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Hard bosons in one dimension

by

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ABSTRACT. — We establish a general criterion relating the absence of Bose-Einstein condensation to specific properties of the one-body reduced density matrix. A general model of hard particles in one dimension is investigated in detail. The results are subsequently applied to hard Bosons in a weak gravitational field. The thermodynamic functions of this system show a singularity at a critical value of the density. However, we prove that this phase transition is not Bose-Einstein condensation.

RÉSUMÉ. — Nous démontrons l'équivalence entre l'absence de condensation de Bose-Einstein et certaines propriétés de la matrice densité réduite à un corps. Nous étudions en détail les propriétés des systèmes unidimensionnels de particules avec cœurs durs. Nous appliquons ces résultats au gaz de Bose avec cœur dur soumis à un champ de gravitation faible. Dans ce modèle, les fonctions thermodynamiques présentent une singularité à une certaine densité critique. Cependant, nous démontrons que cette transition de phase n'est pas la condensation de Bose-Einstein.

1. INTRODUCTION

How stable is a phase transition with respect to a change in the interaction between the microscopic constituents of the system? This question

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is of general interest, but it is of special importance in the case of Bose-Einstein condensation, since this phenomenon was first discovered in non-interacting systems.

Rigorous results on the persistence of Bose-Einstein condensation in the presence of an interaction have proved very difficult to obtain. Beside the mean-field case (see [*1*] [*2*] [*3*]), there is only one situation where the question has been settled (positively) for a whole class of potentials (see section 4 of [*4*]).

In a related article (see [*5*]), the authors adopted another approach to the problem: they considered a one-dimensional Bose gas with attractive boundary conditions and hard core interaction. The corresponding *free* system is known to show Bose-Einstein condensation (see [*6*] [*7*]). The effect of a hard core of diameter a is to replace the configuration space $[0, L]^N$ for N particles in $[0, L]$ by the *accessible region*

$$\Omega^a_{L,N} = \{ (x_1, \ldots, x_N) \in [0, L]^N : | x_i - x_j | \geq a \qquad i \neq j = 1, 2, \ldots, N \} . \quad (1)$$

Accordingly, a boundary condition has to be specified at the contact between the particles. In [*5*] and in the present article, we work with Neumann boundary conditions for the N-particle wave functions Ψ in $S_N(L^2(\Omega^a_{L,N}))$:

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Psi(x_1, \ldots, x_N)\,|_{x_i - x_j = a} = 0 . \quad (2)$$

We refer to the operator $H^a_{L,N}$ on $S_N(L^2(\Omega^2_{L,N}))$ given by

$$-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \lambda\sum_{j=1}^{N}\frac{x_j}{L}$$

with condition (2) as the Hamiltonian of a one-dimensional Bose gas with linear external potential and *Neumann hard cores*. The operator $H^a_{L,N}$ is completely specified only when boundary conditions at 0 and L have been selected.

A remarkable property of $H^a_{L,N}$ is that it can be related to a free Hamiltonian. This can be seen as follows: first note that $\Omega^a_{L,N}$ is a disjoint union of subregions R_π, $\pi \in S_N$, defined by

$$0 \leq x_{\pi(1)} \leq x_{\pi(2)} - a \leq x_{\pi(3)} - 2a \leq \ldots \leq x_{\pi(N)} - (N-1)a \leq L - (N-1)a . \quad (3)$$

Put $D_N(x_1, \ldots, x_N) = (y_1, \ldots, y_N)$ with

$$y_{\pi(j)} = x_{\pi(j)} - a(j-1) \quad \text{in} \quad R_\pi \quad (4)$$

and define

$$D_N : S_N(L^2(\Omega^a_{L,N})) \mapsto S_N(L^2([0, L - a(N-1)]^N)) \quad (5)$$

by

$$D_N\Psi = \Psi \circ D_N^{-1} . \quad (6)$$

Then one can check, see [5], that

$$D_N H^a_{L,N} D_N^{-1} = H^{\text{free}}_{L-a(N-1),N} = \sum_{j=1}^{N} h_{L-a(N-1)}. \tag{7}$$

where the free Hamiltonian in the right hand side of (7) describes N Bosons in $[0, L - a(N - 1)]$ and incorporates a transformed external potential.

