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Existence of the Time Evolution for Schrödinger Operators with Time Dependent Singular Potentials

by

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ABSTRACT. — Let N particles move along fixed paths $y_j(t)$ and cause a time dependent potential. The quantum mechanical description for an additional particle is given by the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\Delta\psi + \sum_{j=1}^{N} q_j v(x - y_j(t))\psi .$$

For a class of trajectories $y_j(t)$ the propagator is constructed for the Coulomb potential $v(x) = \frac{1}{|x|}(x \in \mathbb{R}^3)$ and stronger singularities.

RÉSUMÉ. — Soit N particules se déplaçant sur des trajectoires fixées $y_j(t)$ et produisant un potentiel dépendant du temps. La description quantique d'une particule supplémentaire est donnée par l'équation de Schrödinger

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\Delta\psi + \sum_{j=1}^{N} q_j v(x - y_j(t))\,\psi .$$

Pour une classe de trajectoires $y_j(t)$, on construit le propagateur correspondant au potentiel de Coulomb $v(x) = 1/|x|\ (x \in \mathbb{R}^3)$ et à des potentiels plus singuliers.

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1. INTRODUCTION

There are several possibilities to describe a multi particle system. Often all particles were treated quantum mechanically, which leads to a large number of coordinates. The advantage thereby is a time independent and self-adjoint Hamiltonian and consequently there are no problems to find the unique time evolution for the system by using the functional calculus

$$\psi(t) = e^{-it\mathrm{H}}\psi .$$

In this paper contrarily only such systems are regarded which have N classical and only one quantum mechanical partical, i. e. the first N should move along fixed trajectories $y_j(t)$; $j = 1, \ldots, \mathrm{N}$; and engender a time dependent potential

$$\mathrm{V}(t) = \sum_{j=1}^{\mathrm{N}} q_j v(x - y_j(t)) .$$

which influences the motion of the last one. So the Schrödinger equation for only one particle has to be solved, but with a time dependent Hamiltonian (see [*1*], [*2*], [*14*])

$$\frac{d}{dt}\psi(t) = -i\mathrm{H}(t)\psi(t) \quad ; \quad \mathrm{H}(t) = -\frac{1}{2m}\Delta + \mathrm{V}(t) \quad ; \quad \psi(t) \in \mathrm{L}^2(\mathbb{R}^\nu) .$$

For this reason the time evolution — if it exists — is given by a family of operators $\{ \mathrm{U}(t, s) \}_{t,s}$, called propagator, through the formula

$$\psi(t) = U(t, s)\psi(s) .$$

The following properties are satisfied, if the Hamiltonian is self-adjoint.

(1) $\mathrm{U}(t, s)$ is unitary and strongly continuous in t and s

(2) $\mathrm{U}(t, t) = \mathrm{I}$ for all t

(3) $\mathrm{U}(t, s) = \mathrm{U}(t, r)\,\mathrm{U}(r, s)$ for all t, r and s

But the self-adjointness of $\mathrm{H}(t)$ for all t is not sufficient to show the existence of the time evolution. One needs also some supplementary conditions about the smoothness of $\mathrm{H}(t)$. Kato [*4*]-[*7*] was the first to formulate conditions for the unique existence of solutions in the general case of Banach spaces. Furthermore there are some results of Yosida [*15*]-[*17*] and Tanabe [*12*]. Using quadratic forms Simon [*11*] has obtained a wider class of potentials for which he can solve the Schrödinger equation. But singularities like those in Coulomb and Yukawa potentials for dimension $\nu = 3$ are not included. So here potentials which have the form

$$v(x) = \frac{1}{|x|^\alpha} + v_2(x)$$

will be treated, where $v_2(x)$ is a suitable bounded function. The singular part of the potential $v(x)$ should be homogeneous of degree $\alpha < \min\left\{\frac{\nu}{2}, 2\right\}$, so that it is possible to take time dependent scalars out of this term as a coefficient. In particular the physically important Coulomb and Yukawa forces are included.

2. RESULTS

First we forget about the quantum mechanical particle and look at the N others. We assume that their trajectories are given by

$$y_j(t) = l(t)\mathrm{A}(\varphi(t))y_j \; ; \; y_j \in \mathbb{R}^\nu \; ; j = 1, \ldots, N. \tag{2.1}$$

Here $\mathrm{A}(\varphi)$ should be the matrix of the rotation in a fixed plane with angle φ. Without loss of generality we later take the plane spanned by the directions of the two first coordinates. The functions $l(t)$ and $\varphi(t)$ have to be twice continuously differentiable within the chosen set of times, which can be taken as the whole real line or only as an interval. In addition we consider only those times for which no collisions occur. If one examines the permitted trajectories, it seems to be a strong restriction that $l(t)$ and $\varphi(t)$ should by independent of j and the rotation is around a fixed axis. For N greater than 2 it is really not the description of the general situation. But if one has two classical particles, all systems are admitted, where no external forces act on the particles. After separation of the free motion of the center of mass it is of the form (2.1). In particular

$$\hat{y}_j(t) = \alpha_j + v_j t \; ; \; \alpha_j, v_j \in \mathbb{R}^\nu \; ; j = 1,2$$

is admissible in this sense. For example the whole model describes the transition of an electron from one proton to another, if the influence of the electron on the nuclei of the hydrogen can by neglected. Now we state the main result.

