1,m}). \tag{6.10}$$

This proves (3.5*a*).

As $\sigma_x \to \sigma_y, \sigma_y \to -\sigma_x$ is equivalent to $\gamma \to -\gamma$, (3.5*a*) implies (3.5*b*).

ii) 1 point functions.

As $\varphi^{\pm}$ are translationally invariant and pure

$$\lim_{j\to\infty} \varphi^{\pm}(\sigma_x^{(1)}\sigma_x^{(j)}) = [\varphi^{\pm}(\sigma_x^{(1)})]^2 \tag{6.11}$$

$$\lim_{j\to\infty} \varphi^{\pm}(\sigma_x^{(1)}\sigma_y^{(j)}) = \varphi^{\pm}(\sigma_x^{(1)})\varphi^{\pm}(\sigma_y^{(1)}) \tag{6.12}$$

As remarked above

$$\varphi^{\pm}(\sigma_x^{(1)})_{\substack{\gamma=\gamma_0\\ \lambda=\lambda_0}} = \varphi^{\pm}(\sigma_y^{(1)})_{\substack{\gamma=-\gamma_0\\ \lambda=\lambda_0}}$$

(6.12) and (3.5c) imply either $\varphi^{+}(\sigma_x^{(1)})$ or $\varphi^{+}(\sigma_y^{(1)})$ must vanish. Now we concentrate on the computation of $\varphi^{\pm}(\sigma_x^{(1)})$ in the case $\gamma > 0$. The lefthand side of (6.11) is the determinant of a Tœplitz operator. The limit can be computed by an application of the following formula.

PROPOSITION 2. — Let $X(\theta)$ be a periodic continuous function, $X(\theta + 2\pi) = X(\theta)$, with $|X(\theta)| = 1$ which is analytic as a function of $e^{i\theta}$. Assume further X has the winding number zero, $\log X(\theta) = \sum_n k_n e^{i\theta}$, $k_0 = 0$. (k_n is the nth Fourier coefficient of $\log X(\theta)$.)

$\mathscr{M}(X)$ denotes the multiplication operator of $X(\theta)$ on $L^2(-\pi, \pi)$, then

$$\lim_{N\to\infty} \det{}_N \mathscr{M}(X(\theta)) = \exp\left[\sum_{n=1}^{\infty} n k_n k_{-n}\right], \tag{6.13}$$

where $\det_N A$ denotes the determinant of the restriction of A to the n dimensional subspace spaned by $e^{ik\theta}$ $k = 1, 2, 3, \ , n$.

Remark. — The formula is known as the Szego's formula for Tœplitz determinants. The proof may be found in the chapter X of [*4*].

For our purpose,

$$X(\theta) = \frac{(\cos\theta - \lambda) - i\gamma\sin\theta}{\mu(\theta)} e^{i\theta}. \tag{6.14}$$

If $\gamma > 0$, $X(\theta)$ has the winding number zero,

$$\begin{aligned} X(\theta) &= \left[\frac{(z + z^{-1} - 2\lambda) - \gamma(z - z^{-1})}{(z + z^{-1} - 2\lambda) + \gamma(z - z^{-1})} z^2\right]^{1/2} \\ &= \left[\frac{(1-\gamma)z^2 - 2\lambda z + (1+\gamma)}{(1+\gamma)z^2 - 2\lambda z + (1-\gamma)} z^2\right]^{1/2}, \end{aligned} \tag{6.15}$$

for $z = e^{i\theta}$.

Let α, β be the solutions of the following equation:

$$(1+\gamma)x^2 - 2\lambda x + (1-\gamma) = 0. \tag{6.16}$$

Then,

$$\begin{aligned} (6.16) &= \left[\frac{(1+\gamma)\left(\frac{1}{z} - \alpha\right)\left(\frac{1}{z} - \beta\right)}{(1+\gamma)(z-\alpha)(z-\beta)} z^4\right]^{1/2} \\ &= \left[\frac{(1-\alpha z)(1-\beta z)}{(1-\alpha/z)(1-\beta/z)}\right]^{1/2}. \end{aligned} \tag{6.17}$$

Using the expansion $\log(1-x) = (-1)\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$

$$\log X(\theta) = \sum_{n=-\infty}^{\infty} k_n e^{in\theta},$$

$$k_n = \begin{cases} -\frac{1}{2n}(\alpha^n + \beta^n) & n > 0, \\ 0 & n = 0, \\ \frac{1}{2n}(\alpha^n + \beta^n) & n < 0. \end{cases} \tag{6.18}$$

So we have

$$\sum_{n>0} n k_n k_{-n} = -\sum_{n>0} \frac{1}{4n}\{\alpha^{2n} + \beta^{2n} + 2(\alpha\beta)^n\}$$

$$= 1/4 \log(1-\alpha^2)(1-\beta^2)(1-\alpha\beta)^2. \tag{6.19}$$

Note that $|\alpha|, |\beta| < 1$ for $\gamma > 0$.

Using $\alpha\beta = \frac{1-\gamma}{1+\gamma}$, $\alpha + \beta = \frac{2\lambda}{1+\gamma}$ and Proposition 2, we obtain

$$\lim_{j\to\infty} \varphi^{\pm}(\sigma_x^{(1)}\sigma_x^{(j)}) = (1-\lambda^2)^{1/4}\left[\frac{4\gamma}{(1+\gamma)^2}\right]^{1/2}.$$

As a result

$$\varphi^{+}(\sigma_x^{(j)}) = (1-\lambda^2)^{1/8}\left[\frac{4\gamma}{(1+\gamma)^2}\right]^{1/4} \quad \text{if} \quad \gamma > 0, \tag{6.20}$$

where we have taken the definition of φ^+ (which depends on the choice of an arbitrary sign in the definition of $U(\theta_-)$) such that $\varphi^{+}(\sigma_x^{(j)}) > 0$.

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