It follows from (7) that $H^a_{L,N}$ and $H^{\text{free}}_{L-a(N-1),N}$ have identical spectra. This in turn implies that the free energy density of the gas with Neumann hard cores is closely related to that of the free gas (see [5], and section 3.1 below). Hence the interacting gas will inherit any singularity present in the thermodynamic functions of the free gas. However, it is important to realize that this has no bearing on the condensation properties of the interacting gas. This point is discussed at length in [5], and its origin can be traced to the fact that the unitary equivalence between observables $A \mapsto D_N A D_N^{-1}$ does not map occupation numbers of the interacting system onto occupation numbers of the free gas; see also the remark at the end of section 2.1. Hence, in spite of the equivalence (7), the Bose gas with Neumann hard cores is a genuine interacting model, and the solution of the condensation problem requires a detailed analysis.

The special case where no external potential is present but attractive boundary conditions are imposed at 0, L is studied in [5]; we show that in spite of the presence of a singularity in its free energy density, this model does not show any Bose-Einstein condensation. This shows clearly that, in contrast with the free case (see [8]), there is no connection between condensation and thermodynamic properties in interacting gases.

On the other hand, one could argue that the type of condensation induced in the *free* gas by attractive boundary conditions is rather special, see [7], and that this undermines the value of the results of [5] as a test of the stability of Bose-Einstein condensation. In the present article, we use as our starting point the one dimensional Bose gas in a weak linear external potential (which can be thought of as a gravitation field); the corresponding *free* gas is known to have a phase transition of the same type as the *three dimensional* Bose-Einstein condensation (see [9] [10] [11] and section 3.1 below). We prove that in the presence of Neumann hard cores, a phase transition persists, but Bose-Einstein condensation itself is destroyed.

Whereas in [5] decisive use was made of the explicit form of the ground state wave function of the model, the present analysis is kept as general as possible, and only qualitative features of the wave function (such as decay properties) are used.

All our general results are gathered in section 2. Theorem 1 is a criterion for the absence of Bose-Einstein condensation at zero temperature. It holds for any dimension and any interaction. The other results in section 2 apply only to one-dimensional Neumann hard cores, but they do not involve the explicit form of the wave function, so that they hold irrespective of the presence or absence of the external potential and of the boundary conditions at the end points 0, L.

In section 3, we turn to the hard core gas in a linear external potential. We discuss first the thermodynamics of the system, which is closely related to that of the free gas; the free energy density has a singularity at a critical value of the density. However, we can prove using the results of the previous sections that there is no Bose-Einstein condensation in this model.

2. SOME GENERAL RESULTS

2.1. Condensation and reduced density matrices.

There is a general and very simple link between Bose-Einstein condensation and the one-body reduced density matrix. Consider N interacting Bosons in a box Ω of volume L^ν. Let $\Phi \in S_N(L^2(\Omega^N))$ denote the ground state wave function of the system. At zero temperature, the one-body canonical reduced density matrix is:

$$\rho_L(x, y) = N \int_\Omega dz_1^\nu \dots \int_\Omega dz_{N-1}^\nu \Phi(x, z_1, \dots, z_{N-1})\Phi(y, z_1, \dots, z_{N-1}) . \quad (8)$$

We denote by

$$R_\Phi^L : L^2(\Omega) \mapsto L^2(\Omega) \quad (9)$$

the integral operator with kernel $\rho_L(x, y)$. Note that R_Φ^L is symmetric and trace-class since $\rho_L(x) \equiv \rho_L(x, x)$ is the local density, which integrates to N. The following function turns out to play a distinguished role in our analysis

$$g(x) = \left(\frac{\rho_L(x, x)}{N}\right)^{\frac{1}{2}} . \quad (10)$$

Note that g is a normalised function in the one-particle space $L^2(\Omega)$. On the other hand, for an arbitrary normalised wave function f in $L^2(\Omega)$, the occupation number of the level f is

$$N_f^L = P_f \otimes I \otimes \dots \otimes I + I \otimes P_f \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes P_f \quad (11)$$

where $P_f : L^2(\Omega) \mapsto L^2(\Omega)$ is the orthogonal projection onto f.

Our first result gives several equivalent criteria for the absence of Bose-Einstein condensation ($\langle . \rangle$ denotes the zero-temperature average).