THEOREM. — Let $v_2 : \mathbb{R}^\nu \to \mathbb{R}$ be a bounded function with bounded first derivatives and $\Delta v_2 \in \mathrm{L}^p(\mathbb{R}^\nu) + \mathrm{L}^\infty(\mathbb{R}^\nu)$ for some $p \in [2, \infty) \cap \left(\frac{\nu}{2}, \infty\right)$. Then for $v(x) = 1/|x|^a + v_2(x)$

$$\mathrm{H}(t) = -\frac{1}{2m}\Delta + \sum_{j=1}^{\mathrm{N}} q_j v(x - y_j(t)) \tag{2.2}$$

is self-adjoint on $D(H(t)) = W^{2,2}(\mathbb{R}^\nu)$ for all t if $0 < \alpha < \min\left\{\frac{\nu}{2}, 2\right\}$ and $y_j(t)$ is given as in (2.1). Moreover there exists a unique propagator $U(t, s)$ such that

$$\frac{\partial}{\partial t} U(t, s)\psi = -iH(t)U(t, s)\psi \qquad \forall\, \psi \in D(H_0) \cap D(x^2).$$

The statement of self-adjointness is a corollary of Theorem XIII 96 in [*10*], since $\frac{1}{|x|^\alpha}$ is in $L^p(\mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu)$ with $p = 2$ for $\nu \leq 3$ and $p \in \left(\frac{\nu}{2}, \frac{\nu}{\alpha}\right)$ for $\nu \geq 4$. The proof of the rest of this theorem is given in the next sections. Regarding the uniqueness of the solution it is no restriction to take a compact time interval $[t_1, t_2]$. The propagator corresponding to $v_2 \equiv 0$ will be obtained with the time dependent transformation

$$W(t) = \exp\left[-i\varphi(t)L\right] \exp\left[i\ln(l(t))\, D\right], \tag{2.3}$$

L a generator of rotation, revolving and blowing up the space in accordance with the motion of the N particles. The transformed equation has a new Hamiltonian with a dependence on time, which is more appropriate for finding a solution in spite of the problems of varying domains. Afterwards it is shown how to add the bounded perturbation v_2. For the sake of simplicity we take $m = 1/2$ from here on.

3. SOME SELF-ADJOINT OPERATORS

In this section we introduce some self-adjoint operators, which are needed for solving the transformed equation. The operator generating the rotation in the two first coordinates is denoted by

$$L = X_1P_2 - X_2P_1$$

which is the third component of the angular momentum, if we are in the physical three dimensional space. Analogously we have for the dilatation the symmetrized scalar product of position and momentum

$$D = \frac{1}{2}\{XP + PX\}.$$

Both operators are contained in the new (modified) Hamiltonian

$$H_M(t) = H_0 + f(t)D + k(t)L + h(t)V_M, \tag{3.1}$$

which results from the transformation with $W(t)$ (2.3). The only time dependence is in the real coefficients $f(t)$, $k(t)$ and $h(t)$. The potential has changed to

$$V_M = \sum_{j=1}^{N} \frac{q_j}{|x - y_j|^\alpha}. \tag{3.2}$$

Later we will see that the domain of $H_M(t)$ depends on t. So the smoothness of $H_M(t)$ is given by means of an auxiliary operator valued function $H_S(t)$ with a constant domain

$$H_S(t) = H_M(t) + s(t)x^2 \; ; \; s(t) = (\nu + 2)(f^2(t) + k^2(t) + 1). \tag{3.3}$$

Since we only want to prove the self-adjointness of these operators for any fixed t in this section, we will omit the time here for convenience. The potential V_M should be an operator bounded perturbation of the other terms. So we first set its coefficient h equal to zero.

LEMMA. — For all $f, k \in \mathbb{R}$ the operator $H_0 + fD + kL$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$.

Proof. — We begin by showing the statement on $\mathcal{S}(\mathbb{R}^\nu)$, the Schwartz space of functions of rapid decrease. The unitary group of L can be computed.

$$(e^{-isL}\psi)(x) = \psi(A(s)x) \quad ; \quad \forall s \in \mathbb{R}$$

$$A(s) = \begin{pmatrix} \hat{A}(s) & 0 \\ 0 & I \end{pmatrix} \; ; \; \hat{A}(s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$$

Because the self-adjoint operator D generates the dilatation

$$(e^{-itD}\psi)(x) = e^{-t\frac{\nu}{2}} \psi(e^{-t}x),$$

we see that e^{-isL} commutes with e^{-itD} for all s and t. With

$$\{ e^{-it(fD)} e^{-it(kL)} \}_{t \in \mathbb{R}}$$

we have a strongly continuous one parameter unitary group which leaves $\mathcal{S}(\mathbb{R}^\nu)$ invariant and is strongly continuously differentiable on the same set. Thus the generator $fD + kL$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^\nu)$ (Theorem VIII 10 in [*8*]). Since the Fourier transform of L generates the same rotation in momentum space, the above argument works also for the sum $H_0 + kL$. For $f \neq 0$ we calculate on $\mathcal{S}(\mathbb{R}^\nu)$

$$e^{iH_0/2f}(fD + kL)e^{-iH_0/2f} = H_0 + fD + kL$$

and we have shown the lemma for $\mathcal{S}(\mathbb{R}^\nu)$ because

$$e^{iH_0/2f} : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu).$$

The only reason, why this proof doesn't work with $C_0^\infty(\mathbb{R}^\nu)$, is that in general $e^{i\alpha H_0}\psi \notin C_0^\infty(\mathbb{R}^\nu)$ for $\psi \in C_0^\infty(\mathbb{R}^\nu)$ (see [3]). To complete the proof we use the harmonic oscillator Hamiltonian $H_0 + x^2$, which has the domain $D(H_0 + x^2) = D(H_0) \cap D(x^2)$ and is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$. The following lemma shows that $H_0 + fD + kL$ is operator bounded with respect to $H_0 + x^2$ and so $C_0^\infty(\mathbb{R}^\nu)$ is also a core for the first one.

LEMMA. — Suppose $\psi \in C_0^\infty(\mathbb{R}^\nu)$ and $f, k \in \mathbb{R}$. Then

$$\| D\psi \| + \| L\psi \| \leq \sqrt{\nu + 2}\, \| (H_0 + x^2)\psi \| + \nu \| \psi \| .$$

Proof. — By expanding the square of $H_0 + x^2 = \sum_j (P_j^2 + x_j^2)$ in the scalar product $((H_0 + x^2)\psi, (H_0 + x^2)\psi)$ we get using the canonical commutation relations

$$\| (H_0 + x^2)\psi \|^2 \geq \sum_j \{ (P_j^2\psi, x_j^2\psi) + (x_j^2\psi, P_j^2\psi) \} + 2 \{ \|x_1P_2\psi\|^2 + \|x_2P_1\psi\|^2 \} .$$

Applying the inequality $|y| \leqslant \sum_{j=1}^{\nu} |y_j| \leq \sqrt{\nu}\, |y|$ in $\mathbb{R}^\nu$ we can estimate

$$\begin{aligned}
\| D\psi \| &\leq \sum_{j=1}^{\nu} \frac{1}{2} \| (x_jP_j + P_jx_j)\psi \| \\
&\leq \sqrt{\nu} \left\{ \sum_j \frac{1}{4} \| (x_jP_j + P_jx_j)\psi \|^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{\nu} \left\{ \sum_j \frac{1}{2} [(P_j^2\psi, x_j^2\psi) + (x_j^2\psi, P_j^2\psi)] + \frac{3}{4}\nu \| \psi \|^2 \right\}^{\frac{1}{2}} \\
&\leq \sqrt{\nu} \left\{ \frac{1}{2} \| (H_0 + x^2)\psi \|^2 + \frac{3}{4}\nu \| \psi \|^2 \right\}^{\frac{1}{2}} \\
&\leq \sqrt{\frac{\nu}{2}} \| (H_0 + x^2)\psi \| + \nu \| \psi \| .
\end{aligned}$$

and

$$\| L\psi \| \leq \sqrt{2} \{ \| x_1P_2\psi \|^2 + \| x_2P_1\psi \|^2 \}^{\frac{1}{2}} \leq \| (H_0 + x^2)\psi \| .$$

We are finished with the proof since

$$\sqrt{\frac{\nu}{2}} + 1 \leq \sqrt{2}\left(\frac{\nu}{2} + 1\right)^{\frac{1}{2}} = \sqrt{\nu + 2} . \qquad \square$$

LEMMA. — For all $f,\ k \in \mathbb{R}$, $s = (\nu + 2)(f^2 + k^2 + 1)$ the operator $H_0 + fD + kL + sx^2$ is self-adjoint on $D(H_0) \cap D(x^2)$ and $C_0^\infty(\mathbb{R}^\nu)$ is a core.

Proof. — We check on $C_0^\infty(\mathbb{R}^\nu)$ that

$$\sqrt{s}\, e^{\frac{i}{4}\ln sD}(H_0 + x^2)\, e^{-\frac{i}{4}\ln sD} = H_0 + sx^2 .$$

Since the unitary transformation $e^{\frac{i}{4}\ln sD}$ does not change the operators D and L or the domain of $H_0 + x^2$, the last lemma implies.

$$\|(fD + kL)\psi\| \leq \max\{|f|,|k|\}\left\{\sqrt{\frac{\nu+2}{s}}\,\|(H_0+sx^2)\psi\| + \nu\,\|\psi\|\right\}.$$

The coefficient $\max\{|f|,|k|\}\sqrt{\dfrac{\nu + 2}{s}}$ is smaller than 1 and thus we can apply the Kato-Rellich theorem. □

The next step is the addition of the potential hV_M. To avoid any relation between the coefficients f, k and h, we will show that the operator bound of V_M is arbitrarily small.

LEMMA. — For all $f,\ k,\ h \in \mathbb{R}$ the operators $H_M = H_0\, fD + kL + hV_M$ and $H_s = H_0 + fD + kL + hV_M + sx^2$ are self-adjoint on domains independent of h.

Proof. — H_0 is operator bounded with respect to $H_0 + fD + kL + sx^2$. The relative H_0-bound of V_M is zero, because for $\alpha < \min\left\{\frac{\nu}{2}, 2\right\}$ there is some $p \in [2, \infty) \cap \left(\frac{\nu}{2}, \infty\right)$ with $V_M \in L^p(\mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu)$ (Theorem XIII 96 in [*10*]). Thus for all $a > 0$ there exists a $b > 0$ such that

$$\|V_M\psi\| \leq a\,\|(H_0 + fD + kL + sx^2)\,\psi\| + b\,\|\psi\| .$$

Again the stated properties for H_s follow from the Kato-Rellich theorem. Regarding the other operator a little more work must be done. We have to use that the potential V_M is bounded outside a ball B_δ. Therefore we choose $\varphi \in C_0^\infty(\mathbb{R}^\nu)$ with $\varphi(x) = 1$ for $x \in B_\delta$, $\varphi(x) = 0$ for $x \notin B_{2\delta}$, $\|\varphi\|_\infty = 1$ and $\varphi(x) = \tilde{\varphi}(|x|)$.

$$\|V_M\psi\| \leq \|V_M\varphi\psi\| + \|V_M(1 - \varphi)\|_\infty\,\|\psi\| .$$

Using the first part of the proof we only have to show

$$\|(H_0 + fD + kL + sx^2)\,\varphi\psi\| \leq \text{const}\,\{\,\|(H_0 + fD + kL)\,\psi\| + \|\psi\|\,\}.$$

The compact support of φ can be utilized here to estimate

$$\|sx^2\varphi\psi\| \leq \|sx^2\varphi\|_\infty\,\|\psi\| .$$

Afterwards we want to eliminate the function φ by commuting it with $H_0 + fD + kL$ and applying

$$\| \varphi(H_0 + fD + kL)\psi \| \leq \| (H_0 + fD + kL)\psi \| .$$

The commutators $[D, \varphi]$ and $[L, \varphi]$ are bounded, but

$$[H_0, \varphi] = \{ -2i(\nabla\varphi)P - \Delta\varphi \} .$$

Since $\Delta\varphi$ is bounded it remains to estimate

$$\| (\nabla\varphi)P\psi \| \leq \text{const} \{ \| (H_0 + fD + kL)\psi \| + \| \psi \| \} . \tag{3.4}$$