THEOREM 1. — *The following statements are equivalent* :

i) $$\lim_{L \to \infty} L^{-\nu} \langle N_f^L \rangle = 0 ,$$

for every normalised f in $L^2(\Omega)$.

ii)
$$\lim_{L\to\infty} L^{-\nu}\langle N_g^L \rangle = 0\,,$$
with g as in (10).

iii)
$$\lim_{L\to\infty} L^{-\nu}\| R_\Phi^L \| = 0\,,$$
with R_Φ^L *as in (9).*

iv)
$$\varlimsup_{L\to\infty} L^{-2v}\int_\Omega d^v x\int_\Omega d^v y\,\rho_L^2(x,y) = 0\,,$$
with $\rho_L(x, y)$ *as in (8).*

Proof.
$$L^{-2\nu}\langle N_f^L\rangle^2 = L^{-2\nu}(\Phi, N_f^L\Phi)^2 \tag{12}$$
$$= N^2L^{-2\nu}\left\{\int_\Omega d^\nu z_0\ldots\int_\Omega d^\nu z_N f(z_0)f(z_N)\Phi(z_0,\ldots,z_{N-1})\Phi(z_1,\ldots,z_N)\right\}^2 \tag{13}$$
$$= L^{-2\nu}(f, R_\Phi^L f)^2 \le L^{-2\nu}\|R_\Phi^L\|^2 \tag{14}$$
$$\le L^{-2v}\int_\Omega d^\nu x\int_\Omega d^\nu y\,\rho_L^2(x,y) \tag{15}$$
$$\le L^{-2v}\int_\Omega d^\nu x\int_\Omega d^\nu y\,\rho_L^{\frac{1}{2}}(x,x)\rho_L^{\frac{1}{2}}(y,y)\rho_L(x,y) \tag{16}$$
$$= \rho L^{-\nu}\langle N_g^L\rangle \le \rho L^{-\nu}\|R_\Phi^L\|\,. \tag{17}$$
□

Remarks. — *i)* It is remarkable that it suffices to know that the level g is not macroscopically occupied to conclude that no other level is.

ii) In [5], we used a version of theorem 1 which contained some unnecessarily restrictive hypotheses; note that (16) follows from (15) using the Schwarz inequality and (8).

Our next result gives the general form of $\rho_L(x, y)$ for one-dimensional Neumann hard core gases. The key to the properties of the interacting gas is the fact that, because of (7), the transformation D_N maps the ground state Φ of $H_{L,N}^a$ onto the ground state of a *free* Hamiltonian $H_{L-a(N-1),N}^{\text{free}}$. Hence
$$D_N\Phi = \varphi^{\otimes N} \tag{18}$$

where $\varphi \in L^2[0, L - a(N-1)]$ is the normalised ground state of the one-particle Hamiltonian $h_{L-a(N-1)}$. Using the transformation D_N backwards, we can express Φ, and thus $\rho_L(x, y)$, in terms of φ. We refer to [5] for the proof.

THEOREM 2. — *Consider* N + 1 *Bosons in* [0, L], *interacting through*

Neumann hard cores of diameter a. Let $\rho_L(x, y)$ be the one-body canonical reduced density matrix of this system. Then for $x \leq y$

$$\rho_L(x, y) = \sum_{0 \leq j < k \leq N}{}' I_L^{(j,k)}(x, y) + \sum_{0 \leq j \leq N}{}'' I_L^{(j,j)}(x, y)$$

where

$$I_L^{(j,k)}(x, y) = \frac{(N+1)\varphi(x-ja)\varphi(y-ka)}{j!\,(k-j)!\,(N-k)!}\left[\int_0^{x-ja} dz\varphi^2(z)\right]^j \times \left[\int_{x-(j-1)a}^{y-ka} dz\varphi(z-a)\varphi(z)\right]^{k-j}\left[\int_{y-ka}^{L'} dz\varphi^2(z)\right]^{N-k}$$

with φ as in (18), $L' = L - aN$, and the sums Σ', Σ'', denoting the sums restricted to $ja \leq x$, $(k-j)a \leq y - x - a$, $(N-k)a \leq L - y$ and $ja \leq x$, $(N-j)a \leq L - y$ respectively.

Remark. — This theorem makes it obvious that the unitary equivalence (7) does not extend beyond the thermodynamic level; compare indeed the formulas of theorem 2 with that of the free case $\rho_{L'}(x, y) = \varphi(x)\varphi(y)$.

2.2. **Bounds on the occupation numbers.**

One of the effects of the hard core is to place restrictions on the local density. The following proposition is intuitively obvious, and we shall omit its proof.