We define $\varphi_n(x) := \varphi(2^{-n}x)$ which implies $\varphi_{n+1}(x) = 1$ on the support of φ_n and $\left\| \frac{\partial}{\partial x_j} \varphi_n \right\|_\infty = 2^{-n} \left\| \frac{\partial}{\partial x_j} \varphi \right\|_\infty$. Thus

$$\begin{aligned}
\| (\nabla\varphi_n)P\psi \| &\leq \sum_{j=1}^{\nu} \left\| \left(\frac{\partial}{\partial x_j} \varphi_n \right) \varphi_{n+1} P_j \psi \right\| \\
&\leq \sum_{j=1}^{\nu} \left\| \frac{\partial}{\partial x_j} \varphi_n \right\|_\infty \left\{ \| P_j \varphi_{n+1} \psi \| + \left\| \frac{\partial}{\partial x_j} \varphi_{n+1} \psi \right\| \right\} \\
&\leq \text{const}\, 2^{-n} \{ \| (H_0 + 1)\varphi_{n+1}\psi \| + \| \psi \| \} \\
&\leq \text{const}\, 2^{-n} \{ \| (H_0 + fD + kL + sx^2)\varphi_{n+1}\psi \| + \| \psi \| \} .
\end{aligned}$$

The same procedure as above leads to

$$\begin{aligned}
\| (\nabla\varphi_n)P\psi \| \leq \text{const} \{ & 2^{-n} \| (H_0 + fD + kL)\psi \| + 2^n \| \psi \| \\
& + 2^{-n} \| (\nabla\varphi_{n+1})P\psi \| \} .
\end{aligned} \tag{3.5}$$

The coefficient of $\| \psi \|$ is based on the fact that $\| sx^2 \varphi_{n+1} \|_\infty \sim 2^{2n}$. We can employ the inequality (3.5) iteratively in estimating

$$\| (\nabla\varphi)P\psi \| = \| (\nabla\varphi_0)P\psi \| .$$

The limit $n \to \infty$ exists and shows (3.4). □

While $D(H_s) = D(H_0) \cap D(x^2)$, we do not give the domain of H_M. Even for $k = h = 0$ the domain $D(H_0 + fD)$ varies with the coupling constant f. For $f \neq 0$ the operator $H_0 + fD$ can be studied with the following unitary transformation from $L^2(\mathbb{R}^\nu, d^\nu x)$ to $L^2(\mathbb{R} \times S^{\nu-1}, db\, d\omega)$. For $\psi \in \mathscr{S}(\mathbb{R}^\nu)$ the momentum space wave function $\hat{\psi}$ given by Fourier transform should be taken in polar coordinates to define

$$\tilde{\psi}(b, \omega) := \frac{1}{\sqrt{2\pi |f|}} \int_{\mathbb{R}_+} \frac{dp}{p} p^{\frac{\nu}{2}} p^{i\frac{b}{f}} e^{-i\frac{p^2}{2f}} \hat{\psi}(p, \omega) .$$

So we obtain the $(H_0 + fD)$-space representation, i. e. under the transformation $\psi \rightarrow \tilde{\psi}$ a function of this operator becomes a multiplication operator with this function of b. As a corollary we get that the spectrum of $H_0 + fD$ is absolutely continuous on the whole real line.

Assume that for some $f \neq 0$ the domain of $H_0 + fD$ is equal to that of H_0. One could conclude that $D = \frac{1}{f}(H_0 + fD - H_0)$ is defined on $D(H_0)$ and therefore operator bounded w.r.t. H_0. So for some small f_1 one could apply the Kato-Rellich theorem to H_0 and $f_1 D$. We could obtain $H_0 + f_1 D$ as an operator defined on $D(H_0)$ and bounded from below since $H_0 \geq 0$, which is a contradiction to the result above. Indeed one can prove for an arbitrary pair of different coupling constants that the corresponding domains are not equal. The details can be found in [*13*].

4. SOLUTION OF THE EQUATION WITH A HOMOGENEOUS POTENTIAL

After the self-adjointness problems are settled we turn to solutions of the Schrödinger equation. First we want to treat

$$\frac{d}{dt}\psi(t) = -i(H_0 + f(t)D + k(t)L + h(t)V_M)\psi(t) = -iH_M(t)\psi(t) \qquad (4.1)$$

with continuously differentiable functions f, k and h. The following results are contained in Theorems 4.4.1 and 4.4.2 of Tanabe [*12*]. His formulation is based upon two Banach spaces X and Y with the respective norms $\|\,.\,\|$ and $\|\,.\,\|_Y$. The space Y is densely contained in X and there is some constant c with $\| v \| \leq c \| v \|_Y$ for all $v \in Y$. Furthermore he needs the following assumptions.

i) $\{ A(t) \}_{t\in[0,T]}$ is a family of generators of strongly continuous semigroups in X, which is stable with stability constants M and β.

ii) There exists a family $\{ S(t) \}$ of isomorphic mappings from Y to X. $S(t)$ is strongly continuously differentiable on $[0, T]$ as a function with values in $B(Y, X)$ (the set of all bounded operators from Y to X). There exists a strongly continuous function $B(t)$ with values in $B(X, X)$, such that

$$S(t)\,A(t)\,S^{-1}(t) = A(t) + B(t). \qquad (4.2)$$

iii) $Y \subset D(A(t))$ for each $t \in [0, T]$, so that $A(t) \in B(Y, X)$. The function $A(t)$ of t is continuous in the norm of $B(Y, X)$.

Under these conditions there exists a unique bounded-operator-valued function $U(t, s) \in B(X, X)$ for $0 \leq s \leq t \leq T$ having the properties *a)* $U(t, s)$ is strongly continuous in s and t, $U(s, s) = I$ and

$$\| U(t, s) \| \leq M e^{\beta(t-s)}.$$

b) $U(t, s) = U(t, r) U(r, s)$ for $s \leq r \leq t$.

c) $\frac{\partial}{\partial s} U(t, s) v = - U(t, s) A(s) v$ on $0 \leq s \leq t \leq T$ for each $v \in Y$.

d) $U(t, s) Y \subset Y$; $U(t, s)$ is strongly continuous in Y.

e) For all $v \in Y$ and $s \in [0, T]$, $U(t, s) v$ is continuously differentiable with respect to t in $[s, T]$;

$$\frac{\partial}{\partial t} U(t, s) v = A(t) U(t, s) v.$$

In our application the Banach space X is given as $L^2(\mathbb{R}^\nu)$, but Y must be chosen appropriately. We will see that

$$Y = (D(H_0 + x^2), \| . \|_Y) ; \| \psi \|_Y^2 = \|(H_0 + x^2) \psi \|^2 + \| \psi \|^2$$

is a suitable choice. Regarding (4.1) we have to take

$$A(t) = - i H_M(t). \tag{4.3}$$

The self-adjointness of $H_M(t)$ for all t is stronger than *i*), so that the definition of a stable family can be omited here. From the preceding section it is clear that $Y \subset D(H_M(t))$. Since D, L and V_M are bounded operators from Y to X, we get using the smoothness of f, k and h that $H_M(t)$ is continuous in the norm. Hence the condition *iii*) is satisfied. Finally the family of operators $S(t)$ has to be defined. For that reason we have introduced the self-adjoint operator $H_s(t)$ which has the domain $D(H_0 + x^2)$. It is evident that

$$S(t) := H_s(t) + i \tag{4.4}$$

gives an isomorphic mapping from Y to X for all t (bounded with bounded inverse). The time dependence of $S(t)$ (see (3.1), (3.3)) is contained in the coefficients $f(t)$, $k(t)$, $h(t)$ and $s(t)$, which are continuously differentiable. Thus $S(t)$ is differentiable even in the norm of B(Y, X). We have to check, whether $[S(t), A(t)]$ is bounded from Y to X (compare with (4.2)). Suppose $\varphi \in C_0^\infty(\mathbb{R}^\nu)$ and $\psi \in L^2(\mathbb{R}^\nu)$. With the definitions (3.1), (3.3) we can compute

$$\begin{aligned}
&(H_S(t) \varphi, H_M(t) (H_S(t) + i)^{-1} \psi) - (H_M(t) \varphi, \psi) \\
&\qquad = (iH_M(t) \varphi - 4is(t) D\varphi - 2is(t) f(t) x^2 \varphi, (H_S(t) + i)^{-1} \psi) \\
&\qquad = (i\varphi, H_M(t) (H_S(t) + i)^{-1} \psi) + (\varphi, iB(t) \psi)
\end{aligned}$$

where $B(t) = \{ 4s(t) D + 2s(t) f(t) x^2 \} (H_S(t) + i)^{-1}$ is a bounded-operator-valued function of t, which is continuous in the norm of B(X, X). (Tanabe has proved that $S^{-1}(t)$ is continuous in the norm of B(X, Y) under the assumptions). If $\psi \in D(H_M(t))$, then the above equality gives

$$| (H_S(t) \varphi, H_M(t) (H_S(t) + i)^{-1} \psi) | \leq \text{const} (\psi) \| \varphi \|.$$

Since $C_0^\infty(\mathbb{R}^\nu)$ is a core for $H_S(t)$ one has $H_M(t)(H_S(t)+i)^{-1}\psi \in Y = D(H_S(t))$. Similarly the latter implies $\psi \in D(H_M(t))$. For those ψ we get

$$S(t)\,A(t)\,S^{-1}(t)\,\psi = A(t)\,\psi + B(t)\,\psi .$$

It is not hard to extend the result of Tanabe in the case of self-adjoint operators in Hilbert spaces. One obtains a unitary propagator for all $s, t \in [0, T]$ and *a)-e)* holds correspondingly for all times. We will denote this propagator for $A(t)$ as in (4.3) by $\{ U_M(t, s) \}_{t,s\in[0,T]}$.

The next step to prove the main theorem of the second section is to use the unitary transformation $W(t)$ (see (2.3)). We consider the time dependent potential without the bounded part v_2, which is a multiplication operator with the function

$$[V_1(t)](x) = \sum_{j=1}^{N} \frac{q_j}{|x - y_j(t)|^\alpha}, \tag{4.5}$$

where $\{ y_j(t) \}$ denotes a permitted family of trajectories, i. e. $y_j(t) = l(t)A(\varphi(t))y_j$; $l, \varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $l(t) \neq 0$ for all t in an interval $[t_1, t_2]$. We assume $l(t) > 0$, because otherwise we turn to $\tilde{l}(t) = -l(t)$ and $\tilde{y}_j = -y_j$.