PROPOSITION 1. — *Let $\rho_L(x, x)$ denote the local density of a gas with hard cores of diameter a. Then for any $b > 0$*

$$\int_b^{b+a} dx\rho_L(x, x) \leq 1 .$$

It turns out that without any further knowledge of the function φ, see (18), one can use theorem 2 to derive a relevant bound on the occupation numbers.

PROPOSITION 2. — *Under the assumptions of theorem 2 we have, for every normalised function f in $L^2[0, L]$*

$$\langle N_f^L \rangle \leq 2(N+1)! \sum_{0 \leq j \leq k \leq N} \int_0^{L'} dz_{N+2} \int_0^{z_{N+2}} dz_{N+1} \dots \int_0^{z_2} dz_1$$
$$\varphi(z_{k+2})\varphi(z_{j+1})\,|f(z_{k+2} + ka)|\,|f(z_{j+1} + ja)| \prod_{1 \leq l \leq N+2,\, l \neq j+1,k+2} \varphi^2(z_l) .$$

Proof. — The starting point is the following formula, which follows from theorem 2:

$$\langle N_f^L \rangle = 2\int_0^L dy \int_0^y dx f(x) f(y) \rho_L(x, y) \tag{19}$$

$$= 2\sum_{j=0}^{N} \int_{ja}^{L'+ja} dy \int_{ja}^{y} dx I_L^{(j,j)}(x, y) f(x) f(y)$$

$$+ 2 \sum_{0 \le j < k \le N} \int_{(k+1)a}^{L'+ka} dy \int_{ja}^{y-(k-j+1)a} dx I_L^{(j,k)}(x, y) f(x) f(y) \tag{20}$$

$$\le 2 \sum_{0 \le j \le k \le N} \frac{(N+1)!}{j!(k-j)!(N-k)!} \int_0^{L'} dw \varphi(w) \,|\, f(w+ka) \,|\, \left[\int_w^{L'} du \varphi^2(u)\right]^{N-k}$$

$$\times \int_0^w dz \varphi(z) \,|\, f(z+ja) \,|\, \left[\int_0^z du \varphi^2(u)\right]^j \left[\int_z^w du \varphi^2(u)\right]^{k-j}. \tag{21}$$

In deducing (21) from (20), we use the following estimates

$$\int_{x-(j-1)a}^{y-ka} du \varphi(u-a)\varphi(u) \le \left(\int_{x-(j-1)a}^{y-ka} du \varphi^2(u-a)\right)^{\frac{1}{2}} \left(\int_{x-(j-1)a}^{y-ka} du \varphi^2(u)\right)^{\frac{1}{2}}$$

$$\le \int_{x-ja}^{y-ka} du \varphi^2(u). \tag{22}$$

Consider now the integral over z in (21); performing $k-j$ successive integrations by parts, this becomes:

$$(k-j)! \int_0^w dz_{k+1} \int_0^{z_{k+1}} dz_k \ldots \int_0^{z_{j+2}} dz_{j+1} \varphi^2(z_k) \ldots \varphi^2(z_{j+2}) \varphi^2(z_{j+1})$$

$$\times \,|\, f(z_{j+1} + ja) \,|\, \left[\int_0^{z_{j+1}} du \varphi^2(u)\right]^j. \tag{23}$$

In order to obtain the final result, it suffices to insert (23) in (21) and to transform the integral over w, using $N-k$ successive integrations by parts. □

Proposition 2 can be used to derive a number of useful bounds in any situation where some extra information on φ is available (see section 3.2). The following result is both fairly general and extremely simple.

COROLLARY. — *Suppose that* $|f(x)| \le F(x)$ *where* $F(x)$ *is non-increasing. Then, with* φ *and* L' *as in Theorem 2 :*

$$\langle N_f^L \rangle \le (N+1) \left[\int_0^{L'} dz F(z) \varphi(z)\right]^2.$$