By using the explicit construction of the groups generated by D and L we can compute on Y

$$W(t)\,V_1(t)\,W^{-1}(t) = \frac{1}{l^\alpha(t)} V_M . \tag{4.6}$$

At this point, where the time dependence has to be separated, we crucially use the homogeneity of the potential. In the same way we get

$$W(t)\,H_0\,W^{-1}(t) = \frac{1}{l^2(t)} H_0 . \tag{4.7}$$

Thus, to compensate the factor $\frac{1}{l^2(t)}$, we introduce the following transformation of the time

$$g(t) := \int_{t_1}^{t} \frac{1}{l^2(\tau)} d\tau ,$$

which is a diffeomorphism from $[t_1, t_2]$ to $[0, T]$; $T := \int_{t_1}^{t_2} \frac{1}{l^2(\tau)} d\tau$. We can define

$$U_1(t, s) := W^{-1}(t)\,U_M(g(t), g(s))\,W(s) . \tag{4.8}$$

It is an additional advantage of the set $Y = D(H_0) \cap D(x^2)$ that it is left invariant by $W(t)$. Hence we have no problems to differentiate $U_1(t,s)\psi$ for all $\psi \in Y$ and $t \in [t_1, t_2]$.

$$\begin{aligned}\frac{\partial}{\partial t} U_1(t,s)\psi &= \left\{ i\varphi'(t) L - i\frac{l'(t)}{l(t)} D \right\} U_1(t,s)\psi \\ &\quad + W^{-1}(t)(-iH_M(g(t)))\, U_M(g(t),g(s)) \left(\frac{d}{dt} g(t)\right) W(s)\psi \\ &= \left\{ i(\varphi'(t) - k(g(t))g'(t)) L - i\left(\frac{l'(t)}{l(t)} + f(g(t))g'(t)\right) D \right\} U_1(t,s)\psi \\ &\quad - iW^{-1}(t)\frac{1}{l^2(t)} H_0\, U_M(g(t),g(s))\, W(s)\psi \\ &\quad - iW^{-1}(t)\frac{h(g(t))}{l^2(t)} V_M\, U_M(g(t),g(s))\, W(s)\psi .\end{aligned}$$

We have used that $U_M(g(t), g(s))$ maps Y into Y by property *d*). For the twice continuously differentiable functions l and φ we choose the differentiable functions f, k and h on $[0, T]$ by

$$\begin{aligned} f(t) &:= -\, l(g^{-1}(t))\, l'(g^{-1}(t)) \\ k(t) &:= l^2(g^{-1}(t))\, \varphi'(g^{-1}(t)) \\ h(t) &:= l^{2-\alpha}(g^{-1}(t)) \end{aligned}$$

(g^{-1} denotes the inverse function of g). With (4.6) and (4.7)

$$\frac{\partial}{\partial t} U_1(t,s)\psi = -i\{ H_0 + V_1(t) \} U_1(t,s)\psi . \tag{4.9}$$

In the next theorem we will summarize some characteristics of this propagator. We will denote the self-adjoint operator $H_0 + V_1(t)$ on $D(H_0)$ by $H_1(t)$.

THEOREM. — There exists an unique operator valued function $U_1(t, s)$ (defined by (4.8) for all $t, s \in [t_1, t_2]$ with the following properties

a) $U_1(t, s)$ is unitary and strongly continuous in s and t, $U_1(s, s) = I$.

b) $U_1(t, s) = U_1(t, r)\, U_1(r, s)$ for all $s, r, t \in [t_1, t_2]$; in particular $[U_1(t, s)]^* = U_1(s, t)$.

c) $\frac{\partial}{\partial s} U_1(t,s)\psi = i\, U_1(t,s)\, H_1(s)\psi$ for $t, s \in [t_1, t_2]$ and $\psi \in Y$.

d) $U_1(t, s)[Y] = Y$, $U_1(t, s)$ is strongly continuous in Y.

e) $\frac{\partial}{\partial t} U_1(t,s)\psi = -i H_1(t)\, U_1(t,s)\psi$ for $t, s \in [t_1, t_2]$ and $\psi \in Y$.

Proof. — Since $H_1(t)$ is a symmetric operator on Y for all t, every solution of $\frac{d}{dt}\varphi(t) = -i\,\mathrm{H}(t)\varphi(t)$ is norm preserving. Thus the difference of two solutions $\tilde{\mathrm{U}}_1(t,s)\psi$ and $\mathrm{U}_1(t,s)\psi$, which is also a solution, has norm

$$\| \tilde{\mathrm{U}}_1(t,s)\psi - \mathrm{U}_1(t,s)\psi \| = \| \tilde{\mathrm{U}}_1(s,s)\psi - \mathrm{U}_1(s,s)\psi \| = 0 .$$

It is sufficient for the uniqueness of $\mathrm{U}_1(t,s)$ to take ψ in the dense set Y. The properties *a)* and *b)* are obvious by the definition and the proof of *c)* is similar to that of *e)* which we have done above. The equality in *d)* is a direct consequence of $\mathrm{U_M}(t,s)\mathrm{Y} \subset \mathrm{Y}$, $\mathrm{U_M^*}(t,s)\mathrm{Y} = \mathrm{U_M}(s,t)\mathrm{Y} \subset \mathrm{Y}$ and $e^{i\lambda\mathrm{D}}\mathrm{Y} = e^{i\mu\mathrm{L}}\mathrm{Y} = \mathrm{Y}$.

For any operator M, which is bounded in $\mathrm{L}^2(\mathbb{R}^\nu)$ and leaves Y invariant, one can define $\mathrm{M_Y} = (\mathrm{H}_0 + x^2 + i)\,\mathrm{M}\,(\mathrm{H}_0 + x^2 + i)^{-1}$ as a closed operator on $\mathrm{L}^2(\mathbb{R}^\nu)$. By the closed graph theorem $\mathrm{M_Y}$ is bounded. Thus M is bounded as an operator from Y to Y. Employing this to $\mathrm{W}^{-1}(t)$, $\mathrm{U_M}(g(t), g(s))$ and $\mathrm{W}(s)$ we get the property *d)*, if we can show that the three operators are strongly continuous in Y. For $\mathrm{U_M}(g(t), g(s))$ we refer to the theorem of Tanabe, which has provided the propagator. Suppose $\psi \in \mathrm{Y}$, then

$$\begin{aligned} a)\quad (\mathrm{H}_0 + x^2)\,e^{i\lambda\mathrm{D}}\psi &= e^{i\lambda\mathrm{D}}(e^{2\lambda}\mathrm{H}_0\psi + e^{-2\lambda}x^2\psi) \\ &= e^{i\lambda(\mathrm{D}-2i)}(\mathrm{H}_0\psi) + e^{i\lambda(\mathrm{D}+2i)}(x^2\psi) \\ b)\quad (\mathrm{H}_0 + x^2)\,e^{i\mu\mathrm{L}}\psi &= e^{i\mu\mathrm{L}}(\mathrm{H}_0 + x^2)\psi . \end{aligned}$$