Proof. — Using Proposition 2 and the fact that $F(x)$ is non-increasing we get

$$\langle N_f^L \rangle \leq 2(N+1)! \sum_{0 \leq j \leq k \leq N} \int_0^{L'} dz_{N+2} \int_0^{z_{N+2}} dz_{N+1} \dots \int_0^{z_2} dz_1$$

$$\varphi(z_{k+2}) F(z_{k+2}) \varphi(z_{j+1}) F(z_{j+1}) \prod_{1 \leq l \leq N+2, l \neq j+1, k+2} \varphi^2(z_l) \quad (24)$$

$$= (N+1) \sum_{\pi \in S_{N+2}} \int_0^{L'} dz_{\pi(1)} \int_0^{z_{\pi(1)}} dz_{\pi(2)}$$

$$\dots \int_0^{z_{\pi(N+1)}} dz_{\pi(N+2)} \varphi(z_1) F(z_1) \varphi(z_2) F(z_2) \prod_{3 \leq l \leq N+2} \varphi^2(z_l) \quad (25)$$

$$= (N+1) \left[\int_0^{L'} dz \varphi(z) F(z) \right]^2 . \quad (26)$$

□

3. HARD BOSONS IN A WEAK LINEAR POTENTIAL

3.1. Thermodynamic functions.

In order to show how the results of the previous sections can be exploited in a given situation, we concentrate now on the case where the hard Bosons are submitted to a weak gravitation field. This means that the Hamiltonian of the system is given by ($\lambda > 0$):

$$H_{L,N+1}^a = -\frac{1}{2} \sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2} + \frac{\lambda}{L} \sum_{j=1}^{N+1} x_j \quad \text{on} \quad S_{N+1}(L^2(\Omega_{L,N+1}^a)) \quad (27)$$

with condition (2) and Dirichlet boundary conditions at 0 and L, namely for $j = 1, 2, \dots, N+1$

$$\Psi(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{N+1}) = \Psi(x_1, \dots, x_{j-1}, L, x_{j+1}, \dots, x_{N+1}) = 0. \quad (28)$$

In that case, the transformation (7) maps $H_{L,N+1}^a$ onto

$$\sum_{j=1}^{N+1} \left[h_{L'}(\lambda') + \frac{\lambda a N}{2L} \right] \quad (29)$$

where, as before, $L' = L - aN$ and $\lambda' = \lambda\left(1 - a\frac{N}{L}\right)$ and the one particle Hamiltonian $h_L(\lambda)$ is the operator on $L^2([0, L])$ defined by

$$h_L(\lambda) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\lambda x}{L} \tag{30}$$

with Dirichlet boundary conditions at 0, L.

We know that in the presence of a weak external potential, *free* Bosons exhibit Bose-Einstein condensation in one dimension, see [*9*] [*10*] [*11*]. We are now in a position to test the stability of this phenomenon with respect to a perturbation by a hard core interaction. First of all, let us recall the thermodynamics of the free gas with one-particle Hamiltonian (30).

The most important feature is the existence of a critical density

$$\rho_0^c(\lambda) = \frac{1}{\sqrt{2\pi\lambda}\beta^{3/2}}[g_{3/2}(1) - g_{3/2}(e^{-\beta\lambda})] \tag{31}$$

such that the canonical pressure reads

$$\pi_0(\rho, \lambda) = \frac{1}{\sqrt{2\pi\lambda}\beta^{5/2}}[g_{5/2}(1) - g_{5/2}(e^{-\beta\lambda})] \qquad \rho \geq \rho_0^c(\lambda) \tag{32}$$

and is given implicitely for $\rho < \rho_0^c(\lambda)$ by $\pi_0(\rho, \lambda) = p_0(\mu_0(\rho, \lambda), \lambda)$ where p_0 is the grand canonical pressure

$$p_0(\mu, \lambda) = \frac{1}{\sqrt{2\pi\lambda}\beta^{5/2}}[g_{5/2}(e^{\beta\mu}) - g_{5/2}(e^{\beta(\mu-\lambda)})] \qquad \mu < 0 \tag{33}$$

and $\mu_0(\rho, \lambda)$ can be got by inversion of

$$\rho_0(\mu, \lambda) = \frac{1}{\sqrt{2\pi\lambda}\beta^{3/2}}[g_{3/2}(e^{\beta\mu}) - g_{3/2}(e^{\beta(\mu-\lambda)})] \qquad \mu < 0\,. \tag{34}$$

In the above formulas, we have used the standard notation

$$g_\alpha(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^\alpha}\,. \tag{35}$$

The free energy density $f_0(\rho, \lambda)$ can be obtained using the equation

$$\pi_0(\rho, \lambda) = \rho\frac{\partial}{\partial\rho}f_0(\rho, \lambda) - f_0(\rho, \lambda)\,. \tag{36}$$

This yields in particular (see fig. 1)

$$f_0(\rho, \lambda) = \frac{-1}{\sqrt{2\pi\lambda}\beta^{5/2}}[g_{5/2}(1) - g_{5/2}(e^{-\beta\lambda})] \qquad \rho \geq \rho_0^c(\lambda)\,. \tag{37}$$

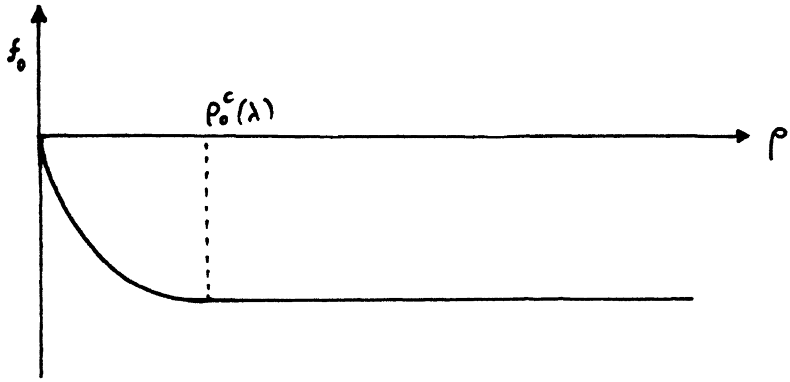

FIG. 1. — The free energy density of the free gas.

One can prove that $\frac{\partial^3 f_0}{\partial \rho^3}$ jumps from zero to the value $-\lambda^2\beta^2$ when ρ decreases through $\rho_0^c(\lambda)$. The phase transition is accompanied by macroscopic occupation of the ground state, see [*10*]. Now, in view of (29), the free energy density $f_a(\rho, \lambda)$ of the model described by the Hamiltonian (27) is given in the thermodynamic limit by

$$f_a(\rho, \lambda) = \frac{\lambda a \rho^2}{2} + (1 - a\rho) f_0\left(\frac{\rho}{1 - a\rho}, \lambda(1 - a\rho)\right). \tag{38}$$

This implies that f_a has a singularity at the critical density $\rho_a^c(\lambda)$ defined as the unique solution of the equation

$$\frac{x}{1 - ax} = \rho_0^c(\lambda(1 - ax)) \qquad x < 1/a. \tag{39}$$

Moreover, one can get from (37) (38) an explicit formula for $f_a(\rho, \lambda)$ in the regime $1/a > \rho \geq \rho_a^c(\lambda)$:

$$f_a(\rho, \lambda) = \frac{\lambda a \rho^2}{2} + \frac{1}{\sqrt{2\pi}\beta^{5/2}} \left[g_{5/2}(e^{-\beta\lambda(1 - a\rho)}) - g_{5/2}(1) \right]. \tag{40}$$

As in the free gas, the third derivative of the free energy has a discontinuity, given by

$$\frac{\partial^3}{\partial \rho^3} f_a(\rho_a^c(\lambda) +, \lambda) - \frac{\partial^3}{\partial \rho^3} f_a(\rho_a^c(\lambda) -, \lambda) = \frac{\beta^2 \lambda^2}{(1 - a\rho_a^c(\lambda))^3}. \tag{41}$$

Remarks. — *i*) The condition $\lambda > 0$ is essential for the existence of the transition since (31) implies that $\rho_0^c(\lambda) \to \infty$ as $\lambda \to 0$.

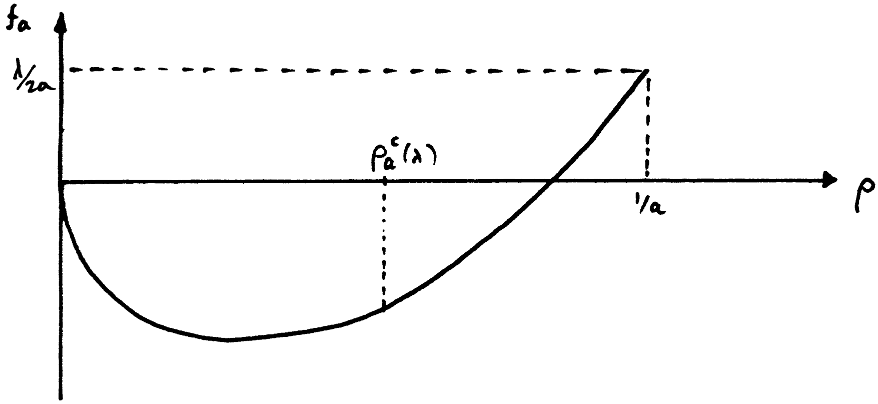

FIG. 2. — The free energy density of the interacting gas.