Therefore $\{ e^{i\lambda\mathrm{D}} \}_\lambda$ and $\{ e^{i\mu\mathrm{L}} \}_\mu$ are strongly continuous groups in Y. Hence $\mathrm{W}^{-1}(t)$ and $\mathrm{W}(s)$ are strongly continuous in Y. □

5. THE BOUNDED PERTURBATION OF THE POTENTIAL

In the preceding sections we have solved the Schrödinger equation with the Hamiltonian

$$\mathrm{H}_1(t) = \mathrm{H}_0 + \sum_{j=1}^{\mathrm{N}} q_j v_1(x - y_j(t))\,;\ v_1(x) = \frac{1}{|x|^\alpha}.$$

We now consider the operator $\mathrm{H}(t) = \mathrm{H}_1(t) + \mathrm{V}_2(t)$ with some appropriate bounded multiplication operator $\mathrm{V}_2(t)$ for all t. The interaction representation allows the treatment of this perturbation. For this we fix some $t_0 \in [t_1, t_2]$ and define

$$\tilde{\mathrm{V}}(t) := \mathrm{U}_1(t_0, t)\,\mathrm{V}_2(t)\,\mathrm{U}_1(t, t_0). \tag{5.1}$$

Suppose there exists a propagator $\{ \tilde{\mathrm{U}}(t,s) \}_{t,s}$ satisfying

$$\frac{\partial}{\partial t}\tilde{\mathrm{U}}(t,s)\psi = -i\,\tilde{\mathrm{V}}(t)\,\tilde{\mathrm{U}}(t,s)\psi \quad \forall \psi \in \mathrm{L}^2(\mathbb{R}^\nu). \tag{5.2}$$

Then we apply the time evolution $U_1(t, t_0)$ again and get

$$U(t, s) := U_1(t, t_0)\,\tilde{U}(t, s)\,U_1(t_0, s)\,. \tag{5.3}$$

If we formally differentiate this transformed propagator, we obtain

$$\begin{aligned}\frac{\partial}{\partial t} U(t, s)\,\psi &= -\,i\,H_1(t)\,U_1(t, t_0)\,\tilde{U}(t, s)\,U_1(t_0, s)\,\psi \\ &\quad + U_1(t, t_0)\,(-\,i\,\tilde{V}(t))\,\tilde{U}(t, s)\,U_1(t_0, s)\,\psi \\ &= -\,i\,H_1(t)\,U(t, s)\,\psi - i\,V_2(t)\,U(t, s)\,\psi\,. \end{aligned} \tag{5.4}$$

The next theorem gives some conditions guaranteeing the existence of $U(t, s)$ and its derivative.

THEOREM. — $\{ V_2(t) \}_{t\in[t_1,t_2]}$ be a family of bounded functions from $\mathbb{R}^\nu$ to $\mathbb{R}$ with

i) $V_2(t)\,[D(H_0)] \subset D(H_0)$ for all t

ii) $V_2(t)$ is strongly continuous in t as a multiplication operator in $L^2(\mathbb{R}^\nu)$.

iii) for all $\psi \in Y$ the function $t \to [H_0, V_2(t)]\,\psi$ is continuous in $L^2(\mathbb{R}^\nu)$.

Then there exists a unitary strongly continuous propagator $U(t, s)$, which satisfies $U(t, s) : Y \to Y$, and $U(t, s)\,\psi$ is continuously differentiable in $L^2(\mathbb{R}^\nu)$ for all $\psi \in Y$ with

$$\frac{\partial}{\partial t} U(t, s)\,\psi = -\,i\,H(t)\,U(t, s)\,\psi\ , \quad H(t) = H_1(t) + V_2(t)\,.$$

Proof. — Since $U_1(t_0, t) = U_1^*(t, t_0)$, the operator $\tilde{V}(t)$ defined in (5.1) is self-adjoint and bounded on $L^2(\mathbb{R}^\nu)$. Moreover by continuity of $V_2(t)$ (condition *ii)*) and of $U_1(t, s)$ in the strong topology of bounded operators in $L^2(\mathbb{R}^\nu)$ the operator $V(t)$ is also strongly continuous in $L^2(\mathbb{R}^\nu)$. Thus we can define

$$\tilde{U}(t, s)\,\psi = \psi + \sum_{n=1}^{\infty} (-i)^n \int_s^t \int_s^{t_1} \dots \int_s^{t_{n-1}} \leq \tilde{V}(t_1) \dots \tilde{V}(t_n)\psi dt_n \dots dt_1 \tag{5.5}$$

This propagator is unitary, strongly continuous and it satisfies (5.2). A solution constructed in this manner is called a Dyson expansion (see Theorem X 69 in [*9*]). Because we want to differentiate $U(t, s)\,\psi$ defined in (5.3) for $\psi \in Y$, we have to verify $\tilde{U}(t, s)\,U_1(t_0, s)\,\psi \in Y$. The propagator $U_1(t_0, s)$ leaves Y invariant, thus it remains to show

$$\tilde{U}(t, s)\,Y \subset Y\,. \tag{5.6}$$

By the hypothesis *i)* and the fact, that the domain $D(x^2)$ is obviously left invariant by the multiplication operator $V_2(t)$, it follows for $\psi \in Y$ that $V_2(t)\,\psi \in Y$. Hence $\tilde{V}(t)$ also maps Y into Y. Now we can try to regard (5.5) as a definition in the Hilbert space Y. Let $\psi \in Y$ and $\varphi := (H_0 + x^2)\psi$, then

$$(H_0 + x^2) V_2(t)\psi = V_2(t)\varphi + [H_0, V_2(t)]\psi$$

is continuous in $L^2(\mathbb{R}^\nu)$ by the assumptions. Thus $V_2(t)$ and consequently $\tilde{V}(t)$ is strongly continuous in Y. The sum in (5.5) converges on Y as well. This yields (5.6). □

As an application of this theorem we will examine the case

$$[V_2(t)](x) = \sum_{j=1}^{N} q_j v_2(x - y_j(t)), \tag{5.7}$$

so that the full Hamiltonian H(t) has the form (2.2). The following lemma completes the proof of the main theorem of Section 2.