ii) A striking difference between f_0 and f_a (see figures 1, 2) is the absence of a flat part in f_a; this means that the abnormal density fluctuations of the free gas have been suppressed by the hard core interaction, see [*2*] [*13*].

iii) We refer to [*5*] for a comment on the seemingly puzzling finiteness of f_a as ρ tends to the closest packing density $1/a$. Note however that here the isothermal compressibility $\left(\rho \frac{\partial}{\partial \rho} \pi_a\right)^{-1} = \left(\rho^2 \frac{\partial^2}{\partial \rho^2} f_a\right)^{-1}$ tends to zero as ρ approaches $1/a$.

3.2. **Absence of Bose-Einstein condensation.**

In this section, we show that the phase transition displayed by our model is not Bose-Einstein condensation, in the sense that even at zero temperature no occupation number N_f^L has a macroscopic average value. The proof is based on proposition 2 and on the properties of φ, the ground state wave function of the one-particle Hamiltonian.

LEMMA 1. — *Let* $\varphi(x)$ *be the ground state wave function of* $h_L(\lambda)$, (*see* (*30*)); *then there exists a constant* A, *independent of* L, *such that*

$$\varphi(x) \leq \frac{A}{L^{1/6}} \exp\left(- x\left(\frac{2\lambda}{L}\right)^{1/3}\right).$$

Proof. — By definition, φ must satisfy

$$-\frac{1}{2} \varphi''(x) + \frac{\lambda x}{L} \varphi(x) = E_L L^{-2/3} \varphi(x) \quad \text{in} \quad [0, L], \qquad \varphi(0) = \varphi(L) = 0 \quad (42)$$

Now, if we put

$$\varphi(x) = \mathrm{L}^{-\frac{1}{6}} u\left(\left(\frac{2\lambda}{\mathrm{L}}\right)^{1/3} x - \gamma_{\mathrm{L}}\right) \tag{43}$$

with

$$\gamma_{\mathrm{L}} = \left(\frac{2}{\lambda^2}\right)^{1/3} \mathrm{E}_{\mathrm{L}} \tag{44}$$

we find that u satisfies Airy's equation

$$\begin{aligned} u''(y) - yu(y) = 0 \quad \text{in} \quad [-\gamma_{\mathrm{L}}, -\gamma_{\mathrm{L}} + (2\lambda\mathrm{L}^2)^{1/3}] \\ u(-\gamma_{\mathrm{L}}) = u(-\gamma_{\mathrm{L}} + (2\lambda\mathrm{L}^2)^{1/3}) = 0 \end{aligned} \tag{45}$$

Hence, the solution is, see [*14*]

$$u(y) = \mathrm{C}_{\mathrm{L}}\left[\mathrm{A}i(y) - \frac{\mathrm{A}i(-\gamma_{\mathrm{L}})}{\mathrm{B}i(-\gamma_{\mathrm{L}})}\mathrm{B}_i(y)\right] \tag{46}$$

where Ai, Bi are the Airy functions. Moreover, the requirement that φ should be the ground state of $h_{\mathrm{L}}(\lambda)$ determines γ_{L} as the smallest solution of the transcendental equation

$$\mathrm{A}i(-\gamma_{\mathrm{L}})\mathrm{B}i(-\gamma_{\mathrm{L}}+(2\lambda\mathrm{L}^2)^{1/3}) = \mathrm{B}i(-\gamma_{\mathrm{L}})\mathrm{A}i(-\gamma_{\mathrm{L}}+(2\lambda\mathrm{L}^2)^{1/3}). \tag{47}$$

As L tends to infinity, $-\gamma_{\mathrm{L}}$ tends from below to the first zero of the function Ai, see [*14*]. One can check that the constant C_{L} which ensures the normalisation of $u(y)$, and thus of $\varphi(x)$, remains bounded as L tends to infinity. One the other hand $\mathrm{A}i(-\gamma_{\mathrm{L}})\mathrm{B}i(y)/\mathrm{B}i(-\gamma_{\mathrm{L}})$ is positive for $y > 0$ and bounded on $[-\gamma_{\mathrm{L}}, 0]$, so that there are constants C, D such that

$$u(y) \leq \mathrm{C}[\mathrm{A}i(y) + e^{-y}] \tag{48}$$

$$\leq \mathrm{D}e^{-y}. \tag{49}$$

In deducing (49) from (48), we used the boundedness and the asymptotic properties of Ai, see [*14*]. The result follows from (43) and (49). □

LEMMA 2. — *Consider the model defined by (27); let φ be the ground state of $h_{\mathrm{L'}}(\lambda')$, see (29), and define g as in (10). Then, for* L *large enough, there is a constant* C *such that for every z in* [0, L'] *and k in* { 0, 1, . . ., N }

$$\int_0^z dx\varphi(x)g(x + ka) \leq \frac{\mathrm{C}}{\mathrm{L}^{1/3}}\int_0^z \frac{dx}{\mathrm{L}^{1/3}} e^{-x\left(\frac{2\lambda}{\mathrm{L}}\right)^{1/3}}.$$

Proof.

$$\int_0^z dx\varphi(x)g(x+ka) = \frac{1}{\sqrt{N}}\sum_{j=0}^{z/a-1}\int_{ja}^{(j+1)a} dx\varphi(x)\rho_{\mathrm{L}}^{\frac{1}{2}}(x+ka, x+ka) \tag{50}$$

$$\leq \frac{1}{\sqrt{\mathrm{N}}}\sum_{j=0}^{z/a-1}\left\{\int_{ja}^{(j+1)a} dx\varphi^2(x)\right\}^{1/2} \tag{51}$$

by virtue of the Schwarz inequality and of Proposition 2. Taking Lemma 1 into account in (51) we get (note that $\lambda'/L' = \lambda/L$):

$$\int_0^z dx\varphi(z)g(x+ka) \leq \frac{A[1-e^{-2a\left(\frac{2\lambda}{L}\right)^{1/3}}]^{\frac{1}{2}}}{(2N)^{\frac{1}{2}}(2\lambda')^{\frac{1}{6}}} \sum_{j=0}^{z/a-1} e^{-ja\left(\frac{2\lambda}{L}\right)^{1/3}} \tag{52}$$

$$= \frac{A2\lambda}{(2N)^{1/2}(2\lambda')^{1/6}} \frac{[1-e^{-2a\left(\frac{2\lambda}{L}\right)^{1/3}}]^{1/2}}{1-e^{-a\left(\frac{2\lambda}{L}\right)^{1/3}}} \sum_{j=0}^{z/a-1} \int_{ja}^{(j+1)a} \frac{dx}{L^{1/3}} e^{-x\left(\frac{2\lambda}{L}\right)^{1/3}} \tag{53}$$

$$\leq \frac{C}{L^{1/3}} \int_0^z \frac{dx}{L^{1/3}} e^{-x\left(\frac{2\lambda}{L}\right)^{1/3}} \tag{54}$$

for L large enough since the expression multiplying the sum in (53) is of order $L^{-1/3}$. □

We can now state the main result of this section: the phase transition displayed by our model is not Bose-Einstein condensation.

THEOREM 3. — *Consider the one-dimensional Bose gas with Neumann hard cores and linear external potential as defined by (27), and let g be as in (10). There exists a constant* B *such that, for* L *large enough*

$$\langle N_g^L \rangle \leq BL^{1/3}.$$

Hence, for every normalised f in $L^2[0, L]$,

$$\langle N_f^L \rangle \leq B'L^{2/3}$$

so that there is no macroscopic occupation of any level.

Proof. — Combining Lemma 2 and Proposition 2, we find, proceeding as in the Corollary:

$$\langle N_g^L \rangle \leq \left(\frac{C}{L^{1/3}}\right)^2 (N+1) \left[\int_0^{L'} \frac{dx}{L^{1/3}} e^{-x\left(\frac{2\lambda}{L}\right)^{1/3}}\right]^2 \tag{55}$$

$$= BL^{1/3}. \tag{56}$$

The second part of the theorem follows from Theorem 1, more precisely from (12), (17). □

Remark. — The question of the interpretation of the phase transition exhibited by our model remains open. In any case, the result of this study can be taken as a warning: the discovery of a singularity in the thermodynamic functions of an interacting Bose gas is no evidence for Bose-Einstein condensation.

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