LEMMA. — Suppose $v_2 : \mathbb{R}^\nu \to \mathbb{R}$ is bounded and

$$\frac{\partial}{\partial x_j} v_2 \in L^\infty(\mathbb{R}^\nu) \qquad j = 1, \ldots, \nu$$

$$\Delta v_2 \in L^p(\mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu) \text{ for some } p \in [2, \infty) \cap \left(\frac{\nu}{2}, \infty\right).$$

(The derivations are taken in the sence of distributions). Then $V_2(t)$ defined in (5.7) satisfies the conditions *i)-iii)* of the preceding theorem.

Proof. — Fix some $\psi \in D(H_0)$, i. e. $\psi, \Delta\psi \in L^2(\mathbb{R}^\nu)$ (in the sence of distributions). Then $v_2\psi \in L^2(\mathbb{R}^\nu)$ and we can consider the distribution

$$\Delta(v_2\psi) = (\Delta v_2)\psi + 2\sum_{j=1}^{\nu}\left(\frac{\partial}{\partial x_j}v_2\right)\left(\frac{\partial}{\partial x_j}\psi\right) + v_2(\Delta\psi). \tag{5.8}$$

By the assumptions $\Delta(v_2\psi) \in L^2(\mathbb{R}^\nu)$ if and only if $(\Delta v_2)\psi \in L^2(\mathbb{R}^\nu)$; moreover there exist two functions $u \in L^p(\mathbb{R}^\nu)$ and $w \in L^\infty(\mathbb{R}^\nu)$ with $u + w = \Delta v_2$. Since $\psi \in D(H_0)$, by Theorem IX 28 in [*9*], $\psi \in L^q(\mathbb{R}^\nu) \cap L^2(\mathbb{R}^\nu)$ for $q = \dfrac{2p}{p-2}$ $\left(\Rightarrow q \in \left[2, \dfrac{2\nu}{\nu - 4}\right) \text{ for } \nu \geq 4 \text{ and } q = \infty \text{ only if } \nu \leq 3\right)$. Therefore the Hölder inequality (Theorem III 1 in [*8*]) shows that $u\psi \in L^2(\mathbb{R}^\nu)$ and

$$\| (\Delta v_2)\psi \|_2 \leq \| u \|_p \| \psi \|_q + \| w \|_\infty \| \psi \|_2 .$$

So v_2 as a multiplication operator maps $D(H_0)$ into $D(H_0)$. Regarding

$$V_2(t) = \sum_{j=1}^{N} q_j \exp\{ - i y_j(t) P \} v_2 \exp\{ i y_j(t) P \} \tag{5.9}$$

we get the same for $V_2(t)$ and have proved *i*). With the representation (5.9) of $V_2(t)$ we get immediately also the condition *ii*). For the condition *iii*) we compute with (5.8) that $[H_0, v_2] = -u - w - 2i \sum_{k=1}^{\nu} \left(\frac{\partial}{\partial x_k} v_2 \right) P_k$ and therefore

$$
\begin{aligned}
[H_0, V_2(t)]\,\psi &= \sum_{j=1}^{N} q_j \exp\{-iy_j(t)P\}\,[H_0, v_2]\exp\{iy_j(t)P\}\,\psi \\
&= -\sum_{j=1}^{N} q_j \exp\{-iy_j(t)P\}u(H_0+i)^{-1}\exp\{iy_j(t)P\}(H_0+i)\psi \\
&\quad -\sum_{j=1}^{N} q_j \exp\{-iy_j(t)P\}\,w\exp\{iy_j(t)P\}\,\psi \\
&\quad -2i\sum_{j=1}^{N}\sum_{k=1}^{\nu} q_j \exp\{-iy_j(t)\,P\}\left(\frac{\partial}{\partial x_k} v_2\right)\exp\{iy_j(t)P\}\,P_k\psi\,.
\end{aligned}
$$

Since $(H_0 + i)^{-1}$ maps $L^2(\mathbb{R}^\nu)$ into $L^q(\mathbb{R}^\nu)$ and the multiplication operator with the function u maps $L^q(\mathbb{R}^\nu)$ into $L^2(\mathbb{R}^\nu)$, the operator $u(H_0 + i)^{-1}$ is bounded in $L^2(\mathbb{R}^\nu)$ and thus $[H_0, V_2(t)]\,\psi$ is continuous. □

We remark that in the proof of this lemma we do not need any restrictions for the trajectories $y_j(t)$ except for their continuity. It is obviously sufficient, if $v_2 \in C^2(\mathbb{R}^\nu, \mathbb{R})$, but we have chosen this weaker condition to include

$$v_2(x) = f(|\,x\,|)\,, \tag{5.10}$$

where $f \in C^2((0, \infty), \mathbb{R})$ with f, f' and f'' bounded on $(0, \infty)$. For $\nu \geq 3$ one can show that the derivatives of v_2 as defined in (5.10) meet the requirements of the lemma. Therefore we also have proven the existence of the time evolution in the case of the Yukawa potential

$$v(x) = \frac{e^{-k|x|}}{|\,x\,|} = \frac{1}{|\,x\,|} + \frac{e^{-k|x|} - 1}{|\,x\,|}\,.$$

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